J. Math. Soc. Japan
Vol. 45, No. 3, 1993

Characteristic Cauchy problems for some non-Fuchsian partial differential operators

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(Received Dec. 2, 1991)
(Revised July 30, 1992)

§ 0. Introduction.

We consider the characteristic Cauchy problems for some non-Fuchsian partial differential operators with real-analytic coefficients. We give some theorems that are similar to the Cauchy-Kovalevskaya theorem and the Holmgren theorem. As a corollary, we obtain some results on the non-existence of null-solutions.

First, we give the definition of (essentially) Fuchsian operators and null-solutions. Consider a partial differential operator

$$P=t^{k}\partial_{t}^{m}+\sum_{j+|\alpha|\leq m,\, j<m}a_{j,\alpha}(t, x)\partial_{t}^{j}\partial_{x}^{\alpha},$$

where k is a non-negative integer and $a_{j,\alpha}$ are smooth (that is, C^{∞} or holomorphic etc.) in a neighborhood of $(0,0)\in \boldsymbol{R}_{t}\times\boldsymbol{R}_{x}^{n}$.

DEFINITION 0.1. (1) A partial differential operator P is called *Fuchsian* (with respect to the hypersurface $\Sigma:=\{(t, x)\,;\,t=0\}$), if P is written in the form

$$P=t^{k}\partial_{t}^{m}+a_{m-1}(x)t^{k-1}\partial_{t}^{m-1}+\cdots+a_{m-k}(x)\partial_{t}^{m-k}$$
$$+\sum_{p<m}\sum_{|\beta|\leq m-p}t^{\alpha(p,\beta)}\partial_{t}^{p}a_{p,\beta}(t, x)\partial_{x}^{\beta}$$

with $\alpha(p, \beta)=\max(0, k+p-m+1)$, where $0\leqq k\leqq m$ and a_{j}, $a_{p,\beta}$ are smooth ([**1**]). If P can be written as $P=t^{h}Q$, where Q is Fuchsian and h is an integer, then P is called *essentially Fuchsian* ([8]). Also, an operator P is called *non-Fuchsian* (resp. *essentially non-Fuchsian*), if P is not Fuchsian (resp. not essentially Fuchsian).

(2) A distribution u in a neighborhood of $(0,0)$ is called a *null-solution* for P at $(0,0)$ with respect to Σ (or rather $\Sigma_{+}=\{t>0\}$), if $Pu=0$ in a neighborhood of $(0,0)$ and $(0,0)\in\operatorname{supp}u\subset\{t\geqq 0\}$.

After a pioneering study by Y. Hasegawa ([4]), M.S. Baouendi and C. Goulaouic ([**1**]) defined Fuchsian partial differential operators, and proved some generalizations of the classical Cauchy-Kovalevskaya theorem and the Holmgren uniqueness theorem for Fuchsian operators. Especially, they proved that if P is a Fuchsian operator with real-analytic coefficients, then there exist no sufficiently smooth null-solutions. On the other hand, for operators whose principal parts are essentially non-Fuchsian, there are many results on the existence of C^∞ null-solutions. (We refer only [12] and [9]. See the references of these papers.)

In [8], the author considered essentially non-Fuchsian operators whose principal parts are essentially Fuchsian under the assumption that the coefficients depend only on t. According to this result, there are some essentially non-Fuchsian operators that have no sufficiently smooth null-solutions. This suggests that there may exist a class of non-Fuchsian operators that have similar properties to Fuchsian operators.

In this article, we consider the characteristic Cauchy problems for some essentially non-Fuchsian operators whose principal parts are essentially Fuchsian. Considering functions that are of C^∞ class with respect to the variable t and holomorphic with respect to x, we give some theorems that are similar to the Cauchy-Kovalevskaya theorem and the Holmgren uniqueness theorem (Theorems 1.5 and 1.6). We also get some results on the non-existence of null-solutions (Theorems 1.7 and 1.8).

NOTATION.

(i) The set of all integers (resp. nonnegative integers) is denoted by $\boldsymbol{Z}$ (resp. $\boldsymbol{N}$). For a real number a, put $[a]:=\max\{n\in\boldsymbol{Z}\,;\, n\leqq a\}$ (Gauss's symbol).

(ii) Put $\vartheta:=t\partial_t$ and $\tilde{\vartheta}:=\partial_t t=\vartheta+1$.

(iii) For a bounded domain Ω in $\boldsymbol{C}^n$, we denote by $\mathcal{O}(\Omega)$ the set of all holomorphic functions on Ω. This is a Fréchet space with the topology of uniform convergence on the compact sets. Put $E(\Omega):=\{\varphi\in C^0(\bar{\Omega})\,;\,\varphi$ is holomorphic in $\Omega\}$. This is a Banach space with the supremum norm. The closure of the entire functions in $E(\Omega)$ is denoted by $F(\Omega)$. The dual space of $E(\Omega)$ (resp. $F(\Omega)$) is denoted by $E'(\Omega)$ (resp. $F'(\Omega)$).

(iv) For two locally convex topological vector spaces X and Y, we denote by $\mathcal{L}(X, Y)$ the space of all continuous linear operators from X to Y, endowed with the topology of uniform convergence on the bounded sets. Put $\mathcal{L}(X):=\mathcal{L}(X, X)$.

§ 1. Statements of the main results.

Let Ω be a bounded domain in $\boldsymbol{C}^n$ that contains the origin 0, and let T be a positive real number. Consider a linear partial differential operator

(1.1)
$$P=\sum_{j+|\alpha|\leqq m} a_{j,\alpha}(t, x)\partial_t^j\partial_x^\alpha ,$$

where $a_{j,\alpha}\in C^\infty([-T, T]; \mathcal{O}(\Omega))$ and $a_{m,0}(t, x)\equiv t^\kappa(\kappa\in \boldsymbol{N})$.

Let $r(j, \alpha)$ be the vanishing order of $a_{j,\alpha}$ on Σ, that is

(1.2)
$$r(j, \alpha):=\sup\{r\in \boldsymbol{N};\ t^{-r}a_{j,\alpha}\in C^\infty([-T, T]; \mathcal{O}(\Omega))\},$$

and put

(1.3)
$$\tilde{a}_{j,\alpha}(t, x):=t^{-r(j,\alpha)}a_{j,\alpha}(t, x).$$

(If $r(j, \alpha)=\infty$, then put $\tilde{a}_{j,\alpha}(t, x)\equiv 0$.) Note that $\tilde{a}_{j,\alpha}\in C^\infty([-T, T]; \mathcal{O}(\Omega))$.

Associating a weight $\omega(j, \alpha):=r(j, \alpha)-j$ to each differential monomial $a_{j,\alpha}(t, x)\partial_t^j\partial_x^\alpha$, we draw a Newton polygon $\Delta(P)$ using the points $(j+|\alpha|, \omega(j, \alpha))$ $(j+|\alpha|\leqq m)$ in (u, v)-plane as follows.

DEFINITION 1.1. (1) Put

$$\Delta(P):=ch\Big(\bigcup_{j+|\alpha|\leqq m}\{(u, v)\in \boldsymbol{R}^2;\ u\leqq j+|\alpha|,\ v\geqq\omega(j, \alpha)\}\Big),$$

where $ch(A)$ denotes the convex hull of A. This is called the *Newton polygon* of P ([**6**], [**13**], [**11**]). (Cf. Definition 2.1.) This is different from the Newton polygon used in [8].

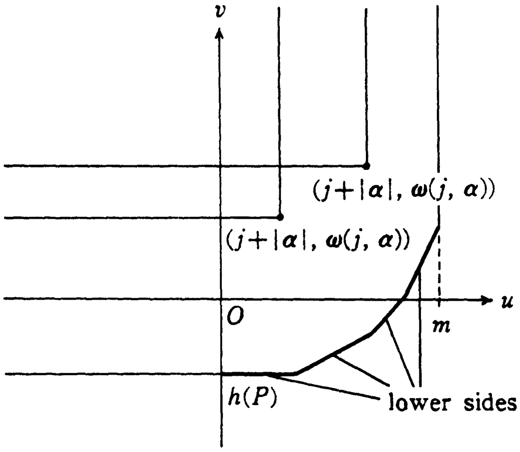

Figure 1. $\Delta(P)$.

(2) Put

$$\hat{V}:=\{(j, \alpha)\in \boldsymbol{N}\times \boldsymbol{N}^n;\ (j+|\alpha|, \omega(j, \alpha)) \text{ is a vertex of } \Delta(P)\}.$$

(3) Put

$$h(P):=\min\{\omega(j, \alpha)\in \boldsymbol{R};\ j+|\alpha|\leqq m\}$$
$$=\min\{v\in \boldsymbol{R};\ (u, v)\in \Delta(P) \text{ for some } u\in \boldsymbol{N}\}.$$

This is the lowest weight of P and called the *height* of $\Delta(P)$.

(4) The boundary of $\Delta(P)\cap([0, \infty)\times \boldsymbol{R})$ is the union of two vertical half-lines and a finite number of compact line segments with distinct slopes. Each of these compact line segments is called a *lower side* of $\Delta(P)$. The set of the slopes of the lower sides of $\Delta(P)$ is denoted by S. For $\mu\in S$, the lower side of $\Delta(P)$ with slope μ is denoted by L_μ. Put

$$I_\mu := \{(j, \alpha)\in \boldsymbol{N}\times \boldsymbol{N}^n ; (j+|\alpha|, \omega(j, \alpha))\in L_\mu\}.$$

(5) For $\mu\in S$, $\mu>0$, we put

$$C_\mu(\lambda ; x) := \sum_{(j,0)\in I_\mu} \tilde{a}_{j,0}(0, x)\lambda^j .$$

If $0\in S$, then we put

$$C_0(\lambda ; x) := \sum_{(j,0)\in I_0} \tilde{a}_{j,0}(0, x)\lambda(\lambda-1)\cdots(\lambda-j+1).$$

If $0\notin S$, then we put $C_0(\lambda ; x)=\tilde{a}_{0,0}(0, x)$. The polynomial C_0 is called the *indicial polynomial* of P. (M. S. Baouendi and C. Goulaouic ([**1**]) called this polynomial "the characteristic polynomial associated with P", in the case of Fuchsian operators.)

(6) If $0\in S$, then we put

$$\Lambda(P) := \sup\{\operatorname{Re}\lambda\in \boldsymbol{R} ; C_0(\lambda ; x)=0 \text{ for some } x\in \Omega\}.$$

If $0\notin S$, then we put $\Lambda(P)=-\infty$.

By the use of these notions, Fuchsian operators are characterized as follows.

PROPOSITION 1.2. *The operator P is Fuchsian if and only if $h(P)\leqq 0$, $S=\{0\}$, and there exist no $(j, \alpha)\in I_0$ such that $\alpha\neq 0$.*

Now, we assume the following conditions.

(A-1) For all $\mu\in S$, there exist no $(j, \alpha)\in I_\mu$ such that $\alpha\neq 0$.

(A-2) If $(j, 0)\in \hat{V}$, then $\tilde{a}_{j,0}(0, 0)\neq 0$.

(A-3) If $\mu\in S$ and $\mu>0$, then all non-zero roots λ of $C_\mu(\lambda ; 0)=0$ satisfy $\operatorname{Re}\lambda<0$.

(A-4) $h(P)\leqq 0$.

(A-5) $C_0(\lambda ; 0)\neq 0$ for $\lambda=|h(P)|, |h(P)|+1, |h(P)|+2, \cdots$.

REMARK 1.3. (1) The assumption (A-1) implies that the principal part P_m of P is essentially Fuchsian.

(2) Note that if $(j, 0)\in \hat{V}$, then $\tilde{a}_{j,0}(0, x)\not\equiv 0$ under the assumption (A-1). Thus, the condition (A-2) is a kind of non-degeneracy at $x=0$. For an arbitrarily fixed $y\in \Omega$, put

$$P_y := \sum_{j+|\alpha|\leq m} a_{j,\alpha}(t, y)\partial_t^j\partial_x^\alpha ,$$

which is a differential operator with coefficients depending only on t. Then, the condition (A-2) is equivalent to that $\Delta(P_y)=\Delta(P)$ for any y in a neighborhood of (0, 0).

(3) The condition (A-4) implies that $C_0(\lambda; x)\equiv 0$ for $\lambda=0, 1, \cdots, |h(P)|-1$. In fact, if $j<|h(P)|$, then $\omega(j, 0)=r(j, 0)-j>-|h(P)|=h(P)$ and hence $(j, 0) \notin I_0$. Further, if $|h(P)|\geqq 1$ then $\Lambda(P)\geqq|h(P)|-1$.

(4) If $0\notin S$, then $h(P)\geqq 0$ and hence (A-4) means that $h(P)=0$. Further, (A-5) follows from (A-2).

(5) We do not exclude Fuchsian operators. Fuchsian operators always satisfy the conditions (A-1)-(A-4).

EXAMPLE 1.4. Consider an operator

$$P=t^\kappa\partial_t^2-t^\nu\partial_x^2+at^p\partial_t+b ,$$

on $\boldsymbol{R}^2$, where $\kappa, \nu, p\in\boldsymbol{N}$, $a, b\in\boldsymbol{C}$. We assume that $\kappa>2p$, $p\geqq 1$, and $b\neq 0$. We have

$$\text{(A-1)} \Longleftrightarrow \nu>\kappa-2 .$$

Under this condition (A-1), there holds that $S=\{\kappa-p-1, p-1\}$. The conditions (A-2) and (A-4) are always satisfied. (Note that $h(P)=0$.) Further, we have

$$\text{(A-3)} \Longleftrightarrow \operatorname{Re} a>0 \text{ and if } p>1 \text{ then } \operatorname{Re}(b/a)>0 ,$$

$$\text{(A-5)} \Longleftrightarrow \text{If } p=1 \text{ then } b/a\notin\{0, -1, -2, \cdots\}.$$

Under the above assumptions, we have the following two theorems, which are the main results of this article.

THEOREM 1.5. *Assume the conditions* (A-1)-(A-5). *Then, there exist a positive* T_0 *and a neighborhood* Ω_0 *of* 0 *in* $\boldsymbol{C}^n$ *for which the following holds:*

For any $f\in C^\infty([0, T]; \mathcal{O}(\Omega))$ *and any* $g_j\in\mathcal{O}(\Omega)$ $(0\leqq j\leqq|h(P)|-1)$, *there exists a unique* $u\in C^\infty([0, T_0]; \mathcal{O}(\Omega_0))$ *satisfying*

$$\text{(CP)} \quad \begin{cases} Pu=f(t, x) & \text{in } [0, T_0]\times\Omega_0 , \\ \partial_t^j u|_{t=0}=g_j(x) & \text{on } \Omega_0 \ (0\leqq j\leqq|h(P)|-1). \end{cases}$$

THEOREM 1.6. *Assume the conditions* (A-1)-(A-5). *Let* $L\in\boldsymbol{N}$ *and* $L>\Lambda(P)$. *If* $u\in C^L([0, T]; \mathcal{D}'(\Omega\cap\boldsymbol{R}^n))$, $Pu=0$ *for* $t>0$ *in a neighborhood of* (0, 0), *and* $\partial_t^j u|_{t=0}=0$ $(0\leqq j\leqq|h(P)|-1)$ *in a neighborhood of* 0, *then* $u=0$ *for* $t>0$ *in a neighborhood of* (0, 0).

Immediately from Theorem 1.6, we have the non-existence of sufficiently smooth null-solutions. We also have the following stronger results on the non-existence of null-solutions, assuming only the conditions (A-1)-(A-3).

THEOREM 1.7. *Assume the conditions* (A-1)-(A-3). *If* $N\in\boldsymbol{N}$ *and* $N>\Lambda(P)$, *then there exist no* C^{N} *null-solutions for* P *at* (0, 0).

THEOREM 1.8. *Assume the conditions* (A-1)-(A-3). *If* $0\notin S$, *then any* $\mathcal{D}'$ *null-solution for* P *at* (0, 0) *has a support included in the initial surface* $\Sigma=\{t=0\}$ *in a neighborhood of* (0, 0). *Further, if* $h(P)=0$ *in addition, then there exist no* $\mathcal{D}'$ *null-solutions for* P *at* (0, 0).

REMARK 1.9. (1) As is well-known, if P is non-Fuchsian, then the solution u in Theorem 1.5 is not necessarily holomorphic in (t, x) even if f, g_j, and the coefficients of P are all holomorphic in (t, x). Also, we cannot replace $[0, T]$ and $[0, T_0]$ with $[-T, T]$ and $[-T_0, T_0]$.

(2) Consider the following condition, which is "opposite" to (A-3):

(A-3)# There exist $\mu\in S$ and $\lambda\in\boldsymbol{C}$ such that $\mu>0$, $\operatorname{Re}\lambda>0$, and $C_{\mu}(\lambda\,;0)=0$.

The author believes that the following holds:

CONJECTURE. *If* P *satisfies the conditions* (A-1), (A-2), *and* (A-3)#, *then there exists a* C^{∞} *null-solution for* P *at* (0, 0).

