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Dolbeault isomorphisms for holomorphic vector bundles over holomorphic fiber spaces and applications

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Introduction.

In the previous papers, we calculated the $\bar{\partial}$-cohomology groups of complex Lie groups ([4], [5]) and some family of weakly pseudoconvex manifolds ([3]), using several kinds of Dolbeault isomorphisms. These manifolds treated in [3], [4] and [5] are weakly pseudoconvex and, in general, noncompact, but they have the structure of the fiber space with base a compact complex manifold and fiber a Stein manifold. In this paper, we extend these Dolbeault isomorphisms to the $\bar{\partial}$-cohomology for holomorphic vector bundles over locally trivial holomorphic fiber spaces whose fibers are Stein manifolds, which is a generalization of the results of [5].

Let M be a locally trivial holomorphic fiber space over a paracompact complex manifold N whose fibers are biholomorphic onto a Stein manifold and $E\rightarrow M$ be a holomorphic vector bundle over M. Let $\Omega_M^r(E)$ be the sheaf of germs of holomorphic r-forms with values in E, $\mathcal{F}$ be the sheaf of germs of C^∞ functions in M which is holomorphic along the fibers, $\mathcal{F}^{r,p}$ be the sheaf of germs of (r, p)-forms with coefficients in $\mathcal{F}$ and $\mathcal{F}^{r,p}(E):=\mathcal{F}^{r,p}\otimes\Omega_M^0(E)$. We get a resolution of sheaves

$$0 \longrightarrow \Omega_M^r(E) \longrightarrow \mathcal{F}^{r,0}(E) \longrightarrow \mathcal{F}^{r,1}(E) \longrightarrow \cdots \longrightarrow \mathcal{F}^{r,q}(E) \longrightarrow 0,$$

where $q=\dim_C N$.

In §1, we obtain vanishing theorems $H^k(M, \mathcal{F}^{r,p}(E))=0$ $(k\geqq 1)$ and a Dolbeault isomorphism for $H^p(M, \Omega_M^r(E))$ $(p\geqq 0)$.

It is known that any complex Lie group has a fibration with base a complex torus and fiber a Stein group ([9]). Using this fact, in §2 we shall apply the results in §1 to the calculation of $H^p(G, \Omega_G^r)$, where G is any complex Lie group.

1. Dolbeault isomorphisms for holomorphic vector bundles.

Throughout this paper, for a complex manifold X and a holomorphic vector bundle E over X, we denote by $\mathcal{E}_X$ the sheaf of germs of C^∞ functions on X

and $\Omega_X^r(E)$ (resp. $\mathcal{E}_X^{r,p}(E)$) the sheaf of germs of holomorphic r-forms (resp. $C^\infty(r, p)$-forms) with values in E.

Let M be a locally trivial holomorphic fiber space over a paracompact complex manifold N of complex dimension q, whose fibers are biholomorphic onto a Stein manifold S of complex dimension s. We denote by π the projection $M \to N$. Let $\{D_\alpha\}$ be a locally finite open covering of N, where D_α is biholomorphic onto a polydisc in $\boldsymbol{C}^q$ for each α, with a family of biholomorphic mappings $i_\alpha: \pi^{-1}(D_\alpha) \to D_\alpha \times S$ and local coordinates $z_\alpha = (z_\alpha^1, \cdots, z_\alpha^q)$ in D_α. Let $\{U_\sigma\}$ be an open covering of S with a local coordinate $w_\sigma = (w_\sigma^1, \cdots, w_\sigma^s)$ in U_σ, for each σ. We sometimes identify $\pi^{-1}(D_\alpha)$ with $D_\alpha \times S$. Then $\{D_\alpha \times U_\sigma\}$ is an open covering of M and $\zeta_{\alpha,\sigma} = (\zeta_{\alpha,\sigma}^1, \cdots, \zeta_{\alpha,\sigma}^{q+s}) := (z_\alpha^1, \cdots, z_\alpha^q, w_\sigma^1, \cdots, w_\sigma^s)$ is a local coordinate in $D_\alpha \times U_\sigma$, for each α and σ. For an open subset $V \subset M$, we put $\mathcal{F}(V) := \{f;\ f$ is of class C^∞ in V and for any $z \in \pi(V)$, $f | \pi^{-1}(z) \cap V$ is holomorphic$\}$. We denote by $\mathcal{F}$ the sheaf defined by the presheaf $\{\mathcal{F}(V)\}$. We call $\mathcal{F}$ the sheaf on the fiber space M holomorphic along the fiber. Put $V_{\alpha,\sigma} := V \cap (D_\alpha \times U_\sigma)$ and

$$\begin{aligned}\mathcal{F}^{r,p}(V) := \{\varphi;\ &\varphi \text{ is a } C^\infty(r, p)\text{-form on } V \text{ and } \varphi | V_{\alpha,\sigma} \\ &= \sum_{I,J} \varphi_{IJ}\, d\zeta_{\alpha,\sigma}^I \wedge d\bar{z}_\sigma^J,\quad \varphi_{IJ} \in \mathcal{F}(V_{\alpha,\sigma}) \text{ for each } \alpha \text{ and } \sigma, \\ &\text{where } I = (i_1, \cdots, i_r),\ J = (j_1, \cdots, j_p), \\ &1 \leqq i_1 < \cdots < i_r \leqq q+s, \text{ and } 1 \leqq j_1 < \cdots < j_p \leqq q\}.\end{aligned}$$

