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The class of second order equations which Riemannian geometry can be applied to

Dedicated to Professor H. Nakagawa on his 60th birthday

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1. Introduction.

Let M be a C^∞ manifold without boundary and $\pi: TM \to M$ the tangent bundle of M. A second order equation on M which is locally expressed by

$$\frac{d^2x^i}{dt^2} = F^i\left(x^1, \cdots, x^n, \frac{dx^1}{dt}, \cdots, \frac{dx^n}{dt}\right)$$

is considered to be a vector field V on TM with $\pi_* V(y)=y$, where $(U; x^1, \cdots, x^n)$ and $(TU; x^1, \cdots, x^n, y^1, \cdots, y^n)$ are local coordinate systems in M and TM, respectively. Let f^t be the flow on TM generated by V. In this paper we study the case that f^t has an invariant hypersurface S in TM such that all fibres S_p are star-shaped hypersurfaces around the origin in T_pM. Then we can define the exponential map at each point $p \in \pi(S)$ by $\mathrm{Exp}_p ty = \pi f^t y$ for any $y \in S_p$. The purpose of the present paper is to show how to give a Riemannian metric on an open set in T_pM on which all rays from the origin are geodesics and how to use it to study the behavior of the trajectories.

In the geometry of geodesics in Riemannian manifolds it is important to estimate the lengths of Jacobi vector fields Y along geodesics $\gamma: [0, a) \to M$ with $Y(0)=0$ which arises from the geodesic variation emanating from $\gamma(0)$. In fact, by making use of the estimate we have proved many results in the theory of Anosov geodesic flows (cf. [**AA**], [**E**]), the estimate of the measure theoretic entropy of geodesic flows (cf. [**P**], [**OS**], [**BW**]), the theory of parallels (for example, the theorem of E. Hopf and L. Green (cf. [**H**], [**G**])), the topological and differentiable sphere theorems (cf. [**CE**]), the study of non-positively and non-negatively curved Riemannian manifolds (cf. [**E**], [**W**]) and so on. We should notice that in the estimation used in the above theory and theorems the Riemannian metric do not need to be defined on the whole manifold, but in the neighborhood of a geodesic in question. For example, conjugate points of $p \in M$ along a geodesic $\gamma: [0, \infty) \to M$ with $\gamma(0)=p$ are independent of the choice of Riemannian metrics g such that all geodesics emanating from p and staying in a neighborhood of γ with respect to the original metric are also geodesics with respect to g. This fact would suggest us that many results proved by making use of the estimates mentioned above are extended to much wider classes.

Roughly speaking, systems satisfying Huygens' principle belong to such classes. Huygens' principle is stated as follows (cf. [**A**]): Let $\Phi_{q_0}(t)$ be the wave front of the point q_0 after time t. For every point q of this front, consider the wave front after time s, $\Phi_q(s)$. Then the wave front of the point q_0 after time $s+t$, $\Phi_{q_0}(s+t)$, will be the envelope of the fronts $\Phi_q(s)$, $q\in\Phi_{q_0}(t)$. This principle corresponds to the Gauss lemma for geodesic spheres in Riemannian manifolds. According to [**A**], Huygens' principle defines a natural contact structure D and the contact flow on S. The flows in our classes must have such natural contact structures also (see Proposition 2.4).

In the paper [**I1**] we deal with the geometry of geodesic flows in the unit tangent bundles as a special case of the following model. Let N be a manifold. Let $f^t: N\to N$ be a flow and $\Pi: E\to N$ a vector bundle with inner product $\langle\cdot,\cdot\rangle_p$ on each fibre E_p. Assume that there exists a connection $\tilde{\nabla}$ along the flow f^t such that

$$V\langle X, Y\rangle=\langle\tilde{\nabla}_V X, Y\rangle+\langle X, \tilde{\nabla}_V Y\rangle$$

for any section $X, Y: N\to E$, where V is a vector field on N generating f^t. The geodesic flow f^t on the unit tangent bundle in the above theories is the case when N is the unit tangent bundle of a Riemannian manifold M, $E=\bigcup_{v\in N}T_{\pi(v)}M$ and $\langle\cdot,\cdot\rangle$ is the original Riemannian metric of M. In the paper [**I2**] we deal with it by putting $E=TN$ and $\langle\cdot,\cdot\rangle$ the natural Riemannian metric on TN induced from the Riemannian metric of M. The main theorem of the present paper is Theorem 3.4 which shows the relation between these specific models for more general flows. Moreover, the condition (1) of Theorem 3.4 is equivalent to the existence of an f^t-invariant complementary distribution of V on S because of Proposition 2.2. This is the expression of Huygens' principle by using V and S.

The idea of this work appeared to the author's mind while he stayed in University of North Carolina at Chapel Hill. He would like to express his hearty thanks to Professor P. Eberlein for accepting his visit, and to Professor C. Ferraris for his discussion there which stimulated him.

2. Complementary f^t-invariant distribution.

Let g be a generalized metric on $\pi(S)$ (cf. [**M**]), namely g_y is by definition an inner product on the tangent space $T_{\pi(y)}M$ for any $y\in S$. Hence, we define a Riemannian metric G on S by

$$G_y(W, Z) = g_y(\pi_* W, \pi_* Z) + g_y(K_y(W), K_y(Z))$$

for any $W, Z \in T_y S$, $y \in S$, where $K_y : T_y TM \to T_{\pi(y)} M$ is the connection map defined in Appendix 1. A vector W with $\pi_* W = 0$ is said to be *vertical*. A vector $W \in T_y S$ is vertical if and only if W is tangent to the fibre $S_{\pi(y)}$. A vector Z with $K(Z) = 0$ is said to be *horizontal*. It follows that $T_y S = \ker K_y \oplus \ker \pi_{*y}$ and $\ker K_y \perp \ker \pi_{*y}$ with respect to the inner product G_y for any $y \in S$, where $\ker K_y$ and $\ker \pi_{*y}$ are the kernels of the linear maps K_y and π_{*y} respectively.

Let $\iota(p) : S_p \to S$ be the inclusion map. Then, we denote $\iota(p)_{*y}(T_y S_p)$ by $T_y S_p$ for any $y \in S_p$.

LEMMA 2.1. *The following are true.*

(1) *$V(y)$ is horizontal for any $y \in S$.*

(2) *$V(y)^\perp \supset T_y S_{\pi(y)}$ for any $y \in S$, where $V(y)^\perp = \{W \in T_y S \mid G_y(W, V(y)) = 0\}$.*

We first show how to give a nice generalized metric on $\pi(S)$ to V if V has a *complementary distribution* D on S which by definition satisfies the condition (1) and (2) in the following proposition.

