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Localization of diffusion processes in one-dimensional random environment[1)]

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Introduction.

Let $X(t, W)$ be a one-dimensional diffusion process starting at 0 and with generator

$$\mathcal{L}_W = \frac{1}{2} e^{W(x)} \frac{d}{dx}\left(e^{-W(x)} \frac{d}{dx}\right), \tag{1}$$

where $\{W(x), x \in \boldsymbol{R}\}$ is a random environment. The process $X(t, W)$ can be constructed from a one-dimensional Brownian motion through a change of scale and time. It is assumed that the Brownian motion (used for the construction of $X(t, W)$) and the random environment $\{W(x)\}$ are independent. *Formally*, $X(t, W)$ is a solution of the stochastic differential equation

$$dX(t) = \text{Brownian differential} - \frac{1}{2} W'(X(t))dt.$$

We are interested in the asymptotic properties of $X(t, W)$ as $t \to \infty$. A result for this type of random environment problem goes back to Sinai [**12**]. When $\{W(x), x \in \boldsymbol{R}\}$ is a Brownian environment, Brox [**1**] introduced the diffusion process $X(t, W)$ as a continuous model of Sinai's random walk ([**12**]) in a Bernoulli environment and obtained the following result of Sinai-type: $(\log t)^{-2} X(t, \cdot) - b(t, \cdot)$ tends to 0 in probability as $t \to \infty$ where $b(t, W)$ is a suitable function depending only on t and the environment $W = W(\cdot)$; moreover, the distribution of $(\log t)^{-2} X(t, \cdot)$ tends to a limit which is the same as the limit distribution in Sinai's case. Kesten [**9**] and Golosov [**5**] obtained the explicit form of the limit distribution (see also [**15**] for some extension). Results of Sinai-type for a wider class of random environments were then obtained by Letchikov [**10**] (for non-simple random walks) and Kawazu, Tamura and Tanaka [**7**], [**8**] (for

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diffusion processes in asymptotically self-similar random environment). A more refined result (localization) was obtained by Golosov [**4**] in the case of Sinai's reflecting random walk. The result is roughly stated as follows: for large n the position of the random walk at time n is localized in a finite neighborhood of a suitable point b_n depending only on n and the environment. The result of this type (localization) was then obtained by Tanaka [**16**] for a diffusion process $X(t, W)$ in Brownian environment. The result asserts the existence of the limit distribution of $X(e^{\lambda}, \cdot)-b_{\lambda}(\cdot)$ as $\lambda\to\infty$ where $b_{\lambda}(W)$ is a suitable function of λ and the environment W alone.

The purpose of this paper is to study the localization of diffusion processes for a considerably wider class of random environments. The random environment $\{W(x), x\in\boldsymbol{R}\}$ we consider is described as follows: When $W(x)$ is observed at integers x, it is a random walk; more precisely, $W(x)$ is a constant on each interval $(n, n+1)$, $n\in\boldsymbol{Z}$, and $\mathfrak{W}^{+}=\{W(n), n\geqq 0\}$ and $\mathfrak{W}^{-}=\{W(-n), n\geqq 0\}$ are independent random walks in $\boldsymbol{R}$. The essential assumption is the existence of common scaling (without centering) of $\mathfrak{W}^{+}$ and $\mathfrak{W}^{-}$ which ensures the convergence in law of the scaled random walks to strictly stable processes. Our main result is stated in § 1. Although we do not discuss here the localization problem in the case of random walks, it will be possible to treat the problem starting from the present model and using optional sampling.

Roughly speaking the outline of our argument is similar to that of [**16**]; however, it is to be noted that there are several points which we had to study anew. For example, in order to know about the probability law of a valley which is essential in our argument, we must study certain time reversals of random walks, conditioned random walks and their relationship (see § 2). The result of this part has also its own interest. To grasp the outline of our method the reader is recommended to proceed to the first subsection of § 4 after only glancing through § 2 and § 3.

In § 1 we state the problem and the result. § 2 is devoted to the study of certain random walk problems as explained above. In § 3 we define a valley of an environment with reference to [**1**] and [**9**] and give some information about the probability law of a valley using the result of § 2. In § 4 we first give an outline of the proof of the main theorem and then prepare some lemmas concerning diffusion processes in a (suitably scaled) environment. We complete the proof of our main theorem in § 5.

§ 1. Statement of the result.

Let $\mathcal{W}$ be the space of step functions $W: \boldsymbol{R}\to\boldsymbol{R}$ with the following properties:

i. W vanishes identically in the open interval $(-1, 1)$.

ii. W is constant in each open interval $(n, n+1)$, $n \in \boldsymbol{Z}$.

iii. W is right continuous on $[0, \infty)$ and left continuous on $(-\infty, 0]$.

Throughout the paper we are given a probability measure $\boldsymbol{Q}$ on $\mathcal{W}$ satisfying the following assumption (A).

Assumption (A). (i) Let $Y_n = W(n) - W(n-1)$, $n \geqq 1$, and $Y_n = W(n) - W(n+1)$, $n \leqq -1$. Then $\{Y_n, n \geqq 1, \boldsymbol{Q}\}$ and $\{Y_n, n \leqq -1, \boldsymbol{Q}\}$ are families of i.i.d. random variables and the families are independent each other.

(ii) The distribution of Y_1 (resp. Y_{-1}) belongs to the domain of attraction with centering constant 0 of some strictly stable distribution μ_+ (resp. μ_-) of index α, $0 < \alpha \leqq 2$, whose density is positive in the whole of $\boldsymbol{R}$. Moreover, the normalizing constants for $W(n) = \sum_{k=1}^{n} Y_k$ and $W(-n) = \sum_{k=1}^{n} Y_{-k}$ (which ensure the convergence in law to μ_+ and μ_-, respectively) can be chosen to be the same.

It is to be noted that μ_+ and μ_- have the same index α. For $W \in \mathcal{W}$ let $X(t, W)$ be a one-dimensional diffusion process with generator $\mathcal{L}_W$ of (1) starting at 0. Following Itô and McKean [**6**] we construct $X(t, W)$ from a one-dimensional Brownian motion $B(t)$ through a change of scale and time (see the beginning of § 4). The probability measure governing this Brownian motion is denoted by $\boldsymbol{P}$. An element W of $\mathcal{W}$ is called an *environment*. We assume that the Brownian motion $B(t)$ and the environment W are independent. Thus the product probability measure $\mathcal{P} = \boldsymbol{P} \otimes \boldsymbol{Q}$ determines the full law of $X(t, \cdot)$.

In general, given a random variable Y taking values in $[0, \infty]$ such that Prob. $\{Y > 0\} > 0$ we define the *renewal function* $R(x)$, $x \geqq 0$, corresponding to Y by

$$R(x) = 1 + \sum_{n=1}^{\infty} \text{Prob.}\{Y_1 + \cdots + Y_n \leqq x\},$$

where Y_k, $k \geqq 1$, are independent copies of Y. Note that $R(x) < \infty$. We denote by $R(x-)$, $x > 0$, the left limit of $R(\cdot)$ at x and put $R(0-) = 1$.

For $W \in \mathcal{W}$ we put

$$\hat{\sigma}^+ = \min\{n \geqq 1 : W(n) \geqq 0\},$$

$$\hat{\sigma}^- = \max\{n \leqq -1 : W(n) \geqq 0\},$$

$$\tau^+ = \min\{n \geqq 1 : W(n) < 0\},$$

$$\tau^- = \max\{n \leqq -1 : W(n) < 0\},$$

and denote by $\hat{R}^+(x)$, $\hat{R}^-(x)$, $R^+(x)$ and $R^-(x)$ the renewal functions corresponding to the random variables $W(\hat{\sigma}^+)$, $W(\hat{\sigma}^-)$, $-W(\tau^+)$ and $-W(\tau^-)$ (the basic probability here is $\boldsymbol{Q}$), respectively. We also denote by $P^+(x, dy)$, $\hat{P}^+(x, dy)$, $P^-(x, dy)$ and $\hat{P}^-(x, dy)$ the distributions of $x + W(1)$, $x - W(1)$, $x + W(-1)$ and

$x-W(-1)$ under $\boldsymbol{Q}$, respectively. We now define for $x\geqq 0$

$$p^{+}(x, dy)=\hat{R}^{+}(x-)^{-1}\hat{P}^{+}(x, dy)\hat{R}^{+}(y-)\mathbf{1}_{(0,\infty)}(y),$$

$$p^{-}(x, dy)=\hat{R}^{-}(x-)^{-1}\hat{P}^{-}(x, dy)\hat{R}^{-}(y-)\mathbf{1}_{(0,\infty)}(y),$$

$$q^{+}(x, dy)=R^{+}(x)^{-1}P^{+}(x, dy)R^{+}(y)\mathbf{1}_{[0,\infty)}(y),$$

$$q^{-}(x, dy)=R^{-}(x)^{-1}P^{-}(x, dy)R^{-}(y)\mathbf{1}_{[0,\infty)}(y).$$

Then, as we shall see in §2, these are Markov transition functions on $[0, \infty)$.

In general a Markov chain with (one-step) transition function $p(x, dy)$ will be called a *p-chain* in this paper.

Let $\mathcal{W}^{+}$ (resp. $\mathcal{W}^{-}$) be the space of step functions $W: \boldsymbol{R}\rightarrow[0, \infty)$ with the following two properties:

1°. $W(0)=0$.

2°. W is constant on each interval $[n, n+1)$ (resp. $(n, n+1]$), $n\in\boldsymbol{Z}$.

We introduce the probability measures $\tilde{\boldsymbol{Q}}^{+}$ and $\tilde{\boldsymbol{Q}}^{-}$ on $\mathcal{W}^{+}$ and $\mathcal{W}^{-}$, respectively, determined by the following conditions:

(1°) $\{W(n), n\geqq 0, \tilde{\boldsymbol{Q}}^{+}\}$ is a q^{+}-chain, $\{W(-n), n\geqq 0, \tilde{\boldsymbol{Q}}^{+}\}$ is a p^{+}-chain and these two chains are independent.

(2°) $\{W(n), n\geqq 0, \tilde{\boldsymbol{Q}}^{-}\}$ is a p^{-}-chain, $\{W(-n), n\geqq 0, \tilde{\boldsymbol{Q}}^{-}\}$ is a q^{-}-chain and these two chains are independent.

Then we can prove that $e^{-W}\in L^{1}(\boldsymbol{R}, dx)$ a.s. with respect to $\tilde{\boldsymbol{Q}}^{+}$ and $\tilde{\boldsymbol{Q}}^{-}$ (see Proposition 2.2 and Corollary 2.2). Next, for each $W\in\mathcal{W}^{+}$ or $\in\mathcal{W}^{-}$ with $e^{-W}\in L^{1}(\boldsymbol{R})$ we introduce a probability measure $\tilde{\boldsymbol{P}}_{W}$ on $\tilde{\Omega}$, the space of real valued continuous functions defined in $[0, \infty)$, as follows: $\tilde{\boldsymbol{P}}_{W}$ is the probability law of the one-dimensional diffusion process with generator (1) and with initial distribution

$$\tilde{\nu}_{W}(dx)=e^{-W(x)}dx\Big/\int_{\boldsymbol{R}}e^{-W(y)}dy\,.$$

Note that $\{\tilde{\omega}(t), t\geqq 0, \tilde{\boldsymbol{P}}_{W}\}$ is a stationary process, where $\tilde{\omega}(t)$ denotes the value of $\tilde{\omega}(\in\tilde{\Omega})$ at time t. We now define probability measure $\tilde{\nu}$ on $\boldsymbol{R}$ and $\tilde{\boldsymbol{P}}$ on $\tilde{\Omega}$ by

$$\tilde{\nu}=\mathfrak{p}\int_{\mathcal{W}^{+}}\tilde{\boldsymbol{Q}}^{+}(dW)\tilde{\nu}_{W}+(1-\mathfrak{p})\int_{\mathcal{W}^{-}}\tilde{\boldsymbol{Q}}^{-}(dW)\tilde{\nu}_{W}\,,$$

$$\tilde{\boldsymbol{P}}=\mathfrak{p}\int_{\mathcal{W}^{+}}\tilde{\boldsymbol{Q}}^{+}(dW)\tilde{\boldsymbol{P}}_{W}+(1-\mathfrak{p})\int_{\mathcal{W}^{-}}\tilde{\boldsymbol{Q}}^{-}(dW)\tilde{\boldsymbol{P}}_{W}\,,$$

where $\mathfrak{p}$ is a constant strictly between 0 and 1 which is defined by (5.11). Finally, for $W\in\mathcal{W}$ let $b_{\lambda}(W)$ be defined by (3.3b) in §3. $b_{\lambda}(W)$ is a function of the environment W alone and does not depend upon the Brownian motion used for the construction of $X(t, W)$. Our main theorem is now stated as follows.

MAIN THEOREM. *The process* $\{X(e^{\lambda}+t, \cdot)-b_{\lambda}(\cdot), t\geqq 0, \mathcal{P}\}$ *converges as* $\lambda\to\infty$ *to the stationary process* $\{\tilde{\omega}(t), t\geqq 0, \tilde{\boldsymbol{P}}\}$ *in the sense of weak convergence of the corresponding probability measures on* $\tilde{\Omega}$ *and hence, in particular, the distribution of* $X(e^{\lambda}, \cdot)-b_{\lambda}(\cdot)$ *converges to* $\tilde{\nu}$ *as* $\lambda\to\infty$. *Moreover, the weak convergence can be strengthened to the convergence with respect to the total variation norm if the probability measures under consideration are restricted to the sub-σ-field* $\tilde{\mathcal{B}}_h=\sigma\{\tilde{\omega}(t): 0\leqq t\leqq h\}$, *h being any positive constant.*

§2. Certain Markov chains attached to random walks.

In this section we introduce several Markov chains attached to a random walk. The results obtained here will be used for proving our main theorem in §3 and §5.

Throughout this section we are given a random walk $\mathfrak{S}=\{S_n, n\geqq 0\}$ where

$$S_0=0, \qquad S_n=X_1+\cdots+X_n \quad (n\geqq 1),$$

X_k, $k\geqq 1$, being independent identically distributed random variables. We often write $S(n)$ instead of S_n. The following will be used:

$$S_n^x = x+S_n, \qquad x\in\boldsymbol{R},$$

$$P(x, dy)=\boldsymbol{P}\{x+X_1\in dy\}, \quad \hat{P}(x, dy)=\boldsymbol{P}\{x-X_1\in dy\}.$$

We shall also consider the dual random walk $\hat{\mathfrak{S}}=\{\hat{S}_n, n\geqq 0\}$ where $\hat{S}_n=-S_n$.