We get the sheaf $\mathcal{F}^{r,p}$ defined by the presheaf $\{\mathcal{F}^{r,p}(V)\}$ for $0 \leqq r \leqq q+s$ and $0 \leqq p \leqq q$. Let $E \to M$ be a holomorphic vector bundle and $\mathcal{O}_M(E)$ the sheaf of germs of holomorphic sections of E. We put $\mathcal{F}(E) := \mathcal{F} \otimes \mathcal{O}_M(E)$ and $\mathcal{F}^{r,p}(E) := \mathcal{F}^{r,p} \otimes \mathcal{O}_M(E)$. For each α, there exists a holomorphic vector bundle E_α over S such that the induced bundle $\pi_\alpha^* E_\alpha$ on $D_\alpha \times S$ by the canonical projection $\pi_\alpha: D_\alpha \times S \to S$ is isomorphic to $E | D_\alpha \times S$. For any open subset $D \subset D_\alpha$ and $U \subset S$,

$$\begin{aligned}H^0(D \times U, \mathcal{F}^{r,p}(E)) &\cong H^0(D \times U, \mathcal{F}^{r,p}(\pi_\alpha^* E_\alpha)) \\ &\cong \oplus_{r'+r''=r} H^0(D, \mathcal{E}_N^{r',p}) \hat{\otimes} H^0(U, \Omega_S^{r''}(E_\alpha)),\end{aligned}$$

where $\hat{\otimes}$ denotes the topological tensor product. We denote by $\mathcal{E}_\alpha^{r',p} \hat{\otimes} \Omega_S^{r''}(E_\alpha)$ the sheaf on $U_\alpha \times S$ defined by the presheaf $\{H^0(D, \mathcal{E}_N^{r',p}) \hat{\otimes} H^0(U, \Omega_S^{r''}(E_\alpha))\}$. Then we have

$$\mathcal{F}^{r,p}(E) | D_\alpha \times S \cong \oplus_{r'+r''=r}\, \mathcal{E}_\alpha^{r',p} \hat{\otimes} \Omega_S^{r''}(E_\alpha), \tag{1.1}$$

for each α. We have an exact sequence of sheaves,

$$0 \longrightarrow \Omega^r(E) \longrightarrow \mathcal{F}^{r,0}(E) \longrightarrow \mathcal{F}^{r,1}(E) \longrightarrow \cdots \longrightarrow \mathcal{F}^{r,q}(E) \longrightarrow 0. \tag{1.2}$$

LEMMA 1.1. $H^k(M, \mathcal{F}^{r,p}(E))=0,\ k\geqq 1$.

PROOF. By (1.1), for each α,

$$H^k(D_\alpha\times S, \mathcal{F}^{r,p}(E)) \cong H^k(D_\alpha\times S, \oplus_{r'+r''=r}\mathcal{E}_\alpha^{r',p}\hat{\otimes}\Omega_S^{r''}(E_\alpha))$$
$$\cong \oplus_{u+v=k,\, r'+r''=r} H^u(D_\alpha, \mathcal{E}_\alpha^{r',p})\hat{\otimes}H^v(S, \Omega_S^{r''}(E_\alpha)) = 0\,,$$

by Künneth's formula ([2]). Hence, $\mathfrak{W}:=\{\pi^{-1}(D_\alpha)\}$ is a Leray covering for $\mathcal{F}^{r,p}(E)$ on M. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to the covering $\{D_\alpha\}$ on N. For $\{f_{\alpha\beta}\}\in Z^1(\mathfrak{W}, \mathcal{F}^{r,p}(E))$ we put $g_\beta(p):=\sum_\alpha\rho_\alpha\cdot\pi(p)f_{\alpha\beta}(p)$ $(p\in \pi^{-1}(D_\beta))$. Then $\{g_\alpha\}\in C^0(\mathfrak{W}, \mathcal{F}^{r,p}(E))$ and $\delta\{g_\alpha\}=\{f_{\alpha\beta}\}$. Hence $H^1(M, \mathcal{F}^{r,p}(E))=0$. Similarly we have $H^k(M, \mathcal{F}^{r,p}(E))=0,\ k\geqq 1$.

By this lemma and (1.2), we obtain the following

THEOREM 1.1. *Let M be a locally trivial holomorphic fiber space over a paracompact complex manifold N of complex dimension q, whose fibers are biholomorphic onto a Stein manifold and $E\to M$ be a holomorphic vector bundle. Then*

$$H^p(M, \Omega_M^r(E)) \cong \frac{\{\varphi\in H^0(M, \mathcal{F}^{r,p}(E))\,;\ \bar\partial\varphi=0\}}{\bar\partial H^0(M, \mathcal{F}^{r,p-1}(E))}$$

for $q\geqq p\geqq 1$.

We note that $H^p(M, \Omega_M^r(E))=0$ for $p\geqq q+1$ by the above theorem and that in case N is a Stein manifold, by the result of B. Jennane [1], $H^p(M, \Omega_M^r(E))=0$ for $p\geqq 2$. Professor Ohsawa kindly pointed out to the authors that M is strongly $(q+1)$-complete for any paracompact complex manifold N. Thus, for any coherent analytic sheaf $\mathcal{A}$ on M, we have $H^p(M, \mathcal{A})=0$ for $p\geqq q+1$.

2. $\bar\partial$ cohomology of complex Lie groups.