PROPOSITION 2.2. *Suppose there exists a* ($\dim S - 1$)*-dimensional distribution D on S such that*

(1) *$D(y) \not\ni V(y)$ for any $y \in S$,*

(2) *$D(y) \supset T_y S_{\pi(y)}$ for any $y \in S$.*

Then there exists a generalized metric g on $\pi(S)$ such that $D(y) = V(y)^\perp$ and $G_y(V(y), V(y)) = 1$ for any $y \in S$.

PROOF. Since $D(y) \supset \ker \pi_* = T_y S_{\pi(y)}$, we have $\dim \pi_* D(y) = \dim M - 1$. And $y \notin \pi_* D(y)$. In fact, if we suppose that $y \in \pi_* D(y)$, then there exists a $W \in T_y S_{\pi(y)} \subset D(y)$ such that $V(y) + W \in D(y)$. Since $T_y S_{\pi(y)} \subset D(y)$, it follows that $V(y) \in D(y)$, contradicting (1).

Let $\tilde{g}$ be a Riemannian metric on M. Define a generalized metric g on $\pi(S)$ by

$$\begin{cases} g_y(y, y) = 1 & \\ g_y(y, \pi_* W) = 0 & \text{for any } W \in D(y) \\ g_y(\pi_* W, \pi_* Z) = \tilde{g}(\pi_* W, \pi_* Z) & \text{for any } W, Z \in D(y) \end{cases}$$

for each $y \in S$. Since V is horizontal, we have $D(y) = V(y)^\perp$ for any $y \in S$ with respect to G. And, $G_y(V(y), V(y)) = 1$ for any $y \in S$. ■

For $y \in S$ let $(a(y), b(y))$ be the maximal interval containing zero on which the integral curve $t \to f^t y$ with $f^0 y = y$ of V is defined.

PROPOSITION 2.3. *Let D be a complementary distribution on S. Then, $(f^t)_*D(y)=D(f^t y)$ for any $y\in S$ and $a(y)<t<b(y)$ if and only if $[V, X](y)\in D(y)$ for any $y\in S$ and $X\in D$, where $[\cdot, \cdot]$ is the Lie bracket.*

PROOF. If D is f^t-invariant, then $(f^{-t})_*X(f^t y)\in D(y)$ for any $y\in S$. Hence, $[V, X](y)\in D(y)$ for any $X\in D$, since

$$[V, X](y) = L_{V(y)}X = \lim_{t\to 0} \frac{(f^{-t})_*X(f^t y)-X(y)}{t},$$

where L is the Lie derivative.

Suppose $[V, X]\in D$ for any $X\in D$. Let (U, φ) be a local coordinate neighborhood in S and let $X_1, \cdots, X_{2(n-1)}$ be a local basis of D on U. Suppose $\omega_1, \cdots, \omega_{2(n-1)}, \omega_{2n-1}$ is the dual basis of $X_1, \cdots, X_{2(n-1)}, V$ on U. Therefore, $D=\{X\,|\,\omega_{2n-1}(X)=0\}$. Let $X\in D$. We have

$$0 = V(\omega_{2n-1}(X)) = (L_V\omega_{2n-1})(X)+\omega_{2n-1}(L_V X).$$

Since $L_V X=[V, X]\in D$, we get $(L_V\omega_{2n-1})(X)=0$. And we have also

$$(L_V\omega_{2n-1})(V) = V(\omega_{2n-1}(V))-\omega_{2n-1}(L_V V) = 0.$$

Therefore, $L_V\omega_{2n-1}=0$. Define Y along $f^t y$ by $Y(f^t y)=(f^t)_*X(y)$ for $a(y)<t<b(y)$. Then,

$$Y(f^{t+s}y) = (f^{s+t})_*X(y) = (f^s)_*(f^t)_*X(y) = (f^s)_*Y(f^t y)$$

for $a(y)<s, t, s+t<b(y)$. Therefore, we have $L_{V(f^t y)}Y=0$, and, hence,

$$V(f^t y)(\omega_{2n-1}(Y)) = (L_{V(f^t y)}\omega_{2n-1})(Y)+\omega_{2n-1}(L_{V(f^t y)}Y) = 0$$

for $a(y)<t<b(y)$. Thus, $\omega_{2n-1}(Y)$ is constant along $f^t y$. Since $\omega_{2n-1}(Y)(y)=0$, it follows that $\omega_{2n-1}(Y)(f^t y)=0$ for $a(y)<t<b(y)$.

We now have $(f^t)_*X(y)\in D(f^t y)$, namely $(f^t)_*D(y)\subset D(f^t y)$ for $a(y)<t<b(y)$. Since $\dim(f^t)_*D(y)=\dim D(f^t y)$, we conclude that $(f^t)_*D(y)=D(f^t y)$ for any $a(y)<t<b(y)$. ■

We proceed to study the relation between the existence of flows f^t having a complementary f^t-invariant distribution D and the shape of S.

Let $(U; x^1, \cdots, x^n)$ be a coordinate neighborhood of M such that S is written as $(x^1, \cdots, x^n, y^1, \cdots, y^{n-1}, H(x^1, \cdots, x^n, y^1, \cdots, y^{n-1}))$ in $\pi^{-1}(U)$, where $(x^1, \cdots, x^n, y^1, \cdots, y^n)$ are the coordinates of the vectors $\sum_{i=1}^n y^i(\partial/\partial x^i)|_{(x^1,\dots,x^n)}$. Then

$$\begin{cases} X^i = \dfrac{\partial}{\partial x^i}+\dfrac{\partial H}{\partial x^i}\dfrac{\partial}{\partial y^n}, & i=1, \cdots, n \\[2ex] X^{n+j} = \dfrac{\partial}{\partial y^j}+\dfrac{\partial H}{\partial y^j}\dfrac{\partial}{\partial y^n}, & j=1, \cdots, n-1 \end{cases}$$

is a basis of the tangent bundle of S in $\pi^{-1}(U)$, and

$$V(x^1, \cdots, x^n, y^1, \cdots, y^{n-1}, H(x^1, \cdots, x^n, y^1, \cdots, y^{n-1}))$$
$$= \sum_{i=1}^{n-1} y^i \frac{\partial}{\partial x^i} + H\frac{\partial}{\partial x^n} + \sum_{j=1}^{n} b^j \frac{\partial}{\partial y^j} = \sum_{i=1}^{n-1} y^i X^i + HX^n + \sum_{j=1}^{n-1} b^j X^{n+j}$$

for some functions $b^j = b^j(x^1, \cdots, x^n, y^1, \cdots, y^{n-1})$ $(j=1, \cdots, n)$ with

$$b^n = \sum_{i=1}^{n-1} y^i \frac{\partial H}{\partial x^i} + H\frac{\partial H}{\partial x^n} + \sum_{j=1}^{n-1} b^j \frac{\partial H}{\partial y^j}.$$