If the coefficients $a_{j,\alpha}$ of P are independent of x, then this conjecture follows from Theorem A (1) in [8].

The assumptions (A-4) and (A-5) have a different nature from (A-1)-(A-3). In Theorem 1.5, the assumptions (A-4) and (A-5) are used only to show that the formal Taylor expansion of the solution u with respect to t is uniquely determined from f and g_j, as we shall see in Section 4. In other words, the Cauchy problem (CP) is reduced to the *flat Cauchy problem* by the use of (A-4) and (A-5), and the flat Cauchy problem is uniquely solvable under the assumptions (A-1)-(A-3). As for Theorem 1.6, the situation is similar.

In Section 4, we shall give some extended results to the reduced problems under the assumptions (A-1)-(A-3) (Theorems 4.3, 4.4, 4.6). From these extended results, the four theorems above easily follow.

In order to give these extended results, we need to consider generalized Newton polygons and some function spaces associated with generalized Newton polygons. These are given in Sections 2 and 3. Our assumptions make it possible to treat the operator P as a kind of perturbation of an ordinary differential operator. In Sections 5 and 6, we investigate ordinary differential operators. Using the results in Section 6, we prove Theorem 4.4 in Section 7. Theorems 4.3 and 4.6 are proved by the use of scales of Banach spaces. In Section 8, we give the unique solvability of abstract equations in scales of Banach spaces.

Finally, we prove Theorems 4.3 and 4.6 in Section 9.

§ 2. Additive group of generalized Newton polygons.

In this section, we define an additive group extended from the Newton polygons and show some basic properties of this group. We begin with a definition of Newton polygons.

DEFINITION 2.1. (1) For a point $(a, b) \in \boldsymbol{N} \times \boldsymbol{R}$, put $\Delta(a, b) := (-\infty, a] \times [b, \infty) \subset \boldsymbol{R}^2$.

(2) A subset Δ of $\boldsymbol{R}^2$ is called a *Newton polygon*, if $\Delta = ch(\bigcup_{j=1}^{M} \Delta(a_j, b_j))$ for a finite number of points $(a_j, b_j) \in \boldsymbol{N} \times \boldsymbol{R}$ $(j=1, \cdots, M)$. Note that $\Delta(a, b)$ is also a Newton polygon.

(3) For a Newton polygon Δ, put $s = s(\Delta) := \max\{u \in \boldsymbol{N}; (u, v) \in \Delta$ for some $v \in \boldsymbol{R}\}$ and $h = h(\Delta) := \min\{v \in \boldsymbol{R}; (u, v) \in \Delta$ for some $u \in \boldsymbol{N}\}$. We call $s(\Delta)$ the *size* of Δ and $h(\Delta)$ the *height* of Δ. Further, put $L(j) = L_\Delta(j) := \min\{v \in \boldsymbol{R}; (j, v) \in \Delta\}$ for $j = 0, 1, \cdots, s(\Delta)$. This is called the *side function* of Δ. A Newton polygon Δ is uniquely determined by its size $s(\Delta)$ and side function L_Δ. Note that $\Delta = ch(\bigcup_{j=0}^{s(\Delta)} \Delta(j, L_\Delta(j)))$ and that $h(\Delta) = L_\Delta(0)$.

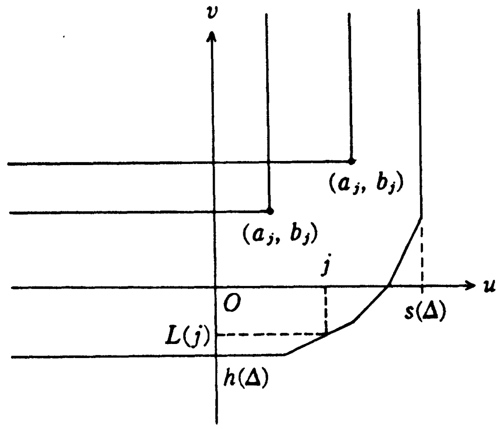

Figure 2. Newton polygon.

REMARK 2.2. Let L be the side function of a Newton polygon Δ with size s. Put $\kappa_j := L(j) - L(j-1)$ $(j=1, \cdots, s)$. Then, there hold

$$0 \leqq \kappa_1 \leqq \cdots \leqq \kappa_s, \tag{2.1}$$

$$L(j) = h + \sum_{l=1}^{j} \kappa_l \quad (j=0, 1, \cdots, s), \qquad \text{where } h = h(\Delta) = L(0). \tag{2.2}$$

Conversely, let $h \in \boldsymbol{R}$, $s \in \boldsymbol{N}$ and let $\kappa_1, \cdots, \kappa_s$ satisfy (2.1). If we define a function L on $\{0, 1, \cdots, s\}$ by (2.2), then there exists a Newton polygon Δ with size s and side function L.

Next, we define an addition of Newton polygons.

DEFINITION 2.3. For two subsets D_1 and D_2 of $\boldsymbol{R}^2$, put

$$D_1+D_2 := \{(u_1+u_2,\ v_1+v_2) \in \boldsymbol{R}^2\,;\ (u_i,\ v_i) \in D_i\ \ (i=1,\ 2)\}. \tag{2.3}$$

PROPOSITION 2.4. (Cf. [**10**, Definition 3.1].) *For $i=1, 2$, let Δ_i be a Newton polygon with size s_i and side function L_i. Put $L_i(j)=\infty$ for $j>s_i$ $(i=1, 2)$. Then, the sum $\Delta_1+\Delta_2$ is a Newton polygon with size s_1+s_2 and side function $L_1\oplus L_2$, where*

$$(L_1\oplus L_2)(j) := \min\{L_1(k)+L_2(j-k)\,;\ k\in\boldsymbol{N},\ 0\leqq k\leqq j\} \tag{2.4}$$

$$\text{for } j\in\boldsymbol{N},\ 0\leqq j\leqq s_1+s_2\,.$$

The unit element of this addition for Newton polygons is $\Delta(0, 0)=(-\infty, 0]\times[0, \infty)$.

PROOF. Put $\kappa_j^{(i)} := L_i(j)-L_i(j-1)$ $(1\leqq j\leqq s_i\,;\ i=1, 2)$. Reorder $\kappa_1^{(1)}, \cdots, \kappa_{s_1}^{(1)}, \kappa_1^{(2)}, \cdots, \kappa_{s_2}^{(2)}$ as $0\leqq\kappa_1\leqq\cdots\leqq\kappa_{s_1+s_2}$. It is easy to show that

$$(L_1\oplus L_2)(j) = L_1(0)+L_2(0)+\sum_{l=1}^{j}\kappa_l \quad (0\leqq j\leqq s_1+s_2)\,. \tag{2.5}$$

Thus, there exists a Newton polygon Δ with size s_1+s_2 and side function $L_1\oplus L_2$, by Remark 2.2.

First, we show $\Delta\subset\Delta_1+\Delta_2$. Take an arbitrary $(u, v)\in\Delta$. If $u\leqq 0$, then there holds trivially that $(u, v)\in\Delta_1+\Delta_2$. Assume that $u\geqq 0$. Since $\Delta=ch(\Delta\cap(\boldsymbol{Z}\times\boldsymbol{R}))$ and since $\Delta_1+\Delta_2$ is convex, we may assume that $u\in\boldsymbol{N}$. We have $0\leqq u\leqq s_1+s_2$, $v\geqq(L_1\oplus L_2)(u)=L_1(k)+L_2(u-k)$ for some k by (2.4). Put $v_1=L_1(k)$, $v_2=v-v_1\geqq L_2(u-k)$. We have $(k, v_1)\in\Delta_1$, $(u-k, v_2)\in\Delta_2$. Hence, we have $(u, v)\in\Delta_1+\Delta_2$.

Next, we show $\Delta_1+\Delta_2\subset\Delta$. Take an arbitrary $(u, v)\in\Delta_1+\Delta_2$. If $u\leqq 0$, then there holds trivially that $(u, v)\in\Delta$. Assume that $u\geqq 0$. We can write as $(u, v)=(u_1+u_2, v_1+v_2)$, where $\pi_i:=(u_i, v_i)\in\Delta_i$ $(i=1, 2)$. Since Δ_i is a Newton polygon, there exist $\pi_i^{(\pm)}\in\Delta_i\cap(\boldsymbol{N}\times\boldsymbol{R})$ such that the point π_i belongs to the compact line segment connecting $\pi_i^{(+)}$ and $\pi_i^{(-)}$. The point $\pi_1+\pi_2$ belongs to the closed parallelogram with the vertices $\pi_i^{(\pm)}$ $(i=1, 2)$. Since Δ is convex, if these four vertices belong to Δ, then $\pi_1+\pi_2$ also belongs to Δ. Thus, we have only to show that $(u, v)\in\Delta$ under the assumption that $u_i\in\boldsymbol{N}$ $(i=1, 2)$. In this case, since $0\leqq u_i\leqq s_i$ and $v_i\geqq L_i(u_i)$ $(i=1, 2)$, we have $v=v_1+v_2\geqq L_1(u_1)+L_2(u_2)\geqq(L_1\oplus L_2)(u_1+u_2)=(L_1\oplus L_2)(u)$. Thus, $(u, v)\in\Delta$. ■

The following two propositions give basic meanings of this addition.

PROPOSITION 2.5. *Let Δ be a Newton polygon with size s and side function L. Put $k_j := L(j)-L(j-1)$ for $1\leqq j\leqq s$. For a non-negative real number k, let $\Delta[k]$ be a Newton polygon with size 1 and side function $L[k]$, where $L[k](0)=0$ and $L[k](1)=k$. Then, there holds $\Delta=\Delta(0, h)+\Delta[k_1]+\cdots+\Delta[k_s]$, where h is the height of Δ.*

PROOF. It is trivial that if $s=0$ then $\Delta=\Delta(0, h)$ for some $h\in\boldsymbol{R}$. Let $s\geqq 1$. Put $L'(j):=L(j)$ $(0\leqq j\leqq s-1)$ and let Δ' be the Newton polygon with size $s-1$ and side function L'. We have $(L'\oplus L[k_s])(0)=L'(0)=L(0)$. For $1\leqq j\leqq s-1$, we have $(L'\oplus L[k_s])(j)=\min\{L'(j), L'(j-1)+k_s\}=\min\{L(j), L(j-1)+k_s\}=L(j)$, since $L(j)-L(j-1)\leqq L(s)-L(s-1)=k_s$. We also have $(L'\oplus L[k_s])(s)=L'(s-1)+k_s=L(s-1)+k_s=L(s)$. Thus, we obtain $L=L'\oplus L[k_s]$, and hence $\Delta=\Delta'+\Delta[k_s]$. By iterating this argument, we get the desired result. ■

PROPOSITION 2.6. (Cf. [7].) *If P_1 and P_2 are differential operators of the form* (1.1), *then there holds $\Delta(P_1P_2)=\Delta(P_1)+\Delta(P_2)$.*

PROOF. Let D be a subset of $\boldsymbol{R}^2$ satisfying that if $(u, v)\in D$ and $v'\geqq v$, then $(u, v')\in D$. For a partial differential operator R, we say $R=O(D)$ if R can be written in the form:

$$R=\sum_{j+|\alpha|\leqq m} t^{v(j,\alpha)}\tilde{a}_{j,\alpha}(t, x)t^j\partial_t^j\partial_x^\alpha, \tag{2.6}$$

$$(j+|\alpha|, v(j, \alpha))\in D,\ \tilde{a}_{j,\alpha}\in C^\infty([-T, T]; \mathcal{O}(\Omega))\quad (j+|\alpha|\leqq m).$$

It is easy to show that

$$\begin{aligned}(2.7)\qquad &(t^{v_1}\tilde{a}_1(t, x)t^{j_1}\partial_t^{j_1}\partial_x^{\alpha_1})(t^{v_2}\tilde{a}_2(t, x)t^{j_2}\partial_t^{j_2}\partial_x^{\alpha_2})\\ &=t^{v_1+v_2}\tilde{a}_1(t, x)\tilde{a}_2(t, x)t^{j_1+j_2}\partial_t^{j_1+j_2}\partial_x^{\alpha_1+\alpha_2}\\ &\quad+O(\Delta(j_1+j_2+|\alpha_1+\alpha_2|, v_1+v_2)\setminus\{(j_1+j_2+|\alpha_1+\alpha_2|, v_1+v_2)\})\\ &=O(\Delta(j_1+j_2+|\alpha_1+\alpha_2|, v_1+v_2)).\end{aligned}$$

Thus, it is almost trivial that $\Delta(P_1P_2)\subset\Delta(P_1)+\Delta(P_2)$.

In order to show $\supset$ part, we have only to show that all the vertices of $\Delta(P_1)+\Delta(P_2)$ belong to $\Delta(P_1P_2)$.

LEMMA 2.7. *If (u, v) is a vertex of $\Delta(P_1)+\Delta(P_2)$, then there exist unique $(u_i, v_i)\in\Delta(P_i)$ $(i=1, 2)$ such that $(u, v)=(u_1, v_1)+(u_2, v_2)$. Further, (u_i, v_i) is a vertex of $\Delta(P_i)$ $(i=1, 2)$.*

PROOF. Assume that (u_i, v_i), $(u_i', v_i')\in\Delta(P_i)$ $(i=1, 2)$ and $(u, v)=(u_1, v_1)+(u_2, v_2)=(u_1', v_1')+(u_2', v_2')$. We can represent $u_1'=u_1+\rho$, $u_2'=u_2-\rho$, $v_1'=v_1+\sigma$, $v_2'=v_2-\sigma$. Hence, there holds $(u\pm\rho, v\pm\sigma)\in\Delta(P_1)+\Delta(P_2)$. Since (u, v) is a vertex of $\Delta(P_1)+\Delta(P_2)$, this implies $\rho=\sigma=0$. Thus, (u_i, v_i) are unique.

Assume that (u_1, v_1) is not a vertex of $\Delta(P_1)$. There exist $\rho\neq 0$ and σ such that $(u_1\pm\rho, v_1\pm\sigma)\in\Delta(P_1)$. Since $(u_2, v_2)\in\Delta(P_2)$, we have $(u\pm\rho, v\pm\sigma)\in\Delta(P_1)+\Delta(P_2)$. This contradicts the assumption that (u, v) is a vertex of $\Delta(P_1)+\Delta(P_2)$. Thus, (u_1, v_1) is a vertex of $\Delta(P_1)$.

Just the same way, we can show that (u_2, v_2) is a vertex of $\Delta(P_2)$. ■

Now, we return to the proof of Proposition 2.6. Assume that (u, v) is a vertex of $\Delta(P_1)+\Delta(P_2)$ and take (u_i, v_i) in the lemma above. We decompose the operator P_i as

$$(2.8)\qquad P_i=\sum_{j+|\alpha|=u_i} t^{v_i}\tilde{a}_{j,\alpha}^{(i)}(t, x)t^j\partial_t^j\partial_x^\alpha+\sum_{j+|\alpha|\neq u_i} a_{j,\alpha}^{(i)}(t, x)t^j\partial_t^j\partial_x^\alpha=P_i'+P_i''.$$

By (2.7), we have

$$(2.9)\qquad P_1'P_2'=t^v\sum_{j+|\alpha|=u}\Big(\sum_{j_1+|\alpha_1|=u_1}\tilde{a}_{j_1,\alpha_1}^{(1)}(t, x)\tilde{a}_{j-j_1,\alpha-\alpha_1}^{(2)}(t, x)\Big)t^j\partial_t^j\partial_x^\alpha$$
$$+O(\Delta(u, v)\setminus\{(u, v)\}).$$

On the other hand, from $P_i''\in O(\Delta(P_i)\setminus\{(u_i, v_i)\})$, we have

$$(2.10)\qquad P_1'P_2''+P_1''P_2'+P_1''P_2''\in O((\Delta(P_1)+\Delta(P_2))\setminus\{(u, v)\}).$$

Since (u_i, v_i) is a vertex of $\Delta(P_i)$, there exists (j_i, α_i) such that $j_i+|\alpha_i|=u_i$ and $\tilde{a}_{j_i\alpha_i}^{(i)}(0, x)\not\equiv 0$ $(i=1, 2)$. Hence, we have

$$(2.11)\qquad \sum_{j+|\alpha|=u}\Big(\sum_{j_1+|\alpha_1|=u_1}\tilde{a}_{j_1,\alpha_1}^{(1)}(0, x)\tilde{a}_{j-j_1,\alpha-\alpha_1}^{(2)}(0, x)\Big)\lambda^j\xi^\alpha$$
$$=\Big(\sum_{j_1+|\alpha_1|=u_1}\tilde{a}_{j_1,\alpha_1}^{(1)}(0, x)\lambda^{j_1}\xi^{\alpha_1}\Big)\Big(\sum_{j_2+|\alpha_2|=u_2}\tilde{a}_{j_2,\alpha_2}^{(2)}(0, x)\lambda^{j_2}\xi^{\alpha_2}\Big)$$
$$\not\equiv 0.$$

This implies that there exists (j, α) such that $j+|\alpha|=u$ and

$$\sum_{j_1+|\alpha_1|=u_1}\tilde{a}_{j_1,\alpha_1}^{(1)}(0, x)\tilde{a}_{j-j_1,\alpha-\alpha_1}^{(2)}(0, x)\not\equiv 0.$$

Thus, from $P_1P_2=P_1'P_2'+P_1'P_2''+P_1''P_2'+P_1''P_2''$, we have $(u, v)\in\Delta(P_1P_2)$. ■

REMARK 2.8. For a complete locally convex topological vector space X, put

$$(2.12)\qquad \mathcal{F}([0, T]; X):=\{v\in C^0([0, T]; X); v(s^q)\in C^\infty([0, T^{1/q}]; X)$$
$$\text{for some } q\in\boldsymbol{N}\setminus\{0\}\}.$$

We can extend Definition 1.1 for the operators whose coefficients belong to $\mathcal{F}([0, T]; \mathcal{O}(\Omega))$. For such operators, the proposition above is also valid.