Let G be a connected complex Lie group with the Lie algebra $\mathfrak{G}$, G^0 the maximal toroidal subgroup of G, K a maximal compact real Lie subgroup of G with the Lie algebra $\mathfrak{K}$, K_C the complex Lie subgroup with the Lie subalgebra $\mathfrak{K}_C:=\mathfrak{K}+\sqrt{-1}\mathfrak{K}$ of $\mathfrak{G}$ and Z the connected center of K_C. By the result of Matsushima [6], G is biholomorphic onto $K_C\times \boldsymbol{C}^a$ and there exists a connected Stein subgroup S_0 of K_C such that the mapping

$$\rho_0:\ Z\times S_0 \ni (x, y)\longmapsto xy\in K_C$$

is a finite covering homomorphism. By the result of Morimoto [7], G^0 is a closed subgroup of Z and $Z\cong G^0\times \boldsymbol{C}^{*r}\times \boldsymbol{C}^u$ for some non-negative integers r and u. Taking a Stein subgroup $S:=\boldsymbol{C}^{*r}\times\boldsymbol{C}^u\times S_0\times\boldsymbol{C}^a$ of $Z\times S_0\times \boldsymbol{C}^a$, we get a finite covering homomorphism

$$\rho:\ G^0\times S\ni(x_0,\ x_1,\ x_2,\ x_3,\ x_4)\longmapsto(\rho_0((x_0,\ x_1,\ x_2),\ x_3),\ x_4)\in G\,.$$

Let $\pi_1:\ G^0\times S\to G^0$ and $\pi_2:\ G^0\times S\to S$ be the canonical projections. We get a homomorphism

$$\pi:\ G^0\times S/\mathrm{Ker}\,\rho\ni(a,\ b)\,\mathrm{Ker}\,\rho\longmapsto b\pi_2(\mathrm{Ker}\,\rho)\in S/\pi_2(\mathrm{Ker}\,\rho) \tag{2.1}$$

for $a\in G^0$, $b\in S$. From this projection π, $G\cong G^0\times S/\mathrm{Ker}\,\rho$ is regarded as a fiber bundle over the Stein group $\tilde{S}:\ =S/\pi_2(\mathrm{Ker}\,\rho)$ whose fiber is isomorphic onto G^0 and the structure group is the finite subgroup $\pi_1(\mathrm{Ker}\,\rho)$ of G_0. Let $\{U_\alpha\}$ be a locally finite Stein open covering of $\tilde{S}$ with a family of biholomorphic mappings $h_\alpha:\ \pi^{-1}(U_\alpha)\to U_\alpha\times G^0$ and local coordinates $w_\alpha=(w_\alpha^1,\ \cdots,\ w_\alpha^l)$ in U_α. We put $G^0=\boldsymbol{C}^n/\Gamma$, where Γ is a discrete lattice of $\boldsymbol{C}^n$ generated by $\boldsymbol{R}$-linearly independent vectors $\{e_1,\ \cdots,\ e_n,\ v_1=(v_{11},\ \cdots,\ v_{1n}),\ \cdots,\ v_q=(v_{q1},\ \cdots,\ v_{qn})\}$ over $\boldsymbol{Z}$ and e_i denotes the i-th unit vector of $\boldsymbol{C}^n$, $i=1,\ \cdots,\ n$. We sometimes identify $\pi^{-1}(U_\alpha)$ with $U_\alpha\times\boldsymbol{C}^n/\Gamma$. We may assume $\det[\mathrm{Im}\,v_{ij};\ 1\leqq i,\ j\leqq q]\neq 0$. We put $v_i=(v_{i1},\ \cdots,\ v_{in}):=\sqrt{-1}e_i$ for $q+1\leqq i\leqq n$, $\beta_{ij}:=\mathrm{Im}\,v_{ij}$ for $1\leqq i,\ j\leqq n$, and $[\gamma_{ij};\ 1\leqq i,\ j\leqq n]:=[\beta_{ij};\ 1\leqq i,\ j\leqq n]^{-1}$. Let $z=(z^1,\ \cdots,\ z^n)$ be the natural coordinate in $\boldsymbol{C}^n$. Putting

$$(z^1,\ \cdots,\ z^n)=\textstyle\sum_{i=1}^n(t^ie_i+t^{n+i}v_i)\,, \tag{2.2}$$

the mapping $\phi:\ \boldsymbol{C}^n\ni z=(z^1,\ \cdots,\ z^n)\mapsto t=(t^1,\ \cdots,\ t^{2n})\in\boldsymbol{R}^{2n}$ induces an isomorphism as a real Lie group $\phi:\ \boldsymbol{C}^n/\Gamma\to\boldsymbol{R}^{2n}/\phi(\Gamma)=\boldsymbol{T}^{n+q}\times\boldsymbol{R}^{n-q}$, where $\boldsymbol{T}^{n+q}$ is a real torus of real dimension $n+q$. We set

$$K_{m,i}:=\textstyle\sum_{j=1}^n v_{ij}m_j-m_{n+i}\quad\text{and}\quad K_m:=\max\{|K_{m,i}|\,;\ 1\leqq i\leqq q\}$$

for $m=(m_1,\ m_2,\ \cdots,\ m_{n+q})\in\boldsymbol{Z}^{n+q}$. Since $\boldsymbol{C}^n/\Gamma$ is toroidal, $K_m>0$ for any $m\in\boldsymbol{Z}^{n+q}\setminus\{0\}$ ([8]).

DEFINITION 2.1. We say that a toroidal group $\boldsymbol{C}^n/\Gamma$ is of finite type if $\boldsymbol{C}^n/\Gamma$ satisfies the following condition:

There exists $a>0$ such that

$$\sup_{m\neq 0}\exp(-a\|m^*\|)/K_m<\infty,\ \text{where}\ \|m^*\|=\max\{|m_i|\,;\ 1\leqq i\leqq n\}\,. \tag{2.3}$$

The condition (2.3) is equivalent to the following condition:

(2.4) For any $\varepsilon>0$ there exists $a>0$ such that

$$\sup_{m\neq 0}\exp(-\varepsilon\|m'\|-a\|m''\|)/K_m<\infty\,,$$

where $\|m'\|=\max\{|m_i|,\ |m_{n+i}|\,;\ 1\leqq i\leqq q\}$. By the results of [4, 10], a toroidal group $\boldsymbol{C}^n/\Gamma$ of finite type satisfies

$$\dim H^p(G^0, \mathcal{O}) = \begin{cases} \dfrac{q!}{(q-p)!p!} & \text{if } 0 \leqq p \leqq q \\ 0 & \text{if } p > q. \end{cases}$$