By making use of $[X^a, X^b]=0$ for $a, b=1, \cdots, 2n-1$ we have

$$\begin{cases} [X^i, V] = \dfrac{\partial H}{\partial x^i} X^n + \sum_{k=1}^{n-1} \dfrac{\partial b^k}{\partial x^i} X^{n+k}, & i=1, \cdots, n \\ [X^{n+j}, V] = X^j + \dfrac{\partial H}{\partial y^j} X^n + \sum_{k=1}^{n-1} \dfrac{\partial b^k}{\partial y^j} X^{n+k}, & j=1, \cdots, n-1. \end{cases}$$

From these formulas we have the following.

PROPOSITION 2.4. *If D is a complementary f^t-invariant distribution, then $\pi_* D(y)$ is naturally isomorphic to $T_y S_{\pi(y)}$ for any $y \in S$. Therefore, $T_y S$ is naturally isomorphic to span$\{y\} \oplus T_y S_{\pi(y)} \oplus T_y S_{\pi(y)}$.*

PROOF. It follows from $X^{n+i} \in D$ and Proposition 2.3 that $Y^i := X^i + (\partial H/\partial y^i) X^n \in D$ since $T_y S_{\pi(y)} = \text{span}\{X^{n+1}, \cdots, X^{2n-1}\}$. Since $\pi_* X^i = \partial/\partial x^i$ for $i=1, \cdots, n$, we have

$$\pi_* Y^i = \frac{\partial}{\partial x^i} + \frac{\partial H}{\partial y^i}\frac{\partial}{\partial x^n}$$

for $i=1, \cdots, n-1$. By natural isomorphism I of $T_{\pi(y)}M$ to $T_y T_{\pi(y)} M$, we get

$$I(\pi_* Y^i) = \frac{\partial}{\partial y^i} + \frac{\partial H}{\partial y^i}\frac{\partial}{\partial y^n} = X^{n+i}.$$

■

We have used the star-shapedness of S_p in T_pM to define a connection map $K: TTM \to TM$. Moreover, the following proposition shows that the star-shapedness is required to the condition in Proposition 2.2 and 2.4.

PROPOSITION 2.5. *If D is a complementary f^t-invariant distribution, then S_p is star-shaped in T_pM for $p \in \pi(S)$.*

PROOF. Suppose S_p is not star-shaped in T_pM for some $p \in \pi(S)$, namely there exist $y \in S$ and $c_1, \cdots, c_{n-1} \in \boldsymbol{R}$ such that

$$\sum_{i=1}^{n-1} y^i \frac{\partial}{\partial y^i} + H\frac{\partial}{\partial y^n} = \sum_{i=1}^{n-1} c_i X^{n+i}.$$

Since $X^{n+i}=\partial/\partial y^i+(\partial H/\partial y^i)(\partial/\partial y^n)$ for $i=1, \cdots, n-1$, we have that $c_i=y^i$ for $i=1, \cdots, n-1$, therefore,

$$H(x^1, \cdots, x^n, y^1, \cdots, y^{n-1})=\sum_{i=1}^{n-1} y^i \frac{\partial H}{\partial y^i}.$$

Since

$$\pi_* V(y)=\sum_{i=1}^{n-1} y^i \frac{\partial}{\partial x^i}+H \frac{\partial}{\partial x^n}=\pi_*(\sum_{i=1}^{n-1} y^i Y^i),$$

the vector $V(y)-\sum_{i=1}^{n-1} y^i Y^i$ is vertical. The conditions $T_y S_p \subset D(y)$ and $\sum_{i=1}^{n-1} y^i Y^i \in D(y)$ imply that $V(y) \in D(y)$, contradicting the condition $V(y) \notin D(y)$. ∎

We conclude this section with showing the relation between the shape of S_p and the existence of V, D.

THEOREM 2.6. *If D is a complementary f^t-invariant distribution, then b^j and H satisfy the equation:*

$$\sum_{j=1}^{n-1} b^j \frac{\partial^2 H}{\partial y^j \partial y^i}=\frac{\partial H}{\partial x^i}+\frac{\partial H}{\partial y^i} \frac{\partial H}{\partial x^n}-\sum_{j=1}^{n-1} y^j \frac{\partial^2 H}{\partial x^j \partial y^i}-H \frac{\partial^2 H}{\partial x^n \partial y^i}$$

for $i=1, \cdots, n-1$. In particular, if $\partial^2 H/\partial y^j \partial y^i$ make a nonsingular matrix, then such b^j ($j=1, \cdots, n-1$) uniquely exist.

PROOF. For $i=1, \cdots, n-1$ we have

$$\begin{aligned}\left[X^i+\frac{\partial H}{\partial y^i} X^n, V\right]=&\frac{\partial H}{\partial x^i} X^n+\sum_{j=1}^{n-1} \frac{\partial b^j}{\partial x^i} X^{n+j} \\ &+\frac{\partial H}{\partial y^i} \frac{\partial H}{\partial y^n} X^n+\frac{\partial H}{\partial y^i} \sum_{j=1}^{n-1} \frac{\partial b^j}{\partial x^i} X^{n+j}-\left(V \frac{\partial H}{\partial y^i}\right) X^n .\end{aligned}$$

Since $X^{n+j} \in D$ for $j=1, \cdots, n-1$, it must follows that

$$\frac{\partial H}{\partial x^i}+\frac{\partial H}{\partial y^i} \frac{\partial H}{\partial x^n}-V \frac{\partial H}{\partial y^i}=0 .$$

This is our equation. ∎

3. Flows whose trajectories are geodesics.

Let S be a hypersurface in the tangent bundle TM of M such that S_p is a star-shaped hypersurface centered at the origin 0 in T_pM for any $p \in \pi(S)$ and let V be a vector field on S such that $\pi_* V(y)=y$ for any $y \in S$. Suppose $K: TTM \to TM$ is a connection map defined from V as in Appendix 1. Let g be a generalized metric on $\pi(S)$ such that $g_y(y, y)=1$ for any $y \in S$ and G the Riemannian metric on S defined as in Section 2. Then, $G(V(y), V(y))=1$ for any $y \in S$.