Let $\mathfrak{N}$ be the set of all Newton polygons, which is a commutative semigroup with the addition above. By the following proposition, we can embed this additive semigroup $\mathfrak{N}$ in an additive group.

PROPOSITION 2.9. *Let Δ, Δ_1, Δ_2 be three Newton polygons. If $\Delta_1+\Delta\subset\Delta_2+\Delta$, then $\Delta_1\subset\Delta_2$. (The converse is trivially valid.) Especially, if $\Delta_1+\Delta=\Delta_2+\Delta$, then $\Delta_1=\Delta_2$. (This is called the cancellation law.)*

PROOF. By Proposition 2.5, we have only to give a proof in the case of $\Delta=\Delta(0, h)$ or $\Delta=\Delta[k]$. In the case of $\Delta(0, h)$, it is trivial.

Assume that $\Delta_1+\Delta[k]\subset\Delta_2+\Delta[k]$, where k is a nonnegative real number. Let s_i (resp. L_i) be the size (resp. side function) of Δ_i ($i=1, 2$). By $\Delta_1+\Delta[k]\subset\Delta_2+\Delta[k]$, we have $s_1+1\leqq s_2+1$, hence $s_1\leqq s_2$. Further, we have

$$(L_1\oplus L[k])(j)\geqq(L_2\oplus L[k])(j)\quad(0\leqq j\leqq s_1+1). \tag{2.13}$$

From this inequality for $j=0$, we have $L_1(0)\geqq L_2(0)$. Now, assume that $L_1(j)\geqq L_2(j)$ holds for $0\leqq j\leqq d$ ($0\leqq d\leqq s_1-1$). From (2.13) for $j=d+1$, we have $L_1(d+1)\geqq\min\{L_2(d+1), L_2(d)+k\}$. If $L_2(d+1)\leqq L_2(d)+k$, then we have $L_1(d+1)\geqq L_2(d+1)$. If $L_2(d+1)>L_2(d)+k$, then we obtain $L_2(d+2)>L_2(d+1)+k$ using (2.1). Hence, from (2.13) for $j=d+2$, we have $L_1(d+1)+k\geqq L_2(d+1)+k$. Thus, we have $L_1(d+1)\geqq L_2(d+1)$. By the induction, we get $L_1(j)\geqq L_2(j)$ for $0\leqq j\leqq s_1$. ■

DEFINITION 2.10. (1) By the cancellation law, the additive semigroup $\mathfrak{N}$ can be embedded in an additive group $\bar{\mathfrak{N}}$ so that the addition in $\mathfrak{N}$ is preserved in $\bar{\mathfrak{N}}$ and any $\bar{\Delta}\in\bar{\mathfrak{N}}$ is a difference of two elements of $\mathfrak{N}$: $\bar{\Delta}=\Delta-\Delta'$ (Δ, $\Delta'\in\mathfrak{N}$). This group $\bar{\mathfrak{N}}$ is uniquely determined by $\mathfrak{N}$ up to isomorphism. An element of $\bar{\mathfrak{N}}$ is called a *generalized Newton polygon*. For Δ_1, Δ_1', Δ_2, $\Delta_2'\in\mathfrak{N}$, there holds $\Delta_1-\Delta_1'=\Delta_2-\Delta_2'$ if and only if $\Delta_1+\Delta_2'=\Delta_2+\Delta_1'$.

(2) For $\bar{\Delta}=\Delta-\Delta'\in\bar{\mathfrak{N}}$ (Δ, $\Delta'\in\mathfrak{N}$), the integer $s(\bar{\Delta})=s(\Delta)-s(\Delta')$ is called the *size* of $\bar{\Delta}$ and the real number $h(\bar{\Delta})=h(\Delta)-h(\Delta')$ is called the *height* of $\bar{\Delta}$. It is almost trivial that these are well-defined.

(3) For two elements $\bar{\Delta}_1$, $\bar{\Delta}_2$ in $\bar{\mathfrak{N}}$, we say $\bar{\Delta}_1\subset\bar{\Delta}_2$, if $\bar{\Delta}_i=\Delta_i-\Delta_i'$ ($i=1, 2$) and $\Delta_1+\Delta_2'\subset\Delta_2+\Delta_1'$, where Δ_i, $\Delta_i'\in\mathfrak{N}$ ($i=1, 2$). It follows from Proposition 2.9 that this is a well-defined order relation in $\bar{\mathfrak{N}}$.

§3. Function spaces associated with generalized Newton polygons.

In this section, we define some function spaces associated with generalized Newton polygons and study some basic properties of these spaces. Let T be a positive number and X be a complete locally convex topological vector space.

All the spaces in this section are considered as subspaces of $\mathcal{D}'((0, T); X)$.

Thus, for example, "$u \in C^0([0, T]; X)$" means that $u \in C^0((0, T); X)$ and u is continuously extendable onto $[0, T]$.

DEFINITION 3.1. (1) Let Δ be a Newton polygon with size s and side function L. Put

$$F_T[\Delta : X] := \{u \in C^s((0, T]; X); t^{L(j)} \vartheta^j(u) \in C^0([0, T]; X) \ (0 \leqq j \leqq s)\}. \tag{3.1}$$

We can endow this space with a natural topology. Especially, if X is a Banach space, then this space is also a Banach space with the norm

$$\|u\|_{T, \Delta, X} := \sum_{j=0}^{s} \sup_{0 \leq t \leq T} \|t^{L(j)} \vartheta^j(u)(t)\|_X . \tag{3.2}$$

(2) Let Δ' be another Newton polygon with size s' and side function L'. Put

$$F_T[\Delta, \Delta' : X] := \{u \in \mathcal{D}'((0, T); X); u = \sum_{j=0}^{s'} t^{L'(j)} \vartheta^j(v_j) \quad \text{for some } v_j \in F_T[\Delta : X] \ (0 \leqq j \leqq s')\}. \tag{3.3}$$

Obviously, we have $F_T[\Delta, \Delta(0, 0) : X] = F_T[\Delta : X]$. When X is a Banach space, we define a norm $\|\cdot\|_{T, \Delta, \Delta', X}$ by

$$\|u\|_{T, \Delta, \Delta', X} := \inf \left\{ \sum_{j=0}^{s'} \|v_j\|_{T, \Delta, X} ; v_j \in F_T[\Delta : X] \ (0 \leqq j \leqq s'),\ u = \sum_{j=0}^{s'} t^{L'(j)} \vartheta^j(v_j) \right\}.$$

(When X is not a Banach space, we do not consider any topology for simplicity, though we can.)

LEMMA 3.2. *Assume that* $\Delta_1, \Delta_1', \Delta_2, \Delta_2' \in \mathfrak{N}$ *and* $\bar{\Delta} := \Delta_1 - \Delta_1' = \Delta_2 - \Delta_2'$.

(1) *There holds that* $F_T[\Delta_1, \Delta_1' : X] = F_T[\Delta_2, \Delta_2' : X]$. *Thus, we can define*

$$F_T[\bar{\Delta} : X] := F_T[\Delta_1, \Delta_1' : X]. \tag{3.4}$$

(2) *If X is a Banach space, then the norms* $\|u\|_{T, \Delta_1, \Delta_1', X}$ *and* $\|u\|_{T, \Delta_2, \Delta_2', X}$ *in the space* $F_T[\bar{\Delta} : X]$ *are equivalent to each other. Further, the space* $F_T[\bar{\Delta} : X]$ *is a Banach space with these equivalent norms.*

To prove this lemma, we use a result on ordinary differential operators. Hence, the proof is delayed until Section 5.

EXAMPLE 3.3. (1) There holds $C^0([0, T]; X) = F_T[\Delta(0, 0) : X]$. Moreover, if we put

$$C^N_{flat}([0, T]; X) := \{u \in C^N([0, T]; X); \partial_t^j u|_{t=0} = 0 \ (0 \leqq j \leqq N-1)\}, \tag{3.5}$$

then there holds

$$C^N_{flat}([0, T]; X) = F_T[\Delta(N, -N) : X]. \tag{3.6}$$

(2) The space $F_T[\Delta(l, 0): X]$ is equal to the space $C_l^0([0, T], X)$ in [**1**]. The space $F_T[-\Delta(l, 0): X]$ is equal to the space $C_{-l}^0([0, T], X)$ in [**2**].

REMARK 3.4. We can also define some function spaces similar to $F_T[\bar{\Delta}: X]$, based on other various spaces such as L_p spaces instead of $C^0([0, T]; X)$. In fact, J. Elschner ([3]) used weighted Sobolev spaces X_p^ρ similar to $F_T[\Delta: \boldsymbol{C}]$ $(\Delta\in\mathfrak{N})$, without referring to Newton polygons.

We give some properties of the spaces $F_T[\Delta: X]$ and $F_T[\bar{\Delta}: X]$. The first two lemmas are almost trivial and those proofs are omitted.

LEMMA 3.5. *Let Δ be a Newton polygon with size s and side function L. For $u\in\mathcal{D}'((0, T); X)$, the following nine conditions are equivalent to each other.*

(1) $u\in F_T[\Delta: X]$.

(2) $t^{L(j)}\tilde{\vartheta}^j(u)\in C^0([0, T]; X)$ $(0\leqq j\leqq s)$.

(3) $t^{L(j)+j}\partial_t^j(u)\in C^0([0, T]; X)$ $(0\leqq j\leqq s)$.

(4) $\vartheta^j(t^{L(j)}u)\in C^0([0, T]; X)$ $(0\leqq j\leqq s)$.

(5) $\tilde{\vartheta}^j(t^{L(j)}u)\in C^0([0, T]; X)$ $(0\leqq j\leqq s)$.

(6) $\partial_t^j(t^{L(j)+j}u)\in C^0([0, T]; X)$ $(0\leqq j\leqq s)$.

(7) *For any positive integer r, if $j_i\geqq 0$ $(0\leqq i\leqq r)$, $j_1+\cdots+j_r\leqq s$ and $\rho_1+\cdots+\rho_r\geqq L(j_1+\cdots+j_r)$, then $t^{\rho_1}\vartheta^{j_1}\cdots t^{\rho_r}\vartheta^{j_r}(u)\in C^0([0, T]; X)$.*

(8) *For any positive integer r, if $j_i\geqq 0$ $(0\leqq i\leqq r)$, $j_1+\cdots+j_r\leqq s$ and $\rho_1+\cdots+\rho_r\geqq L(j_1+\cdots+j_r)$, then $t^{\rho_1}\tilde{\vartheta}^{j_1}\cdots t^{\rho_r}\tilde{\vartheta}^{j_r}(u)\in C^0([0, T]; X)$.*

(9) *For any positive integer r, if $j_i\geqq 0$ $(0\leqq i\leqq r)$, $j_1+\cdots+j_r\leqq s$ and $\rho_1+\cdots+\rho_r\geqq L(j_1+\cdots+j_r)+j_1+\cdots+j_r$, then $t^{\rho_1}\partial_t^{j_1}\cdots t^{\rho_r}\partial_t^{j_r}(u)\in C^0([0, T]; X)$.*

Also for $F_T[\Delta, \Delta': X]$ $(\Delta, \Delta'\in\mathfrak{N})$, we have a similar lemma, which we omit to state.

LEMMA 3.6. *If $\bar{\Delta}_1, \bar{\Delta}_2\in\bar{\mathfrak{N}}$ and $\bar{\Delta}_1\subset\bar{\Delta}_2$, then $F_T[\bar{\Delta}_1: X]\supset F_T[\bar{\Delta}_2: X]$.*

LEMMA 3.7. *If $\varphi\in\mathcal{F}([0, T]; \mathcal{L}(X, Y))$ and $u\in F_T[\bar{\Delta}: X]$, then $\varphi u\in F_T[\bar{\Delta}: Y]$. Especially, if $h(\bar{\Delta})\geqq 0$ then $\mathcal{F}([0, T]; X)\subset F_T[\bar{\Delta}: X]$.*

PROOF. First, note that $\vartheta^j(\varphi)\in\mathcal{F}([0, T]; \mathcal{L}(X, Y))$ for any j. Hence, if $\bar{\Delta}=\Delta\in\mathfrak{N}$, then the lemma is almost trivial. Let $\bar{\Delta}=\Delta-\Delta'$, where $\Delta, \Delta'\in\mathfrak{N}$. Let s' (resp. L') be the size (resp. side function) of Δ'. Since $u\in F_T[\bar{\Delta}: X]=F_T[\Delta, \Delta': X]$, we have $u=\sum_{j=0}^{s'}t^{L'(j)}\vartheta^j(v_j)$ for some $v_j\in F_T[\Delta: X]$. We prove that $\varphi u\in F_T[\Delta, \Delta': Y]$ by the induction on s'.

If $s'=0$, then $\Delta'=\Delta(0, h)$ for some $h\in\boldsymbol{R}$. Hence, $\Delta-\Delta'=\{(u, v-h); (u, v)\in\Delta\}\in\mathfrak{N}$ and $F_T[\Delta, \Delta': X]=F_T[\Delta-\Delta': X]$. Thus, it is already proved.

Assume that we have proved for $s'-1$. Put $u'=\sum_{j=0}^{s'-1}t^{L'(j)}\vartheta^j(v_j)$. Then, by the induction hypothesis, we have $\varphi u'\in F_T[\Delta, \Delta': Y]$. Now, we have

$$(3.7)\qquad t^{L'(s')}\vartheta^{s'}(\varphi v_{s'})=t^{L'(s')}\sum_{l=0}^{s'}\binom{s'}{l}\vartheta^{s'-l}(\varphi)\vartheta^{l}(v_{s'})$$
$$=\varphi t^{L'(s')}\vartheta^{s'}(v_{s'})+\sum_{l=0}^{s'-1}\binom{s'}{l}\vartheta^{s'-l}(\varphi)t^{L'(s')}\vartheta^{l}(v_{s'}).$$

Since $\varphi v_{s'}\in F_T[\Delta:Y]$, we have $t^{L'(s')}\vartheta^{s'}(\varphi v_{s'})\in F_T[\Delta,\Delta':Y]$. On the other hand, by the induction hypothesis, we have $\vartheta^{s'-l}(\varphi)t^{L'(s')}\vartheta^{l}(v_{s'})\in F_T[\Delta,\Delta':Y]$ $(0\leqq l\leqq s'-1)$. Hence, by (3.7), we have $\varphi t^{L'(s')}\vartheta^{s'}(v_{s'})\in F_T[\Delta,\Delta':Y]$. Thus, we have $\varphi u\in F_T[\Delta,\Delta':Y]$. ■

LEMMA 3.8. *For $\bar{\Delta}\in\mathfrak{N}$ and $a\in\boldsymbol{R}$, there holds $t^a\in F_T[\bar{\Delta}:X]$ if and only if $a\geqq -h(\bar{\Delta})$.*

PROOF. The if part is easy. We prove the only-if part. Suppose $t^a\in F_T[\Delta,\Delta':X]$. Put $h=h(\Delta)$, $h'=h(\Delta')$, and $s'=s(\Delta')$. We may assume that $a\neq h'-h-1$ without loss of generality. Since $\Delta\supset\Delta(0,h)$ and $\Delta'\subset\Delta(s',h')$, we have $F_T[\Delta,\Delta':X]\subset F_T[\Delta(0,h),\Delta(s',h'):X]$. Hence, we can write $t^a=t^{h'}\sum_{j=0}^{s'}\vartheta^j(t^{-h}u_j)$, where $u_j\in C^0([0,T];X)$. Using $\vartheta^j\cdot t^b=t^b(\tilde{\vartheta}-1+b)^j$, we get $t^{a-h'+h}=\sum_{j=0}^{s'}\tilde{\vartheta}^j(v_j)$, where $v_j\in C^0([0,T];X)$. For any $v\in C^0([0,T];X)$, there exists $w\in C^0([0,T];X)$ such that $\tilde{\vartheta}(w)=v$. Hence, we get the equation $t^{a-h'+h}=\tilde{\vartheta}^{s'}(v)$ for some $v\in C^0([0,T];X)$. Solving this equation, we have $v=(a-h'+h+1)^{-s'}t^{a-h'+h}+\sum_{j=0}^{s'-1}C_jt^{-1}(\log t)^j$ for some constants C_j $(0\leqq j\leqq s'-1)$. Since $v\in C^0([0,T];X)$, we have $a-h'+h\geqq 0$, that is $a\geqq -h(\bar{\Delta})$. ■

Last, we define another function space.