2.1. Time reversals of random walks. Let

$$\tau=\min\{n\geqq 1: S_n<0\} \tag{2.1}$$

and assume that

$$\tau<\infty \quad \text{a.s.} \tag{A.1}$$

We consider the time reversal

$$(0, S_{\tau-1}-S_\tau, S_{\tau-2}-S_\tau, \cdots, S_1-S_\tau, -S_\tau), \tag{2.2}$$

which is regarded as a random variable taking values in the path space

$$\{w=(w(0), w(1), \cdots, w(l)): w(0)=0, 0<w(l)=\min\{w(k): 1\leqq k\leqq l\}, l\geqq 0\}.$$

We then take independent copies w_k, $k\geqq 1$, of the random variable (2.2) and write $w_k=(w_k(0), w_k(1), \cdots, w_k(l_k))$, $k\geqq 1$. One of the Markov chains we consider is defined by

(2.3) $$U_n = \begin{cases} w_1(n) & \text{for } 0 \leqq n \leqq l_1, \\ \sum_{j=1}^{k-1} w_j(l_j) + w_k\left(n - \sum_{j=1}^{k-1} l_j\right) & \text{for } \sum_{j=1}^{k-1} l_j < n \leqq \sum_{j=1}^{k} l_j, \quad (k \geqq 2). \end{cases}$$

Let

$$\xi(x) = \begin{cases} 1 & \text{for } x=0, \\ \boldsymbol{E}\{\sum_{0 \leqq n < \tau} \mathbf{1}_{[0,x)}(S_n)\} & \text{for } x>0, \end{cases}$$

and define $p(x, dy)$ by

(2.4) $$p(x, dy) = \xi(x)^{-1}\hat{P}(x, dy)\xi(y)\mathbf{1}_{(0,\infty)}(y).$$

The following theorem is proved in Tanaka [17] (see also Lemma 6 of Golosov [4]).

THEOREM 2.1. *Under the assumption* (A.1) *$p(x, dy)$ is a Markov transition function on $[0, \infty)$ and the process $\{U_n, n \geqq 0\}$ defined by* (2.3) *is a Markov chain on $[0, \infty)$ with transition function $p(x, dy)$, namely, a p-chain.*

We are now going to give another expression of $\xi(x)$. Put

(2.5) $$\hat{\sigma} = \min\{n \geqq 1 : S_n \geqq 0\}$$

and let $\tilde{R}(x)$ be the renewal function corresponding to the random variable $S(\hat{\sigma})$. The convention $S_\infty = S(\infty) = \infty$ is used throughout our argument. Thus $S(\hat{\sigma})$ is understood to be ∞ when $\hat{\sigma} = \infty$.

PROPOSITION 2.1. *Under the assumption* (A.1),

$$\xi(x) = \tilde{R}(x-), \qquad \text{for } x \geqq 0.$$

PROOF. It is known that (for example, see Feller [3: XVIII. 3] or Chung [2: 8.4]) for $0 \leqq r < 1$

(2.6) $$1 - \boldsymbol{E}\{r^{\hat{\sigma}} \cdot e^{\sqrt{-1}\lambda S(\hat{\sigma})}\} = \exp\left(-\sum_{n=1}^{\infty} \frac{r^n}{n} \boldsymbol{E}\{e^{\sqrt{-1}\lambda S(n)} ; S(n) \geqq 0\}\right),$$

(2.7) $$\boldsymbol{E}\{\sum_{0 \leqq n < \tau} r^n e^{\sqrt{-1}\lambda S(n)}\} = \exp\left(\sum_{n=1}^{\infty} \frac{r^n}{n} \boldsymbol{E}\{e^{\sqrt{-1}\lambda S(n)} ; S(n) \geqq 0\}\right).$$

We therefore have

(2.8) $$\boldsymbol{E}\{\sum_{0 \leqq n < \tau} r^n e^{\sqrt{-1}\lambda S(n)}\} = (1 - \boldsymbol{E}\{r^{\hat{\sigma}} \cdot e^{\sqrt{-1}\lambda S(\hat{\sigma})}\})^{-1}, \quad 0 \leqq r < 1.$$

Let $F_r(dx)$ and $G_r(dx)$ be the measures on $[0, \infty)$ defined by

$$F_r(A) = \boldsymbol{E}\{r^{\hat{\sigma}} \cdot \mathbf{1}_A(S(\hat{\sigma}))\},$$
$$G_r(A) = \boldsymbol{E}\{\sum_{0 \leqq n < \tau} r^n \mathbf{1}_A(S(n))\},$$

respectively, for any Borel subset A of $[0, \infty)$. Then (2.8) means that the Fourier transform of the measure $G_r(dx)$ equals the Fourier transform of the measure $\sum_{n=0}^{\infty} F_r^{n*}$, where F_r^{n*} is the n-fold convolution of F_r and F_r^{0*} is the δ-distribution at 0. Thus we have

$$G_r([0, x)) = \sum_{n=0}^{\infty} F_r^{n*}([0, x)), \qquad x>0,\ 0\leqq r<1.$$

Now letting $r \uparrow 1$ in the above we obtain $\xi(x)=\tilde{R}(x-)$. □

COROLLARY 2.1.

$$p(x, dy) = \tilde{R}(x-)^{-1}\hat{P}(x, dy)\tilde{R}(y-)\mathbf{1}_{(0,\infty)}(y), \qquad x\geqq 0.$$

Next we consider the random time

$$\sigma = \min\{n\geq 1 : S_n \leqq 0\}. \tag{2.9}$$

We exclude the trivial case where $X_k=0$, a.s.. Then $\sigma<\infty$ a.s. if and only if (A.1) is satisfied. So under (A.1) time reversal can also be defined in terms of σ; in fact, we can define a process $\{\bar{U}_n, n\geqq 0\}$ exactly in the same way as we defined $\{U_n, n\geqq 1\}$ but with the replacement of τ by σ. The Markovian property of $\{\bar{U}_n, n\geqq 0\}$ can be proved by a method similar to that in [**17**]; however, here we give another proof based on Theorem 2 and Corollary 2.1. Put

$$\hat{\tau} = \min\{n\geqq 1 : S_n>0\}, \tag{2.10}$$

and let $\hat{R}(x)$ be the renewal function corresponding to $S(\hat{\tau})$. We then define $\bar{p}(x, dy)$ by

$$\bar{p}(x, dy) = \hat{R}(x)^{-1}\hat{P}(x, dy)\hat{R}(y)\mathbf{1}_{[0,\infty)}(y). \tag{2.11}$$

THEOREM 2.2. *Under the assumption* (A.1) $\bar{p}(x, dy)$ *is a Markov transition function on* $[0, \infty)$ *and the process* $\{\bar{U}_n, n\geqq 0\}$ *is a* $\bar{p}$*-chain.*

PROOF. Taking it for granted that $\bar{p}(x, dy)$ is a Markov transition function (the proof is omitted), we prove that $\{\bar{U}_n, n\geqq 0\}$ is a $\bar{p}$-chain. For each $\varepsilon>0$ we consider the random walk $S_n^{(\varepsilon)}=X_1^{(\varepsilon)}+\cdots+X_n^{(\varepsilon)}=S_n-n\varepsilon$, where $X_k^{(\varepsilon)}=X_k-\varepsilon$ and put

$$\tau^{(\varepsilon)} = \min\{n\geqq 1:\ S_n^{(\varepsilon)}<0\} = \min\{n\geqq 1:\ S_n<n\varepsilon\}.$$

We can define $\{U_n^{(\varepsilon)}, n\geqq 0\}$ in terms of $S_n^{(\varepsilon)}$ and $\tau^{(\varepsilon)}$ exactly in the same way as we defined $\{U_n, n\geqq 0\}$. Since $\tau^{(\varepsilon)} \uparrow \sigma$ as $\varepsilon \downarrow 0$, we can easily see that for any bounded continuous functions $f_1, \cdots, f_n$ on $[0, \infty)$

$$\lim_{\varepsilon\downarrow 0} \boldsymbol{E}\{f_1(U_1^{(\varepsilon)})f_2(U_2^{(\varepsilon)})\cdots f_n(U_n^{(\varepsilon)})\} = \boldsymbol{E}\{f_1(\bar{U}_1)f_2(\bar{U}_2)\cdots f_n(\bar{U}_n)\}. \tag{2.12}$$

According to Theorem 2.1 and Corollary 2.1, $\{U_n^{(\varepsilon)}, n\geqq 0\}$ is a Markov chain on

$[0, \infty)$ and its transition function is given by

$$p^{(\varepsilon)}(x, dy) = \tilde{R}^{(\varepsilon)}(x-)^{-1}\hat{P}(x, dy)\tilde{R}^{(\varepsilon)}(y-)\mathbf{1}_{(0,\infty)}(y),$$

where $\hat{P}^{(\varepsilon)}(x, dy)=\boldsymbol{P}\{x-X_1^{(\varepsilon)}\in dy\}=\hat{P}(x+\varepsilon, dy)$ and $\tilde{R}^{(\varepsilon)}(x)$ is the renewal function corresponding to $S^{(\varepsilon)}(\hat{\sigma}^{(\varepsilon)})$, $\hat{\sigma}^{(\varepsilon)}$ being defined by

$$\hat{\sigma}^{(\varepsilon)} = \min\{n\geqq 1:\ S_n^{(\varepsilon)}\geqq 0\} = \min\{n\geqq 1:\ S_n\geqq n\varepsilon\}.$$

Note that $S^{(\varepsilon)}(\hat{\sigma}^{(\varepsilon)})=\infty$ when $\hat{\sigma}^{(\varepsilon)}=\infty$. We are now going to prove that

$$\tilde{R}^{(\varepsilon)}(x-) \longrightarrow \hat{R}(x) \quad \text{as } \varepsilon\downarrow 0 \ \text{ for each } x\geqq 0. \tag{2.13}$$

For this purpose take independent copies $Y_k^{(\varepsilon)}$, $k\geqq 1$, of $S^{(\varepsilon)}(\hat{\sigma}^{(\varepsilon)})$ and write

$$\tilde{R}^{(\varepsilon)}(x-) = 1+\sum_{n=1}^{\infty}\tilde{R}_n^{(\varepsilon)}(x-), \qquad \tilde{R}_n^{(\varepsilon)}(x-) = \boldsymbol{P}\{Y_1^{(\varepsilon)}+\cdots+Y_n^{(\varepsilon)}<x\}.$$

It is easy to see that there exist x_0 and $\theta\in(0, 1)$ such that

$$\tilde{R}_1^{(\varepsilon)}(x_0-) = \boldsymbol{P}\{S^{(\varepsilon)}(\hat{\sigma}^{(\varepsilon)})<x_0\} < \theta$$

for all sufficiently small $\varepsilon>0$. Now let $x>0$ be given and put $\nu=[x/x_0]+1$. Then

$$\tilde{R}_{\nu n}^{(\varepsilon)}(x-) \leqq \nu\tilde{R}_n^{(\varepsilon)}\left(\frac{x}{\nu}-\right) \leqq \nu\tilde{R}_n^{(\varepsilon)}(x_0-) \leqq \nu\tilde{R}_1^{(\varepsilon)}(x_0-)^n < \nu\theta^n ,$$

Since $\tilde{R}_n^{(\varepsilon)}(x-)$ is decreasing in n,

$$\tilde{R}_n^{(\varepsilon)}(x-) < c\rho^n \qquad \text{for all sufficiently small } \varepsilon>0, \tag{2.14}$$

where $c=\nu/\theta$ and $0<\rho=\theta^{1/\nu}<1$. On the other hand, since

$$\begin{cases} \hat{\sigma}^{(\varepsilon)}\downarrow\hat{\tau} & \text{as } \varepsilon\downarrow 0, \\ \boldsymbol{P}\{S^{(\varepsilon)}(\hat{\sigma}^{(\varepsilon)})=S(\hat{\tau})-\hat{\tau}\varepsilon,\ \hat{\tau}<\infty\}\to\boldsymbol{P}\{\hat{\tau}<\infty\} & \text{as } \varepsilon\downarrow 0, \end{cases}$$

$\tilde{R}_n^{(\varepsilon)}(x-)$ tends to $\hat{R}_n(x)$ as $\varepsilon\downarrow 0$ for each $n\geqq 1$ and $x\geqq 0$ where $\hat{R}_n(x)=\boldsymbol{P}\{\hat{Y}_1+\cdots+\hat{Y}_n\leqq x\}$, $\hat{Y}_k$, $k\geqq 1$, being independent copies of $S(\hat{\tau})$. This fact combined with (2.14) implies (2.13).

Finally, for a function f on $[0, \infty)$ we denote by f^0 the function defined on the whole of $\boldsymbol{R}$ by $f^0(x)=f(x)$ for $x>0$ and $f^0(x)=0$ for $x\leqq 0$. Since $\{U_n^{(\varepsilon)}, n\geqq 0\}$ is a $p^{(\varepsilon)}$-chain, for any continuous functions $f_1, \cdots, f_m$ on $[0, \infty)$ with compact supports we have

$$\begin{aligned}
&\boldsymbol{E}\{f_1(U_1^{(\varepsilon)})f_2(U_2^{(\varepsilon)})\cdots f_m(U_m^{(\varepsilon)})\}\\
&=\int_{(0,\infty)}\hat{P}^{(\varepsilon)}(0,dx_1)f_1(x_1)\int_{(0,\infty)}\hat{P}^{(\varepsilon)}(x_1,dx_2)f_2(x_2)\int_{(0,\infty)}\cdots\\
&\qquad\cdots\int_{(0,\infty)}\hat{P}^{(\varepsilon)}(x_{m-1},dx_m)f_m(x_m)\tilde{R}^{(\varepsilon)}(x_m-)\\
&=\boldsymbol{E}\{f_1^0(-S_1^{(\varepsilon)})f_2^0(-S_2^{(\varepsilon)})\cdots f_m^0(-S_m^{(\varepsilon)})\tilde{R}_-^{(\varepsilon)}(-S_m^{(\varepsilon)})\}\\
&=\boldsymbol{E}\{f_1^0(-S_1+\varepsilon)f_2^0(-S_2+2\varepsilon)\cdots f_m^0(-S_m+m\varepsilon)\tilde{R}_-^{(\varepsilon)}(-S_m+m\varepsilon)\}
\end{aligned}$$

where $\tilde{R}_-^{(\varepsilon)}(x)=\tilde{R}^{(\varepsilon)}(x-)$ for $x>0$ and $\tilde{R}_-^{(\varepsilon)}(x)=0$ for $x\leqq 0$. Making use of (2.13) together with the monotone property and the right continuity of renewal functions, we can prove that $\tilde{R}_-^{(\varepsilon)}(x+m\varepsilon)$ converges to $\hat{R}(x)$ as $\varepsilon\downarrow 0$ for each $x\geqq 0$. Therefore we have

$$(2.15)\qquad \begin{aligned}
&f_1^0(-S_1+\varepsilon)f_2^0(-S_2+2\varepsilon)\cdots f_m^0(-S_m+m\varepsilon)\tilde{R}_-^{(\varepsilon)}(-S_m+m\varepsilon)\\
&\longrightarrow\begin{cases} f_1^{\#}(-S_1)f_2^{\#}(-S_2)\cdots f_m^{\#}(-S_m)\hat{R}(-S_m), & \text{if } -S_m\geqq 0,\\ 0, & \text{if } -S_m<0,\end{cases}
\end{aligned}$$

as $\varepsilon\downarrow 0$, where $f_k^{\#}(x)=f_k(x)$ for $x\geqq 0$ and $f_k^{\#}(x)=0$ for $x<0$. Since the support of f_m^0 is compact, the convergence of (2.15) is bounded. We thus have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\boldsymbol{E}\{f_1(U_1^{(\varepsilon)})f_2(U_2^{(\varepsilon)})\cdots f_m(U_m^{(\varepsilon)})\}\\
&\qquad=\boldsymbol{E}\{f_1^{\#}(-S_1)f_2^{\#}(-S_2)\cdots f_m^{\#}(-S_m)\hat{R}(-S_m)\}.
\end{aligned}$$

This combined with (2.12) implies

$$\begin{aligned}
&\boldsymbol{E}\{f_1(\bar{U}_1)f_2(\bar{U}_2)\cdots f_m(\bar{U}_m)\}\\
&=\text{the right hand side of (2.15)}\\
&=\int_{[0,\infty)}\bar{p}(0,dx_1)f_1(x_1)\int_{[0,\infty)}\bar{p}(x_1,dx_2)f_2(x_2)\int_{[0,\infty)}\cdots\int_{[0,\infty)}\bar{p}(x_{m-1},dx_m)f_m(x_m),
\end{aligned}$$

as was to be proved. □

2.2. Conditioned random walks. For $\lambda>0$ and $x\geqq 0$ we define $\eta_\lambda(x)$ and $\bar{\eta}_\lambda(x)$ for $0\leqq x<\lambda$ by

(2.16) $\eta_\lambda(x)=$ the probability that the random walk S_n^x hits $[\lambda,\infty)$ before it hits $(-\infty,0)$,

(2.17) $\bar{\eta}_\lambda(x)=$ the probability that the random walk S_n^x hits $[\lambda,\infty)$ before it hits $(-\infty,0]$.