For $g \in \pi^{-1}(U_\alpha)$, we put $h_\alpha(g) = (w_\alpha^1(g), \cdots, w_\alpha^l(g), [z_\alpha(g)]) \in U_\alpha \times \boldsymbol{C}^n/\Gamma$, $[z_\alpha(g)] = (z_\alpha^1(g), \cdots, z_\alpha^n(g)) + \Gamma \in \boldsymbol{C}^n/\Gamma$ and $[t_\alpha(g)] = \phi([z_\alpha(g)]) = (t_\alpha^1(g), \cdots, t_\alpha^{2n}(g)) + \phi(\Gamma) \in \boldsymbol{T}^{n+q} \times \boldsymbol{R}^{n-q}$. From (2.2), we have

$$z_\alpha^i = t_\alpha^i + \textstyle\sum_{j=1}^n t_\alpha^{n+j} v_{ji}, \qquad i=1, \cdots, n. \tag{2.5}$$

For $g \in \pi^{-1}(U_\alpha \cap U_\beta)$,

$$h_\alpha h_\beta^{-1}(w_\beta(g), [z_\beta(g)]) = (w_\alpha(g), f_{\alpha\beta}[z_\beta(g)]) = (w_\alpha(g), [z_\alpha(g)]), \tag{2.6}$$

where $f_{\alpha\beta}: U_\alpha \cap U_\beta \to \pi_1(\mathrm{Ker}\,\rho) \subset \boldsymbol{C}^n/\Gamma$ is holomorphic. We put $\phi \circ f_{\alpha\beta} = (f_{\alpha\beta}^1, \cdots, f_{\alpha\beta}^{2n}) + \phi(\Gamma) \subset \boldsymbol{T}^{n+q} \times \boldsymbol{R}^{n-q}$. Since $\pi_1(\mathrm{Ker}\,\rho)$ is a finite subgroup, $f_{\alpha\beta}$ is locally constant and $f_{\alpha\beta}^{n+i} = 0$ for $i = q+1, \cdots, n$.

From (2.6) we have

$$\begin{aligned} & t_\alpha^i = t_\beta^i + f_{\alpha\beta}^i + n_{\alpha\beta}^i \quad \text{for some integer } n_{\alpha\beta}^i \ (i=1, \cdots, n+q) \\ & \text{and} \quad t_\alpha^{n+j} = t_\beta^{n+j} \ (j = q+1, \cdots, n). \end{aligned} \tag{2.7}$$

From (2.5) and (2.7), it follows that

$$\begin{aligned} & z_\alpha^i = z_\beta^i + f_{\alpha\beta}^i + n_{\alpha\beta}^i + \textstyle\sum_{j=1}^q (f_{\alpha\beta}^{n+j} + n_{\alpha\beta}^{n+j} v_{ji}), \\ & dz_\alpha^i = dz_\beta^i \quad \text{and} \quad d\bar{z}_\alpha^i = d\bar{z}_\beta^i, \quad (i = 1, \cdots, n). \end{aligned} \tag{2.8}$$

Hence dz_α^i and $d\bar{z}_\alpha^i$ are global 1-forms on G, $i=1, \cdots, n$. We get the following fibration of G([**9**]). We denote by π_q the projection $\boldsymbol{C}^n \ni (z^1, \cdots, z^n) \mapsto (z^1, \cdots, z^q) \in \boldsymbol{C}^q$. Since $\pi_q(e_i)$, $\pi_q(v_i)$ $(1 \leqq i \leqq q)$ are $\boldsymbol{R}$-linearly independent, π_q induces the $\boldsymbol{C}^{*n-q}$-principal bundle

$$\pi_q: \boldsymbol{C}^n/\Gamma \ni z + \Gamma \longmapsto \pi_q(z) + \Gamma^* \in \boldsymbol{T}_{\boldsymbol{C}}^q := \boldsymbol{C}^q/\Gamma^*$$

over the complex q-dimensional torus $\boldsymbol{T}_{\boldsymbol{C}}^q$, where $\Gamma^* := \pi_q(\Gamma)$. We also denote by π_q the projection $\boldsymbol{C}^n/\Gamma \times S \ni (a, b) \mapsto \pi_q(a) \in \boldsymbol{T}_{\boldsymbol{C}}^q$. Then $\mathrm{Ker}\,\pi_q \cong \boldsymbol{C}^{*n-q} \times S$ is a closed Stein subgroup of $\boldsymbol{C}^n/\Gamma \times S$. Thus π_q defines a fiber bundle $G \to \boldsymbol{T}_{\boldsymbol{C}}^q/\pi_q(\mathrm{Ker}\,\rho)$ over the complex torus $\boldsymbol{T}_{\boldsymbol{C}}^q/\pi_q(\mathrm{Ker}\,\rho)$, with fiber which is isomorphic onto the closed Stein subgroup $\rho(\mathrm{Ker}\,\pi_q) \subset G$. In the fibration given by (2.1), for each α, π_q sends $g \in \pi^{-1}(U_\alpha)$ to $(z_\alpha^1(g), \cdots, z_\alpha^q(g)) + \Gamma^* + \pi_q(\mathrm{Ker}\,\rho) \in \boldsymbol{T}_{\boldsymbol{C}}^q/\pi_q(\mathrm{Ker}\,\rho)$. Hence $(z_\alpha^1, \cdots, z_\alpha^q)$ defines a local coordinate in $\boldsymbol{T}_{\boldsymbol{C}}^q/\pi_q(\mathrm{Ker}\,\rho)$ and $(w_\alpha^1, \cdots, w_\alpha^l, z_\alpha^{q+1}, \cdots, z_\alpha^n)$ defines a local coordinate in the fibers of $\pi_q: G \to \boldsymbol{T}_{\boldsymbol{C}}^q/\pi_q(\mathrm{Ker}\,\rho)$. Let $\mathcal{F}$ be the sheaf on the fiber bundle $\pi_q: G \to \boldsymbol{T}_{\boldsymbol{C}}^q/\pi_q(\mathrm{Ker}\,\rho)$, which is holomorphic along the fibers. Then