Let N be a submanifold in S and let $\varphi^N : \bigcup_{y\in N}\{\{y\}\times(a(y), b(y))\}\to M$ be the map given by $\varphi^N(y, t)=\pi f^t y$ for any $y\in N$ and $a(y)<t<b(y)$. Let $N_0\subset \bigcup_{y\in N}\{\{y\}\times(a(y), b(y))\}$ be the set of all points where φ_*^N is injective. Obviously, $\varphi_*^N((\partial/\partial t)|_{(y,t)})=f^t y$ and $\varphi_*^N(X)=\pi_*(f^t)_*X$ for any $X\in T_yN\subset T_{(y,t)}N_0$. We define a Riemannian metric H^N on N_0 by

$$H^N_{(y,t)}(X, Y) = g_{f^t y}(\varphi_*^N X, \varphi_*^N Y)$$

for any $X, Y\in T_{(y,t)}N_0$. Let ∇^N be the Levi-Civita connection of H^N on N_0.

In the special case that $N=S_p$ for a point $p\in\pi(S)$ we use a different convention. Let $C_p=\{ty\,|\,0\leqq t<b(y),\ y\in S_p\}$ be the cone in T_pM spaned by S_p. Let $\mathrm{Exp}_p : C_p\to M$ be the map given by $\mathrm{Exp}_p ty=\pi f^t y$ for any $0\leqq t<b(y)$ and $y\in S_p$. Thus, $\mathrm{Exp}_p ty=\varphi^N(y, t)$ for any $0\leqq t<b(y)$ and $y\in S_p$. We call Exp_p the exponential map at $p\in\pi(S)$. This map Exp_p can be regarded as the usual one as follows. Since S_p is star-shaped in T_pM, we can extend the vector field V on S to $C=\bigcup_{p\in\pi(S)}C_p$ by putting $\bar{V}(sy)=s(h_s)_*V(y)$ as seen in Appendix 1 Then the flow $\bar{f}^t$ generated from $\bar{V}$ satisfies that $\bar{f}^t(z)=h_s(f^{st}y)$ for any $z=sy\in C$ $(y\in S)$ and $0\leqq t<b(y)/s$. This is because

$$\frac{d}{dt}h_s(f^{st}y) = (h_s)_*(sV(f^{st}y)) = \bar{V}(h_s(f^{st}y))$$

for any $0\leqq st<b(y)$. Therefore, we have that

$$\mathrm{Exp}_p z = \pi f^s y = \pi(h_s(f^s y)) = \pi\bar{f}^1(z)$$

for any $z=sy\in C_p$ $(y\in S_p)$. We see in Appendix 1 that Exp_p is of class C^1 at the origin 0 and of class C^∞ on $C_p-\{0\}$. Since S is star-shaped at 0, it follows that Exp_{p*0} is nonsingular. This implies that there exists a maximal neighborhood U_p of 0 in C_p such that $\mathrm{Exp}_{p*}|U_p-\{0\}$ is nonsingular. If $\psi: C_p-\{0\}\to\bigcup_{y\in S_p}\{\{y\}\times(0, b(y))\}$ is the map given by $\psi(ty)=(y, t)$, then $\mathrm{Exp}_p =\varphi^{S_p}\circ\psi$, and, hence, $\mathrm{Exp}_{p*}(y_{ty})=f^t y$ and $\mathrm{Exp}_{p*}(tX)_{ty}=\pi_*(f^t)_*X$ for any $X\in T_yS_p$. Define a Riemannian metric H^p on $U_p-\{0\}$ by

$$H^p_z(X, Y) = g_{f^s y}(\mathrm{Exp}_{p*}X, \mathrm{Exp}_{p*}Y)$$

for any $X, Y\in T_zT_pM$, where $z=sy\in U_p-\{0\}$ $(y\in S_p)$. Let ∇^p be the Levi-Civita connection of H^p on $U_p-\{0\}$. The relation between H^p and H^{S_p} is that

$$H^p_{ty}(X, Y) = H^{S_p}_{(y,t)}(\psi_*X, \psi_*Y)$$

for any $X, Y\in T_{ty}T_pM$ $(ty\in U_p-\{0\})$, namely $\psi: U_p-\{0\}\to(S_p)_0$ is an isometry (possibly not surjective).

The following relation between the connection ∇^N and the Levi-Civita connection $\tilde{\nabla}$ of G will play important roles in this section.

LEMMA 3.1.

$$H^N_{(y,t)}\left(X, \nabla^N_{\partial/\partial t}\frac{\partial}{\partial t}\right) = G_{f^t y}((f^t)_* X, \tilde{\nabla}_{V(f^t y)} V)$$

for any $X \in T_y N$ *and* $(y, t) \in N_0$.

PROOF. Since $G_{f^t y}(V(f^t y), V(f^t y))=1$ for all $a(y)<t<b(y)$, we have

$$G_{f^t y}(\tilde{\nabla}_{V(f^t y)}(f^t)_* X, V(f^t y)) = G_{f^t y}(\tilde{\nabla}_{(f^t)_* X} V, V(f^t y)) = 0 .$$

Therefore,

$$\frac{d}{dt}(G_{f^t y}((f^t)_* X, V(f^t y))) - G_{f^t y}((f^t)_* X, \tilde{\nabla}_{V(f^t y)} V) = 0$$

for any $a(y)<t<b(y)$. The first term can be computed as follows.

$$\begin{aligned}
\frac{d}{dt}(G_{f^t y}((f^t)_* X, V(f^t))) &= \frac{d}{dt}(g_{f^t y}(\varphi^N_* X, f^t y)) = \frac{d}{dt}\left(H^N_{(y,t)}\left(X, \frac{\partial}{\partial t}\right)\right) \\
&= H^N_{(y,t)}\left(\nabla^N_{\partial/\partial t} X, \frac{\partial}{\partial t}\right) + H^N_{(y,t)}\left(X, \nabla^N_{\partial/\partial t}\frac{\partial}{\partial t}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial s} H^N_{(c(s),t)}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) + H^N_{(y,t)}\left(X, \nabla^N_{\partial/\partial t}\frac{\partial}{\partial t}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial s} g_{f^t c(s)}(f^t c(s), f^t c(s)) + H^N_{(y,t)}\left(X, \nabla^N_{\partial/\partial t}\frac{\partial}{\partial t}\right) = H^N_{(y,t)}\left(X, \nabla^N_{\partial/\partial t}\frac{\partial}{\partial t}\right),
\end{aligned}$$

where $c: (-\varepsilon, \varepsilon) \to N$ is a curve with $\dot{c}(0)=X$, and we used the facts that

(1) V is horizontal,
(2) $\pi_* V(y)=y$ for any $y \in S$,
(3) $\varphi_{*f^t y}(\partial/\partial t)=f^t y$. ∎

Since $\mathrm{Exp}_p(ty)=\varphi^{S_p}(y, t)$ for any $y \in S_p$ and $0<t<b(y)$, we have the following as a special case.