We also put $\eta_\lambda(x)=\bar{\eta}_\lambda(x)=1$ for $x\geqq\lambda$. By "S_n^x hits A" we understand that S_n^x

hits A at some $n\geqq 1$. We can define Markov transition functions $q_\lambda(x, dy)$ and $\bar{q}_\lambda(x, dy)$ on $[0, \infty)$ by

(2.18) $$q_\lambda(x, dy)=\begin{cases}\eta_\lambda(x)^{-1}P(x, dy)\eta_\lambda(y)\mathbf{1}_{[0,\infty)}(y) & \text{for } 0\leqq x<\lambda,\\ \delta_x(dy) & \text{for } x\geqq\lambda,\end{cases}$$

(2.19) $$\bar{q}_\lambda(x, dy)=\begin{cases}\bar{\eta}_\lambda(x)^{-1}P(x, dy)\bar{\eta}_\lambda(y)\mathbf{1}_{(0,\infty)}(y) & \text{for } 0\leqq x<\lambda,\\ \delta_x(dy) & \text{for } x\geqq\lambda,\end{cases}$$

where δ_x denotes the probability measure concentrated at x. Note that each point of $[\lambda, \infty)$ is a trap for q_λ-chain as well as for $\bar{q}_\lambda$-chain.

The main purpose of this subsection is to define q-chain and $\bar{q}$-chain (conditioned random walks) by showing the existence of the limit $q_\lambda(x, dy)$ and $\bar{q}_\lambda(x, dy)$ as $\lambda\to\infty$. For this purpose we need the assumption

(A.2) $$\hat{\tau}<\infty \quad \text{a.s.},$$

where $\hat{\tau}$ is defined by (2.10).

We denote by $R(x)$ (resp. $\bar{R}(x)$) the renewal function corresponding to $-S_\tau$ (resp. $-S_\sigma$), where τ (resp. σ) is defined by (2.1) (resp. (2.9)). We also put

$$\hat{\sigma}_\lambda=\min\{n\geqq 1:\ S_n\geqq\lambda\}.$$

Then we have the following theorem.

THEOREM 2.3. *Under the assumption* (A.2) *we have for each* $x\geqq 0$

(i) $$\eta_\lambda(x)\sim R(x)\boldsymbol{P}\{\hat{\sigma}_\lambda<\tau\}, \qquad \lambda\to\infty,$$

(ii) $$\bar{\eta}_\lambda(x)\sim \bar{R}(x-)\boldsymbol{P}\{\hat{\sigma}_\lambda<\sigma\}, \qquad \lambda\to\infty.$$

PROOF. We give the proof of (ii). Define a random variable $Y(\lambda)$ by

$$Y(\lambda)=\begin{cases}-S_\sigma & \text{if } \sigma<\hat{\sigma}_\lambda,\\ \infty & \text{if } \sigma>\hat{\sigma}_\lambda,\end{cases}$$

and then take a sequence of random variables Y_k, $k\geqq 1$, in such a way that

(a) the distribution of Y_1 equals that of $Y(\lambda-x)$,

(b) for $n\geqq 1$, the conditional distribution of Y_{n+1} under the condition that $\{Y_k, 1\leqq k\leqq n\}$ is given equals the distribution of $Y(\lambda-x+y)$ where $y=Y_1+\cdots+Y_n$.

Then

(2.20) $$\bar{\eta}_\lambda(x)=\boldsymbol{P}\{Y_1=\infty\}+\sum_{n=1}^{\infty}\boldsymbol{P}\{Y_1+\cdots+Y_n<x, Y_{n+1}=\infty\}.$$

Next, let Y_k', $k\geqq 1$, be independent copies of $Y(\lambda-x)$, let Y_k'', $k\geqq 1$, be independent copies of $Y(\lambda)$ and assume that $\{Y'_k, k\geqq 1\}$ and $\{Y''_k, k\geqq 1\}$ are also independent. Then, using the fact that $Y(\lambda)\geqq Y(\lambda')$ for $\lambda<\lambda'$, we can easily prove that

$$\boldsymbol{P}\{Y''_1=\infty\}\leqq \boldsymbol{P}\{Y_1=\infty\}=\boldsymbol{P}\{Y'_1=\infty\},$$
$$\boldsymbol{P}\{Y'_1+\cdots+Y'_n<x,\ Y''_{n+1}=\infty\}\leqq \boldsymbol{P}\{Y_1+\cdots+Y_n<x,\ Y_{n+1}=\infty\}$$
$$\leqq \boldsymbol{P}\{Y''_1+\cdots+Y''_n<x,\ Y'_{n+1}=\infty\}.$$

Therefore, (2.20) yields

$$\boldsymbol{P}\{Y''_1=\infty\}[1+\sum_{n=1}^{\infty}\boldsymbol{P}\{Y'_1+\cdots+Y'_n<x\}]$$
$$\leqq \bar{\eta}_\lambda(x)\leqq \boldsymbol{P}\{Y'_1=\infty\}[1+\sum_{n=1}^{\infty}\boldsymbol{P}\{Y''_1+\cdots+Y''_n<x\}],$$

that is,

$$\boldsymbol{P}\{\hat{\sigma}_\lambda<\sigma\}\bar{R}_{\lambda-x}(x-)\leqq \bar{\eta}_\lambda(x)\leqq \boldsymbol{P}\{\hat{\sigma}_{\lambda-x}<\sigma\}\bar{R}_\lambda(x-), \tag{2.21}$$

where $\bar{R}_\lambda(x)$ is the renewal function corresponding to $Y(\lambda)$. On the other hand,

$$\boldsymbol{P}\{\hat{\sigma}_\lambda<\sigma\}=\boldsymbol{E}\{\bar{\eta}_\lambda(S(\hat{\sigma}_{\lambda-x}));\ \hat{\sigma}_{\lambda-x}<\sigma\}\geqq \bar{\eta}_\lambda(\lambda-x)\boldsymbol{P}\{\hat{\sigma}_{\lambda-x}<\sigma\},$$

that is,

$$\bar{\eta}_\lambda(\lambda-x)\boldsymbol{P}\{\hat{\sigma}_{\lambda-x}<\sigma\}\leqq \boldsymbol{P}\{\hat{\sigma}_\lambda<\sigma\}\leqq \boldsymbol{P}\{\hat{\sigma}_{\lambda-x}<\sigma\}.$$

Since $\bar{\eta}_\lambda(\lambda-x)$ coincides with the probability that S_n hits $[x, \infty)$ before it hits $(-\infty, -(\lambda-x)]$, $\bar{\eta}_\lambda(\lambda-x)$ tends to $p=\boldsymbol{P}\{S_n$ hits $[x, \infty)\}$ as $\lambda\to\infty$, but $p=1$ by (A.2). Therefore, $\boldsymbol{P}\{\hat{\sigma}_{\lambda-x}<\sigma\}\sim\boldsymbol{P}\{\hat{\sigma}_\lambda<\sigma\}$ as $\lambda\to\infty$. Moreover, it is easy to see that $\bar{R}_\lambda(x-)$ tends to $\bar{R}(x-)$ as $\lambda\to\infty$. Thus (2.21) implies the assertion (ii) of the theorem. The assertion (i) can be proved similarly. □

REMARK 2.1. From the above proof we see that the convergence of $\eta_\lambda(x)/\boldsymbol{P}\{\hat{\sigma}_\lambda<\tau\}$ to $R(x)$ is uniform on each bounded x-interval. A similar statement also holds for $\bar{\eta}_\lambda(x)$.

We now introduce q-chain and $\bar{q}$-chain. Put for $x\geqq 0$

$$q(x, dy)=R(x)^{-1}P(x, dy)R(y)\mathbf{1}_{[0,\infty)}(y), \tag{2.22}$$

$$\bar{q}(x, dy)=\bar{R}(x-)^{-1}P(x, dy)\bar{R}(y-)\mathbf{1}_{(0,\infty)}(y). \tag{2.23}$$

As we shall see soon, these are Markov transition functions on $[0, \infty)$. Theorem 2.3 now means that

$$\begin{cases}\lim_{\lambda\to\infty} q_\lambda\text{-chain}=q\text{-chain},\\ \lim_{\lambda\to\infty}\bar{q}_\lambda\text{-chain}=\bar{q}\text{-chain}.\end{cases}$$

In this sense, q-chain and $\bar{q}$-chain may be regarded as "conditioned random walks". They are also denoted by $q(\mathfrak{S})$-chain and $\bar{q}(\mathfrak{S})$-chain to stress the basic random walk $\mathfrak{S}$.

In the preceding subsection we introduced p-chain and $\bar{p}$-chain. When we want to stress the underlying random walk $\mathfrak{S}$, they are denoted by $p(\mathfrak{S})$-chain and $\bar{p}(\mathfrak{S})$-chain, respectively. Thus we can also consider $p(\hat{\mathfrak{S}})$-chain and $\bar{p}(\hat{\mathfrak{S}})$-chain. Then by (2.11) and Corollary 2.1 the transition functions of these chains are given by (2.23) and (2.22), respectively. Therefore we obtain the following theorem.

THEOREM 2.4. *Under the assumption* (A.2)

$$\begin{cases} \bar{p}(\hat{\mathfrak{S}})\text{-chain} = q(\mathfrak{S})\text{-chain}, \\ p(\hat{\mathfrak{S}})\text{-chain} = \bar{q}(\mathfrak{S})\text{-chain}. \end{cases}$$

2.3. Convergence of $\sum \exp(-U_n)$. In this subsection we introduce the following condition:

(A.3) $$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \boldsymbol{P}\{S_k \geqq 0\} = \kappa_0 \qquad \text{for some constant } \kappa_0 \in (0, 1).$$

We also introduce the condition:

(A.4) The distribution of X_1 is either a symmetric distribution not concentrated at $\{0\}$ or (if it is not symmetric) belongs to the domain of attraction with centering constant 0 of a strictly stable distribution whose density is positive on the whole of $\boldsymbol{R}$.

It is easy to see that (A.4) implies (A.3).

LEMMA 2.1. *Assume* (A.3) *and let* κ *be a constant such that* $\kappa_0 < \kappa < 1$. *Then*

$$\boldsymbol{P}\{\tau > n\} \leqq \text{const.}\, n^{\kappa-1}, \qquad n \geqq 1,$$

where const. *does not depend on* n.

PROOF. We use the Baxter-Spitzer formula (e.g. see [**14**])

(2.24) $$\sum_{n=0}^{\infty} \boldsymbol{P}\{\tau > n\} s^n = \exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \boldsymbol{P}\{S_n \geqq 0\}\right\}, \qquad |s| < 1.$$

Applying Lemma 1 of Rogozin [**11**] it is easily seen that (A.3) implies

(2.25) $$\exp\left\{\sum_{n=1}^{\infty} \frac{s^n}{n} \boldsymbol{P}\{S_n \geqq 0\}\right\} \sim (1-s)^{-\kappa_0} L\left(\frac{1}{1-s}\right) \qquad \text{as } s \uparrow 1,$$

where $L(x)$ is a slowly varying function at infinity. Thus applying the Tauberian Theorem (e.g. see [**3**: p. 447]), (2.24) implies

$$P\{\tau>n\} \sim \frac{1}{\Gamma(\kappa_0)} n^{\kappa_0-1} L(n) \qquad \text{as } n\to\infty .$$

This implies the assertion of the lemma. □

REMARK 2.2. (A.3) implies (A.1) and (A.2). In fact, if we suppose $P\{\tau=\infty\}>0$, then $\lim_{n\to\infty} S_n=\infty$ a.s. and consequently

$$\frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{(-\infty,0)}(S_k) \longrightarrow 0 \qquad \text{a.s.}$$

contradicting (A.3).

PROPOSITION 2.2. *Under the assumption* (A.3) *we have*

$$\sum_{n=1}^{\infty} \exp(-U_n) < \infty \quad a.s. \quad and \quad \sum_{n=1}^{\infty} \exp(-\bar{U}_n) < \infty \quad a.s..$$

PROOF. To prove the convergence of the first series put $H_k=w_k(l_k)$ where $w_k(l_k)$ is in (2.3). Then we have

$$\text{(2.26)} \qquad \sum_{n=1}^{\infty} \exp(-U_n) = \sum_{k=1}^{\infty} \exp[-(H_1+\cdots+H_{k-1})]\{e^{-w_k(1)}+\cdots+e^{-w_k(l_k)}\}$$

$$\leqq \sum_{k=1}^{\infty} l_k \exp[-(H_1+\cdots+H_{k-1})]$$

with the convention that $H_1+\cdots+H_{k-1}=0$ for $k=1$. On the other hand,

$$P\{l_k \exp[-(H_1+\cdots+H_{k-1})]>k^{-2}\}$$

$$= P\{l_k>k^{-2}\exp[H_1+\cdots+H_{k-1}]\}$$

$$\leqq \text{const.}\, E\{k^{-2(\kappa-1)}\exp[(\kappa-1)(H_1+\cdots+H_{k-1})]\} \quad (\text{since } l_k \overset{d}{=} \tau)$$

$$\leqq \text{const.}\, k^2\theta^{k-1},$$

where $\kappa_0<\kappa<1$, $\theta=E\{\exp[(\kappa-1)H_1]\}<1$ and $\overset{d}{=}$ means the equality in distribution. Note that we used Lemma 2.1 to derive the above inequality. Therefore, by the Borel-Cantelli lemma we have

$$P\{l_k \exp[-(H_1+\cdots+H_{k-1})]\leqq k^{-2} \quad \text{for all sufficiently large } k\}=1,$$

and consequently $\sum_{n=1}^{\infty}\exp(-U_n)<\infty$ a.s.. To prove the convergence of the second series it is enough to note that $P\{\sigma>n\}\leqq P\{\tau>n\}$ and to do similarly. □

By Theorem 2.4 combined with Proposition 2.2 we obtain

COROLLARY 2.2. *Let* $\{V_n,\ n\geqq 0\}$ *and* $\{\bar{V}_n,\ n\geqq 0\}$ *be a q-chain and* $\bar{q}$*-chain, respectively. Then under the assumption* (A.3) *we have*

$$\sum_{n=0}^{\infty} \exp(-V_n) < \infty \quad a.s. \qquad and \qquad \sum_{n=0}^{\infty} \exp(-\bar{V}_n) < \infty \quad a.s..$$

A similar result for a q_λ-chain will be need in the section 5. Now we use the following lemma.