(2.9)
$$\varphi \in H^0(G, \mathcal{F}) \text{ if and only if } \varphi \text{ is of class } C^\infty \text{ and } \frac{\partial\varphi}{\partial\bar{w}_\alpha^i}=0,$$
$$\frac{\partial\varphi}{\partial\bar{z}_\alpha^j}=0, \ (i=1, \cdots, l, \ j=q+1, \cdots, n) \text{ for each } \alpha.$$

We put $\varphi_\alpha := \varphi|\pi^{-1}(U_\alpha)$. We expand φ_α on $\pi^{-1}(U_\alpha)\cong U_\alpha\times\boldsymbol{C}^n/\Gamma$: $\varphi_\alpha(w_\alpha, t_\alpha) = \sum_{m\in Z^{n+q}} a_\alpha^m(w_\alpha, t''_\alpha)\exp(2\pi\sqrt{-1}\langle m, t'_\alpha\rangle)$, where $t'_\alpha := (t_\alpha^1, \cdots, t_\alpha^{n+q})$, $t''_\alpha := (t_\alpha^{n+q+1}, \cdots, t_\alpha^{2n})$, $\langle m, t'_\alpha\rangle := \sum_{i=1}^{n+q} m_i t_\alpha^i$ and $a_\alpha^m \in H^0(U_\alpha\times\boldsymbol{R}^{n-q}, \mathcal{E})$. We put $\varphi_\alpha^m(w_\alpha, t_\alpha) := a_\alpha^m(w_\alpha, t''_\alpha)\exp(2\pi\sqrt{-1}\langle m, t'_\alpha\rangle)$. Then $\varphi_\alpha = \sum_{m\in Z^{n+q}}\varphi_\alpha^m$. From (2.5) we get

(2.10)
$$\frac{\partial\varphi_\alpha^m}{\partial\bar{z}_\alpha^i} = \begin{cases} \pi\sum_{k=1}^q \gamma_{ik} K_{m,k}\varphi_\alpha^m(w_\alpha, t_\alpha) & (1\leqq i\leqq q) \\ \sqrt{-1}\left(\pi m_i a_\alpha^m(w_\alpha, t''_\alpha)+\dfrac{1}{2}\dfrac{\partial a_\alpha^m(w_\alpha, t''_\alpha)}{\partial t_\alpha^{n+i}}\right)\exp(2\pi\sqrt{-1}\langle m, t'_\alpha\rangle) & \\ & (q+1\leqq i\leqq n). \end{cases}$$

From (2.9) and (2.10), we can write

(2.11) $$\varphi_\alpha^m(w_\alpha, t_\alpha) := c_\alpha^m(w_\alpha)\exp(-2\pi\textstyle\sum_{i=q+1}^n m_i t_\alpha^{n+i})\exp(2\pi\sqrt{-1}\langle m, t'_\alpha\rangle),$$

where c_α^m are holomorphic function in U_α. In particular,

(2.12) $\varphi\in H^0(G, \mathcal{F})$ is holomorphic if and only if $\varphi_\alpha=\varphi_\alpha^0=c_\alpha^0$ for each α.

Similarly to [Lemma 7 in [**3**]], we have the following

LEMMA 2.1. *Let* $\{c^m(w_\alpha);\ m\in\boldsymbol{Z}^{n+q}\}$ *be a sequence of holomorphic functions in* U_α *and let* $\varphi := \sum_{m\in Z^{n+q}} c^m(w_\alpha)\exp(-2\pi)\sum_{i=q+1}^n m_i t_\alpha^{n+i})\exp(2\pi\sqrt{-1}\langle m, t'_\alpha\rangle)$ *be a formal series in* $U_\alpha\times\boldsymbol{C}^n/\Gamma$. *Then* φ *converges to a function in* $H^0(U_\alpha\times\boldsymbol{C}^n/\Gamma, \mathcal{F})$ *if and only if for any compact subset* A *of* U_α *and* $R>0$ *and any* $k>0$, $\sup_{w\in A}\{|c^m(w)|\,\|m\|^k R^{\|m'\|};\ m\in\boldsymbol{Z}^{n+q}\}<+\infty$, *where* $\|m\| := \max\{|m_i|;\ 1\leqq i\leqq n+q\}$ *and* $\|m'\| := \max\{|m_j|;\ q+1\leqq j\leqq n\}$.