COROLLARY 3.2.

$$H^p_{ty}(tX, \nabla^p_{y_{ty}} y) = H^{S_p}_{(y,t)}\left(X, \nabla^{S_p}_{\partial/\partial t}\frac{\partial}{\partial t}\right) = G_{f^t y}((f^t)_* X, \tilde{\nabla}_{V(f^t y)} V)$$

for any $X \in T_y S_p$ *and* $ty \in U_p - \{0\}$ $(y \in S_p,\ t>0)$.

The following is a direct application of Corollary 3.2.

COROLLARY 3.3. *Let* $y \in S$. *Then,* $\tilde{\nabla}_{V(y)} V$ *is horizontal if* $\nabla^p_{y_{ty}} y$ *(or* $\nabla^{S_p}_{\partial/\partial t}(\partial/\partial t)$*) is bounded near* $ty=0$ *(or* $(y, 0)$*, respectively).*

PROOF. Let $X \in T_p S_{\pi(y)}$. By construction of H^p it follows that H^p is bounded near 0 along ty. Hence, we have

$$G_y(X, \tilde{\nabla}_{V(y)}V) = \lim_{t\to 0} H^p_{ty}(tX, \nabla^p_{y_{ty}}y) = 0$$

■

The following theorem shows the condition equivalent to the existence of complementary f^t-invariant distribution.

THEOREM 3.4. *The following are equivalent.*

(1) $(f^t)_*(V(y)^\perp)=V(f^t y)^\perp$ *for any* $y\in S$ *and* $a(y)<t<b(y)$.

(2) *There exists a positive continuous function* $\varepsilon: S\to \boldsymbol{R}^+$ *such that*

$$(f^t)_*(T_y S_{\pi(y)}) \subset V(f^t y)^\perp$$

for any $0\leqq t<\varepsilon(y)$.

(3) *The trajectories* $t\to f^t y$ *are geodesics in* (S, G) *for all* $y\in S$.

(4) *For any submanifold* N *in* S *it follows that the curves* $t\to(y, t)$ *are geodesics in* (N_0, H^{N_0}), *where* $y\in N$.

(5) *For any* $p\in\pi(S)$ *it follows that the curves* $t\to ty$ *are geodesics for all* $y\in S_p$ *in* $(U_p-\{0\}, H^p)$.

PROOF. (2)⇒(1): We first prove this for $0<t<b(y)$. Let $\varepsilon_0= 1/2 \min\{\varepsilon(f^s y)\,|\,0\leqq s\leqq t\}$. Then it follows that $(f^u)_* T_{f^s y}S_{\pi(f^s y)}\subset V(f^{u+s}y)^\perp$ for any $0\leqq u\leqq \varepsilon_0$ and any $0\leqq s\leqq t$. Let $T_z S=\mathrm{span}\{V(z)\}\oplus W(z)\oplus T_z S_{\pi(z)}$ denote an orthonormal decomposition of $T_z S$ at $z\in S$. Let $z\in f^{[0,t]}y$. By assumption we have $(f^u)_*(f^s)_* T_z S_{\pi(z)}\subset V(f^{u+s}z)^\perp$ for any $0\leqq s,\ u,\ s+u\leqq\varepsilon_0$. Let $\bar\varepsilon(z)>0$ be such that $(\mathrm{Exp}_{\pi(z)})_{*hz}$ is nonsingular for any $0\leqq h\leqq\bar\varepsilon(z)$. Take an arbitrary s with $0<s<\bar\varepsilon(z)$, $0<u+s<\varepsilon_0$ and any $X\in V(f^s z)^\perp$. Then, we can choose $w\in W(f^s z)$, $\eta\in T_{f^s z}S_{\pi(f^s z)}$ such that $X=w+\eta$. Since $(\mathrm{Exp}_{\pi(z)})_{*sz}$ is nonsingular, we have $w_0\in T_z S_{\pi(z)}$ such that $(\mathrm{Exp}_{\pi(z)})_{*sz} s w_0=\pi_*(f^s)_* w_0=\pi_* w$. Therefore, there exists an $\eta_1\in T_{f^s z}S_{\pi(f^s z)}$ such that $(f^s)_* w_0=w+\eta_1$. Hence, $w+\eta=(f^s)_* w_0-\eta_1+\eta$. Thus, $(f^u)_*(w+\eta)=(f^u)_*(f^s)_* w_0-(f^u)_*\eta_1+(f^u)_*\eta$. Since $(f^u)_*(f^s)_* w_0\in V(f^{u+s}z)^\perp$, $(f^u)_*\eta_1\in V(f^u(f^s z))^\perp$ and $(f^u)_*\eta\in V(f^u(f^s z))^\perp$, we have $(f^u)_*(w+\eta)\in V(f^{u+s}z)^\perp$, namely $(f^u)_*(V(f^s z)^\perp)\subset V(f^{u+s}z)^\perp$. Taking s to zero, we get $(f^u)_*(V(z)^\perp)\subset V(f^u z)^\perp$ for any $0\leqq u\leqq\varepsilon_0$. Taking the dimension in consideration, we have $(f^u)_*(V(z)^\perp)=V(f^u z)^\perp$ for any $0\leqq u\leqq\varepsilon_0$. Let $0=u_1<u_2<\cdots<u_{n-1}<u_n=t$ be a partition with $u_{i+1}-u_i<\varepsilon_0$ for $i=1, \cdots, n-1$. Then we have

$$f^t(V(y)^\perp) = (f^{u_n-u_{n-1}})_* \cdots (f^{u_2-u_1})_*(V(y)^\perp) = V(f^t y)^\perp .$$

For the case that $a(y)<t<0$ we get

$$(f^{-t})_*(V(f^t y)^\perp) = V(y)^\perp .$$

and, therefore, $V(f^t y)^\perp=(f^t)_*(V(y)^\perp)$. This completes the proof of this part.