LEMMA 2.2. *Under the assumption* (A.4), *we have*

$$\limsup_{\lambda\to\infty} \eta_\lambda(0)^{-1}\eta_{\lambda/2}(0) < \infty . \tag{2.27}$$

PROOF. First we note that $\liminf_{\lambda\to\infty} \eta_\lambda(\lambda/2)>0$. In fact, under the condition (A.4) we have

$$\begin{cases} \eta_\lambda(\lambda/2) \geqq 1/2 & \text{if } X_1 \text{ is symmetrically distributed,} \\ \lim_{\lambda\to\infty} \eta_\lambda(\lambda/2) = \text{const.}>0 & \text{otherwise,} \end{cases}$$

because the process $\{\lambda^{-1}S_{[\varphi(\lambda)t]}, t\geqq 0\}$ converges in law to a stable process as $\lambda\to\infty$ where $\varphi(\lambda)$ is a suitable regularly varying function at infinity with index α, $0<\alpha\leqq 2$. Next using the strong Markov property we have

$$\begin{aligned} \eta_\lambda(0) &= \boldsymbol{P}\{\hat{\sigma}_\lambda<\tau\} = \boldsymbol{E}\{\eta_\lambda(S(\hat{\sigma}_{\lambda/2}));\ \hat{\sigma}_{\lambda/2}<\tau\} \\ &\geqq \eta_\lambda(\lambda/2)\boldsymbol{P}\{\hat{\sigma}_{\lambda/2}<\tau\} = \eta_\lambda(\lambda/2)\eta_{\lambda/2}(0), \end{aligned}$$

and consequently the left hand side of (2.27) is bounded by $\{\liminf_{\lambda\to\infty} \eta_\lambda(\lambda/2)\}^{-1} <\infty$. □

Let $\{V_\lambda(n), n\geqq 0\}$ be a q_λ-chain with $V_\lambda(0)=0$. Note that any point in $[\lambda, \infty)$ is a trap for this chain. We put $\sigma_\lambda(0)=0$ and for $k=1, 2, \cdots$

$$\begin{aligned} \sigma_\lambda(k) &= \min\{n>\sigma_\lambda(k-1): V_\lambda(n)\leqq \inf_{m>n} V_\lambda(m)\}, \\ H_\lambda(k) &= \max\{V_\lambda(n): \sigma_\lambda(k-1)<n\leqq\sigma_\lambda(k)\}, \\ N_\lambda &= \begin{cases} \max\{k\geqq 1: H_\lambda(1), \cdots, H_\lambda(k)<\lambda/2\} & \text{if } H_\lambda(1)<\lambda/2, \\ 0 & \text{if } H_\lambda(1)\geqq\lambda/2. \end{cases} \end{aligned}$$

In the following proposition $\{V_n, n\geqq 0\}$ is a q-chain with $V_0=0$.

PROPOSITION 2.3. *If* (A.4) *is satisfied, then*

$$\sum_{n=1}^{\sigma_\lambda(N_\lambda)} \exp\{-V_\lambda(n)\} \longrightarrow \sum_{n=1}^{\infty} \exp\{-V_n\} \qquad \textit{in law as } \lambda\to\infty .$$

PROOF. For $k=1, 2, \cdots$ we put

$$\begin{aligned} Z_\lambda(k) &= \prod_{j=1}^{k} \mathbf{1}_{[0,\lambda/2)}(H_\lambda(j)) \sum_{\sigma_\lambda(k-1)<n\leqq\sigma_\lambda(k)} \exp[-V_\lambda(n)], \\ Z(k) &= \sum_{\sigma(k-1)<n\leqq\sigma(k)} \exp[-V_n], \end{aligned}$$

where $\sigma(0)=0$ and $\sigma(k)=\min\{n>\sigma(k-1): V_n\leqq \inf_{m>n} V_m\}$, $k\geqq 1$. Then

$$\sum_{n=1}^{\sigma_\lambda(N_\lambda)} \exp[-V_\lambda(n)] = \sum_{k=1}^{\infty} Z_\lambda(k), \qquad \sum_{n=1}^{\infty} \exp[-V_n] = \sum_{k=1}^{\infty} Z(k).$$

Therefore, to prove the proposition it is enough to show that

(2.28) for each k the joint distribution of $Z_\lambda(1), \cdots, Z_\lambda(k)$ converges to that of $Z(1), \cdots, Z(k)$ as $\lambda\to\infty$;

(2.29) there exist $\lambda_0>0$ and $\theta\in(0, 1)$ such that for any $\lambda>\lambda_0$
 (i) $\boldsymbol{P}\{Z_\lambda(k)>k^{-2}\} \leqq \text{const.}\, k^2\theta^{k-1}$,
 (ii) $\boldsymbol{P}\{Z(k)>k^{-2}\} \leqq \text{const.}\, k^2\theta^{k-1}$.

To show (2.29) (ii). we note that $\{V_n, n\geqq 0\}$ is equivalent in law to the $\bar{p}$-chain $\{\bar{U}_n, n\geqq 0\}$ (Theorem 2.4). Then, recalling the proof of Proposition 2.2, we see that $Z(k)$ is identical in law to

(2.30) $$\exp[-(\bar{H}_1+\cdots+\bar{H}_{k-1})]\{\exp[-\bar{w}_k(1)]+\cdots+\exp[-\bar{w}_k(\bar{l}_k)]\}$$

which is a term similar to those appearing in (2.26) (note that $\bar{H}_i=\bar{w}_i(\bar{l}_i)$). Therefore as in the proof of Proposition 2.2 we see that (2.29) (ii) holds. To proceed to the proof of (2.28) and (i) of (2.29), first we note that

(2.31) $$\eta_\lambda(0)^{-1}\eta_{\lambda/2}(0) \leqq c \qquad \text{for all sufficiently large } \lambda$$

with some positive constant c (Lemma 2.2). Now for any Borel subset A of $(0, \infty)^k$ consider the event $\Gamma=\{(Z_\lambda(1), \cdots, Z_\lambda(k))\in A\}$. Since $\sigma_\lambda(k)\geqq k$, $\Gamma=\bigcup_{n=k}^{\infty}[\Gamma\cap\{\sigma_\lambda(k)=n\}]$. Moreover, from the fact that $Z_\lambda(k)>0$ on Γ we can see that $H_\lambda(k)<\lambda/2$ on Γ and hence $V_\lambda(j)<\lambda/2$ (for all $1\leqq j\leqq n$) on $\Gamma\cap\{\sigma_\lambda(k)=n\}$. Therefore, each event $\Gamma\cap\{\sigma_\lambda(k)=n\}$ can be expressed as

$$\{(V_\lambda(1), \cdots, V_\lambda(n))\in A_n\cap[0, \lambda/2)^n,\ V_\lambda(n)\leqq \inf_{m>n} V_\lambda(m)\}$$

with a *suitable* Borel subset A_n of $[0, \infty)^n$. Note that A_n can be chosen to be *independent* of λ. By the definition of q_λ-chain we have

(2.32) $$\boldsymbol{P}\{(V_\lambda(1), \cdots, V_\lambda(n))\in A_n\cap[0, \lambda/2)^n,\ V_\lambda(n)\leqq \inf_{m>n} V_\lambda(m)\}$$

$$= \int_{A_n\cap[0,\lambda/2)^n} \eta_\lambda(0)^{-1}P(0, dx_1)P(x_1, dx_2)\cdots P(x_{n-1}, dx_n)\eta_\lambda(x_n)$$

$$\times\, \eta_\lambda(x_n)^{-1}\Big\{\int_{[\lambda,\infty)} P(x_n, dy_1)+\int_{(x_n,\lambda)} P(x_n, dy_1)\int_{[\lambda,\infty)} P(y_1, dy_2)$$

$$+\int_{(x_n,\lambda)} P(x_n, dy_1)\int_{(x_n,\lambda)} P(y_1, dy_2)\int_{[\lambda,\infty)} P(y_2, dy_3)+\cdots\Big\}$$

$$= \int_{A_n\cap[0,\lambda/2)^n} \eta_\lambda(0)^{-1}P(0, dx_1)P(x_1, dx_2)\cdots P(x_{n-1}, dx_n)\eta_{\lambda-x_n}(0).$$

Denote by $I_{\lambda,n}$ the last integral of the above and let

$$I_n = \int_{A_n} P(0, dx_1)P(x_1, dx_2)\cdots P(x_{n-1}, dx_n).$$

Then by (2.31), $I_{\lambda,n}$ is dominated by $c\cdot I_n$ for all sufficiently large λ, say for $\lambda>\lambda_0$ (independent of n), and $I_{\lambda,n}$ converges to I_n as $\lambda\to\infty$, because $\eta_\lambda(0)^{-1}\eta_{\lambda-x_n}(0)\to 1$ for each fixed $x_n\geqq 0$, similarly to Lemma 2.2 or end of proof of Theorem 2.3. Therefore

$$\lim_{\lambda\to\infty} \boldsymbol{P}\{(Z_\lambda(1), \cdots, Z_\lambda(k))\in A\} = \lim_{\lambda\to\infty} \sum_{n=k}^{\infty} I_{\lambda,n} = \sum_{n=k}^{\infty} I_n .$$

But as in (2.32) we can see that $\sum_{n=k}^{\infty} I_n$ equals $\boldsymbol{P}\{(Z(1), \cdots, Z(k))\in A\}$ and hence (2.28) holds. Finally, taking $A=(0, \infty)\times\cdots\times(0, \infty)\times(k^{-2}, \infty)$ we obtain

$$\boldsymbol{P}\{Z_\lambda(k)>k^{-2}\} \leqq c\cdot \boldsymbol{P}\{Z(k)>k^{-2}\} \leqq \text{const.}\, k^2\theta^{k-1},\ \lambda>\lambda_0 ,$$

so that, (i) of (2.29) holds. □

§ 3. Some properties of the random environment $(\mathcal{W}, \boldsymbol{Q})$.

Let $\boldsymbol{Q}$ be a probability measure on $\mathcal{W}$ satisfying the assumption (A) in § 1. We introduce another probability space $(\boldsymbol{W}, \mathbf{Q})$ as follows. $\boldsymbol{W}$ is the space of functions $W: \boldsymbol{R}\to\boldsymbol{R}$ with the following three properties.

(i) $W(0)=0$.

(ii) W is right continuous and has left limits on $[0, \infty)$.

(iii) W is left continuous and has right limits on $(-\infty, 0]$.

$\mathbf{Q}$ is the probability measure on $\boldsymbol{W}$ determined by the following (i) and (ii):

(i) $\{W(t), t\geqq 0, \mathbf{Q}\}$ is the strictly stable process such that the distribution of $W(1)$ is μ_+ and $\{W(-t), t\geqq 0, \mathbf{Q}\}$ is the strictly stable process such that the distribution of $W(-1)$ is μ_- (for the meaning of $\mu_\pm$ see (ii) of Assumption (A)).

(ii) $\{W(t), t\geqq 0, \mathbf{Q}\}$ and $\{W(-t), t\geqq 0, \mathbf{Q}\}$ are independent.

The assumption (A) implies the existence of a regularly varying function $\varphi(\lambda)$ at infinity with exponent α such that the distributions of $\lambda^{-1}W(\varphi(\lambda))$ and $\lambda^{-1}W(-\varphi(\lambda))$ under $\boldsymbol{Q}$ converge to μ_+ and μ_- as $\lambda\to\infty$, respectively (e.g. see Feller [**3**: XVII. 5]). Therefore by a theorem of Skorohod [**13**] the process $\{\lambda^{-1}W(\varphi(\lambda)x), x\in\boldsymbol{R}, \boldsymbol{Q}\}$ converges in law to $\{W(x), x\in\boldsymbol{R}, \mathbf{Q}\}$ as $\lambda\to\infty$. Fixing such a regularly varying function $\varphi(\lambda)$, $\lambda>0$, we define a scaling map $\Phi(\lambda)$, $\lambda>0$, from $\boldsymbol{W}$ into itself by

$$(3.1) \qquad (\Phi_\lambda W)(x) = \lambda^{-1}W(\varphi(\lambda)x), \qquad x\in\boldsymbol{R}.$$

For $W\in\boldsymbol{W}$ and $x\in\boldsymbol{R}$ we say that W is *oscillating* at x if the following

(3.2a) and (3.2b) hold.

(3.2a) $\sup_{(x, x+\varepsilon)} W > W(x+)$ and $\inf_{(x, x+\varepsilon)} W < W(x+)$, for any $\varepsilon>0$.

(3.2b) $\sup_{(x-\varepsilon, x)} W > W(x-)$ and $\inf_{(x-\varepsilon, x)} W < W(x-)$, for any $\varepsilon>0$.

Here $\sup_I W$ and $\inf_I W$ stand for $\sup_{y\in I} W(y)$ and $\inf_{y\in I} W(y)$, respectively, for a subset $I\subset \boldsymbol{R}$. W is said to take a local maximum (resp. local minimum) at x if $\sup_{(x-\varepsilon, x+\varepsilon)} W=W^*(x)$ (resp. $\inf_{(x-\varepsilon, x+\varepsilon)} W=W_*(x)$) for some $\varepsilon>0$ where $W^*(x)=W(x+)\vee W(x-)$ and $W_*(x)=W(x+)\wedge W(x-)$.

Let $\boldsymbol{W}^\#$ be the set of elements W ($\in \boldsymbol{W}$) satisfying the following conditions

i) $\sup\{W(x): x\geqq 0\}=\sup\{W(x): x\leqq 0\}=\infty$.

ii) If W is discontinuous at $x\in\boldsymbol{R}$, then W is oscillating at x.

iii) For any open set $G\subset\boldsymbol{R}$, $\#\{x\in G: W(x)=\sup_G W\}\leqq 1$ and $\#\{x\in G: W(x)=\inf_G W\}\leqq 1$.

iv) W does not take a local minimum at 0.

REMARK 3.1. By ii) we see that $W\in\boldsymbol{W}^\#$ can take local maxima or local minima only at continuity points of W.

LEMMA 3.1. $\mathbf{Q}\{\boldsymbol{W}^\#\}=1$.