Let $\Omega^r := \Omega_G^r$ be the sheaf of germs of holomorphic r-forms on G. By Theorem 1.1, we have an isomorphism

(2.13) $$F: H^p(G, \Omega^r) \cong \frac{\{\varphi\in H^0(G, \mathcal{F}^{r,p});\ \bar\partial\varphi=0\}}{\bar\partial H^0(G, \mathcal{F}^{r,p-1})}$$

for $p\geqq 1$. Let $\varphi\in H^0(G, \mathcal{F}^{r,p})$ be a $\bar\partial$-closed form $(1\leqq p\leqq q)$. We put $\varphi_\alpha := \varphi|\pi^{-1}(U_\alpha)$, for each α. For $1\leqq i_1<\cdots<i_{r'}\leqq l$, $1\leqq j_1<\cdots<j_{r''}\leqq n$ and $1\leqq k_1<\cdots<k_p\leqq q$, put $I := (i_1, \cdots, i_{r'})$, $J := (j_1, \cdots, j_{r''})$ and $K := (k_1, \cdots, k_p)$. We can write $\varphi_\alpha = \sum_{I,J,K,r'+r''=r}\varphi_{\alpha,IJK}\,dw_\alpha^I\wedge dz_\alpha^J\wedge d\bar{z}_\alpha^K$, where $\varphi_{\alpha,IJK}\in H^0(\pi^{-1}(U_\alpha), \mathcal{F})$. From (2.11) we write $\varphi_{\alpha,IJK} = \sum_{m\in Z^{n+q}} c_{\alpha,IJK}^m(w_\alpha)\exp(-2\pi\sum_{i=q+1}^n m_i t_\alpha^{n+i})\exp(2\pi\sqrt{-1}\langle m, t'_\alpha\rangle)$, where $c_{\alpha,IJK}^m$ are holomorphic functions in U_α. Put $\varphi_{\alpha,IJK} = c_{\alpha,IJK}^m(w_\alpha)\times\exp(-2\pi\sum_{i=q+1}^n m_i t_\alpha^{n+i})\exp(2\pi\sqrt{-1}\langle m, t'_\alpha\rangle)$ and $\varphi_\alpha^m := \sum_{I,J,K,r'+r''=r}\varphi_{\alpha,IJK}^m\,dw_\alpha^I\wedge dz_\alpha^J\wedge d\bar{z}_\alpha^K$. Then $\varphi_\alpha = \sum_{m\in Z^{n+q}}\varphi_\alpha^m$. For each I and J, $\varphi_{\alpha,IJ} := \sum_K \varphi_{\alpha,IJK}\,d\bar{z}_\alpha^K$

is a $\bar{\partial}$-closed $(0, p)$-form in $\pi^{-1}(U_\alpha)$. Let $m \in \boldsymbol{Z}^{n+q} \setminus \{0\}$ and $s(m) := \min\{s\,;\, |K_{m,s}| = K_m,\ 1 \leqq s \leqq q\}$. For $1 \leqq k_1 < \cdots < k_{p-1} \leqq q$ we put $K' := (k_1, \cdots, k_{p-1})$. We put

$$c^{m,s}_{\alpha, IJK'} := \sum_{k=1}^{n} \beta_{sk} c^{m}_{\alpha, IJk\,k_1 \cdots k_{p-1}}(w_\alpha), \quad \text{and}$$

$$d^{m}_{\alpha, IJK'}(w_\alpha) := \frac{c^{m,s(m)}_{\alpha, IJK'}(w_\alpha)}{\pi K_{m,s(m)}}. \tag{2.14}$$

Further, put

$$\psi^{m}_{\alpha, IJ} := \sum_{K'} d^{m}_{\alpha, IJK'}(w_\alpha) \exp\left(-2\pi \sum_{i=q+1}^{n} m_i t_\alpha^{n+i}\right) \exp\left(2\pi\sqrt{-1}\langle m, t'_\alpha\rangle\right) d\bar{z}_\alpha^{K'},$$

and $\psi^m_\alpha := (-1)^r \sum_{I, J, K', r'+r''=r} \psi^m_{\alpha, IJK'}\, dw^I_\alpha \wedge dz^J_\alpha \wedge d\bar{z}^{K'}_\alpha$. Then, $\psi^m_{\alpha, IJ} \in H^0(\pi^{-1}(U_\alpha), \mathcal{F}^{0, p-1})$ and $\psi^m_\alpha \in H^0(\pi^{-1}(U_\alpha), \mathcal{F}^{r, p-1})$. From the similar calculation to the previous paper ([4]), we get

$$\bar{\partial}\psi^m_{\alpha, IJ} = \varphi^m_{\alpha, IJ}, \qquad \bar{\partial}\psi^m_\alpha = \varphi^m_\alpha \tag{2.15}$$

for $m \in \boldsymbol{Z}^{n+q} \setminus \{0\}$ and

$$\varphi_\alpha = \varphi^0_\alpha + \sum_{m \in Z^{n+q} \setminus \{0\}} \bar{\partial}\psi^m_\alpha\,. \tag{2.16}$$

In $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$, from (2.7) and (2.8), we have

$$\varphi^m_\alpha = \varphi^m_\beta \qquad \text{for all} \quad m \in \boldsymbol{Z}^{n+q}. \tag{2.17}$$

From (2.15) and (2.17), we get in $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$,

$$\varphi^0_\alpha = \varphi^0_\beta \ \text{and} \ \bar{\partial}\psi^m_\alpha = \bar{\partial}\psi^m_\beta \ \text{for} \ m \in \boldsymbol{Z}^{n+q} \setminus \{0\}. \tag{2.18}$$