(1)⇒(3): We have to show that $G_y(X, \tilde\nabla_{V(y)}V)=0$ for any $y\in S$ and any $X\in T_y S$. Since $G(V, V)=1$ on S, we have $G_y(\tilde\nabla_{V(y)}V, V(y))=0$. Let $X\in V(y)^\perp$.

Then, by using $(f^t)_*V(y)^\perp = V(f^t y)^\perp$, we have

$$0 = X(G(V, V)) = 2G_y(\tilde{\nabla}_X V, V(y)) = 2G_y(\tilde{\nabla}_{V(y)}(f^t)_*X, V(y))$$
$$= 2\{V(y)(G((f^t)_*X, V)) - G_y(X, \tilde{\nabla}_{V(y)}V)\} = -2G_y(X, \tilde{\nabla}_{V(y)}V).$$

This completes the proof of this part.

(3)⇒(4): We have only to prove that $\nabla^N_{\partial/\partial t}(\partial/\partial t)=0$. Since $H^N(\partial/\partial t, \partial/\partial t)=1$ on N_0, we have $H^N(\partial/\partial t, \nabla^N_{\partial/\partial t}(\partial/\partial t))=0$ on N_0. Let $X \in T_y N$. By Lemma 3.1 and the assumption (3), we get $H^N(X, \nabla^N_{\partial/\partial t}(\partial/\partial t))=0$ on N_0. These imply that $\nabla^N_{\partial/\partial t}(\partial/\partial t)=0$ on N_0.

(4)⇒(5): Since $\phi: (U_p - \{0\}, H^p) \to ((S_p)_0, H^{S_p})$ given by $\phi(ty)=(y, t)$ is an isometry, we see that (5) is a special case of the condition (4) because $\phi_*(y)= \partial/\partial t$.

(5)⇒(2): The condition (5) implies $\nabla^p_y y=0$ where y is the vector field on $U_p - \{0\}$ given by $y_{ty}=y$ for any $y \in S_p$. Hence, by Corollary 3.2, we see that $G_{f^t y}((f^t)_*X, \tilde{\nabla}_{V(f^t y)}V)=0$ for any $y \in S$, any $X \in T_y S_p$ and any t with $ty \in U_{\pi(y)} - \{0\}$. This implies that $(f^t)_*(T_y S_{\pi(y)}) \subset V(f^t y)^\perp$ for any $y \in S$ and $ty \in U_{\pi(y)} - \{0\}$, because

$$V(f^t y)(G((f^t)_*X, V)) = G_{f^t y}(\tilde{\nabla}_{V(f^t y)}(f^t)_*X, V(f^t y)) = \frac{1}{2}(f^t)_*X(G(V, V))=0$$

and $G_y(X, V(y))=0$ for any $X \in T_y S_{\pi(y)}$. Since the functions $b, s_1: S \to \boldsymbol{R}^+$ are lower semi-continuous as seen in Appendix 2, we can find a positive function $\varepsilon: S \to \boldsymbol{R}^+$ in the condition (2). ■

We conclude this section with a remark on singular points of Exp_p. Let $\gamma_y: [0, b(y)) \to M$ denote $\gamma_y(t) = \pi f^t y$ for any $y \in S$. We say that $\gamma_y(s)$ is a conjugate point to p along γ_y if $(f^s)_* T_y S_{\pi(y)} \cap T_{f^s y} S_{\pi(f^s y)} \neq \{0\}$. It follows that p is a conjugate point to $\gamma_y(s)$ along γ_y if so $\gamma_y(s)$ is to p. Since $\mathrm{Exp}_p ty = \pi f^t y$ for any $y \in S_p$ and $0 \leqq t < b(y)$, we have that $(\mathrm{Exp}_p)_{*sy}$ is singular if and only if $\gamma_y(s)$ is a conjugate point to p along γ_y.

According to Appendix 2. let $s_1: S \to \boldsymbol{R}^+$ be defined by

$$s_1(y) = \inf\{s>0 \mid \gamma_y(s) \text{ is a conjugate point to } p \text{ along } \gamma_y\}.$$

We call $\gamma_y(s_1(y))$ the first conjugate point to p along γ_y. Thus, we have

$$U_p = \{ty \mid y \in S_p \text{ and } 0 \leqq t < \min\{s_1(y), b(y)\}\}.$$

COROLLARY 3.5. *Assume that one of the conditions in Theorem* 3.4 *is true. Let* $y \in S_p$. *Then* $\gamma_y(s)$ *is a first conjugate point to* p *along* γ_y *if and only if there exists a nontrivial Jacobi vector field* Y *along the geodesic* $t \to ty$ *in* $(U_p - \{0\}, H^p)$ *such that* $\lim_{t \to +0} H^p_{ty}(Y(t), Y(t))=0$ *and* $\lim_{t \to s-0} H^p_{ty}(Y(t), Y(t))=0$.

PROOF. The curves $t \to ty$ are geodesics in $(U_p - \{0\}, H^p)$ for all $y \in S_p$, so a vector field $t \to tz =: Y_z(t)$ along the geodesic ty is a Jacobi vector field for any $z \in T_y S_p$. Since

$$H^p_{ty}(Y_z(t), Y_z(t)) = g_{f^t y}((\mathrm{Exp}_p)_{*ty} tz, (\mathrm{Exp}_p)_{*ty} tz) = g_{f^t y}(\pi_*(f^t)_* z, \pi_*(f^t)_* z)$$

for any $0 \leqq t < b(y)$, this corollary is true. ■

A direct consequence is the following.

COROLLARY 3.6. *Assume that one of the conditions in Theorem* 3.4 *is true. If* $(U_p - \{0\}, H^p)$ *has nonpositive radial sectional curvature centered at the origin, then there is no conjugate point to* p *along* γ_y *for any* $y \in S_p$.