PROOF. Since (3.2a) holds $\mathbf{Q}$-a.s. when x is replaced by an arbitrary (non-negative) stopping time with respect to the process $\{W(t), t\geqq 0, \mathbf{Q}\}$ and since points of discontinuity of $\{W(t), t\geqq 0\}$ can be sorted out by a sequence of such stopping times, we see that (3.2a) holds for any discontinuity points $x\geqq 0$ of W $\mathbf{Q}$-a.s. Considering the process $\{W(t-T-)-W(-T-), 0\leqq t\leqq T, \mathbf{Q}\}$ instead of $\{W(t), t\geqq 0, \mathbf{Q}\}$ where $T>0$ is arbitrary but fixed, we see that (3.2a) holds for any discontinuity point $x\leqq 0$ of W $\mathbf{Q}$-a.s.. Since (3.2b) can be discussed similarly, we see that $\mathbf{Q}$-almost all W have the property ii) in the definition of $\boldsymbol{W}^\#$. We next prove that $\mathbf{Q}$-almost all W have the property iii). In order to prove the first part of iii) holds $\mathbf{Q}$-a.s., it is enough to prove that $\mathbf{Q}\{\Lambda_{abcd}\}=0$ for any rational numbers $a<b<c<d$, where $\Lambda_{abcd}=\{W\in\boldsymbol{W}: \sup_{(a,b)} W=\sup_{(c,d)} W\}$. Set $m=(b+c)/2$, $X_1=\sup_{a<x<b}\{W(x)-W(b)\}$, $Y_1=W(b)-W(m)$, $X_2=\sup_{c<x<d}\{W(x)-W(c)\}$ and $Y_2=W(c)-W(m)$. Then X_1, Y_1, X_2 and Y_2 are independent. Since the distributions of Y_1 and Y_2 have densities, so do the distributions of X_1+Y_1 and X_2+Y_2. Therefore

$$\mathbf{Q}\{\sup_{(a,b)} W=\sup_{(c,d)} W\} = \mathbf{Q}\{X_1+Y_1+W(m) = X_2+Y_2+W(m)\}$$
$$= \mathbf{Q}\{X_1+Y_1 = X_2+Y_2\} = 0.$$

Similarly the second part of iii) holds $\mathbf{Q}$-a.s.. □

For $W\in \boldsymbol{W}^{\#}$, we give the definition of a valley of W. A part $\{W(x),\ a\leqq x\leqq c\}$ of W is called a *valley* of W if

(i) $a<c$,

(ii) there exists $b\in(a, c)$ such that

$$\begin{cases} W(a)>W(x)>W(b) & \text{for every } x\in(a, b), \\ W(c)>W(x)>W(b) & \text{for every } x\in(b, c), \end{cases}$$

(iii) W is continuous at a, b and c.

(iv) $H_{-}\equiv\sup\{W(y)-W(x):\ a\leqq x\leqq y\leqq b\}<W(c)-W(b)$,

$H_{+}\equiv\sup\{W(x)-W(y):\ b\leqq x\leqq y\leqq c\}<W(a)-W(b)$.

For simplicity, we write (a, b, c) instead of $\{W(x),\ a\leqq x\leqq c\}$. $A=H_{+}\vee H_{-}$ is called the *inner directed ascent* and $D=(W(a)-W(b))\wedge(W(c)-W(b))$ is the *depth* of the valley.

In our discussions a valley (a, b, c) with $a<0<c$ plays a particularly important role. Kesten [**9**] gave another description of such a valley. Following [**9**] we define $(a_{\lambda}, b_{\lambda}, c_{\lambda})$ for $W\in\boldsymbol{W}$ and $\lambda>0$. Set

$$c_{\lambda}^{+}=\inf\{x>0: W^{*}(x)-\inf_{[0,x)}W_{*}\geqq\lambda\},\quad c_{\lambda}^{-}=\sup\{x<0: W^{*}(x)-\inf_{(x,0]}W_{*}\geqq\lambda\},$$

$$V_{\lambda}^{+}=\inf_{[0,c_{\lambda}^{+}]}W_{*}+\lambda,\qquad V_{\lambda}^{-}=\inf_{[c_{\lambda}^{-},0]}W_{*}+\lambda,$$

$$b_{\lambda}^{+}=\inf\{x\geqq 0: W_{*}(x)=V_{\lambda}^{+}-\lambda\},\quad b_{\lambda}^{-}=\sup\{x\leqq 0: W_{*}(x)=V_{\lambda}^{-}-\lambda\},$$

$$M_{\lambda}^{+}=\sup_{[0,b_{\lambda}^{+}]}W^{*},\qquad M_{\lambda}^{-}=\sup_{[b_{\lambda}^{-},0]}W^{*},$$

$$a_{\lambda}^{+}=\inf\{x\geqq 0: W^{*}(x)=M_{\lambda}^{+}\},\qquad a_{\lambda}^{-}=\sup\{x\leqq 0: W^{*}(x)=M_{\lambda}^{-}\},$$

$$e_{\lambda}^{+}=\inf\{x\geqq c_{\lambda}^{+}: W_{*}(x)\geqq W^{*}(c_{\lambda}^{+})+\lambda/2 \text{ or } W^{*}(x)\leqq W^{*}(c_{\lambda}^{+})-\lambda/2\},$$

$$e_{\lambda}^{-}=\sup\{x\leqq c_{\lambda}^{-}: W_{*}(x)\geqq W^{*}(c_{\lambda}^{-})+\lambda/2 \text{ or } W^{*}(x)\leqq W^{*}(c_{\lambda}^{-})-\lambda/2\},$$

$$d_{\lambda}^{+}=\inf\{x\geqq c_{\lambda}^{+}: W^{*}(x)=\sup_{(c_{\lambda}^{+},e_{\lambda}^{+})}W^{*}\},\quad d_{\lambda}^{-}=\sup\{x\leqq c_{\lambda}^{-}: W^{*}(x)=\sup_{(e_{\lambda}^{-},c_{\lambda}^{-})}W^{*}\}.$$

Let a_{λ}, b_{λ} and c_{λ} be measurable functions on $\boldsymbol{W}$ defined as follows:

$$a_{\lambda}=a_{\lambda}(W)=\begin{cases} a_{\lambda}^{-} & \text{if } M_{\lambda}^{+}\vee V_{\lambda}^{+}<M_{\lambda}^{-}, \\ d_{\lambda}^{-} & \text{if } M_{\lambda}^{+}\vee V_{\lambda}^{+}\geqq M_{\lambda}^{-} \text{ and } M_{\lambda}^{+}\vee V_{\lambda}^{+}<V_{\lambda}^{-}, \\ d_{\lambda}^{-} & \text{if } M_{\lambda}^{+}\vee V_{\lambda}^{+}>M_{\lambda}^{-}\vee V_{\lambda}^{-}, \\ -1 & \text{otherwise,} \end{cases}\tag{3.3a}$$

$$
\text{(3.3b)} \qquad b_\lambda = b_\lambda(W) = \begin{cases} b_\lambda^+ & \text{if } M_\lambda^+ \vee V_\lambda^+ < M_\lambda^- \vee V_\lambda^-, \\ b_\lambda^- & \text{if } M_\lambda^+ \vee V_\lambda^+ > M_\lambda^- \vee V_\lambda^-, \\ 0 & \text{otherwise,} \end{cases}
$$

$$
\text{(3.3c)} \qquad c_\lambda = c_\lambda(W) = \begin{cases} a_\lambda^+ & \text{if } M_\lambda^- \vee V_\lambda^- < M_\lambda^+, \\ d_\lambda^+ & \text{if } M_\lambda^- \vee V_\lambda^- \geqq M_\lambda^+ \text{ and } M_\lambda^- \vee V_\lambda^- < V_\lambda^+, \\ d_\lambda^+ & \text{if } M_\lambda^- \vee V_\lambda^- > M_\lambda^+ \vee V_\lambda^+, \\ 1 & \text{otherwise.} \end{cases}
$$

Note that $(a_\lambda, b_\lambda, c_\lambda)$ is not necessarily a valley; however, if $W \in \boldsymbol{W}^\#$, then $(a_\lambda, b_\lambda, c_\lambda)$ is a valley of W with $A < \lambda < D$ (see [**9**], [**15**] or [**7**]; see also Figure 1). The reason why we used d_λ^+ and d_λ^- for defining a_λ and c_λ is to make W continuous at a_λ and c_λ. The centering constant $b_\lambda(\cdot)$ appearing in our main theorem is the b_λ defined by (3.3b) in terms of b_λ^+ and b_λ^-. We note that b_λ^+ can also be characterized as the beginning of the first descending ladder excursion with height $\geqq \lambda$ of $\{W(x),\ x \geqq 0\}$. Here $\mathcal{E} = \{W(x) - W(x_1),\ x_1 \leqq x \leqq x_2\}$, $0 \leqq x_1 < x_2$, is called a *descending ladder excursion* of $\{W(x),\ x \geqq 0\}$ if

$$
\text{(3.4)} \qquad \begin{cases} W(x_1) < \inf_{[0,\, x_1 - \varepsilon]} W & \text{for any } \varepsilon \in (0,\ x_1) \text{ when } x_1 > 0, \\ W(x_1) \leqq W(x) & \text{for any } x \in (x_1,\ x_2), \\ W(x_1) > \inf_{[x_2,\, x_2 + \varepsilon]} W & \text{for any } \varepsilon > 0, \end{cases}
$$

and $H(\mathcal{E}) = \sup_{[x_1, x_2]} \{W(x) - W(x_1)\}$ is called its *height*. In particular, if $W \in \boldsymbol{W}$ and

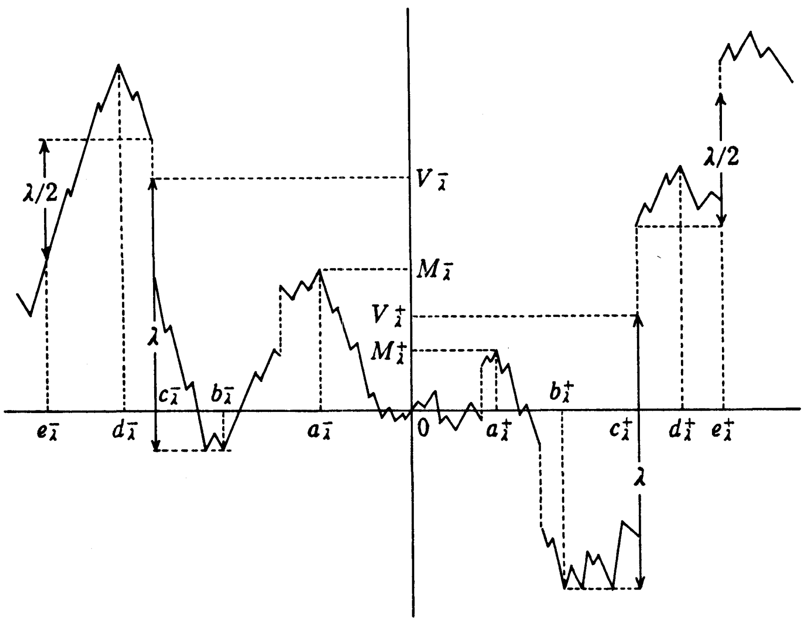

Figure 1. Here $b_\lambda = b_\lambda^+$, $a_\lambda = a_\lambda^-$ and $c_\lambda = d_\lambda^+$.

if x_1, x_2 are integers with $0 \leqq x_1 < x_2$, (3.4) yields

$$(3.4)' \quad \begin{cases} W(x_1) < W(n) & \text{for any } n \in [0, x_1) \cap \boldsymbol{Z}, \\ W(x_1) \leqq W(n) & \text{for any } n \in (x_1, x_2) \cap \boldsymbol{Z}, \\ W(x_1) > W(x_2). \end{cases}$$

In this case $\{W(n) - W(x_1),\ n \in [x_1, x_2] \cap \boldsymbol{Z}\}$ is also called a *descending ladder excursion.*

The following lemma can be proved easily.

LEMMA 3.2. *For each* $\lambda > 0$ *there exists* $\boldsymbol{W}_\lambda^\# \subset \boldsymbol{W}^\#$ *with* $\mathbf{Q}\{\boldsymbol{W}_\lambda^\#\} = 1$ *and with the following property: For any* $W \in \boldsymbol{W}_\lambda^\#$ *and for any sequence* $\{W_n, n \geqq 1\}$ *in* $\boldsymbol{W}$ *converging to* W *with respect to the Skorohod topology,* $a_\lambda(W_n)$, $b_\lambda(W_n)$ *and* $c_\lambda(W_n)$ *converge to* $a_\lambda(W)$, $b_\lambda(W)$ *and* $c_\lambda(W)$, *respectively.*

For $W^+ \in \mathcal{W}^+$, $W^- \in \mathcal{W}^-$ and $\lambda > 0$ (recall that $\mathcal{W}^+$ (resp. $\mathcal{W}^-$) is the space of nonnegative step functions satisfying 1° and 2° of § 1), we introduce the following notation:

$$(3.5) \quad \begin{cases} \rho_\lambda^+ = \rho_\lambda^+(W^+) = \min\{n \geqq 0 : W^+(n) \geqq \lambda\}, \\ \rho_\lambda^- = \rho_\lambda^-(W^-) = \max\{n \leqq 0 : W^-(n) \geqq \lambda\}, \end{cases}$$

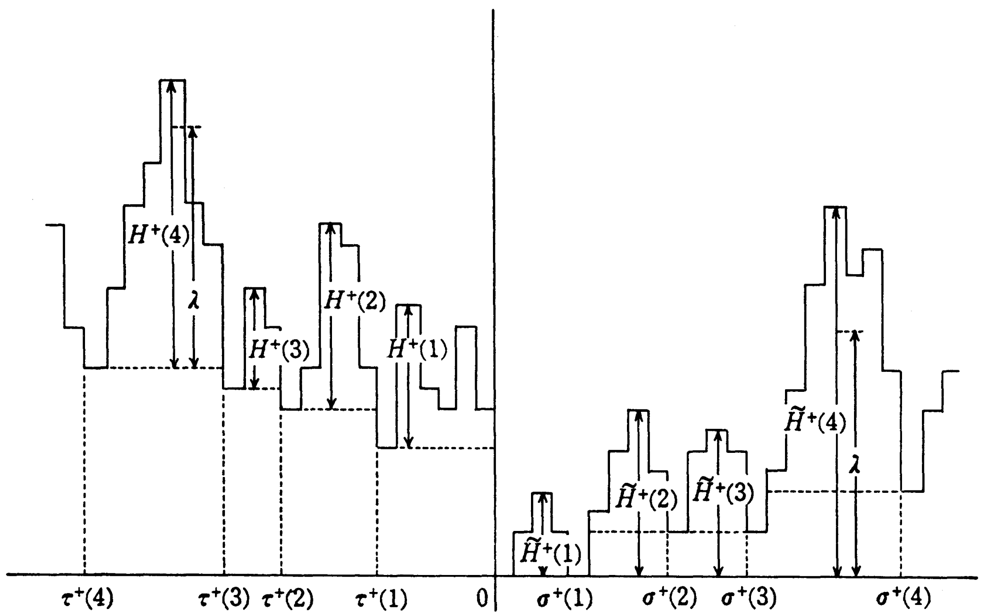

Figure 2. Here $\tau^+(N_\lambda^+) = \tau^+(3)$ and $\sigma^+(\tilde{N}_\lambda^+) = \sigma^+(3)$ ($\sigma^+(k)$, $\tilde{H}^+(k)$, $\tilde{N}_\lambda^+$ are defined in (4.10), (4.11) aad (4.12) by using ε instead of λ).

$$(3.6)\quad \begin{cases} \tau^+(0)=\tau^+(0, W^+)=0, \quad \tau^-(0)=\tau^-(0, W^-)=0, \\ \tau^+(k)=\tau^+(k, W^+)=\max\{n<\tau^+(k-1): W^+(n)<\inf\{W^+(j): j<n\}\}, \ k\geqq 1, \\ \tau^-(k)=\tau^-(k, W^-)=\min\{n>\tau^-(k-1): W^-(n)<\inf\{W^-(j): j>n\}\}, \ k\geqq 1, \end{cases}$$

$$(3.7)\quad \begin{cases} H^+(k)=H^+(k, W^+)=\max\{W^+(n)-W^+(\tau^+(k)): \tau^+(k)\leqq n\leqq \tau^+(k-1)\}, \ k\geqq 1, \\ H^-(k)=H^-(k, W^-)=\max\{W^-(n)-W^-(\tau^-(k)): \tau^-(k-1)\leqq n\leqq \tau^-(k)\}, \ k\geqq 1, \end{cases}$$

$$(3.8)\quad \begin{cases} N_\lambda^+=N_\lambda^+(W^+)=\max\{k\geqq 1: \max_{1\leqq j\leqq k} H^+(j)<\lambda\}, \\ N_\lambda^-=N_\lambda^-(W^-)=\max\{k\geqq 1: \max_{1\leqq j\leqq k} H^-(j)<\lambda\}. \end{cases}$$

In (3.5) and (3.6) the convention $\min\phi=-\max\phi=\infty$ is used while in (3.7) and (3.8) $\max\phi=0$.