Thus $\varphi^0 := \varphi^0_\alpha$ is a holomorphic (r, p)-form on G. We put $\Phi^m := \bar{\partial}\psi^m_\alpha \in H^0(G, \mathcal{F}^{r,p})$ and $\Phi^{m,1} := \delta(\{\psi^m_\alpha\}) = \{\psi^m_\beta - \psi^m_\alpha\} \in Z^1(\{\pi^{-1}(U_\alpha)\}, \mathcal{F}^{r,p-1})$. Since $\bar{\partial}\Phi^{m,1} = 0$, similarly to getting (2.14), we have $\Psi^{m,1} \in C^1(\{\pi^{-1}(U_\alpha)\}, \mathcal{F}^{r,p-2})$ such that $\Phi^{m,1} = \bar{\partial}\Psi^{m,1}$. Continuing this argument, we get $\Phi^{m,p} \in Z^p(\{\pi^{-1}(U_\alpha)\}, \Omega^r)$ and $F([\Phi^{m,p}]) = [\Phi^m]$, where F is the isomorphism in (2.13). Since $H^0(\pi^{-1}(U_\alpha), \Omega^r) \cong \bigoplus_{r'+r''=r} H^0(U_\alpha, \Omega^{r'}_{S}) \otimes \boldsymbol{C}\{dz^J_\alpha\,;\, |J| = r''\}$ and dz^i_α are global 1-forms on G for $1 \leqq i \leqq n$, we have $H^p(\pi^{-1}(U_\alpha), \Omega^r) \cong \bigoplus_{r'+r''=r} H^p(\{U_\alpha\}, \Omega^{r'}_{\tilde{S}}) \otimes \boldsymbol{C}\{dz^J_\alpha\,;\, |J| = r''\} \cong \bigoplus_{r'+r''=r} \cong H^p(\tilde{S}, \Omega^{r'}_{\tilde{S}}) \otimes \boldsymbol{C}\{dz^J_\alpha\,;\, |J| = r''\} = 0$. Then we get $\Psi^m \in H^0(G, \mathcal{F}^{r,p-1})$ satisfying

$$\Phi^m = \bar{\partial}\Psi^m\,. \tag{2.19}$$

From (2.16), (2.18) and (2.19), we have the following proposition.

PROPOSITION 2.1. *Let $\varphi \in H^0(G, \mathcal{F}^{r,p})$ be a $\bar{\partial}$-closed form ($r \geqq 0$, $q \geqq p \geqq 1$). Then we have a holomorphic (r, p)-form $\varphi^0 = \sum_{I, J, K, |I|+|J|=r} c^0_{\alpha, IJK}\, dw^I_\alpha \wedge dz^J_\alpha \wedge d\bar{z}^K_\alpha$ on G and a $(r, p-1)$-form $\Psi^m \in H^0(G, \mathcal{F}^{r,p-1})$ for each $m \in \boldsymbol{Z}^{n+q} \setminus \{0\}$ satisfying $\varphi = \varphi^0 + \sum_{m \in Z^{n+q} \setminus \{0\}} \bar{\partial}\Psi^m$.*

In Proposition 2.1, $\psi := \sum_{m \in Z^{n+q} \setminus \{0\}} \Psi^m$ gives a formal solution of the $\bar{\partial}$-closed form $\varphi - \varphi^0$. To study the convergence of the formal solution, we have the following

LEMMA 2.2. *Let* $\varphi_\alpha=\sum_{I,J,K,|I|+|J|=r}\varphi_{\alpha,IJK}dw_\alpha^I\wedge dz_\alpha^J\wedge d\bar{z}_\alpha^K\in H^0(U_\alpha\times \boldsymbol{C}^n/\Gamma, \mathcal{E}^{r,p})$ *be a* $\bar\partial$*-closed form* $(r\geqq 0,\ q\geqq p\geqq 1)$. *If* $\boldsymbol{C}^n/\Gamma$ *is finite type, then we have* $\psi_\alpha\in H^0(U_\alpha\times\boldsymbol{C}^n/\Gamma, \mathcal{E}^{r,p-1})$ *satisfying* $\varphi_\alpha=\varphi_\alpha^0+\bar\partial\psi_\alpha$.

PROOF. We put $\varphi_\alpha=\sum_{m\in Z^{n+q}\setminus\{0\}}\varphi_\alpha^m$ and $\varphi_\alpha^m=\sum_{I,J,K,r'+r''=r}c_{\alpha,IJK}^m(w_\alpha)$ $\exp(-2\pi\sum_{i=q+1}^n m_i t_\alpha^{n+i})\exp(2\pi\sqrt{-1}\langle m, t'_\alpha\rangle)dw_\alpha^I\wedge dz_\alpha^J\wedge d\bar{z}_\alpha^K$. By Lemma 2.1, for any compact subset A of U_α, any $R>0$ and any $k>0$, we have

$$\sup_{w\in A}\{|c_{\alpha,IJK}^m(w)|\,\|m\|^k R^{\|m'\|};\ m\in \boldsymbol{Z}^{n+q}\}<+\infty. \tag{2.20}$$

By (2.15), for each $m\in\boldsymbol{Z}^{n+q}\setminus\{0\}$, we have $\psi_\alpha^m=(-1)^r\sum_{I,J,K',r'+r''=r}d_{\alpha,IJK'}^m(w_\alpha)$ $\exp(-2\pi\sum_{i=q+1}^n m_i t_\alpha^{n+i})\exp(2\pi\sqrt{-1}\langle m, t'_\alpha\rangle)dw_\alpha^I\wedge dz_\alpha^J\wedge d\bar{z}_\alpha^{K'}$ satisfying $\varphi_\alpha^m=\bar\partial\psi_\alpha^m$, where $d_{\alpha,IJK'}^m$ are given by (2.14). From (2.3) and (2.20), we have $\sup_{w\in A}$ $\{|d_{\alpha,IJK'}^m(w)|\,\|m\|^k R^{\|m'\|};\ m\in\boldsymbol{Z}^{n+q}\}<+\infty$, for any compact subset A of U_α, any $R>0$ and any $k>0$. Then, $\psi_\alpha=\sum_{m\in Z^{n+q}\setminus\{0\}}\psi_\alpha^m$ converges in $H^0(U_\alpha\times\boldsymbol{C}^n/\Gamma$, $\mathcal{E}^{r,p-1})$ and $\varphi_\alpha=\varphi_\alpha^0+\bar\partial\psi_\alpha$. By (2.13), $H^p(G,\Omega^r)$ has a quotient topology of the Fréchet space $\{\varphi\in H^0(G,\mathcal{E}^{r,p});\ \bar\partial\varphi=0\}$, for $p\geqq 1$.