DEFINITION 3.9. We put

$$\mathcal{G}_{\Lambda,T}(X):=\bigcup_{a>\Lambda,\,N\in\boldsymbol{N}}F_T[-\Delta(N,a):X].$$

REMARK 3.10. For $\bar{\Delta}\in\mathfrak{N}$ and $a\in\boldsymbol{R}$, there holds the following:
(1) $t^a\in\mathcal{G}_{\Lambda,T}(X)$ if and only if $a>\Lambda$,
(2) $F_T[\bar{\Delta}:X]\subset\mathcal{G}_{\Lambda,T}(X)$ if and only if $h(\bar{\Delta})<-\Lambda$.

§4. Reduction of the problem and extended results.

In this section, we reduce the Cauchy problem considered in the main theorems to the flat Cauchy problem or a similar problem using the assumptions (A-1), (A-4) and (A-5). Further, we give some extended results to such problems under the assumptions (A-1)-(A-3).

Consider a partial differential operator P of the form (1.1) and assume the condition (A-1). The operator P can be written as

(4.1) $$P=t^{h(P)}\tilde{P}=t^{h(P)}\{C_0(\vartheta;x)+tq(t,x;\vartheta,\partial_x)\},$$

where $q(t,x;\lambda,\xi)$ is a polynomial of (λ,ξ) with $C^\infty([-T,T];\mathcal{O}(\Omega))$ coefficients.

Let $u=\sum_{j=0}^\infty u_j(x)t^j$, $f=\sum_{j=0}^\infty f_j(x)t^j$ be the formal Taylor expansions with respect to t. In the space of formal power series with respect to t, the equation $Pu=f$ is equivalent to the equation $P(\sum_{j\geqq|h(P)|}u_jt^j)=f-P(\sum_{j=0}^{|h(P)|-1}u_jt^j)$, and hence to the following infinite number of equations under the assumption (A-4). (See Remark 1.3 (3). Also note that $\vartheta(t^j)=jt^j$.)

(4.2) $$C_0(j;x)u_j=f_{j-|h(P)|}+\sum_{l=0}^{j-1}R_{j,l}(x;\partial_x)u_l$$
$$(j=|h(P)|,\ |h(P)|+1,\ |h(P)|+2,\ \cdots),$$

where $R_{j,l}$ are some partial differential operators with respect to x with $\mathcal{O}(\Omega)$ coefficients. Now, we assume the condition (A-5). By reducing Ω if necessary, we may assume

(A-5)′ $\quad C_0(\lambda;x)\neq 0$ on Ω, for $\lambda=|h(P)|,\ |h(P)|+1,\ |h(P)|+2,\ \cdots$.

Then, u_j $(j\geqq|h(P)|)$ are uniquely determined by $u_0,u_1,\cdots,u_{|h(P)|-1}$, and f_j $(j=0,1,2,\cdots)$.

Take $v\in C^\infty([-T,T];\mathcal{O}(\Omega))$ that has the formal Taylor expansion $\sum_{j=0}^\infty u_j(x)t^j$. If we set $u=v+\tilde{u}$ and $f=Pv+\tilde{f}$, then $\tilde{u}$ and $\tilde{f}$ are *flat* on $\{t=0\}$, that is all the derivatives vanish on $\{t=0\}$. Thus, Theorem 1.5 has been reduced to the following proposition.

PROPOSITION 4.1. *Assume the conditions* (A-1)-(A-3). *Then, there exist a positive* T_0 *and a neighborhood* Ω_0 *of* 0 *in* $\boldsymbol{C}^n$ *for which the following holds:*

For any $f\in C^\infty([0,T];\mathcal{O}(\Omega))$ *that is flat on* $\{t=0\}$, *there exists a unique* $u\in C^\infty([0,T_0];\mathcal{O}(\Omega_0))$ *that is flat on* $\{t=0\}$ *and satisfies*

(4.3) $$Pu=f(t,x)\quad in\ [0,T_0]\times\Omega_0.$$

By a similar argument, Theorem 1.6 is reduced to the following:

PROPOSITION 4.2. *Assume the conditions* (A-1)-(A-3). *Let* $L>\Lambda(P)$. *If* $u\in t^L\times C^0([0,T];\mathcal{D}'(\Omega\cap\boldsymbol{R}^n))$ *and* $Pu=0$ *for* $t>0$ *in a neighborhood of* (0, 0), *then* $u=0$ *for* $t>0$ *in a neighborhood of* (0, 0).

Now, we give some extensions of these propositions in the form of the following three theorems. By the reduction argument made above, Theorems 1.5 and 1.6 follow from these three theorems.

In the following, we assume that the coefficients of P belong to $\mathcal{F}([0,T];\mathcal{O}(\Omega))$. (See Remark 2.8.) We also assume the conditions (A-1), (A-2), and (A-3).

The first theorem is the unique solvability in the class $\mathcal{G}_{\Lambda,T}(\mathcal{O}(\Omega))$.

THEOREM 4.3. *There exist a positive* T_0 *and a neighborhood* Ω_0 *of* 0 *in* $\boldsymbol{C}^n$ *for which the following holds:*

For any $f\in\mathcal{G}_{\Lambda(P)+h(P),T}(\mathcal{O}(\Omega))$, *there exists a unique* $u\in\mathcal{G}_{\Lambda(P),T_0}(\mathcal{O}(\Omega_0))$ *satisfying* $Pu=f$ *on* $(0, T_0)\times\Omega_0$.

The second theorem is the regularity of solutions with respect to t.

THEOREM 4.4. *There exists a neighborhood* Ω_0 *of* 0 *in* $\boldsymbol{C}^n$ *such that, for any* $T'\in(0, T]$ *and any subdomain* Ω' *of* Ω_0, *the following holds:*

Assume that $u\in\mathcal{G}_{\Lambda(P),T'}(\mathcal{O}(\Omega'))$. *If* $\bar{\Delta}\in\mathfrak{N}$, $h(\bar{\Delta})<-(\Lambda(P)+h(P))$, *and* $Pu\in F_{T'}[\bar{\Delta}:\mathcal{O}(\Omega')]$, *then* $u\in F_{T'}[\bar{\Delta}+\Delta(P):\mathcal{O}(\Omega')]$.

Especially, if $N\in\boldsymbol{N}$, $N>\Lambda(P)+h(P)$, *and* $Pu\in C^N_{flat}([0, T'];\mathcal{O}(\Omega'))$, *then* $u\in C^{N'}_{flat}([0, T'];\mathcal{O}(\Omega'))$, *where* $N'=\min\{N, [N-h(P)]\}$.

REMARK 4.5. Note that the condition $h(\bar{\Delta})<-(\Lambda(P)+h(P))$ means that $F_{T'}[\bar{\Delta}+\Delta(P):\mathcal{O}(\Omega')]\subset\mathcal{G}_{\Lambda(P),T'}(\mathcal{O}(\Omega'))$.

The last theorem is the uniqueness in a wider space.

THEOREM 4.6. *If* $u\in\mathcal{G}_{\Lambda(P),T}(\mathcal{D}'(\Omega\cap\boldsymbol{R}^n))$ *and* $Pu=0$ *for* $t>0$ *in a neighborhood of* $(0, 0)$, *then* $u=0$ *for* $t>0$ *in a neighborhood of* $(0, 0)$.

Theorem 1.7 easily follows from Theorem 4.6. We now prove Theorem 1.8 from Theorem 4.6.

PROOF OF THEOREM 1.8. Assume that $0\notin S$. Also assume that $u\in\mathcal{D}'((-T, T)\times(\Omega\cap\boldsymbol{R}^n))$ satisfies $Pu=0$ in a neighborhood of $(0, 0)$ and $\operatorname{supp} u\subset\{t\geqq 0\}$. Since u is a distribution of finite order in a smaller neighborhood of $(0, 0)$, we may assume that $u_{|t>0}\in\mathcal{G}_{-\infty,T}(\mathcal{D}'(\Omega\cap\boldsymbol{R}^n))$. Since $0\notin S$, we have $\Lambda(P)=-\infty$ and hence Theorem 4.6 implies that $u=0$ in $(0, T_1)\times U_1$ for some $T_1>0$ and an open neighborhood U_1 of $0\in\boldsymbol{R}^n$. Thus, $\operatorname{supp} u\cap((-T_1, T_1)\times U_1)\subset\{t=0\}$.

Now, assume $h(P)=0$ in addition. Since $\operatorname{supp} u\cap((-T_1, T_1)\times U_1)\subset\{t=0\}$, the distribution $u|_{(-T_1,T_1)\times U_1}$ can be written as $u=\sum_{j=0}^{M} a_j(x)\delta^{(j)}(t)$ for some $M\in\boldsymbol{N}$ and $a_j\in\mathcal{D}'(U_1)$ $(j=0, 1, \cdots, M)$. From (4.1), the operator P can be written as $P=C_0(\vartheta; x)+tq(t, x;\vartheta, \partial_x)$. Since $0\notin S$, the indicial polynomial $C_0(\lambda; x)=\alpha(x)$ is independent of λ and does not vanish in a neighborhood of $x=0$ by the assumption (A-2). Thus, there holds

$$\{\alpha(x)+tq(t, x;\vartheta, \partial_x)\}\left\{\sum_{j=0}^{M} a_j(x)\delta^{(j)}(t)\right\}=0 \qquad \text{on } (-T_1, T_1)\times U_1. \tag{4.4}$$

By this equation and the facts that

$$\vartheta^l(\delta^{(j)}(t))=(-j-1)^l\delta^{(j)}(t), \tag{4.5}$$

$$t^{l}\delta^{(j)}(t)=\begin{cases}(-j)(-j+1)\cdots(-j+l-1)\delta^{(j-l)}(t) & (j\geqq l),\\ 0 \quad (j<l),\end{cases} \tag{4.6}$$

we can easily show that $a_j(x)\equiv 0$ in a neighborhood of $x=0$ ($j=0, 1, \cdots, M$). ∎

In Sections 7 and 9, we shall prove the extended results given above.

§ 5. Ordinary differential operators 1.

In this section, we consider ordinary differential operators on function spaces defined in Section 3 and prove Lemma 3.2.

We consider an ordinary differential operator

$$\mathcal{P}=\sum_{j=0}^{m} a_j(t)\partial_t^j, \tag{5.1}$$

where $a_j\in\mathcal{F}([0, T]; \boldsymbol{C})$ $(0\leqq j\leqq m)$ and $a_m(t)\equiv t^{\kappa}$ $(\kappa\in\boldsymbol{Q}, \kappa\geqq 0)$.

As is stated in Remark 2.8, we can define $\Delta(\mathcal{P})$, $\hat{V}$, $h(\mathcal{P})$, S, I_μ, C_μ, $\Lambda(\mathcal{P})$ for $\mathcal{P}$ by Definition 1.1. The conditions (A-1) and (A-2) are automatically satisfied.

The following is well-known. (Cf. [**5**], [**3**, Chapter 2].)

PROPOSITION 5.1. (1) *The operator $\mathcal{P}$ can be decomposed as follows:*

$$\mathcal{P}=t^{h}(t^{k_1}\vartheta-\lambda_1(t))\cdots(t^{k_m}\vartheta-\lambda_m(t)). \tag{5.2}$$

Here, $h, k_1, \cdots, k_m\in\boldsymbol{Q}$, $k_j\geqq 0$, $\lambda_j\in\mathcal{F}([0, T]; \boldsymbol{C})$, and if $k_j>0$ then $\lambda_j(0)\neq 0$ $(1\leqq j\leqq m)$.

(2) *The numbers $h, k_1, \cdots, k_m, \lambda_1(0), \cdots, \lambda_m(0)$ are determined as follows:*

Let Γ be the side function of $\Delta(\mathcal{P})$. Renumber k's and λ's as $k_1\leqq k_2\leqq\cdots\leqq k_m$. Then, there holds

$$h=h(\mathcal{P}), \qquad k_j=\Gamma(j)-\Gamma(j-1) \quad (1\leqq j\leqq m).$$

Especially, $S=\{k_1, \cdots, k_m\}$. Further, if $\mu\in S$ and $\mu>0$, then the nonzero roots of $C_\mu(\lambda)=0$ are $\lambda_j(0)$ for j such that $k_j=\mu$, with repetitions according to multiplicity. If $0\in S$, then the roots of $C_0(\lambda)=0$ are $\lambda_j(0)$ for j such that $k_j=0$, with repetitions according to multiplicity.

By this proposition, the condition (A-3) means that "if $k_j>0$, then $\operatorname{Re}\lambda_j(0)<0$" in (5.2). Further, by the definition of $\Lambda(\mathcal{P})$, we have "if $k_j=0$, then $\operatorname{Re}\lambda_j(0)\leqq\Lambda(\mathcal{P})$".

REMARK 5.2. Even if $a_j\in C^\infty([0, T]; \boldsymbol{C})$, the functions λ_j do not necessarily belong to $C^\infty([0, T]; \boldsymbol{C})$. This is the reason why we consider $\mathcal{F}([0, T]; \boldsymbol{C})$ from the beginning.

Now, we can prove the following proposition.

PROPOSITION 5.3. *Assume the condition* (A-3). *Let* $T' \in (0, T]$ *and let* X *be a complete locally convex topological vector space.*

(1) *The operator* $\mathcal{P}$ *is a bijection from* $\mathcal{G}_{\Lambda(\mathcal{P}), T'}(X)$ *to* $\mathcal{G}_{\Lambda(\mathcal{P})+h(\mathcal{P}), T'}(X)$.

(2) *Let* $\bar{\Delta} \in \bar{\mathfrak{N}}$ *and* $h(\bar{\Delta}) < -(\Lambda(\mathcal{P}) + h(\mathcal{P}))$. *Then, the operator* $\mathcal{P}$ *is a bijection from* $F_{T'}[\bar{\Delta} + \Delta(\mathcal{P}) : X]$ *to* $F_{T'}[\bar{\Delta} : X]$.

(3) *If* X *is a Banach space, then the operator* $\mathcal{P}$ *in* (2) *is an isomorphism and the operator norm of* $\mathcal{P}^{-1}$ *can be estimated by a constant that is independent of* T' *and* X. (*This constant may depend on* $\bar{\Delta}$. *In order to consider the operator norm, we fix a decomposition* $\bar{\Delta} = \Delta - \Delta'$ ($\Delta, \Delta' \in \mathfrak{N}$) *and take the norms* $\|\cdot\|_{T', \Delta+\Delta(\mathcal{P}), \Delta', X}$ *and* $\|\cdot\|_{T', \Delta, \Delta', X}$.)

Note that we have not yet proved Lemma 3.2. Hence, we prove this proposition for $F_T[\Delta, \Delta' : X]$ instead of $F_T[\bar{\Delta} : X]$. We prove Lemma 3.2 after the proof of this proposition. First, we consider a factor with $k_j > 0$.

LEMMA 5.4. *Let* $k \in \boldsymbol{Q}$, $k > 0$, $\lambda \in \mathcal{F}([0, T]; \boldsymbol{C})$, *and* $\operatorname{Re} \lambda(0) < 0$. *Consider* $\mathcal{R} = t^k \vartheta - \lambda(t)$. *For any* $T' \in (0, T]$ *and any complete locally convex topological vector space* X, *there holds the following:*

(i) *If* $u \in \mathcal{G}_{-\infty, T'}(X)$ *satisfies that* $\mathcal{R}u = 0$ *on* $(0, T')$, *then* $u = 0$ *on* $(0, T')$.

(ii) *For any* $a \in \boldsymbol{R}$ *and* $f \in t^a \times C^0([0, T']; X)$, *there exists a unique* $u \in t^a \times C^0([0, T']; X)$ *satisfying* $\mathcal{R}u = f$ *on* $(0, T')$. *Further, for any seminorm* $\|\cdot\|$ *of* X, *there holds the estimate:*

$$\frac{1}{t^a}\|u(t)\| \leqq C \sup_{0<s\leqq t} \frac{1}{s^a}\|f(s)\| \quad (0 < t \leqq T')$$

with a constant C *that is independent of* T', X, *and* $\|\cdot\|$.

PROOF. We can take $\Lambda \in \mathcal{F}([0, T]; \boldsymbol{C})$ and $b \in \boldsymbol{C}$ as

$$\left(\frac{\Lambda(t)}{t^k} + b \log t\right)' = \frac{\lambda(t)}{t^{k+1}}. \tag{5.3}$$

Since $\Lambda(0) = -\lambda(0)/k$, there holds $\operatorname{Re} \Lambda(0) > 0$.