Denote the random walks $\{W(n), n\geqq 0, \boldsymbol{Q}\}$ and $\{W(-n), n\geqq 0, \boldsymbol{Q}\}$ by $\mathfrak{W}^+$ and $\mathfrak{W}^-$, respectively. Let $\boldsymbol{Q}_\lambda^+$ and $\boldsymbol{Q}_\lambda^-$ be the probability measure on $\mathcal{W}^+$ and $\mathcal{W}^-$, respectively, determined by the following conditions.

(i) $\{W^+(n), n\geqq 0, \boldsymbol{Q}_\lambda^+\}$ is a $q_\lambda(\mathfrak{W}^+)$-chain and $\{W^+(-n), n\geqq 0, \boldsymbol{Q}_\lambda^+\}$ is a $p(\mathfrak{W}^+)$-chain, respectively, and $\{W^+(n), n\geqq 0, \boldsymbol{Q}_\lambda^+\}$ and $\{W^+(-n), n\geqq 0, \boldsymbol{Q}_\lambda^+\}$ are independent.

(ii) $\{W^-(n), n\geqq 0, \boldsymbol{Q}_\lambda^-\}$ is a $p(\mathfrak{W}^-)$-chain and $\{W^-(-n), n\geqq 0, \boldsymbol{Q}_\lambda^-\}$ is a $q_\lambda(\mathfrak{W}^-)$-chain, respectively, and $\{W^-(n), n\geqq 0, \boldsymbol{Q}_\lambda^-\}$ and $\{W^-(-n), n\geqq 0, \boldsymbol{Q}_\lambda^-\}$ are independent.

LEMMA 3.3. (i) *Under* $\boldsymbol{Q}$ $\{W(b_\lambda^++n)-W(b_\lambda^+), -b_\lambda^+\leqq n\leqq c_\lambda^+-b_\lambda^+\}$ *and* $\{W(b_\lambda^--n)-W(b_\lambda^-), b_\lambda^-\leqq n\leqq b_\lambda^--c_\lambda^-\}$ are independent.

(ii) $\{W(b_\lambda^++n)-W(b_\lambda^+), -b_\lambda^+\leqq n\leqq c_\lambda^+-b_\lambda^+, \boldsymbol{Q}\}$

$$\overset{d}{=} \{W^+(n), \tau^+(N_\lambda^+)\leqq n\leqq \rho_\lambda^+, \boldsymbol{Q}_\lambda^+\}.$$

(iii) $\{W(b_\lambda^--n)-W(b_\lambda^-), b_\lambda^-\leqq n\leqq b_\lambda^--c_\lambda^-, \boldsymbol{Q}\}$

$$\overset{d}{=} \{W^-(-n), -\tau^-(N_\lambda^-)\leqq n\leqq -\rho_\lambda^-, \boldsymbol{Q}_\lambda^-\}.$$

PROOF. Since (i) is obvious, we prove (ii). For $W\in\mathcal{W}$ and $\lambda>0$ set

$$\tau(0)=0,\ \tau(k)=\min\{n>\tau(k-1): W(n)<W(\tau(k-1))\},\ k\geqq 1,$$

$$H(0)=0,\ H(k)=\max\{W(n)-W(\tau(k-1)): \tau(k-1)\leqq n\leqq \tau(k)\},\ k\geqq 1,$$

$$N_\lambda=\max\{k\geqq 0: \max_{1\leqq j\leqq k} H(j)<\lambda\}.$$

Then the descending ladder excursions

$$(0, W(\tau(k-1)+1)-W(\tau(k-1)), \cdots, W(\tau(k))-W(\tau(k-1))),\quad k\geqq 1,$$

are independent; $N_\lambda+1$ is the index of the first excursion with height $H(\cdot)\geqq\lambda$;

$\tau(N_\lambda)=b_\lambda^\dagger$. Therefore, $\{W(n), 0\leqq n\leqq b_\lambda^\dagger\}$ and $\{W(b_\lambda^\dagger+n)-W(b_\lambda^\dagger), n\geqq 0\}$ are independent. Consider the reversed excursions

$$w_k = (0, W(\tau(k)-1)-W(\tau(k)), \cdots, \\ W(\tau(k-1)+1)-W(\tau(k)), W(\tau(k-1))-W(\tau(k))).$$

Then w_k, $k\geqq 1$, are i.i.d. and by Theorem 2.1 the process defined by piecing together these w_k as in (2.3) is a $p(\mathfrak{W}^+)$-chain. Since $(w_{N_\lambda}, w_{N_{\lambda-1}}, \cdots, w_1) \stackrel{d}{=} (w_1, w_2, \cdots, w_{N_\lambda})$, the process $\{W(b_\lambda^\dagger+n)-W(b_\lambda^\dagger), -b_\lambda^\dagger\leqq n\leqq 0, \boldsymbol{Q}\}$ is equivalent to $\{W^+(n), n\leqq 0, \boldsymbol{Q}_\lambda^\dagger\}$ considered up to a certain random time but this random time is nothing but $\tau^+(N_\lambda^\dagger)$. Next we show that

$$\{W(b_\lambda^\dagger+n)-W(b_\lambda^\dagger), 0\leqq n\leqq c_\lambda^\dagger-b_\lambda^\dagger, \boldsymbol{Q}\} \stackrel{d}{=} \{W^+(n), 0\leqq n\leqq \rho_\lambda^\dagger, \boldsymbol{Q}_\lambda^\dagger\}.$$

For any Borel sets $A_k\subset[0, \lambda)$, $0\leqq k\leqq n-1$, and $A_n\subset[\lambda, \infty)$ we have

$$\begin{aligned} &\boldsymbol{Q}\{W(b_\lambda^\dagger+k)-W(b_\lambda^\dagger)\in A_k, 0\leqq k\leqq n, c_\lambda^\dagger-b_\lambda^\dagger=n\} \\ &= \boldsymbol{Q}\{W(k)\in A_k, 0\leqq k\leqq n\}/\boldsymbol{Q}\{H(1)\geqq\lambda\} \\ &= \delta_0(A_0)\int_{A_1} q_\lambda(0, dx_1)\int_{A_2} q_\lambda(x_1, dx_2)\cdots\int_{A_n} q_\lambda(x_{n-1}, dx_n) \\ &= \boldsymbol{Q}_\lambda^\dagger\{W^+(k)\in A_k, 0\leqq k\leqq n, \rho_\lambda^\dagger=n\}, \end{aligned}$$

where $q_\lambda(x, dy)$ is the transition function of $q_\lambda(\mathfrak{W}^+)$-chain. This proves (ii). The statement (iii) can be proved similarly. □

In what follows, for $W\in\boldsymbol{W}$ and $\lambda>0$, we put

$$a^\lambda = a_1(\Phi_\lambda W), \quad b^\lambda = b_1(\Phi_\lambda W), \quad c^\lambda = c_1(\Phi_\lambda W). \tag{3.9}$$

We then have

$$a_\lambda(W) = \varphi(\lambda)a^\lambda, \quad b_\lambda(W) = \varphi(\lambda)b^\lambda, \quad c_\lambda(W) = \varphi(\lambda)c^\lambda. \tag{3.10}$$

§ 4. Diffusion processes in the random environment $(\mathcal{W}, \boldsymbol{Q})$.

Let $\Omega=\{\omega: [0, \infty)\to\boldsymbol{R};$ continuous and $\omega(0)=0\}$ and let $\boldsymbol{P}$ be the Wiener measure on Ω. Let $B(t)=\omega(t)$ be the value of ω $(\in\Omega)$ at $t\geqq 0$. Then $\{B(t), t\geqq 0, \boldsymbol{P}\}$ is a one-dimensional Brownian motion starting at 0. For $W\in\boldsymbol{W}$, we set

$$S(x) = \int_0^x \exp(W(y))dy, \qquad A(s) = \int_0^s \exp(-2W(S^{-1}(B(u)))du,$$

$$X(t, W) = S^{-1}(B(A^{-1}(t))),$$

where S^{-1} and A^{-1} are the inverse functions of S and A, respectively. Then $\{X(t, W), t\geqq 0, \boldsymbol{P}\}$ is a diffusion process with generator $\mathcal{L}_W=(1/2)e^{W(x)}(d/dx)\cdot(e^{-W(x)}(d/dx))$ starting at 0. Then the following scaling relation holds (see [8]): For fixed λ and W

$$\{X(t, \lambda\Phi_\lambda W), t\geqq 0, \boldsymbol{P}\} \stackrel{d}{=} \{\varphi(\lambda)^{-1}X(\varphi(\lambda)^2 t, W), t\geqq 0, \boldsymbol{P}\}. \tag{4.1}$$

Let $\boldsymbol{P}_W$ be the probability measure on Ω induced by the diffusion process $X(\cdot, W)$.

We set $\tilde{\Omega}=C([0, \infty)\to\boldsymbol{R})$ and write $\tilde{\omega}(t)$ for the value of $\tilde{\omega}$ $(\in\tilde{\Omega})$ at t. For $a<c$ and $W\in\boldsymbol{W}$ we denote by $\nu_{W[a,c]}$ the probability measure on $[a, c]$ defined by

$$\nu_{W[a,c]}(dx) = e^{-W(x)}dx\Big/\int_a^c e^{-W(y)}dy. \tag{4.2}$$

We then denote by $\boldsymbol{P}_{W[a,c]}$ the probability measure on $\tilde{\Omega}$ governing the diffusion process with state space $[a, c]$, with (local) generator $\mathcal{L}_W$, with reflecting barriers at a and c and with initial distribution $\nu_{W[a,c]}$. Note that this reflecting diffusion is a stationary diffusion process. It is also noted that $\boldsymbol{P}_{W[a,c]}$ is concentrated on the closed subset $\tilde{\Omega}_{a,c}=C([0, \infty)\to[a, c])$ of $\tilde{\Omega}$.

4.1. Outline of the proof of the main theorem. Here we give an outline of the proof (the following 1° and 2° are discussed in the next subsection while 3° and 4° are discussed in § 5).

1°. We consider the asymptotic behavior of the diffusion process $\{X(e^\lambda+t,W), t\geqq 0, \boldsymbol{P}_W\}$ in the environment W under $\boldsymbol{Q}$ as $\lambda\to\infty$. By the scaling relation (4.1), $X(e^\lambda+\cdot, W)$ is equal to $\varphi(\lambda)X(\varphi(\lambda)^{-2}(e^\lambda+\cdot), \lambda\Phi_\lambda W)$ in law and by Assumption (A), $(\Phi_\lambda W, \boldsymbol{Q})$ converges to $(W, \mathbf{Q})$ in law as $\lambda\to\infty$. Then using the estimate of the exit time of $[a_1(\Phi_\lambda W), c_1(\Phi_\lambda W)]$ for $X(\cdot, \lambda\Phi_\lambda W)$ (see Lemma 4.1) and the coupling property (4.6), we see that the asymptotic distribution of the diffusion $X(\varphi(\lambda)^{-2}(e^\lambda+\cdot), \lambda\Phi_\lambda W)$, as $\lambda\to\infty$, is equal to the asymptotic distribution of the stationary reflecting diffusion in the environment $\lambda\Phi_\lambda W$ on the state space $[a_1(\Phi_\lambda W), c_1(\Phi_\lambda W)]$.

2°. Next we divide the space $\mathcal{W}$ into two events, that is, $\{b_1(\Phi_\lambda W)=b_1^+(\Phi_\lambda W)\}$ and $\{b_1(\Phi_\lambda W)=b_1^-(\Phi_\lambda W)\}$, and we consider the asymptotic behavior of the stationary reflecting diffusion on each of the events. We consider the case of $\{b_1(\Phi_\lambda W)=b_1^+(\Phi_\lambda W)\}$. Since the invariant probability measure for the stationary reflecting diffusion in the environment $\lambda\Phi_\lambda W$ under $\boldsymbol{Q}$ converges in law to the δ-distribution at $b_1^+(W)$ under $\mathbf{Q}$, when studying the asymptotic property of this stationary reflecting diffusion as $\lambda\to\infty$ one may restrict its state space to a much smaller interval containing $b_1^+(\Phi_\lambda W)$ (see Lemma 4.4).

3°. By the scaling relation Lemma 5.2 and Lemma 5.3, we change the environment $\lambda\Phi_\lambda W$ to W again to consider the asymptotic distribution of the stationary reflecting diffusion with random centering $b_\lambda^\dagger(W)$ on the restricted state space. Then, by virtue of Lemma 3.3, the study of the distribution of this diffusion in the environment W under $\boldsymbol{Q}$ turns to that of the distribution of a stationary reflecting diffusion in the environment W under the probability $\boldsymbol{Q}_\lambda^\dagger \otimes \boldsymbol{Q}_\lambda^-$ on $\mathcal{W}^+ \times \mathcal{W}^-$. Since the restricted state space of the stationary reflecting diffusion can be small enough, by using the renewal property of the environment W under $\boldsymbol{Q}_\lambda^\dagger \otimes \boldsymbol{Q}_\lambda^-$ we see that the event $\{b_\lambda = b_\lambda^\dagger\}$ and the diffusion become asymptotically independent as $\lambda \to \infty$ ((5.13)). The probability of the event $\{b_\lambda = b_\lambda^\dagger\}$ evaluated under $\boldsymbol{Q}_\lambda^\dagger \otimes \boldsymbol{Q}_\lambda^-$ converges to $\mathfrak{p} = \mathbf{Q}\{b_1 = b_1^\dagger\}$ as $\lambda \to \infty$ ((5.16)).

4°. The final step is to study the limit distribution of the stationary diffusion in the environment W under $\boldsymbol{Q}_\lambda^\dagger$. This can be done by proving the formula (5.18). To derive (5.18) we use Proposition 2.3 and the following facts: the restricted state space expands to $\boldsymbol{R}$ as $\lambda \to \infty$, $\boldsymbol{Q}_\lambda^\dagger$ converges to $\tilde{\boldsymbol{Q}}^+$, as $\lambda \to \infty$ (Theorem 2.3 and Remark 2.1), and e^{-W} is in L^1 a.s. under $\tilde{\boldsymbol{Q}}^+$ (Corollary 2.2).