4. Metric structure.

In this section we assume that $\pi(S) = M$ and V is complete in S. Let M have a Riemannian metric $\tilde{g}$. Since S_p is a star-shaped hypersurface centered at the origin, we can find a generalized metric g on M as seen in Proposition 2.2 if there is a complementary distribution D. We assume that D is f^t-invariant in this section. In order to define a pseudo-distance (see Lemma 4.1) on M we need a function $L : TM - \{0\} \to \boldsymbol{R}^+ \cup \{0\}$ which is defined as follows:

$$\begin{cases} L(z) = \lambda = \sqrt{g_y(z, z)} & \text{if } z = \lambda y \ (\lambda > 0) \text{ and } y \in S_p, \\ L(z) = 0 & \text{if } z \notin C_{\pi(y)} - \{0\}. \end{cases}$$

The function L is continuous on $TM - \{0\}$. Any two points p and q in M can be joined by a piecewise smooth curve, since M is connected. Let

$$\lambda(c) := \int_a^b L(\dot{c}(t))dt$$

which is called the *length* of a curve $c : [a, b] \to M$. It follows that $\lambda(c)$ is independent of the choice of an oriented parameter of c. Thus, we define a function $d : M \times M \to \boldsymbol{R}^+ \cup \{0\}$ by

$$d(p, q) := \inf\{\lambda(c) \mid c \text{ is a piecewise smooth curve from } p \text{ to } q\}.$$

Since we do not assume the symmetry of the function L, this function $d(\cdot, \cdot)$ is not necessarily symmetric ([**B2**], [**BM**]).

LEMMA 4.1. *The following are true.*
(1) $d(p, q) \geqq 0$ *for any* $p, q \in M$.
(2) $d(p, q) = 0$ *if* $p = q$.
(3) $d(p, q) \leqq d(p, r) + d(r, q)$ *for any* $p, q, r \in M$.
(4) $d(p, q) = 0$ *implies* $p = q$ *if all* S_p *are closed hypersurfaces in* T_pM,

namely compact hypersurfaces without boundary.

Let $B_p^+(t)=\{q\in M \mid d(p, q)\leqq t\}$ and $B_p^-(t)=\{q\in M \mid d(q, p)\leqq t\}$. In general, $B_p^+(t)\neq B_p^-(t)$. The property (3) in Lemma 4.1 implies that $B_p^+(t)\supset B_{\gamma_y(s)}^+(u)$ for $t=s+u$, $0\leqq s, u$, where $y\in S_p$. If $B_p^+(t)$ and $B_{\gamma_y(s)}^+(u)$ are differentiable at $\gamma_y(t)$, then their tangent space is $\pi_* D(f^t y)$. This is connected with Huygens' principle ([**A**]).

We shall pay our attention to geometry in $\mathrm{Exp}_p(U_p-\{0\})$. We say that $c:[a, b]\to M$ is *an underlying curve* from $c(a)=\mathrm{Exp}_p(y)$ to $c(b)=\mathrm{Exp}_p(z)$ in $\mathrm{Exp}_p(U_p-\{0\})$ if there exists a curve $\bar{c}:[a, b]\to U_p-\{0\}$ such that $\bar{c}(a)=y$, $\bar{c}(b)=z$ and $\mathrm{Exp}_p\bar{c}(t)=c(t)$ for any $a\leqq t\leqq b$. As seen before any curve $t\to ty$ in $U_p-\{0\}$ is a geodesic in $(U_p-\{0\}, H^p)$ for any $y\in S_p$. This does not mean that $\gamma_y:[a, b]\to M$ has minimum length in the set of all underlying curves from $\gamma_y(a)$ to $\gamma_y(b)$ in $\mathrm{Exp}_p(U_p-\{0\})$. However we have the following.

LEMMA 4.2. *If S_q is convex in T_qM for any $q\in M$, then any curve $\gamma_y:[a, b]\to M$ has minimum length $b-a$ in the set of all underlying curves c from $\gamma_y(a)$ to $\gamma_y(b)$ in* $\mathrm{Exp}_p(U_p-\{0\})$ *with $L(\dot{c})\neq 0$, where $y\in S_p$.*

PROOF. Let $c:[a, b]\to M$ be a C^∞ underlying curve from $c(a)=\gamma_y(a)$ to $c(b)=\gamma_y(b)$ in $\mathrm{Exp}_p(U_p-\{0\})$ and let $\bar{c}(t):=\mathrm{Exp}_p^{-1}c(t)$ for any $a\leqq t\leqq b$. Define $v:[a, b]\to S_p$ and $s:[a, b]\to \boldsymbol{R}$ by $\bar{c}(t)=s(t)v(t)$ for any $a\leqq t\leqq b$, and we have that $c(t)=\gamma_{v(t)}(s(t))=\pi f^{s(t)}v(t)$ for any $a\leqq t\leqq b$. Therefore,

$$\dot{c}(t)=\dot{s}(t)f^{s(t)}v(t)+\pi_*(f^{s(t)})_*\dot{v}(t).$$

for any $a\leqq t\leqq b$. Since $L(\dot{c}(t))\neq 0$ for any $a\leqq t\leqq b$, there is the unique vector $y(t)\in S_{c(t)}$ such that $\dot{c}(t)=L(\dot{c}(t))y(t)$ for each $a\leqq t\leqq b$. We can have the decomposition of $y(t)$ as

$$y(t)=\alpha(t)f^{s(t)}v(t)+w(t)$$

where $w(t)\in\pi_*V(f^{s(t)}v(t))^\perp=T_{f^{s(t)}v(t)}S_{c(t)}$ for any $a\leqq t\leqq b$. Since all S_q are convex, we have $\alpha(t)\leqq 1$ for any $a\leqq t\leqq b$. Since

$$L(\dot{c}(t))(\alpha(t)f^{s(t)}v(t)+w(t))=\dot{s}(t)f^{s(t)}v(t)+\pi_*(f^{s(t)})_*\dot{v}(t)$$

for any $a\leqq t\leqq b$, and $(f^{s(t)})_*\dot{v}(t)\in V(f^{s(t)}v(t))^\perp$, we have $L(\dot{c}(t))\alpha(t)=\dot{s}(t)$ for any $a\leqq t\leqq b$. Thus we get $L(\dot{c}(t))\geqq\dot{s}(t)$ for any $a\leqq t\leqq b$ because $L(\dot{c}(t))>0$. Hence we finally have

$$\lambda(c)=\int_a^b L(\dot{c}(t))dt\geqq\int_a^b\dot{s}(t)dt=s(b)-s(a)=b-a.$$

■

This lemma states that any point q can be joined from any point p by a broken geodesic γ with minimum length if there is a piecewise smooth curve $c:[a, b]\rightarrow M$ with $c(a)=p$, $c(b)=q$, $L(\dot{c}(t))\neq 0$ for any $a\leqq t\leqq b$ with minimum length. It is natural to ask when a point q can be joined from a point p by a geodesic segment with minimum length. The spaces with this property are studied in the book [**BP**].