PROOF OF (i). Let $u \in \mathcal{G}_{-\infty, T'}(X)$ and $\mathcal{R}u = 0$ on $(0, T')$. It is easy to see that $u(t) = t^b e^{\Lambda(t)/t^k} u_0$ for some $u_0 \in X$. Since $\operatorname{Re} \Lambda(0) > 0$, the function $t^b e^{\Lambda(t)/t^k}$ is not extendable to $t = 0$ as a distribution. Hence, we have $u = 0$.

PROOF OF (ii). For $f \in t^a \times C^0([0, T']; X)$, put

$$\mathcal{S}[f](t) = t^b e^{\Lambda(t)/t^k} \int_0^t e^{-\Lambda(s)/s^k} s^{-b} \frac{1}{s^{k+1}} f(s) ds. \tag{5.4}$$

Since $\operatorname{Re} \Lambda(0) > 0$, this is well-defined and $\mathcal{S}[f] \in C^0((0, T']; X)$. Further, it is easy to see that $\mathcal{R}\mathcal{S}[f](t)=f(t)$ on $(0, T')$. Put $\Phi(t)=\mathrm{Re}\,\Lambda(t)/t^k$. We want to use the following estimate:

SUBLEMMA 5.5. *For any $p\in\boldsymbol{R}$, there exists a constant C_p for which the following estimate holds:*

$$e^{\Phi(t)}\int_0^t e^{-\Phi(s)}s^{p-1}ds \leqq C_p t^{p+k} \quad (t\in(0, T]). \tag{5.5}$$

POOF. Since $\Phi'(t)=\dfrac{\mathrm{Re}\,\tilde{\lambda}(t)}{t^{k+1}}$ where $\tilde{\lambda}(t)=\lambda(t)-bt^k$, we have

$$\begin{aligned}e^{\Phi(t)}\int_0^t e^{-\Phi(s)}s^{p-1}ds &= e^{\Phi(t)}\int_0^t e^{-\Phi(s)}(-\Phi'(s))\frac{-1}{\mathrm{Re}\,\tilde{\lambda}(s)}s^{p+k}ds \\ &= e^{\Phi(t)}\Big(\Big[e^{-\Phi(s)}\frac{-1}{\mathrm{Re}\,\tilde{\lambda}(s)}s^{p+k}\Big]_0^t+\int_0^t e^{-\Phi(s)}\Big(\frac{1}{\mathrm{Re}\,\tilde{\lambda}(s)}s^{p+k}\Big)'ds\Big) \\ &\leqq \frac{t^{p+k}}{|\mathrm{Re}\,\tilde{\lambda}(t)|}+Ce^{\Phi(t)}\int_0^t e^{-\Phi(s)}s^{p+k-1}ds\end{aligned} \tag{5.6}$$

in a neighborhood of $t=0$ for some constant C. Because

$$e^{\Phi(t)}e^{-\Phi(s)} \leqq C' \quad (0<s\leqq t\leqq T)$$

for some constant C', we can get the desired estimate. ∎

Now, we return to the proof of Lemma 5.4. Let $\|\cdot\|$ be an arbitrary semi-norm of X. By (5.4) and the sublemma above, we have

$$\begin{aligned}\|\mathcal{S}[f](t)\| &\leqq t^{\mathrm{Re}\,b}e^{\Phi(t)}\int_0^t e^{-\Phi(s)}s^{-\mathrm{Re}\,b}\frac{1}{s^{k+1}}\|f(s)\|ds \\ &\leqq t^{\mathrm{Re}\,b}e^{\Phi(t)}\int_0^t e^{-\Phi(s)}s^{-\mathrm{Re}\,b}\frac{1}{s^{k+1}}s^a ds \sup_{0<s\leqq t}\frac{1}{s^a}\|f(s)\| \\ &\leqq Ct^a \sup_{0<s\leqq t}\frac{1}{s^a}\|f(s)\| \quad (0<t\leqq T')\end{aligned} \tag{5.7}$$

for some constant C that is independent of T', X, and $\|\cdot\|$. Thus, if we prove that $\lim_{t\to+0}t^{-a}\mathcal{S}[f](t)$ exists in X, then the proof of (ii) is completed.

Since $f\in t^a\times C^0([0, T']; X)$, we can write $f(t)=t^a(g_0+g(t))$, where $g_0\in X$, $g\in C^0([0, T']; X)$, and $g(0)=0$. As for g, by the estimate (5.7), we have $t^{-a}\mathcal{S}[t^a g](t)\to 0$ $(t\to+0)$. On the other hand, by an integration by parts using (5.3), we can easily prove that

$$\mathcal{S}[t^a g_0](t) = -\frac{t^a}{\lambda(t)}g_0+\mathcal{S}\Big[t^{k+1}\Big(\frac{t^a}{\lambda(t)}\Big)'g_0\Big](t). \tag{5.8}$$

By the estimate (5.7), $t^{-a}\times\mathcal{S}[t^{k+1}(t^a/\lambda(t))'g_0](t)$ converges to 0 as $t\to+0$, and hence we have $t^{-a}\mathcal{S}[t^a g_0](t)\to -g_0/\lambda(0)$ as $t\to+0$. ∎

COROLLARY 5.6. *Let $\Delta\in\mathfrak{N}$. For any $T'\in(0, T]$ and any complete locally convex topological vector space X, the operator $\mathfrak{R}$ in Lemma 5.4 is an isomorphism from $F_{T'}[\Delta+\Delta(\mathfrak{R}):X]$ to $F_{T'}[\Delta:X]$. Further, if X is a Banach space, then the operator norm of $\mathfrak{R}^{-1}$ can be estimated by a constant that is independent of T' and X.*

PROOF. Note that $\Delta(\mathfrak{R})=\Delta[k]$, which is defined in Proposition 2.5. It is almost trivial that $\mathfrak{R}$ is a continuous injective operator from $F_{T'}[\Delta+\Delta[k]:X]$ to $F_{T'}[\Delta:X]$. Let s (resp. L) be the size (resp. side function) of Δ. We prove the surjectivity by the induction on s.

If $s=0$, then $\Delta=\Delta(0, h)$ for some $h\in\boldsymbol{R}$. Hence, $F_{T'}[\Delta:X]=t^{-h}\times C^0([0, T'];X)$. By Lemma 5.4, for any $f\in t^{-h}\times C^0([0, T'];X)$, there exists $u\in t^{-h}\times C^0([0, T'];X)$ such that $\mathfrak{R}u=f$. By this equation and Lemma 3.7, we have $t^k\vartheta(u)\in t^{-h}\times C^0([0, T'];X)$. This implies $u\in F_{T'}[\Delta(0, h)+\Delta[k]:X]$, which proves the case $s=0$.

We assume that the corollary holds for $s-1$ $(s\geqq 1)$. Let $f\in F_{T'}[\Delta:X]$, that is $t^{L(j)}\vartheta^j(f)\in C^0([0, T'];X)$ for $0\leqq j\leqq s$. The condition $u\in F_{T'}[\Delta+\Delta[k]:X]$ means that

$$(5.9)\qquad t^{L(j)}\vartheta^j(u),\ t^{L(j)+k}\vartheta^{j+1}(u)\in C^0([0, T'];X)\qquad \text{for } 0\leqq j\leqq s\,.$$

By the induction hypothesis, we have

$$(5.10)\qquad t^{L(j)}\vartheta^j(u),\ t^{L(j)+k}\vartheta^{j+1}(u)\in C^0([0, T'];X)\qquad \text{for } 0\leqq j\leqq s-1\,.$$

Now, put $\vartheta(u)=v$. From the equation $\mathfrak{R}u=f$, we have

$$(5.11)\qquad \mathfrak{R}v=\tilde{f}:=\vartheta(f)+\vartheta(\lambda)u-kt^k\vartheta(u)\,.$$

It is easy to see that

$$(5.12)\qquad t^{L(j+1)}\vartheta^j(\tilde{f})\in C^0([0, T'];X)\quad (0\leqq j\leqq s-1)\,.$$

Put $\tilde{L}(j):=L(j+1)$ $(0\leqq j\leqq s-1)$. Let $\tilde{\Delta}$ be the Newton polygon with size $s-1$ and side function $\tilde{L}$. The condition (5.12) means that $\tilde{f}\in F_T[\tilde{\Delta}:X]$. From Equation (5.11), we have $v\in F_T[\tilde{\Delta}+\Delta[k]:X]$ by the induction hypothesis. This implies that

$$(5.13)\qquad t^{\tilde{L}(s-1)}\vartheta^{s-1}(v),\quad t^{\tilde{L}(s-1)+k}\vartheta^s(v)\in C^0([0, T'];X)\,,$$

and hence

$$(5.14)\qquad t^{L(s)}\vartheta^s(u),\quad t^{L(s)+k}\vartheta^{s+1}(u)\in C^0([0, T'];X)\,.$$

The conditions (5.10) and (5.14) imply (5.9), that is $u\in F_{T'}[\Delta+\Delta[k]:X]$.

The continuity of $\mathfrak{R}^{-1}$ and the latter part of the corollary are clear from the argument above. ■

Next, we consider a factor with $k_j=0$.

LEMMA 5.7. *Let $\lambda \in \mathcal{F}([0, T]; \boldsymbol{C})$. Consider $\mathcal{R}=\vartheta-\lambda(t)$. For any $T' \in (0, T]$ and any complete locally convex topological vector space X, there holds the following:*

(i) *If $u \in \mathcal{G}_{\mathrm{Re}\,\lambda(0), T'}(X)$ satisfies that $\mathcal{R}u=0$ on $(0, T')$, then $u=0$ on $(0, T')$.*

(ii) *For any $a>\mathrm{Re}\,\lambda(0)$ and $f \in t^a \times C^0([0, T']; X)$, there exists a unique $u \in t^a \times C^0([0, T']; X)$ satisfying $\mathcal{R}u=f$ on $(0, T')$. Further, for any seminorm $\|\cdot\|$ of X, there holds the estimate:*

$$\frac{1}{t^a}\|u(t)\| \leqq C \sup_{0<s\leqq t} \frac{1}{s^a}\|f(s)\| \quad (0<t\leqq T')$$

with a constant C that is independent of T', X, and $\|\cdot\|$.

PROOF. Put $\tilde{\lambda}(t) := \lambda(t)-\lambda(0)$ and take $\Lambda \in \mathcal{F}([0, T]; \boldsymbol{C})$ as $\Lambda(0)=0$ and $\Lambda'(t) = \tilde{\lambda}(t)/t$.

PROOF OF (i). Let $u \in \mathcal{G}_{\mathrm{Re}\,\lambda(0), T'}(X)$ and $\mathcal{R}u=0$ on $(0, T')$. It is easy to see that $u(t)=t^{\lambda(0)}e^{\Lambda(t)}u_0$ for some $u_0 \in X$. Since $t^{\lambda(0)}e^{\Lambda(t)} \notin \mathcal{G}_{\mathrm{Re}\,\lambda(0), T'}(\boldsymbol{C})$, we have $u=0$.

PROOF OF (ii). For $f \in t^a \times C^0([0, T']; X)$ $(a>\mathrm{Re}\,\lambda(0))$, put

$$\mathcal{S}[f](t) = t^{\lambda(0)}e^{\Lambda(t)}\int_0^t e^{-\Lambda(s)}s^{-\lambda(0)-1}f(s)ds\,. \tag{5.15}$$

This is well-defined and $\mathcal{S}[f] \in C^0((0, T']; X)$. Further, it is easy to see that $\mathcal{R}\mathcal{S}[f](t)=f(t)$ on $(0, T')$. Put $\Psi(t)=\mathrm{Re}\,\Lambda(t)$.

Let $\|\cdot\|$ be an arbitrary seminorm of X. By (5.15), we have

$$\begin{aligned}\|\mathcal{S}[f](t)\| &\leqq t^{\mathrm{Re}\,\lambda(0)}C'\int_0^t s^{-\mathrm{Re}\,\lambda(0)-1+a}ds \sup_{0<s\leqq t}\frac{1}{s^a}\|f(s)\| \\ &\leqq Ct^a \sup_{0<s\leqq t}\frac{1}{s^a}\|f(s)\| \quad (0<t\leqq T')\end{aligned}$$

for some constants C' and C that are independent of T', X, and $\|\cdot\|$. By a similar argument to that in the proof of Lemma 5.4, we can prove that $\lim_{t\to+0} t^{-a}\mathcal{S}[f](t)$ exists in X, and hence the proof of (ii) is completed. ■

COROLLARY 5.8. *Let $\Delta \in \mathfrak{N}$ and $h(\Delta)<-\mathrm{Re}\,\lambda(0)$. For any $T' \in (0, T]$ and any complete locally convex topological vector space X, the operator $\mathcal{R}$ in Lemma 5.7 is an isomorphism from $F_{T'}[\Delta+\Delta(\mathcal{R}): X]$ to $F_{T'}[\Delta: X]$. Further, if X is a Banach space, then the operator norm of $\mathcal{R}^{-1}$ can be estimated by a constant that is independent of T' and X.*

PROOF. Note that $\Delta(\mathcal{R})=\Delta(1, 0)$. It is almost trivial that $\mathcal{R}$ is a continuous injective operator from $F_{T'}[\Delta+\Delta(1, 0): X]$ to $F_{T'}[\Delta: X]$. Let s (resp. L) be the size (resp. side function) of Δ. We prove the surjectivity by the induction on s.

If $s=0$, then $\Delta=\Delta(0, h)$ for some $h<-\operatorname{Re}\lambda(0)$. Hence, $F_{T'}[\Delta: X]=t^{-h}\times C^0([0, T']; X)$. By Lemma 5.7, for any $f\in t^{-h}\times C^0([0, T']; X)$, there exists $u\in t^{-h}\times C^0([0, T']; X)$ such that $\mathfrak{R}u=f$. By this equation and Lemma 3.7, we have $\vartheta(u)\in t^{-h}\times C^0([0, T']; X)$. This implies $u\in F_{T'}[\Delta(0, h)+\Delta(1, 0): X]$, which proves the case $s=0$.

We assume that the corollary holds for $s-1$ $(s\geqq 1)$. Let $f\in F_{T'}[\Delta: X]$, that is, $t^{L(j)}\vartheta^j(f)\in C^0([0, T']; X)$ for $0\leqq j\leqq s$. The condition $u\in F_{T'}[\Delta+\Delta(1, 0): X]$ means that

$$(5.16)\qquad t^{L(0)}u \in C^0([0, T']; X) \quad\text{and}$$
$$t^{L(j-1)}\vartheta^j(u) \in C^0([0, T']; X) \qquad \text{for } 1\leqq j\leqq s+1\,.$$

By the induction hypothesis, we have

$$(5.17)\qquad t^{L(0)}u \in C^0([0, T']; X) \quad\text{and}$$
$$t^{L(j-1)}\vartheta^j(u) \in C^0([0, T']; X) \qquad \text{for } 1\leqq j\leqq s.$$

Now, from the equation $\vartheta(u)=f+\lambda(t)u$, we have

$$\vartheta^{s+1}(u) = \vartheta^s(f)+\sum_{l=0}^{s}\binom{s}{l}\vartheta^{s-l}(\lambda)\vartheta^l(u)\,,$$

and hence by (5.17) we have

$$(5.18)\qquad t^{L(s)}\vartheta^{s+1}(u) \in C^0([0, T']; X)\,.$$

The conditions (5.17) and (5.18) imply (5.16), that is $u\in F_{T'}[\Delta+\Delta(1, 0): X]$.

The continuity of $\mathfrak{R}^{-1}$ and the latter part of the corollary are clear from the argument above. ■

At this stage, by Proposition 5.1, we have proved Proposition 5.3 (2), (3) in the case of $\tilde{\Delta}=\Delta\in\mathfrak{N}$. In order to prove the general case, we give the following lemma:

LEMMA 5.9. *Let $\mathfrak{R}=t^k\vartheta-\lambda(t)$ be the operator given in Lemma* 5.4 (*the case of* $k>0$) *or Lemma* 5.7 (*the case of* $k=0$). *Let Δ, $\Delta'\in\mathfrak{N}$. In the case of $k=0$, assume $h(\Delta)-h(\Delta')<-\operatorname{Re}\lambda(0)$. For any $T'\in(0, T]$ and any complete locally convex topological vector space X, the operator $\mathfrak{R}$ is a bijection from $F_{T'}[\Delta+\Delta(\mathfrak{R}), \Delta': X]$ to $F_{T'}[\Delta, \Delta': X]$. Further, if X is a Banach space, then the operator $\mathfrak{R}$ is an isomorphism and the operator norm of $\mathfrak{R}^{-1}$ can be estimated by a constant that is independent of T' and X.*

PROOF. It is easy to see that $\mathcal{R}$ is an injective operator from $F_{T'}[\Delta+\Delta(\mathcal{R}), \Delta' : X]$ to $F_{T'}[\Delta, \Delta' : X]$. In order to prove the surjectivity, we use the following sublemma:

SUBLEMMA 5.10. *Let $p\in\boldsymbol{R}$ and $j\in\boldsymbol{N}$. If $A\in\boldsymbol{R}$ is taken sufficiently large, then there exist $a_0, \cdots, a_j, \mu\in\mathcal{F}([0, T]; \boldsymbol{C})$ such that*

$$\mathcal{R}t^pQ = t^p(\vartheta-A)^j(\mathcal{R}+\mu(t)), \qquad \textit{where } Q=\sum_{l=0}^{j} a_l(t)\vartheta^l .$$

Further, $\mu(0)=0$ in the case of $k>0$ and $\mu(0)=p$ in the case of $k=0$.