4.2. Some lemmas. We consider a coupling of diffusion processes. Let us define the measurable functions R, T and $\hat{T}$ of $(\omega, \tilde{\omega}, W) \in \Omega \times \tilde{\Omega} \times \boldsymbol{W}$ as follows:

$$R = R(\omega, \tilde{\omega}) = \inf\{t \geqq 0 : \omega(t) = \tilde{\omega}(t)\}, \tag{4.3}$$

$$T = T(\omega, \tilde{\omega}, W) = \inf\{t \geqq R : \omega(t) \notin (a_1(W), c_1(W))\}, \tag{4.4}$$

$$\hat{T} = \hat{T}(\omega, \tilde{\omega}, W) = \inf\{t \geqq R : \tilde{\omega}(t) \notin (a_1(W), c_1(W))\}. \tag{4.5}$$

For $\lambda > 0$, we write $T^\lambda = T(\omega, \tilde{\omega}, \Phi_\lambda W)$ and $\hat{T}^\lambda = \hat{T}(\omega, \tilde{\omega}, \Phi_\lambda W)$ for simplicity. We also write $\boldsymbol{P}_W^\lambda = \boldsymbol{P}_{\lambda\Phi_\lambda W}$, $\hat{\boldsymbol{P}}_W^\lambda = \boldsymbol{P}_{\lambda\Phi_\lambda W[a^\lambda, c^\lambda]}$ and $\mathbf{P}_W^\lambda = \boldsymbol{P}_W^\lambda \otimes \hat{\boldsymbol{P}}_W^\lambda$. Recall (3.9) for a^λ and c^λ. Then for each $t > 0$ we can regard $\mathbf{P}_W^\lambda\{R < t < T^\lambda\}$ as a random variable defined on the probability space $(\mathcal{W}, \boldsymbol{Q})$. Now if we set

$$\omega'(t) = \begin{cases} \omega(t) & \text{for } 0 \leqq t \leqq R, \\ \tilde{\omega}(t) & \text{for } t > R, \end{cases}$$

then for each fixed W

$$\{\omega(t), 0 \leqq t \leqq T^\lambda, \mathbf{P}_W^\lambda\} \overset{d}{=} \{\omega'(t), 0 \leqq t \leqq \hat{T}^\lambda, \mathbf{P}_W^\lambda\}. \tag{4.6}$$

LEMMA 4.1. *Let $\gamma(\lambda)$ be an arbitrary function satisfying $\gamma(\lambda) \to 1$ as $\lambda \to \infty$. Then*

$$\begin{aligned} &\mathbf{P}_W^\lambda\{R < \exp(\lambda\gamma(\lambda)) < T^\lambda(\cdot, \cdot, W)\} \\ &= \mathbf{P}_W^\lambda\{R < \exp(\lambda\gamma(\lambda)) < \hat{T}^\lambda(\cdot, \cdot, W)\} \longrightarrow 1 \end{aligned}$$

in probability with respect to $\boldsymbol{Q}$ as $\lambda \to \infty$.

PROOF. Let $\{\lambda_n, n \geqq 1\}$ be any sequence of increasing positive numbers with

$\lambda_n\to\infty$. Since $\{\Phi_\lambda W, Q\}\to\{W, \mathbf{Q}\}$ in law as $\lambda\to\infty$, an application of Skorohod's realization theorem of almost sure convergence entails the existence of a probability space $(\bar{\Omega}, \bar{\mathcal{B}}, \bar{Q})$ and $\boldsymbol{W}$-valued random variables $\overline{W}$, $\overline{W}_n$, $n\geqq 1$, defined on $(\bar{\Omega}, \bar{\mathcal{B}}, \bar{Q})$ such that (i) $\{\overline{W}_n, \bar{Q}\}\stackrel{d}{=}\{\Phi_{\lambda_n}W, Q\}$, $\{\overline{W}, \bar{Q}\}\stackrel{d}{=}\{W, \mathbf{Q}\}$ and (ii) $\overline{W}_n\to\overline{W}$ (in the Skorohod topology), $\bar{Q}$-a.s. as $n\to\infty$. By Lemma 3.1, we can assume that $\overline{W}\in\boldsymbol{W}^{\#}$. Set $T_n=T(\overline{W}_n)$, $\hat{T}_n=\hat{T}(\overline{W}_n)$ and

$$\mathbf{P}_n = \boldsymbol{P}_{\lambda_n\overline{W}_n}\otimes\boldsymbol{P}_{\lambda_n\overline{W}_n[a_1(\overline{W}_n), c_1(\overline{W}_n)]}.$$

Then the following is known (Kawazu-Tamura-Tanaka [**7**; p. 179]; this is originally due to Brox [**1**] who discussed the Brownian environment case):

$$\begin{aligned}&\mathbf{P}_n\{R<\exp[\lambda_n\gamma(\lambda_n)]<T_n\}\\ &=\mathbf{P}_n\{R<\exp[\lambda_n\gamma(\lambda_n)]<\hat{T}_n\}\longrightarrow 1\quad(\bar{Q}\text{-a.s.})\qquad\text{as } n\to\infty.\end{aligned}$$

This implies our assertion. □

We denote by $\tilde{\mathcal{B}}_t$ the σ-algebra on $\tilde{\Omega}$ generated by the sets $\{\tilde{\omega}:\tilde{\omega}(s)\leqq x\}$, $0\leqq s\leqq t$, $x\in\boldsymbol{R}$ and put $\tilde{\mathcal{B}}=\bigvee_{t\geq 0}\tilde{\mathcal{B}}_t$. Let θ_t, $t\geqq 0$, be the shift operator on $\tilde{\Omega}$ defined by $(\theta_t\tilde{\omega})(s)=\tilde{\omega}(t+s)$, $s\geqq 0$. For $\lambda>0$, we define $\Psi_\lambda:\tilde{\Omega}\to\tilde{\Omega}$ by

$$(\Psi_\lambda\tilde{\omega})(t)=\varphi(\lambda)\tilde{\omega}(\varphi(\lambda)^{-2}t),\qquad t\geqq 0. \tag{4.7}$$

For $\tilde{\omega}\in\tilde{\Omega}$ and $x\in\boldsymbol{R}$, $\tilde{\omega}+x$ denotes the path whose value at time t is $\tilde{\omega}(t)+x$. Since an element ω of Ω is also an element of $\tilde{\Omega}$, $\theta_t\omega$, $\Psi_\lambda\omega$ and $\omega+x$ are well-defined. Set $s(\lambda)=\varphi(\lambda)^{-2}e^{\lambda}$. Then the scaling relation (4.1) implies for each fixed W

$$\{(\theta_{\exp(\lambda)}\omega)(t), t\geqq 0, \boldsymbol{P}_W\}\stackrel{d}{=}\{(\Psi_\lambda\theta_{s(\lambda)}\omega)(t), t\geqq 0, \boldsymbol{P}_{\lambda\Phi_\lambda W}\}. \tag{4.8}$$

LEMMA 4.2. *For any fixed positive constant h and $\Gamma\in\tilde{\mathcal{B}}_h$*

$$\boldsymbol{E}^Q[\boldsymbol{P}_W\{\theta_{\exp(\lambda)}\omega-b_\lambda(W)\in\Gamma\}]=\boldsymbol{E}^Q[\hat{\boldsymbol{P}}_W^\lambda\{\tilde{\omega}-b^\lambda\in\Psi_\lambda^{-1}(\Gamma)\}]+\varepsilon_\lambda,$$

where $\varepsilon_\lambda\to 0$ uniformly in $\Gamma\in\tilde{\mathcal{B}}_h$ as $\lambda\to\infty$.

PROOF. We put $t(\lambda)=s(\lambda)+\varphi(\lambda)^{-2}h$. Then both $s(\lambda)$ and $t(\lambda)$ are of the form $\exp(\lambda\gamma(\lambda))$ with $\gamma(\lambda)\to 1$ as $\lambda\to\infty$. We have

$$\begin{aligned}&\boldsymbol{E}^Q[\boldsymbol{P}_W\{\theta_{\exp(\lambda)}\omega-b_\lambda(W)\in\Gamma\}]\\ &=\boldsymbol{E}^Q[\boldsymbol{P}_W^\lambda\{\Psi_\lambda\theta_{s(\lambda)}\omega-\varphi(\lambda)b^\lambda\in\Gamma\}]\qquad\text{(by (3.10) and (4.8))}\\ &=\boldsymbol{E}^Q[\mathbf{P}_W^\lambda\{\theta_{s(\lambda)}\omega-b^\lambda\in\Psi_\lambda^{-1}(\Gamma),\ R<s(\lambda)<t(\lambda)<T^\lambda\}]+\varepsilon_\lambda'\quad\text{(by Lemma 4.1)}\\ &=\boldsymbol{E}^Q[\mathbf{P}_W^\lambda\{\theta_{s(\lambda)}\tilde{\omega}-b^\lambda\in\Psi_\lambda^{-1}(\Gamma),\ R<s(\lambda)<t(\lambda)<\hat{T}^\lambda\}]+\varepsilon_\lambda'\qquad\text{(by (4.6))}\end{aligned}$$

$$= \boldsymbol{E}^Q[\hat{\boldsymbol{P}}^\lambda_W\{\theta_{s(\lambda)}\tilde{\omega}-b^\lambda\in\Psi_\lambda^{-1}(\Gamma)\}]+\varepsilon_\lambda \qquad \text{(by Lemma 4.1)}$$
$$= \boldsymbol{E}^Q[\hat{\boldsymbol{P}}^\lambda_W\{\tilde{\omega}-b^\lambda\in\Psi_\lambda^{-1}(\Gamma)\}]+\varepsilon_\lambda \qquad \text{(stationarity)},$$

where ε'_λ and ε_λ converge to 0 uniformly in Γ as $\lambda\to\infty$. This proves the Lemma. □

To proceed let $W\in\boldsymbol{W}$, $0<\varepsilon<1$ and define $u^+_\varepsilon(W)$ to be the beginning of the last descending ladder excursion of $\{W(x), x\geqq 0\}$ with height $\geqq\varepsilon$ before $b^+_1(W)$. Recalling the definitions in (3.6), (3.7), (3.8), for $\lambda>0$ and $W\in\Phi_\lambda\mathcal{W}$ we have

$$(4.9)\qquad u^+_\varepsilon(W)=\begin{cases}\varphi(\lambda)^{-1}\tau^+(N^+_\varepsilon(\hat{W})+1,\hat{W})+b^+_1(W), & \text{if } N^+_\varepsilon(\hat{W})\geqq 1,\\ 0\qquad\qquad , & \text{if } N^+_\varepsilon(\hat{W})=0,\end{cases}$$

where $\hat{W}\in\mathcal{W}^+$ is defined by

$$\hat{W}(x)=W(0\vee(b^+_1+\varphi(\lambda)^{-1}x)\wedge c^+_1)-W(b^+_1),\qquad x\in\boldsymbol{R}.$$

Note that the graph of $\hat{W}$ is illustrated by Figure 2 with $\lambda=\varepsilon$. We next define $v^+_\varepsilon(W)$ for $W\in\boldsymbol{W}$ and $\varepsilon\in(0,1)$. Put

$$\bar{c}^+_\varepsilon(W)=\inf\{x\geqq b^+_1: W^*(x)-W_*(b^+_1)\geqq\varepsilon\},$$
$$v^+_\varepsilon(W)=\sup\{x\leqq\bar{c}^+_\varepsilon: W_*(x)\leqq\inf_{[\bar{c}^+_\varepsilon,c^+_1]}W_*\}.$$

To define $v^+_{\varepsilon,\lambda}(W)$ for $W\in\Phi_\lambda\mathcal{W}$ we need the following notation: for $W^+\in\mathcal{W}^+$ and $\varepsilon>0$,

$$(4.10)\qquad \begin{cases}\sigma^+(0)=\sigma^+(0,W^+)=0,\\ \sigma^+(k)=\sigma^+(k,W^+)=\min\{n>\sigma^+(k-1): W^+(n)\leqq\inf\{W^+(j): j>n\}\}\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k\geqq 1,\end{cases}$$

$$(4.11)\qquad \tilde{H}^+(k)=\tilde{H}^+(k,W^+)=\max\{W^+(n): \sigma^+(k-1)\leqq n\leqq\sigma^+(k)\},\quad k\geqq 1,$$

$$(4.12)\qquad \tilde{N}^+_\varepsilon=\tilde{N}^+_\varepsilon(W^+)=\max\{k\geqq 1: \max_{1\leqq j\leqq k}\tilde{H}^+(j)<\varepsilon\}.$$

We define $v^+_{\varepsilon,\lambda}(W)$ by

$$(4.13)\qquad v^+_{\varepsilon,\lambda}(W)=\varphi(\lambda)^{-1}\sigma^+(\tilde{N}^+_\varepsilon(\hat{W}),\hat{W})+b^+_1(W),\qquad W\in\Phi_\lambda\mathcal{W}.$$

Then we see that for $W\in\Phi_\lambda\mathcal{W}$

$$v^+_{\varepsilon,\lambda}(W)=v^+_\varepsilon(W)-\varphi(\lambda)^{-1}.$$

Furthermore we use the following notation: for $W\in\mathcal{W}$

$$b^{+,\lambda}=b^+_1(\Phi_\lambda W),\quad u^{+,\lambda}_\varepsilon=u^+_\varepsilon(\Phi_\lambda W),\quad v^{+,\lambda}_\varepsilon=v^+_{\varepsilon,\lambda}(\Phi_\lambda W).$$

LEMMA 4.3. (i) *For any* $\varepsilon\in(0,1)$, (4.14) $$\lim_{\delta\downarrow 0}\varliminf_{\lambda\to\infty} \boldsymbol{Q}\{u_\varepsilon^{+,\lambda}<b^{+,\lambda}-\delta\}=1,$$

(4.15) $$\lim_{\delta\downarrow 0}\varliminf_{\lambda\to\infty} \boldsymbol{Q}\{v_\varepsilon^{+,\lambda}>b^{+,\lambda}+\delta\}=1.$$

(ii) *For any* $\delta>0$,

(4.16) $$\lim_{\varepsilon\downarrow 0}\varlimsup_{\lambda\to\infty} \boldsymbol{Q}\{u_\varepsilon^{+,\lambda}<b^{+,\lambda}-\delta\}=0,$$

(4.17) $$\lim_{\varepsilon\downarrow 0}\varlimsup_{\lambda\to\infty} \boldsymbol{Q}\{v_\varepsilon^{+,\lambda}>b^{+,\lambda}+\delta\}=0.$$

PROOF. Noting that $\{\Phi_\lambda W, \boldsymbol{Q}\}$ converges in law to $\{W, \mathbf{Q}\}$ as $\lambda\to\infty$ by the assumption (A), we have (4.14) and (4.16). By a consideration similar to Lemma 3.2, we see that there exists a subset $\overline{\boldsymbol{W}}_1^\#$ of $\boldsymbol{W}^\#$ with $\mathbf{Q}\{\overline{\boldsymbol{W}}_1^\#\}=1$ such that for any $W\in\overline{\boldsymbol{W}}_1^\#$ and for any sequence $\{W_{\lambda_n}, n\geqq 1\}$ with $W_{\lambda_n}\in\Phi_{\lambda_n}\mathcal{W}$ which converges to W in the Skorohod topology, $v_{\varepsilon,\lambda_n}^+(W_{\lambda_n})$ converges to $v_\varepsilon^+(W)$ as $n\to\infty$. This fact combined with the relation between v_ε^+ and $v_{\varepsilon,\lambda}^+$ implies (4.15) and (4.17). □

LEMMA 4.4. *Let h be any fixed positive constant. Then, for $\Gamma\in\tilde{\mathcal{B}}_h$ and $\varepsilon>0$,*

$$\boldsymbol{E}^Q[\hat{\boldsymbol{P}}_W^\lambda\{\tilde{\omega}-b^\lambda\in\Gamma\}, b^\lambda=b^{+,\lambda}]$$
$$=\boldsymbol{E}^Q[\boldsymbol{P}_{\lambda\Phi_\lambda W[u_\varepsilon^{+,\lambda}, v_{1/2}^{+,\lambda}]}\{\tilde{\omega}-b^\lambda\in\Gamma\}, b^\lambda=b^{+,\lambda}]+o(1),$$

where $o(1)$ is a term which tends to 0 uniformly in $\Gamma\in\tilde{\mathcal{B}}_h$ as $\lambda\to\infty$ (and which may depend on h and ε).