THEOREM 2.1. *Let* G *be a connected complex Lie group of complex dimension* $n+l$ *and* $G^0=\boldsymbol{C}^n/\Gamma$ *the maximal toroidal subgroup of* G *of complex dimension* n. *Then the following statements* (1), (2), (3) *and* (4) *are equivalent.*

(1) G^0 *is of finite type.*

$$(2)\quad H^p(G,\Omega^r)\cong\begin{cases}\bigoplus_{r'+r''=r}H^0(\tilde S,\Omega_{\tilde S}^{r'})\otimes\boldsymbol{C}\{dz^{j_1}\wedge\cdots\wedge dz^{j_{r''}}\wedge d\bar z^{k_1}\wedge\cdots\wedge d\bar z^{k_p};\\ 1\leqq j_1<\cdots<j_{r''}\leqq q,\ 1\leqq k_1<\cdots<k_p\leqq q\} & \text{for}\quad 1\leqq p\leqq q\\ 0 & \text{for}\quad p\geqq q+1,\end{cases}$$

where $\tilde S=G/G^0$ *is a Stein group and* dz^j *are global* 1*-forms on* $G^0=\boldsymbol{C}^n/\Gamma(j=1,\cdots,n)$.

(3) $H^p(G,\Omega^r)$ *has a Hausdorff topology, for any* $p, r\geqq 0$.

(4) $\bar\partial H^0(G,\mathcal{E}^{r,p-1})$ *is a closed subspace of the Fréchet space* $H^0(G,\mathcal{E}^{r,p})$ *for* $p\geqq 1$ *and* $r\geqq 0$.

PROOF. Assume (1) holds. Let $\varphi\in H^0(G,\mathcal{E}^{r,p})$ be a $\bar\partial$-closed form for $p\geqq 1$ and $r\geqq 0$. We put $\varphi_\alpha:=\varphi|\pi^{-1}(U_\alpha)$, for each α. By Lemma 2.2, we have holomorphic (r,p)-forms φ_α^0 and $\psi_\alpha\in H^0(\pi^{-1}(U_\alpha),\mathcal{E}^{r,p-1})$ satisfying $\varphi_\alpha=\varphi_\alpha^0+\bar\partial\psi_\alpha$. By (2.18) $\varphi^0=\varphi_\alpha^0$ is a holomorphic (r,p)-form on G and $\Phi:=\bar\partial\psi_\alpha\in H^0(G,\mathcal{E}^{r,p})$ is a closed (r,p)-form. Then similarly to getting (2.19) and by Lemma 2.2, we have $\Psi\in H^0(G,\mathcal{E}^{r,p-1})$ satisfying $\varphi=\varphi^0+\bar\partial\Psi$. Combining this with (2.12), we get (2). It is obvious that (2)⇒(3)⇒(4). Finally we prove (4)⇒(1). Suppose G^0 is not of finite type. Then by (2.4), there exists $\varepsilon>0$ such that we can choose a sequence $\{m_\mu;\ \mu\geqq 1\}$ in $Z^{n+q}\setminus\{0\}$ satisfying $\exp(-\varepsilon\|m'_\mu\|-\mu\|m''_\mu\|)/K_{m_\mu}>\mu$ for any $\mu\geqq 1$. Put

$$\delta^m := \begin{cases} \exp(-\varepsilon\|m'_\mu\|-\mu\|m''_\mu\|)/K_{m_\mu} & m=m_\mu \text{ for some } \mu\geqq 1 \\ 0 & \text{otherwise}. \end{cases}$$

For each α, we put

$$\phi^m_\alpha := \left(\sum_\gamma \exp(2\pi\sqrt{-1}\langle m,\ f'_{\gamma\alpha}\rangle)\right)\delta^m \exp\left(-2\pi\sum_{i=q+1}^n m_i t_\alpha^{n+i}\right) \times\exp(2\pi\sqrt{-1}\langle m,\ t'_\alpha\rangle)$$

in $\pi^{-1}(U_\alpha)$. From (2.7), in $\pi^{-1}(U_\alpha)\cap\pi^{-1}(U_\beta)$, $\phi^m_\alpha=\phi^m_\beta$. Then we have $\phi^m\in H^0(G, \mathcal{F})$ such that $\phi^m|\pi^{-1}(U_\alpha)=\phi^m_\alpha$. By (2.10), we have $\bar\partial\phi^m_\alpha=\sum_{j=1}^q(\pi\sum_{i=1}^q\gamma_{ij}K_{m,i}\phi^m_\alpha)d\bar z^j_\alpha$. By Lemma 2.1, we see $\sum_{m\in Z^{n+q}\setminus\{0\}}\bar\partial\phi^m_\alpha$ converges to a form $\varphi\in H^0(G, \mathcal{F}^{0,1})$. By the choice of the sequence $\{m_\mu\}$, the formal series $\sum_{m\in Z^{n+q}\setminus\{0\}}\phi^m$ cannot converge to any function in $H^0(G, \mathcal{F})$. Suppose $\varphi=\bar\partial\lambda$ for some $\lambda=\sum_{m\in Z^{n+q}\setminus\{0\}}\lambda^m$. Then we can see $\lambda^m=\phi^m$ for $m\neq 0$. It is a contradiction. Then $\varphi=\lim_{N\to\infty}(\bar\partial\sum_{\|m\|<N}\phi^m)$ belongs not to $\bar\partial H^0(G, \mathcal{F}^{0,0})$, but to the closure of $\bar\partial H^0(G, \mathcal{F}^{0,0})$. This contradicts the statement (4).

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