PROOF. Since $\mathcal{R}t^p=t^p(\mathcal{R}+pt^k)$, we may assume that $p=0$. If $j=0$, then this sublemma is trivial.

Let $j=1$ and put $Q=\vartheta+\nu$. The relation $\mathcal{R}Q=(\vartheta-A)(\mathcal{R}+\mu)$ means

$$(5.19)\qquad \begin{cases} t^k\nu = (k-A)t^k+\mu , \\ t^k\vartheta(\nu)-\lambda\nu = -\vartheta(\lambda)+\vartheta(\mu)-A(\mu-\lambda). \end{cases}$$

These are equivalent to

$$(5.20)\qquad \begin{cases} \mu = t^k(\nu-k+A), \\ ((k-A)t^k+\lambda)\nu = \vartheta(\lambda)+(k-A)^2t^k-A\lambda . \end{cases}$$

It is easy to see that, if A is sufficiently large, then $(k-A)t^k+\lambda(t)\neq 0$ on $[0, T]$. Thus, we can take μ and ν satisfying these conditions and there holds $\mu(0)=0$.

Let $j\geqq 2$. For $\mu\in\mathcal{F}([0, T]; \boldsymbol{C})$ such that $\mu(0)=0$, the operator $\mathcal{R}+\mu$ also satisfies the condition for $\mathcal{R}$. Hence, using the result for $j=1$, we get a series of μ's and Q's as follows:

$$(5.21)\qquad \begin{cases} \mathcal{R}Q_1 = (\vartheta-A)(\mathcal{R}+\mu_1) \\ (\mathcal{R}+\mu_1)Q_2 = (\vartheta-A)(\mathcal{R}+\mu_2) \\ \qquad\cdots \qquad \cdots \\ (\mathcal{R}+\mu_{j-1})Q_j = (\vartheta-A)(\mathcal{R}+\mu_j) . \end{cases}$$

From these, we get $\mathcal{R}Q_1\cdots Q_j=(\vartheta-A)^j(\mathcal{R}+\mu_j)$. ■

Now, we return to the proof of Lemma 5.9. Let $f\in F_{T'}[\Delta, \Delta' : X]$. Let s' be the size and L' be the side function of Δ'. Take a sufficiently large $A\in\boldsymbol{R}$. We can write as

$$f = \sum_{j=0}^{s'} t^{L'(j)}(\vartheta-A)^j(g_j), \quad g_j \in F_{T'}[\Delta : X] \ (0\leqq j\leqq s') .$$

Consider $\mathcal{R}u_j=t^{L'(j)}(\vartheta-A)^j(g_j)$. Take Q and μ in the sublemma above for $p=L'(j)$. Putting $u_j=t^{L'(j)}Q(v_j)$, we have only to solve $(\mathcal{R}+\mu)(v_j)=g_j$. By

Corollary 5.6 or 5.8, there exists a solution $v_j \in F_{T'}[\Delta+\Delta(\mathscr{R}): X]$. Since $u_j = t^{L'(j)}Q(v_j) \in F_{T'}[\Delta+\Delta(\mathscr{R}), \Delta': X]$, we have $u=\sum_{j=0}^{s'} u_j \in F_{T'}[\Delta+\Delta(\mathscr{R}), \Delta': X]$. Thus, we get the surjectivity. We can also show the latter part of the lemma by tracing the argument above with the consideration of norms. ■

From Proposition 5.1 and Lemma 5.9, Proposition 5.3 (2) follows in the sense that the operator $\mathscr{P}$ is a bijection from $F_{T'}[\Delta+\Delta(\mathscr{P}), \Delta': X]$ to $F_{T'}[\Delta, \Delta': X]$. Proposition 5.3 (3) also follows. Proposition 5.3 (1) follows from Proposition 5.3 (2) and Definition 3.9.

Last, we prove Lemma 3.2.

PROOF OF LEMMA 3.2. (1) Let $\Delta_1, \Delta_1', \Delta_2, \Delta_2' \in \mathfrak{N}$ and $\bar{\Delta}=\Delta_1-\Delta_1'=\Delta_2-\Delta_2'$. Let s_i (resp. s_i') be the size and L_i (resp. L_i') be the side function of Δ_i (resp. Δ_i') ($i=1, 2$). Put $h := h(\Delta_2)$, $k_j := L_2(j)-L_2(j-1)$, and

$$\mathscr{P}_2 := t^h \prod_{j=1}^{s_2} (t^{k_j}\vartheta + A), \tag{5.22}$$

where A is a sufficiently large real constant. As already proved, $\mathscr{P}_2$ is a bijection from $F_T[\Delta_1+\Delta_2: X]$ to $F_T[\Delta_1: X]$ and satisfies $\Delta(\mathscr{P}_2)=\Delta_2$. Now, if $u \in F_T[\Delta_1, \Delta_1': X]$, then $u=\sum_{j=0}^{s_1'} t^{L_1'(j)}\vartheta^j(v_j)$ for some $v_j \in F_T[\Delta_1: X]$. We can write as $v_j=\mathscr{P}_2(w_j)$ for some $w_j \in F_T[\Delta_1+\Delta_2: X]$. Since $\Delta(\mathscr{P}_2)=\Delta_2$, it is easy to see that we can write as

$$u = \sum_{j=0}^{s_1'+s_2} t^{(L_1'\oplus L_2)(j)}\vartheta^j(h_j), \quad h_j \in F_T[\Delta_1+\Delta_2: X] \quad (0 \leqq j \leqq s_1'+s_2). \tag{5.23}$$

This means that $u \in F_T[\Delta_1+\Delta_2, \Delta_1'+\Delta_2: X]$.

Conversely, if $u \in F_T[\Delta_1+\Delta_2, \Delta_1'+\Delta_2: X]$, then u can be written as (5.23). For any j, there exists $l=l(j)$ such that $(L_1' \oplus L_2)(j)=L_1'(j-l)+L_2(l)$. Hence,

$$\begin{aligned} t^{(L_1'\oplus L_2)(j)}\vartheta^j(h_j) &= t^{L_1'(j-l)}t^{L_2(l)}\vartheta^{j-l}\vartheta^l(h_j) \\ &= t^{L_1'(j-l)}(\vartheta-L_2(l))^{j-l}t^{L_2(l)}\vartheta^l(h_j). \end{aligned}$$

Since $t^{L_2(l)}\vartheta^l(h_j) \in F_T[\Delta_1: X]$, we have $t^{(L_1'\oplus L_2)(j)}\vartheta^j(h_j) \in F_T[\Delta_1, \Delta_1': X]$.

Thus, we have proved that $F_T[\Delta_1, \Delta_1': X]=F_T[\Delta_1+\Delta_2, \Delta_1'+\Delta_2: X]$. By the same way, we have $F_T[\Delta_2, \Delta_2': X]=F_T[\Delta_2+\Delta_1, \Delta_2'+\Delta_1: X]$. Since $\Delta_1'+\Delta_2=\Delta_2'+\Delta_1$, there holds $F_T[\Delta_1, \Delta_1': X]=F_T[\Delta_2, \Delta_2': X]$ as a set.

(2) Assume that X is a Banach space. As in (1), there exists an ordinary differential operator $\mathscr{P}_1'$ that is a bijection from $F_T[\Delta_1+\Delta_1', \Delta_1': X]$ to $F_T[\Delta_1, \Delta_1': X]$ and satisfies $\Delta(\mathscr{P}_1')=\Delta_1'$. Since $F_T[\Delta_1+\Delta_1', \Delta_1': X]=F_T[\Delta_1: X]$ as a set by (1), we can define another norm $\|u\|_* := \|\mathscr{P}_1'^{-1}u\|_{T,\Delta_1,X}$ of $F_T[\Delta_1, \Delta_1': X]$. Obviously, the space $F_T[\Delta_1, \Delta_1': X]$ is a Banach space with this norm.

First, we prove that the norm $\|\cdot\|_*$ is equivalent to the norm $\|\cdot\|_{T,\Delta_1,\Delta_1',X}$.

Let $u\in F_T[\Delta_1, \Delta'_1 : X]$ and put $w=\mathcal{P}_1'^{-1}u\in F_T[\Delta_1 : X]$. Since we can write as $\mathcal{P}'_1=\sum_{j=0}^{s'_1} t^{L'_1(j)}\vartheta^j\alpha_j$ for some $\alpha_j\in\mathcal{F}([0, T]; \boldsymbol{C})$, we have $u=\sum_{j=0}^{s'_1} t^{L'_1(j)}\vartheta^j(\alpha_j w)$. Hence,

$$\|u\|_{T,\Delta_1,\Delta'_1,X} \leqq \sum_{j=0}^{s'_1} \|\alpha_j w\|_{T,\Delta_1,X} \leqq C\|w\|_{T,\Delta_1,X} = C\|u\|_*$$

for some constant C.

On the other hand, assume that $u=\sum_{j=0}^{s'_1} t^{L'_1(j)}\vartheta^j(v_j)$ where $v_j\in F_T[\Delta_1 : X]$. Since the operator $\mathcal{P}_1'^{-1}t^{L'_1(j)}\vartheta^j$ is a closed transform in the Banach space $F_T[\Delta_1 : X]$, this operator is bounded by the closed graph theorem, and hence there exist $w_j\in F_T[\Delta_1 : X]$ such that

$$t^{L'_1(j)}\vartheta^j(v_j) = \mathcal{P}'_1(w_j) \quad \text{and} \quad \|w_j\|_{T,\Delta_1,X} \leqq C\|v_j\|_{T,\Delta_1,X}$$

for some constant C. From $u=\mathcal{P}'_1(\sum_{j=0}^{s'_1} w_j)$, we have

$$\|u\|_* = \|\sum_{j=0}^{s'_1} w_j\|_{T,\Delta_1,X} \leqq \sum_{j=0}^{s'_1} \|w_j\|_{T,\Delta_1,X} \leqq C\sum_{j=0}^{s'_1} \|v_j\|_{T,\Delta_1,X}\,.$$

By the definition of $\|\cdot\|_{T,\Delta_1,\Delta'_1,X}$, we have

$$\|u\|_* \leqq C\|u\|_{T,\Delta_1,\Delta'_1,X}\,.$$

Thus, the norm $\|\cdot\|_*$ is equivalent to the norm $\|\cdot\|_{T,\Delta_1,\Delta'_1,X}$, and hence the space $F_T[\bar{\Delta} : X]=F_T[\Delta_1, \Delta'_1 : X]$ is a Banach space with the norm $\|\cdot\|_{T,\Delta_1,\Delta'_1,X}$. By just the same way, the space $F_T[\bar{\Delta} : X]$ is also a Banach space with the norm $\|\cdot\|_{T,\Delta_2,\Delta'_2,X}$. Because these two norms define topologies finer than the induced topology from $\mathcal{D}'((0, T); X)$, these two norms are equivalent to each other, by the closed graph theorem.

§6. Ordinary differential operators 2.

In this section, we consider ordinary differential operators with the parameter x.

As in Section 1, let Ω be a bounded domain in $\boldsymbol{C}^n$ containing the origin 0. We consider an ordinary differential operator

$$\mathcal{P} = \sum_{j=0}^{m} a_j(t, x)\partial_t^j\,, \tag{6.1}$$

where $a_j\in\mathcal{F}([0, T]; \mathcal{O}(\Omega))$ $(0\leqq j\leqq m)$ and $a_m(t, x)\equiv t^\kappa$ $(\kappa\in\boldsymbol{Q}, \kappa\geqq 0)$.

As is stated in Remark 2.8, we can define $\Delta(\mathcal{P})$, $\hat{V}$, $h(\mathcal{P})$, S, I_μ, C_μ, $\Lambda(\mathcal{P})$ for $\mathcal{P}$ by Definition 1.1. The condition (A-1) is automatically satisfied.

Let $H(\Omega)$ denote one of the two Banach spaces $E(\Omega)$ and $F(\Omega)$.

THEOREM 6.1. *Assume the conditions* (A-2) *and* (A-3).

(1) *There exists an open neighborhood* Ω_0 *of* $0 \in C^n$ *such that, for any subdomain* Ω' *of* Ω_0 *and for any* $T' \in (0, T]$, *the following holds:*

1. *The operator* $\mathcal{P}$ *is a bijection from* $\mathcal{G}_{\Lambda(\mathcal{P}),T'}(H(\Omega'))$ *to* $\mathcal{G}_{\Lambda(\mathcal{P})+h(\mathcal{P}),T'}(H(\Omega'))$.

2. *Let* $\bar{\Delta} \in \mathfrak{N}$ *and* $h(\bar{\Delta}) < -(\Lambda(\mathcal{P}) + h(\mathcal{P}))$. *Then, the operator* $\mathcal{P}$ *is an isomorphism from* $F_{T'}[\bar{\Delta} + \Delta(\mathcal{P}) : H(\Omega')]$ *to* $F_{T'}[\bar{\Delta} : H(\Omega')]$.

3. *The operator norm of* $\mathcal{P}^{-1}$ *in* 2 *can be estimated by a constant that is independent of* T' *and* Ω'. (*As in Proposition* 5.3, *we fix a decomposition of* $\bar{\Delta}$ *to consider the operator norm. This constant may also depend on* $\bar{\Delta}$.)

(2) *For any* $\bar{\Delta} \in \mathfrak{N}$ *satisfying* $h(\bar{\Delta}) < -(\Lambda(\mathcal{P}) + h(\mathcal{P}))$, *there exists an open neighborhood* Ω_0 *of* $0 \in C^n$ *such that, for any subdomain* Ω' *of* Ω_0 *and for any* $T' \in (0, T]$, *the following holds:*

The operator $\mathcal{P}$ *is an isomorphism from* $F_{T'}[\bar{\Delta} + \Delta(\mathcal{P}) : H'(\Omega')]$ *to* $F_{T'}[\bar{\Delta} : H'(\Omega')]$. *Further, the operator norm of* $\mathcal{P}^{-1}$ *can be estimated by a constant that is independent of* T' *and* Ω'. (*We also fix a decomposition of* $\bar{\Delta}$ *to consider the operator norm.*)

REMARK 6.2. Note that the domain Ω_0 can be taken independently of $\bar{\Delta}$ in (1). As for H', the author could not prove the independence of Ω_0 on $\bar{\Delta}$. Thus, (2) is rather weak compared with (1). It is, however, sufficient for our purpose, that is the proof of Theorem 4.6. The results (1) shall be used in the proof of Theorem 4.3 and Theorem 4.4.

PROOF OF THEOREM 6.1. First note that, if Ω is convex, then $\mathcal{O}(\Omega) \subset F(\Omega')$ for any Ω' such that $\bar{\Omega}' \subset \Omega$. Hence, by reducing Ω if necessary, we may assume that the coefficients a_j belong to $\mathcal{F}([0, T]; F(\Omega))$ $(0 \leq j \leq m)$.

Put

$$\mathcal{P}_0 := \sum_{j=0}^{m} a_j(t, 0)\partial_t^j \quad \text{and} \quad \tilde{\mathcal{P}} := \mathcal{P} - \mathcal{P}_0 .$$

The operator $\mathcal{P}_0$ satisfies the condition (A-3) and there holds $\Delta(\mathcal{P}_0) = \Delta(\mathcal{P})$.

Assume that $\bar{\Delta} \in \mathfrak{N}$, $h(\bar{\Delta}) < -(\Lambda(\mathcal{P}) + h(\mathcal{P}))$, $T' \in (0, T]$ and that Ω' is a subdomain of Ω. Let X denote one of the spaces $H(\Omega')$ and $H'(\Omega')$. By Proposition 5.3, the operator $\mathcal{P}_0$ is an isomorphism from $F_{T'}[\bar{\Delta} + \Delta(\mathcal{P}) : X]$ to $F_{T'}[\bar{\Delta} : X]$. Since the coefficients of $\tilde{\mathcal{P}}$ vanish at $x = 0$, there exists an open neighborhood Ω_0 of $x = 0$ such that, if $\Omega' \subset \Omega_0$, then the operator norm of $\tilde{\mathcal{P}}\mathcal{P}_0^{-1}$ as a transform in $F_{T'}[\bar{\Delta} : X]$ is less than 1. Since $\mathcal{P} = (Id + \tilde{\mathcal{P}}\mathcal{P}_0^{-1})\mathcal{P}_0$, the operator $\mathcal{P}$ is an isomorphism from $F_{T'}[\bar{\Delta} + \Delta(\mathcal{P}) : X]$ to $F_{T'}[\bar{\Delta} : X]$. It also follows from Proposition 5.3 that the operator norm of $\mathcal{P}^{-1}$ can be estimated by a constant that is independent of T' and Ω'. Thus, (2) has been proved and (1)-2 and 3 has also been proved except that Ω_0 can be taken independently of $\bar{\Delta}$.