LEMMA 4.3. *Assume that S_r is closed and convex in T_rM for any $r\in M$. If a point $q\in M$ can be joined from $p\neq q$ by a piecewise smooth curve $c:[a, b]\rightarrow M$, $c(a)=p$, $c(b)=q$ with $\lambda(c)<s_1(p)$, then there exists a $y\in S_p$ such that $\gamma_y:[0, d]\rightarrow M$ is a geodesic segment from p to q with minimum length.*

PROOF. Since Exp_p is nonsingular in $U_p-\{0\}$, we can lift c to $U_p-\{0\}$, namely there exists a piecewise smooth curve $\bar{c}:[a, b]\rightarrow U_p$ such that $\mathrm{Exp}_p\bar{c}(t)=c(t)$ for any $a\leqq t\leqq b$. In fact, $\bar{c}$ does not reach the boundary of U_p, since $\lambda(c)<s_1(p)$ and Lemma 4.2. We have a $y\in S_p$ such that $\bar{c}(b)=\alpha y$ for some $\alpha>0$. Then, $\gamma_y:[0, \alpha]\rightarrow M$ is a desired geodesic segment. ■

In the case when $s_1(r)=\infty$ for any $r\in M$ we have the following.

COROLLARY 4.4. *Assume that S_r is closed and convex in T_rM for any $r\in M$ and any point has no conjugate point along any geodesic. Then, any point $q\in M$ can be joined from $p\neq q$ by a geodesic segment. Moreover, there exists the unique geodesic segment from any point p to any point q in M if M is simply connected.*

We conclude this section with the case where S_r are strictly convex in T_rM. In the following "*finitely compact*" by definition means that bounded closed sets are compact.

COROLLARY 4.5 ([**B1**], [**B2**], [**BM**]). *Assume that S_r is closed and strictly convex in T_rM for any $r\in M$ and the distance $d(\cdot, \cdot)$ is finitely compact. Then, M becomes a G-space with possibly nonsymmetric distance in the sense of Busemann.*

5. Appendix 1. Connection map.

Let M, TM, V and S be as in Introduction. Let $h_\alpha: TM\rightarrow TM$ be a map given by $h_\alpha(y)=\alpha y$, where $\alpha>0$. We define a vector field $\bar{V}$ on the cone spaned by S as $\bar{V}(\alpha y)=\alpha(h_\alpha)_*(V|S)(y)$ for any $\alpha>0$ and $y\in S$. Then, $\bar{V}$ is of class C^∞ on $C-\{0\}$. Let $(U; x^1, \cdots, x^n)$ be a local coordinate system of M and $(TU; x^1, \cdots, x^n, y^1, \cdots, y^n)$ a local coordinate system of TM such that a vector $y=\sum_{i=1}^n y^i(\partial/\partial x^i)|_{(x^1,\dots,x^n)}$ has coordinate $(x, y)=(x^1, \cdots, x^n, y^1, \cdots, y^n)$. In this coordinate neighborhood we assume that

$$\bar{V}(y)=\sum_{i=1}^{n}y^{i}\frac{\partial}{\partial x^{i}}\Big|_{(x,y)}+\sum_{i=1}^{n}b^{i}(x,y)\frac{\partial}{\partial y^{i}}\Big|_{(x,y)}$$

for some functions $b^i: TU\cap C\to \boldsymbol{R}$. Then,

$$b^{i}(x,\alpha y)=\alpha^{2}b^{i}(x,y)$$

for any $\alpha>0$. This shows that $\bar{V}$ is at least of class C^1 at the origin. Let $P: C\to S$ be a projection such that $P(w)$ is the intersection of S and the ray passing through $w\in C$ from the origin. As seen in the paper [**D**] we can find functions $\Gamma^i_{jk}: S\to\boldsymbol{R}$, $i, j, k=1,\cdots, n$ such that

$$b^{i}(x,y)=-\sum_{i,j,k}\Gamma^{i}_{jk}(P(y))y^{j}y^{k}$$

for any $y\in T_xU\cap C$. Actually we have

$$\Gamma^{i}_{jk}(y)=-\frac{1}{2}\frac{\partial^{2}}{\partial y^{j}\partial y^{k}}b^{i}(x,y)$$

where $y\in S_x$. At last we can define a connection map $K_y: T_yTM\to T_xM$ $(\pi(y)=x)$ by

$$K_{y}(x,y,X,Y)=(x,Y+\Gamma(P(y))(y,X))$$

where

$$\Gamma(P(y))(y,X)=\sum_{i=1}^{n}\Big(\sum_{j,k}\Gamma^{i}_{jk}(P(y))y^{j}X^{k}\Big)\frac{\partial}{\partial x^{i}}\Big|_{x}$$

in $TU\cap C$. In fact, $\Gamma(P(y))(y, X)$ depends on the choice of the coordinate system, but $Y+\Gamma(P(y))(y, X)$ is independent of the choice of the coordinate system $(U; x^1,\cdots, x^n)$.

6. Appendix 2. Domains of flows.

Let $a, b: S\to\boldsymbol{R}$ denote the functions given by

$$a(y)=\inf\{s\,|\,f^{t}\text{ is defined for all }s<t\leqq 0\},$$

$$b(y)=\sup\{s\,|\,f^{t}\text{ is defined for all }0\leqq t<s\}.$$

It follows that a is upper semi-continuous and b is lower semi-continuous, namely

$$\overline{\lim}_{y\to y_0}a(y)\leqq a(y_0)$$

$$\underline{\lim}_{y\to y_0}b(y)\geqq b(y_0)$$

for any $y_0\in S$. Therefore, we can find continuous function $\underline{a}$ and $\bar{b}$ on S such that $a<\underline{a}<0$ and $0<\bar{b}<b$.

Let $s_1: S\to\boldsymbol{R}^+$ be the function such that $\gamma_y(s_1(y))$ is the first conjugate point to $\pi(y)$ along γ_y. By definition, $\bar{s}_1$ is lower semi-continuous. Thus, we can find a continuous function $\bar{s}_1 : S \to \boldsymbol{R}$ such that $0 < \bar{s}_1 < s_1$.

The function $\varepsilon : S \to \boldsymbol{R}^+$ given by $\varepsilon(y) = \min\{\bar{s}_1(y), \bar{b}(y)\}$ is continuous on S.

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