Now, take a fixed open neighborhood Ω_0 of 0 in C^n such that $\bar{\Omega}_0 \subset \Omega$ and the following holds:

(A-2)° If $(j, 0)\in\hat{V}$, then $\tilde{a}_{j,0}(0, x)\neq 0$ on $\bar{\Omega}_0$.

(A-3)° If $\mu\in S$ and $\mu>0$, then all non-zero roots λ of $C_\mu(\lambda; x)=0$ $(x\in\bar{\Omega}_0)$ satisfy $\mathrm{Re}\,\lambda<0$.

Assume that $\bar{\Delta}\in\mathfrak{N}$, $h(\bar{\Delta})<-(\Lambda(\mathcal{P})+h(\mathcal{P}))$, $T'\in(0, T]$ and that Ω' is a subdomain of Ω_0. We can prove the following:

CLAIM. *The operator $\mathcal{P}$ is an isomorphism from $F_{T'}[\bar{\Delta}+\Delta(\mathcal{P}): H(\Omega')]$ to $F_{T'}[\bar{\Delta}: H(\Omega')]$.*

We have only to prove the surjectivity. Let $f\in F_{T'}[\bar{\Delta}: H(\Omega')]$. For any fixed $x_0\in\bar{\Omega}'$, put

$$\mathcal{P}_{x_0}:=\sum_{j=0}^{m} a_j(t, x_0)\partial^j\,.$$

Note that $\Delta(\mathcal{P}_{x_0})=\Delta(\mathcal{P})$ and $\mathcal{P}_{x_0}$ satisfies the conditions (A-2) and (A-3). By Proposition 5.3, there exists a unique $u_{x_0}\in F_{T'}[\bar{\Delta}+\Delta(\mathcal{P}): \boldsymbol{C}]$ such that $\mathcal{P}_{x_0}u_{x_0}=f(t, x_0)$. Put $u_0(t, x):=u_x(t)$ $(x\in\bar{\Omega}')$.

LEMMA 6.3. *There exists a finite number of points $x_i\in\bar{\Omega}_0$ $(i=1, 2, \cdots, M)$ and open neighborhoods Ω_i of x_i such that*

(i) $\bigcup_{i=1}^{M}\Omega_i\supset\bar{\Omega}_0$,

(ii) *the operator $\mathcal{P}$ is an isomorphism from $F_{T'}[\bar{\Delta}+\Delta(\mathcal{P}): H(\Omega'')]$ to $F_{T'}[\bar{\Delta}: H(\Omega'')]$ for any $T'\in(0, T]$ and any subdomain Ω'' of Ω_i $(i=1, \cdots, M)$.*

PROOF. For any $x_0\in\bar{\Omega}_0$, consider x_0 as the origin. By the same argument as above, there exists an open neighborhood Ω_{x_0} of x_0 such that the operator $\mathcal{P}$ is an isomorphism from $F_{T'}[\bar{\Delta}+\Delta(\mathcal{P}): H(\Omega'')]$ to $F_{T'}[\bar{\Delta}: H(\Omega'')]$ for any $T'\in(0, T]$ and any subdomain Ω'' of Ω_{x_0}. Since $\bigcup_{x_0}\Omega_{x_0}\supset\bar{\Omega}_0$, we have the lemma by the compactness of $\bar{\Omega}_0$. ∎

By this lemma, the equation $\mathcal{P}u=f|_{\overline{\Omega_i\cap\Omega'}}$ has a unique solution $u=u_i\in F_{T'}[\bar{\Delta}+\Delta(\mathcal{P}): H(\Omega_i\cap\Omega')]$. Since $u=u_i(t, x_0)$ is also a solution of $\mathcal{P}_{x_0}u=f(t, x_0)$ for $x_0\in\overline{\Omega_i\cap\Omega'}$, we have $u_0(t, x)=u_i(t, x)$ for $x\in\overline{\Omega_i\cap\Omega'}$, by the uniqueness. Thus, we have

$$u_0|_{\overline{\Omega_i\cap\Omega'}}\in F_{T'}[\bar{\Delta}+\Delta(\mathcal{P}): H(\Omega_i\cap\Omega')] \quad (i=1, \cdots, M).$$

This implies

$$u_0\in F_{T'}[\bar{\Delta}+\Delta(\mathcal{P}): H(\Omega')]\,.$$

Thus, Claim is proved, and hence (1)-2 and 3 are proved. (1)-1 follows from (1)-2 and Definition 3.9. ∎

§7. Proof of the regularity theorem.

In this section, we give a proof of Theorem 4.4.

Consider a differential operator P of the form (1.1) with the coefficients in $\mathcal{F}([0, T]; \mathcal{O}(\Omega))$. First, we rewrite the operator P. Let Γ be the side function of $\Delta(P)$. By $t^j\partial_t^j=\vartheta(\vartheta-1)\cdots(\vartheta-j+1)$, we can write the operator P in the following form:

$$P=\sum_{j+|\alpha|\leq m} b_{j,\alpha}(t, x)t^{\Gamma(j+|\alpha|)}\vartheta^j\partial_x^\alpha, \tag{7.1}$$

where $b_{j,\alpha}\in\mathcal{F}([0, T]; \mathcal{O}(\Omega))$, and $b_{m,0}(t, x)\equiv 1$.

Further, we have $C_\mu(\lambda; x)=\sum_{(j,\Gamma(j))\in L_\mu} b_{j,0}(0, x)\lambda^j$ for $\mu\in S$ (even if $\mu=0$). Under this representation, the conditions (A-1) and (A-2) are equivalent to

(A-1)′ $b_{j,\alpha}(0, x)\equiv 0$ if $\alpha\neq 0$.

(A-2)′ If $(j, 0)\in\hat{V}$, then $b_{j,0}(0, 0)\neq 0$.

Assume the conditions (A-1), (A-2), and (A-3). Put

$$\mathcal{P}=\sum_{j=0}^{m} b_{j,0}(t, x)t^{\Gamma(j)}\vartheta^j \quad\text{and}\quad C=\sum_{\alpha\neq 0} b_{j,\alpha}(t, x)t^{\Gamma(j+|\alpha|)}\vartheta^j\partial_x^\alpha. \tag{7.2}$$

By the assumption (A-1), we have $\Delta(P)=\Delta(\mathcal{P})$ and $\Lambda(P)=\Lambda(\mathcal{P})$. Further, the operator $\mathcal{P}$ also satisfies the conditions (A-2) and (A-3). Since $\mathcal{O}(\Omega')=\text{projlim}_{\overline{\Omega''}\cap\Omega'} E(\Omega'')$, Theorem 6.1 (1) implies that there exists an open neighborhood Ω_0 of 0 in $\boldsymbol{C}^n$ such that, for any $T'\in(0, T]$ and any subdomain Ω' of Ω_0, the operator $\mathcal{P}$ is a bijection as follows:

$$\begin{array}{lccc} \mathcal{P}: & \mathcal{G}_{\Lambda(P),T'}(\mathcal{O}(\Omega')) & \xrightarrow{\sim} & \mathcal{G}_{\Lambda(P)+h(P),T'}(\mathcal{O}(\Omega')) \\ & \cup & & \cup \\ \mathcal{P}: & F_{T'}[\bar{\Delta}+\Delta(P):\mathcal{O}(\Omega')] & \xrightarrow{\sim} & F_{T'}[\bar{\Delta}:\mathcal{O}(\Omega')] \end{array} \tag{7.3}$$

$$\text{if } \bar{\Delta}\in\mathfrak{N} \text{ and } h(\bar{\Delta})<-(\Lambda(P)+h(P)).$$

Also by the assumption (A-1), we can write the operator C in the form

$$C=\sum_{j\leq m-1} c_j(t, x; \partial_x)t^{\Gamma(j)+1/q}\vartheta^j, \tag{7.4}$$

where $q\in\boldsymbol{N}\setminus\{0\}$ and $c_j(t, x; \partial_x)$ are differential operators with respect to x with coefficients in $\mathcal{F}([0, T]; \mathcal{O}(\Omega))$. Hence, for any $\bar{\Delta}\in\mathfrak{N}$, any $T'\in(0, T]$, and any subdomain Ω' of Ω_0, we have

$$C: F_{T'}[\bar{\Delta}+\Delta(P):\mathcal{O}(\Omega')] \longrightarrow F_{T'}[\bar{\Delta}+\Delta_0:\mathcal{O}(\Omega')], \tag{7.5}$$

where $\Delta_0:=\Delta[k]+\Delta(0, -1/q)$ for $k:=\Gamma(m)-\Gamma(m-1)\geq 0$.

Now, assume that $\bar{\Delta}\in\mathfrak{N}$, $h(\bar{\Delta})<-(\Lambda(P)+h(P))$, $u\in\mathcal{G}_{\Lambda(P),T'}(\mathcal{O}(\Omega'))$ and that $Pu\in F_{T'}[\bar{\Delta}:\mathcal{O}(\Omega')]$. Since $u\in\mathcal{G}_{\Lambda(P),T'}(\mathcal{O}(\Omega'))$, we have $u\in F_{T'}[-\Delta(N, a):\mathcal{O}(\Omega')]$

for some $N\in\boldsymbol{N}$ and $a>\Lambda(P)$, by Definition 3.9.

LEMMA 7.1. *For any $r\in\boldsymbol{N}$, there holds*

$$u\in F_{T'}[\bar{\Delta}+\Delta(P):\mathcal{O}(\Omega')]+F_{T'}[-\Delta(N, a)+r\Delta_0:\mathcal{O}(\Omega')]. \tag{7.6}$$

Here, $r\Delta_0$ denotes the sum of r copies of Δ_0.

PROOF. We prove by the induction on r. For $r=0$, this is trivial. Assume (7.6) for $r=k$. By (7.5), we have

$$\mathcal{C}u\in F_{T'}[\bar{\Delta}+\Delta_0:\mathcal{O}(\Omega')]+F_{T'}[-\Delta(N, a)+(k+1)\Delta_0-\Delta(P):\mathcal{O}(\Omega')].$$

Since $F_{T'}[\bar{\Delta}+\Delta_0:\mathcal{O}(\Omega')]\subset F_{T'}[\bar{\Delta}:\mathcal{O}(\Omega')]$, we have

$$f-\mathcal{C}u\in F_{T'}[\bar{\Delta}:\mathcal{O}(\Omega')]+F_{T'}[-\Delta(N, a)+(k+1)\Delta_0-\Delta(P):\mathcal{O}(\Omega')].$$

By the equation $\mathcal{P}u=f-\mathcal{C}u$ and (7.3), we have

$$u\in F_{T'}[\bar{\Delta}+\Delta(P):\mathcal{O}(\Omega')]+F_{T'}[-\Delta(N, a)+(k+1)\Delta_0:\mathcal{O}(\Omega')].$$

Thus, (7.6) holds for $r=k+1$. ∎

Since

$$r\Delta_0=((-\infty, 0]\times[-r/q, \infty))\cup\{(u, v)\in\boldsymbol{R}^2; u\leqq r, v\geqq -r/q+ku\}$$

there holds $-\Delta(N, a)+r\Delta_0\supset\bar{\Delta}+\Delta(P)$ for sufficiently large r. Hence, we have

$$u\in F_{T'}[\bar{\Delta}+\Delta(P):\mathcal{O}(\Omega')].$$

The latter part of the theorem follows easily by Example 3.3-(1).

§ 8. Abstract equations.

In this section, we solve equations of the form $v+Qv=f$ in scales of Banach spaces.

Let $\{X_s\}_{0<s<s_1}$ be an *increasing scale* of Banach spaces. That is, for $0<s<r<s_1$, the Banach space X_s is embedded in X_r and the operator norm of the embedding does not exceed 1. We denote the norm of X_s by $\|\cdot\|_s$.

We consider an operator

$$Q=t^{1/q}\sum_{i,j=0}^{m}C_{m-j}^{(i)}(t)\tilde{\vartheta}^{j-m}\mathcal{E}_i, \tag{8.1}$$

where $q, m\in\boldsymbol{N}\setminus\{0\}$ and the operators $C_{m-j}^{(i)}(t)$, $\mathcal{E}_i$ satisfy the following four conditions:

(B-1) $C_{m-j}^{(i)}\in\bigcap_{0<s<r<s_1}C^0([0, T]; \mathcal{L}(X_s, X_r))$ $(i, j=0, 1, \cdots, m)$.

(B-2) There exists a constant M such that

$$\|C_{m-j}^{(i)}(t)(w)\|_r \leqq \frac{M}{(r-s)^{m-j}}\|w\|_s$$

for any $w \in X_s$, $0<s<r<s_1$, $t \in [0, T]$, $0 \leqq i$, $j \leqq m$.

(B-3) $\mathcal{E}_i \in \mathcal{L}(C^0([0, T']; X_s))$ for any $T' \in (0, T]$ and any $s \in (0, s_1)$ $(0 \leqq i \leqq m)$.

(B-4) $A := \sup_{T', s, i} \|\mathcal{E}_i\|_{\mathcal{L}(C^0([0,T']; X_s))} < \infty$.

REMARK 8.1. The operator $\tilde{\vartheta} = \vartheta + 1$ is an isomorphism:

$$F_T[\Delta(1, 0) : X](\subset C^0([0, T]; X)) \xrightarrow{\sim} C^0([0, T]; X)$$

for any T and X. Hence, $\tilde{\vartheta}^{j-m}$ $(j \leqq m)$ is a continuous transform in $C^0([0, T']; X_s)$ for any T' and s. Thus, the operator Q is a well-defined continuous operator from $C^0([0, T']; X_s)$ to $C^0([0, T']; X_r)$, if $0<s<r<s_1$ and $T' \in (0, T]$. Further, since

$$(\tilde{\vartheta}^{-l} g)(t) = \int_{[0,1]^l} g(\sigma_1 \cdots \sigma_l t) d\sigma_1 \cdots d\sigma_l \quad (l \geqq 1),$$

there holds the following for any Banach space X, any $l \in \boldsymbol{N}$, and any $p \in \boldsymbol{R}$ with $p \geqq 0$.

(8.2) If $g \in C^0([0, T]; X)$ and $\|g(t)\|_X \leqq C_0 t^p$, then $\|(\tilde{\vartheta}^{-l} g)(t)\|_X \leqq \dfrac{C_0}{(p+1)^l} t^p$.

This estimate plays an important role in the proof of the theorem below.

For the operator Q, we consider the equation $v + Qv = f$.

THEOREM 8.2. *Let $0<s_0<s_1 \leqq 1$. For any $s \in (s_0, s_1)$, there exists $\rho \in (0, T]$ such that the following holds:*

For any $f \in C^0([0, T]; X_{s_0})$, there exists a unique $v \in C^0([0, \rho]; X_s)$ satisfying $v + Qv = f$.

In order to prove the theorem, we prepare a lemma, which is proved by a similar method to the proof of Proposition 2 in [**1**].

LEMMA 8.3. *Let $w_0 \in \bigcap_{s_0<s<s_1} C^0([0, T]; X_s)$ be given and assume that*

$$K := \sup_{s_0<s<s_1,\, 0 \leq t \leq T} \|w_0(t)\|_s < \infty \,. \tag{8.3}$$

If we put

$$w_{p+1} = -Qw_p \quad (p=0, 1, \cdots), \tag{8.4}$$

then we have $w_p \in \bigcap_{s_0<s<s_1} C^0([0, T]; X_s)$ $(p=0, 1, \cdots)$ and

$$\|w_p(t)\|_s \leqq K\left(\frac{C_0 t^{1/q}}{(s-s_0)^m}\right)^p \quad (p=0, 1, \cdots) \tag{8.5}$$

for any $s \in (s_0, s_1)$ and $t \in [0, T]$, where $C_0 := (m+1)^2 MAe^{qm}$.

PROOF. We show the estimate by the induction on p. For $p=0$, the estimate is trivial. Assume the estimate (8.5) for p $(p\geqq 0)$. By (8.1) and (B-2), we have

$$\|w_{p+1}(t)\|_s \leqq t^{1/q}\sum_{i,j=0}^{m}\frac{M}{\eta^{m-j}}\|\tilde{\vartheta}^{j-m}\mathcal{E}_i w_p(t)\|_{s-\eta},$$

for any $s\in(s_0, s_1)$ and $\eta\in(0, s-s_0)$.

By the assumption (B-4) and the induction hypothesis, we have

$$\|\mathcal{E}_i w_p(t)\|_{s-\eta} \leqq A\sup_{\tau\in[0,t]}\|w_p(\tau)\|_{s-\eta} \leqq AK\frac{C_0^p t^{p/q}}{(s-\eta-s_0)^{mp}}.$$

By (8.2), we have

$$\|\tilde{\vartheta}^{j-m}\mathcal{E}_i w_p(t)\|_{s-\eta} \leqq AK\frac{C_0^p t^{p/q}}{(s-\eta-s_0)^{mp}(p/q+1)^{m-j}}.$$

Thus, we have

$$\|w_{p+1}(t)\|_s \leqq (m+1)AKM\sum_{j=0}^{m}\frac{C_0^p t^{(p+1)/q}}{\eta^{m-j}(s-s_0-\eta)^{mp}}\left(\frac{q}{p+q}\right)^{m-j}.$$

If $p\geqq 1$, then we take

$$\eta=(s-s_0)\frac{q}{p+q}.$$

We have $s-s_0-\eta=(s-s_0)p/(p+q)$ and hence

$$\frac{1}{\eta^{m-j}(s-s_0-\eta)^{mp}}\left(\frac{q}{p+q}\right)^{m-j}=\left(\frac{p+q}{p}\right)^{mp}\frac{1}{(s-s_0)^{m(p+1)-j}}.$$

Since $s-s_0\leqq 1$ and $\{(p+q)/p\}^p\leqq e^q$, we have

$$\|w_{p+1}(t)\|_s \leqq (m+1)^2 AKM\frac{C_0^p t^{(p+1)/q}e^{qm}}{(s-s_0)^{m(p+1)}}. \tag{8.6}$$

If $p=0$, then $(s-s_0)q/(p+q)=s-s_0$, which is not admissible as a value of η, hence the argument above is not valid. In this case, we take $\eta=(s-s_0)(1-\varepsilon)$ $(\varepsilon\in(0, 1))$. By letting $\varepsilon\downarrow 0$, we also get the estimate (8.6).

Since $C_0=(m+1)^2 MAe^{qm}$, we have the estimate (8.5) for $p+1$. ■

PROOF OF THEOREM 8.2. First, in order to prove the existence of the solution, we put

$$v_0=0,\quad v_p=f-Qv_{p-1}\quad(p=1, 2, \cdots). \tag{8.7}$$

Since $f\in C^0([0, T]; X_{s_0})$ and Q is an operator from $C^0([0, T]; X_s)$ to $C^0([0, T]; X_r)$ for $0<s<r<s_1$, we have $v_p\in C^0([0, T]; X_s)$ $(p\geqq 0)$ for any $s\in(s_0, s_1)$. Put $K:=\sup_{\tau\in[0,T]}\|f(\tau)\|_{s_0}$.

Since $v_{p+1}-v_p=-Q(v_p-v_{p-1})$, we have the following estimate by Lemma 8.3.

$$(8.8)\qquad \|v_{p+1}(t)-v_p(t)\|_s \leqq K\Big(\frac{C_0t^{1/q}}{(s-s_0)^m}\Big)^p \quad (p\geqq 0,\ s_0<s<s_1,\ 0\leqq t\leqq T).$$

By this estimate, $v:=\lim_{p\to\infty} v_p$ exists in $C^0([0, \rho]; X_s)$, if $C_0\rho^{1/q}<(s-s_0)^m$.

Next, we prove the uniqueness. Let $v_i\in C^0([0, \rho]; X_s)$ $(i=1, 2)$, where $\rho\in(0, T]$ and $s\in(s_0, s_1)$. Assume that $v_i=f-Qv_i$ $(i=1, 2)$. Since $v_1-v_2=-Q(v_1-v_2)$, we have by Lemma 8.3 that there exists a constant $\tilde{K}$ such that

$$\|v_1(t)-v_2(t)\|_{s'} \leqq \tilde{K}\Big(\frac{C_0t^{1/q}}{(s'-s)^m}\Big)^p$$

for any $p\geqq 0$, $s'\in(s, s_1)$, $t\in[0, \rho]$. From this estimate, it follows that, if ρ is sufficiently small, then $v_1=v_2$. ■

§ 9. Proof of extended results.

In this section, we prove the results in Section 4, using the results in the preceding sections.

First, we define two scales of Banach spaces as in [1]. (Notations are slightly different.) Let B be an arbitrary bounded open set in $\boldsymbol{R}^n$ such that $\bar{B}\subset\Omega$. For $s>0$, we set $B_s=\bigcup_{a\in B} B(a, s)$, where $B(a, s):=\{z\in\boldsymbol{C}^n; |z-a|<s\}$. Take an arbitrary $s_1\in(0, 1]$. (Later, B and s_1 shall be taken suitably.) The system $(F(B_s))_{0<s<s_1}$ is a decreasing scale of Banach spaces. Since the embedding of $F(B_r)$ into $F(B_s)$ $(0<s<r<s_1)$ has a dense image, $F'(B_s)$ is also embedded in $F'(B_r)$ with embedding of norm $\leqq 1$. Thus, the system $(F'(B_s))_{0<s<s_1}$ is an increasing scale of Banach spaces. We denote by $(X_s)_{0<s<s_1}$ one of the increasing scales $(F(B_{s_1-s}))_{0<s<s_1}$ and $(F'(B_s))_{0<s<s_1}$. The norm of X_s is denoted by $\|\cdot\|_s$.

Next, consider the operator P given in Section 4 and assume the conditions (A-1), (A-2), and (A-3).

Rewriting P as (7.1), we can define $\mathcal{P}$ and $\mathcal{C}$ by (7.2). By the assumptions, there holds $\Delta(P)=\Delta(\mathcal{P})$, $\Lambda(P)=\Lambda(\mathcal{P})$, and the operator $\mathcal{P}$ also satisfies (A-2) and (A-3), which implies that we can apply Theorem 6.1 to $\mathcal{P}$. Further, as for the operator $\mathcal{C}$, the following holds:

PROPOSITION 9.1. (1) *The operator* $\mathcal{C}$ *can be written as follows:*

$$(9.1)\qquad \mathcal{C} = t^{1/q}\sum_{i=0}^{m-1}\sum_{j=i}^{m-1} C_{m-j}^{(i)}(t)\tilde{\vartheta}^{j-i}t^{\Gamma(m-i)},$$

where $q\in\boldsymbol{N}\setminus\{0\}$, Γ *is the side function of* $\Delta(P)$, *and*

$$C_{m-j}^{(i)}(t) = C_{m-j}^{(i)}(t, x; \partial_x) = \sum_{|\alpha|\leqq m-j} c_{j,\alpha}^{(i)}(t, x)\partial_x^\alpha,$$

$$\text{with } c_{j,\alpha}^{(i)}\in\mathcal{F}([0, T]; \mathcal{O}(\Omega)).$$

(2) *The operators* $C_{m-j}^{(i)}(t)$ *satisfy the condition* (B-2) *in Section* 8. (*We put*

$C_{m-j}^{(i)}=0$ *unless* $0\leqq i\leqq j\leqq m-1$.)

(3) *Put* $\Delta_0:=\Delta[k]+\Delta(0, -1/q)$ *for* $k:=\Gamma(m)-\Gamma(m-1)\geqq 0$ *as in Section* 7. *For any* $\bar{\Delta}\in\mathfrak{N}$, *there holds*

$$\mathcal{C}: F_{T'}[\bar{\Delta}+\Delta(P): X_s] \longrightarrow F_{T'}[\bar{\Delta}+\Delta_0: X_r] \quad (0<s<r<s_1).$$

PROOF. (1) By (7.2) and (A-1)′ in Section 7, we can write as

$$\mathcal{C} = \sum_{\alpha\neq 0} \tilde{b}_{j,\alpha}(t, x)t^{\Gamma(j+|\alpha|)+1/q}\vartheta^j\partial_x^\alpha$$

$$= t^{1/q}\sum_{i=1}^{m-1}\sum_{j=0}^{m-i-1}\Big(\sum_{|\alpha|=m-i-j}\tilde{b}_{j,\alpha}(t, x)\partial_x^\alpha\Big)t^{\Gamma(m-i)}\vartheta^j$$

for some $q\in\boldsymbol{N}\setminus\{0\}$ and $\tilde{b}_{j,\alpha}\in\mathcal{F}([0, T]; \mathcal{O}(\Omega))$. Since $t^a\vartheta^j=(\vartheta-a)^jt^a$ and $\vartheta=\tilde{\vartheta}-1$, the operator $\mathcal{C}$ can be written in the form (9.1).

It is well-known that the estimate in (B-2) follows from the Cauchy's integral formula.

Since the operator $\mathcal{C}$ can be also written as (7.4), we have (3) in the same way as (7.5). ■

Take Ω_0 in Theorem 6.1 for $\bar{\Delta}=\Delta(0, 0)$. Put

$$\mathcal{E}_i:=\tilde{\vartheta}^{m-i}t^{\Gamma(m-i)}\mathcal{P}^{-1} \quad (i=0, 1, \cdots, m).$$

If $\bar{B}_{s_1}\subset\Omega_0$, then the operators $\mathcal{E}_i$ satisfy the conditions (B-3) and (B-4) in Section 8. Thus, the operator

$$\mathcal{Q}:=\mathcal{C}\mathcal{P}^{-1}=t^{1/q}\sum_{i,j=0}^{m} C_{m-j}^{(i)}(t)\tilde{\vartheta}^{j-m}\mathcal{E}_i \tag{9.2}$$

satisfies the conditions on $\mathcal{Q}$ in Section 8.

Now, we prove the following proposition, by which Theorem 4.3 and Theorem 4.6 are proved.

PROPOSITION 9.2. *For any* $\bar{\Delta}\in\mathfrak{N}$ *satisfying* $h(\bar{\Delta})<-(\Lambda(P)+h(P))$, *there exists an open neighborhood* Ω_1 *of* 0 *in* $\boldsymbol{C}^n$ *for which, if* B *and* s_1 *satisfy* $\bar{B}_{s_1}\subset\Omega_1$, *then the following holds:*

Let $s_0\in(0, s_1)$ *and* $s\in(s_0, s_1)$. *There exists* $\rho\in(0, T]$ *such that, for any* $f\in F_T[\bar{\Delta}: X_{s_0}]$, *there exists a unique* $u\in F_\rho[\bar{\Delta}+\Delta(P): X_s]$ *satisfying* $Pu=f$.

Further, in the case of $X_s=F(B_{s_1-s})$, *we can take* Ω_1 *and* ρ *independently of* $\bar{\Delta}$. *Hence, in this case, for any* $f\in\mathcal{G}_{\Lambda(P),T}(X_{s_0})$, *there exists a unique* $u\in\mathcal{G}_{\Lambda(P)+h(P),\rho}(X_s)$ *satisfying* $Pu=f$.

PROOF. By considering $\tilde{P}:=t^{h(\bar{\Delta})}Pt^{-h(\bar{\Delta})-h(P)}$, we can assume that $h(\bar{\Delta})=h(P)=0>\Lambda(P)$, without loss of generality. (Note that $\Lambda(\tilde{P})=\Lambda(P)+h(\bar{\Delta})+h(P)$.)

Take $M\in\boldsymbol{N}$ as $\bar{\Delta}+M\Delta_0\supset\Delta(0, 0)$. By Theorem 6.1, there exists an open neighborhood $\Omega_1 \subset \Omega_0$ of 0 such that, if $\bar{B}_{s_1} \subset \Omega_1$, then the operator $\mathcal{P}$ is a bijection from $F_{T'}[\bar{\Delta}+r\Delta_0+\Delta(\mathcal{P}) : X_s]$ to $F_{T'}[\bar{\Delta}+r\Delta_0 : X_s]$ for any $T' \in (0, T]$, $s \in (s_0, s_1)$, and for $r=0, 1, \cdots, M$. Note that Ω_1 can be taken independently of $\bar{\Delta} \in \mathfrak{N}$ in the case of $X_s = F(B_{s_1-s})$. Assume that $\bar{B}_{s_1} \subset \Omega_1$.

1. PROOF OF EXISTENCE. Let $f \in F_T[\bar{\Delta} : X_{s_0}]$ for some $s_0 \in (0, s_1)$. We can take $w_1 \in F_T[\bar{\Delta}+\Delta(P) : X_{s_0}]$ such that $\mathcal{P}w_1 = f$. By Proposition 9.1 (3), we have $f_1 := -Cw_1 \in F_T[\bar{\Delta}+\Delta_0 : X_s]$ for any $s \in (s_0, s_1)$. By $u = w_1 + u_1$, the equation $Pu = f$ is reduced to $Pu_1 = f_1$. By iterating this reduction, the equation $Pu = f$ is reduced to

$$Pu_M = f_M \in \bigcap_{s_0<s<s_1} F_T[\bar{\Delta}+M\Delta_0 : X_s] \subset \bigcap_{s_0<s<s_1} C^0([0, T]; X_s).$$

Take an arbitrary $s_0' \in (s_0, s_1)$ and consider $Pu = f \in C^0([0, T]: X_{s_0'})$. By putting $\mathcal{P}u = v$, this equation is written as $v + Qv = f$. By Theorem 8.2, for any $s \in (s_0', s_1)$, there exists $\rho \in (0, T]$ such that there exists a unique $v \in C^0([0, \rho] : X_s)$ satisfying $v + Qv = f$. Since s_0' is arbitrary, we obtain the existence of solution in the proposition.

2. PROOF OF UNIQUNESS. Let $u \in F_\rho[\bar{\Delta}+\Delta(P) : X_s]$ satisfy $Pu = 0$. By an argument similar to that in Section 7, we can show that $u \in \bigcap_{s' \in (s, s_1)} C^0([0, \rho] : X_{s'})$. By the uniqueness in Theorem 8.2, if ρ is sufficiently small, then we have $u = 0$. ∎

If Ω' is sufficiently small, then there holds $\mathcal{O}(\Omega) \subset F(\Omega')$. Hence, Theorem 4.3 follows trivially from the proposition above, by Definition 3.9.

We prove Theorem 4.6, by a similar argument to the proof of Theorem 4 in [1].

PROOF OF THEOREM 4.6. Let $u \in \mathcal{G}_{\Lambda(P), T}(\mathcal{D}'(\Omega \cap \boldsymbol{R}^n))$ and $Pu = 0$ in $(0, T) \times \tilde{B}$, where $\tilde{B}$ is an open neighborhood of 0 in $\boldsymbol{R}^n$. There exists $N \in \boldsymbol{N}$ and $a > \Lambda(P)$ such that $u \in F_T[-\Delta(N, a) : \mathcal{D}'(\Omega \cap \boldsymbol{R}^n)]$.

Take Ω_1 in Proposition 9.2 for $\bar{\Delta} = -\Delta(N, a) - \Delta(P)$. We may assume that $\bar{\tilde{B}} \subset \Omega_1$.

Take an open neighborhood ω of 0 in $\boldsymbol{R}^n$ such that $\bar{\omega} \subset \tilde{B}$ and take $\varphi \in C_0^\infty(\tilde{B})$ such that $\varphi = 1$ near ω. Put

$$v := \varphi u \in F_T[-\Delta(N, a) : \mathcal{E}'(\tilde{B})] \subset F_T[-\Delta(N, a) : F'(\tilde{B}_s)] \quad (s>0).$$

If we put $f := Pv$, then we have $f \in F_T[-\Delta(N, a) - \Delta(P) : \mathcal{E}'(B)]$, where $B := \tilde{B} \setminus \bar{\omega}$. Hence, we have $f \in F_T[-\Delta(N, a) - \Delta(P) : F'(B_s)]$ for any $s > 0$.

Fix a sufficiently small $s > 0$. By Proposition 9.2, there exist $\rho \in (0, T]$ and $u_0 \in F_\rho[-\Delta(N, a) : F'(B_s)]$ such that $Pu_0 = f$. Since $F'(B_s) \subset F'(\tilde{B}_s)$, both of u_0 and v are solutions of the equation $Pu = f$ in $F_\rho[-\Delta(N, a) : F'(\tilde{B}_s)]$. Hence, by the uniqueness in Proposition 9.2, there exists $\rho' \in (0, \rho]$ such that $u_0 = v$ in

$F_{\rho'}[-\Delta(N, a): F'(\tilde{B}_s)]$. This means that

$$v \in F_{\rho'}[-\Delta(N, a): F'(B_s)] \cap F_{\rho'}[-\Delta(N, a): \mathcal{E}'(\tilde{B})] .$$

Since $\tilde{\vartheta}^N$ is a bijection from $C^0([0, \rho']: X)$ to $F_{\rho'}[-\Delta(N, 0): X]$, we can take $\tilde{v} \in C^0([0, \rho']; F'(B_s)) \cap C^0([0, \rho']; \mathcal{E}'(\tilde{B}))$ such that $t^a \tilde{\vartheta}^N(\tilde{v}) = v$. Since $\tilde{v}(t) \in F'(B_s)$ $(t \in [0, \rho'])$, we obtain $\tilde{v}(t)|_{\omega_1} = 0$ for any $t \in [0, \rho']$ by Lemma 5 in [**1**], where ω_1 is an open neighborhood of 0 in $\boldsymbol{R}^n$ such that $\bar{\omega}_1 \cap \bar{B}_s = \emptyset$. Thus, we obtain $v = 0$ in $(0, \rho') \times \omega_1$, which implies that $u = 0$ in $(0, \rho') \times \omega_1$. ■

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