PROOF. As was already remarked, $\{\Phi_\lambda W, \boldsymbol{Q}\}$ converges to $\{W, \mathbf{Q}\}$ in law as $\lambda\to\infty$. Let $\{\lambda_n, n\geqq 1\}$ be an arbitrary sequence of positive numbers with $\lambda_n\to\infty$ as $n\to\infty$. By Skorohod's realization theorem of almost sure convergence, there exist a probability space $(\bar{\Omega}, \bar{\mathcal{B}}, \bar{\boldsymbol{Q}})$ and $\boldsymbol{W}$-valued random variables $\overline{W}$, $\overline{W}_n$, $n\geqq 1$, defined on $(\bar{\Omega}, \bar{\mathcal{B}}, \bar{\boldsymbol{Q}})$ such that (i) $\{\overline{W}_n, \bar{\boldsymbol{Q}}\}\stackrel{d}{=}\{\Phi_{\lambda_n}W, \boldsymbol{Q}\}$, $\{\overline{W}, \bar{\boldsymbol{Q}}\}\stackrel{d}{=}\{W, \mathbf{Q}\}$ and (ii) $\overline{W}_n\to\overline{W}$ (in the Skorohod topology), $\bar{\boldsymbol{Q}}$-a.s., as $n\to\infty$. From now on, we consider the case where $b_1(\overline{W})=b_1^+(\overline{W})$. Set $a_{(n)}=a_1(\overline{W}_n)$, $b_{(n)}=b_1(\overline{W}_n)$, $c_{(n)}=c_1(\overline{W}_n)$, $b_{(n)}^+=b_1^+(\overline{W}_n)$ and $c_{(n)}^+=c_1^+(\overline{W}_n)$. Then, by Lemma 3.2, $a_{(n)}$, $b_{(n)}$, $c_{(n)}$, $b_{(n)}^+$ and $c_{(n)}^+$ converge ($\bar{\boldsymbol{Q}}$-a.s.) to $a_1(\overline{W})$, $b_1(\overline{W})$, $c_1(\overline{W})$, $b_1^+(\overline{W})$, $c_1^+(\overline{W})$, respectively, as $n\to\infty$. Let δ be any positive number and set $\bar{a}_n=0\vee(b_{(n)}^+-\delta)$, $\bar{c}_n=c_{(n)}^+\wedge(b_{(n)}^++\delta)$, $\bar{a}=0\vee(b_1^+(\overline{W})-\delta)$ and $\bar{c}=c_1^+(\overline{W})\wedge(b_1^+(\overline{W})+\delta)$. Then, $\bar{a}_n\to\bar{a}$ and $\bar{c}_n\to\bar{c}$ ($\bar{\boldsymbol{Q}}$-a.s.). Let ν_n be the probability distribution defined by (4.2) but with the replacement of W and $[a, c]$ by $\lambda_n\overline{W}_n$ and $[a_{(n)}, c_{(n)}]$, respectively. Then, since $b_1^+(\overline{W})$ is a point of local minimum for $\overline{W}$, we see that

(4.18) ν_n converges weakly to the δ-distribution at $b_1^+(\overline{W})$.

Set

$$\bar{T} = \inf\{t \geqq 0 : \tilde{\omega}(t) \notin (\bar{a}_n, \bar{c}_n)\}.$$

Then, Proposition I-3 in [7: p. 179] combined with (4.18) implies

$$(4.19) \qquad \boldsymbol{P}_{\lambda_n \bar{W}_n[a_{(n)}, c_{(n)}]}\{\bar{T} > h\} \longrightarrow 1, \qquad \boldsymbol{P}_{\lambda_n \bar{W}_n[\bar{a}_n, \bar{c}_n]}\{\bar{T} > h\} \longrightarrow 1,$$

as $n \to \infty$ $\bar{\boldsymbol{Q}}$-a.s. on $\{b_1(\bar{W}) = b_1^+(\bar{W})\}$. Therefore writing

$$\kappa_n = \int_{\bar{a}_n}^{\bar{c}_n} \exp\{-\lambda_n \bar{W}_n(x)\} dx \Big/ \int_{a_{(n)}}^{c_{(n)}} \exp\{-\lambda_n \bar{W}_n(x)\} dx,$$

we have, on the set $\{b_1(\bar{W}) = b_1^+(\bar{W})\}$,

$$\begin{aligned}
(4.20) \qquad & \boldsymbol{P}_{\lambda_n \bar{W}_n[a_{(n)}, c_{(n)}]}\{\tilde{\omega} - b_{(n)} \in \Gamma\} \\
&= \boldsymbol{P}_{\lambda_n \bar{W}_n[a_{(n)}, c_{(n)}]}\{\tilde{\omega} - b_{(n)}^+ \in \Gamma,\ \bar{T} > h\} + o(1) \qquad \text{(by (4.19))} \\
&= \kappa_n \boldsymbol{P}_{\lambda_n \bar{W}_n[\bar{a}_n, \bar{c}_n]}\{\tilde{\omega} - b_{(n)}^+ \in \Gamma,\ \bar{T} > h\} + o(1) \\
&= \boldsymbol{P}_{\lambda_n \bar{W}_n[\bar{a}_n, \bar{c}_n]}\{\tilde{\omega} - b_{(n)}^+ \in \Gamma,\ \bar{T} > h\} + o(1) \qquad \text{(since } \kappa_n \to 1) \\
&= \boldsymbol{P}_{\lambda_n \bar{W}_n[\bar{a}_n, \bar{c}_n]}\{\tilde{\omega} - b_{(n)}^+ \in \Gamma\} + o(1) \qquad \text{(by (4.19))},
\end{aligned}$$

where $o(1)$ is a term (which may depend on δ and h and differ from place to place) such that the integral of its absolute value over the set $\{b_1(\bar{W}) = b_1^+(\bar{W})\}$ with respect to $\bar{\boldsymbol{Q}}$ tends to 0 uniformly in $\Gamma \in \mathscr{B}_h$ as $n \to \infty$. Using Lemma 3.1, we see that $\boldsymbol{1}_{\{b_{(n)} = b_{(n)}^+\}} \to \boldsymbol{1}_{\{b_1(\bar{W}) = b_1^+(\bar{W})\}}$ $\bar{\boldsymbol{Q}}$-a.s. as $n \to \infty$. Therefore (4.20) yields

$$\begin{aligned}
(4.21) \qquad & \boldsymbol{P}_{\lambda_n \bar{W}_n[a_{(n)}, c_{(n)}]}\{\tilde{\omega} - b_{(n)} \in \Gamma\} \boldsymbol{1}_{\{b_{(n)} = b_{(n)}^+\}} \\
&= \boldsymbol{P}_{\lambda_n \bar{W}_n[\bar{a}_n, \bar{c}_n]}\{\tilde{\omega} - b_{(n)}^+ \in \Gamma\} \boldsymbol{1}_{\{b_{(n)} = b_{(n)}^+\}} + o(1),
\end{aligned}$$

which again yields

$$\begin{aligned}
(4.22) \qquad & \boldsymbol{E}^Q[\hat{\boldsymbol{P}}_W^\lambda\{\tilde{\omega} - b^\lambda \in \Gamma\},\ b^\lambda = b^{+,\lambda}] \\
&= \boldsymbol{E}^Q[\boldsymbol{P}_{\lambda \Phi_\lambda W[-,-]}\{\tilde{\omega} - b^{+,\lambda} \in \Gamma\},\ b^\lambda = b^{+,\lambda}] + o(1),
\end{aligned}$$

where $[-, -] = [0 \vee (b^{+,\lambda} - \delta), (b^{+,\lambda} + \delta) \wedge c^{+,\lambda}]$ and $o(1)$ means a term which converges to 0 uniformly in $\Gamma \in \mathscr{B}_h$ as $\lambda \to \infty$, δ and h being fixed. By arguments similar to those used for deriving (4.20), (4.21) and (4.22), we have

$$\begin{aligned}
(4.23) \qquad & \boldsymbol{E}^Q[\boldsymbol{P}_{\lambda \Phi_\lambda W[-,-]}\{\tilde{\omega} - b^{+,\lambda} \in \Gamma\},\ b^\lambda = b^{+,\lambda},\ u_\varepsilon^{+,\lambda} < b^{+,\lambda} - \delta,\ v_{1/2}^{+,\lambda} > b^{+,\lambda} + \delta] \\
&= \boldsymbol{E}^Q[\boldsymbol{P}_{\lambda \Phi_\lambda W[\cdot,\cdot]}\{\tilde{\omega} - b^{+,\lambda} \in \Gamma\},\ b^\lambda = b^{+,\lambda}, \\
&\qquad\qquad u_\varepsilon^{+,\lambda} < b^{+,\lambda} - \delta,\ v_{1/2}^{+,\lambda} > b^{+,\lambda} + \delta] + o(1),
\end{aligned}$$

where $[\cdot, \cdot] = [u_\varepsilon^{+,\lambda}, v_{1/2}^{+,\lambda}]$. (4.23) and Lemma 4.3(i) finally imply the assertion of Lemma 4.4. □

§ 5. The proof of the main theorem.

In addition to the notation of § 3 and § 4 we use the following notation for given $W^+\in\mathcal{W}^+$, $W^-\in\mathcal{W}^-$ and $\lambda>0$:

$$\bar{b}_\lambda^+ = -\tau^+(N_\lambda^+), \qquad \bar{b}_\lambda^- = -\tau^-(N_\lambda^-), \tag{5.1}$$

$$\begin{cases} \bar{M}_\lambda^+ = \max\{W^+(n)-W^+(\tau^+(N_\lambda^+)) : \tau^+(N_\lambda^+)\leqq n\leqq 0\}, \\ \bar{M}_\lambda^- = \max\{W^-(n)-W^-(\tau^-(N_\lambda^-)) : 0\leqq n\leqq \tau^-(N_\lambda^-)\}, \end{cases} \tag{5.2}$$

$$\bar{V}_\lambda^+ = \lambda-W^+(\tau^+(N_\lambda^+)), \qquad \bar{V}_\lambda^- = \lambda-W^-(\tau^-(N_\lambda^-)), \tag{5.3}$$

$$\bar{b}_\lambda=\begin{cases} \bar{b}_\lambda^+ & \text{if } \bar{M}_\lambda^+\vee\bar{V}_\lambda^+<\bar{M}_\lambda^-\vee\bar{V}_\lambda^-, \\ \bar{b}_\lambda^- & \text{if } \bar{M}_\lambda^+\vee\bar{V}_\lambda^+>\bar{M}_\lambda^-\vee\bar{V}_\lambda^-, \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

For $\lambda>0$ fixed we associate with each fixed $W\in\mathcal{W}$ an element $\tilde{W}$ of $\mathcal{W}^+$ defined by

$$\tilde{W}(x) = W(0\vee(b_\lambda^+ + x)\wedge c_\lambda^+)-W(b_\lambda^+), \qquad x\in\boldsymbol{R}, \tag{5.5}$$

(the suffix λ in $\tilde{W}$ is suppressed). Then from Lemma 3.3 and its proof we have following.

LEMMA 5.1. *For any $\varepsilon>0$ and $\lambda>0$ the joint distribution of*

$$\{W(b_\lambda^+ + x)-W(b_\lambda^+),\ -b_\lambda^+\leqq x\leqq c_\lambda^+ - b_\lambda^+\}$$

and $\{b_\lambda^+, M_\lambda^+, V_\lambda^+, \tau^+(N_{\varepsilon\lambda}^+(\tilde{W})+1, \tilde{W}), \sigma^+(\tilde{N}_{\lambda/2}(\tilde{W}), \tilde{W})\}$ under $\boldsymbol{Q}$ is equal to the joint distribution of $\{W^+(x), \tau^+(N_\lambda^+)\leqq x\leqq \rho_\lambda^+\}$ and $\{\bar{b}_\lambda^+, \bar{M}_\lambda^+, \bar{V}_\lambda^+, \tau^+(N_{\varepsilon\lambda}^+ +1), \sigma_+(\tilde{N}_{\lambda/2})\}$ under $\boldsymbol{Q}_\lambda^+$.

LEMMA 5.2. *Let $W\in\boldsymbol{W}$ be fixed and let $u<v$. Then for any $\Gamma\in\mathfrak{B}$ we have*

$$\boldsymbol{P}_{\lambda\Phi_\lambda W[u,v]}\{\tilde{\omega}-b^{+,\lambda}\in\Psi_\lambda^{-1}(\Gamma)\} = \boldsymbol{P}_{W[\varphi(\lambda)u,\varphi(\lambda)v]}\{\tilde{\omega}-b_\lambda^+\in\Gamma\}.$$

PROOF. Since $b_\lambda^+=\varphi(\lambda)b^{+,\lambda}$, we have $\Psi_\lambda(\tilde{\omega}-b^{+,\lambda})=\Psi_\lambda\tilde{\omega}-b_\lambda^+$. Noting this equality we see that the assertion of the lemma is another expression of the following fact (which itself can be proved easily): Under $\boldsymbol{P}_{\lambda\Phi_\lambda W[u,v]}$ the process $\{(\Psi_\lambda\tilde{\omega})(t), t\geqq 0\}$ $(=\{\varphi(\lambda)\tilde{\omega}(\varphi(\lambda)^{-2}t), t\geqq 0\})$ is a reflecting diffusion on $[\varphi(\lambda)u, \varphi(\lambda)v]$ with local generator $\mathcal{L}_W$ and initial distribution $\nu_{W[\varphi(\lambda)u,\varphi(\lambda)v]}$. □

LEMMA 5.3. *Let $\tilde{W}$ be defined by* (5.5). *Then*

$$\varphi(\lambda)u_\varepsilon^{+,\lambda} = \tau^+(N_{\lambda/2}^+(\tilde{W})+1, \tilde{W})+b_\lambda^+ \qquad \textit{if } N_{\lambda\varepsilon}^+(\tilde{W})\geqq 1, \tag{5.6a}$$

$$\varphi(\lambda)v_{1/2}^{+,\lambda} = \sigma^+(\tilde{N}_{\lambda/2}^+(\tilde{W}))+b_\lambda^+. \tag{5.6b}$$

PROOF. To prove the lemma it is enough to apply (4.9) and (4.13) to $u_{\varepsilon}^{+,\lambda}=u_{\varepsilon}^{+}(\Phi_{\lambda}W)$ and $v_{1/2}^{+,\lambda}=v_{1/2,\lambda}^{+}(\Phi_{\lambda}W)$, respectively. □

Let h be a positive constant and let $\Gamma\in\tilde{\mathscr{B}}_{h}$. Then by Lemma 4.2 we have, as $\lambda\to\infty$,

$$\begin{aligned}(5.7)\qquad \mathscr{P}\{X(e^{\lambda}+\cdot, W)-b_{\lambda}(W)\in\Gamma\} &= \boldsymbol{E}^{Q}[\boldsymbol{P}_{W}\{\theta_{\exp(\lambda)}\omega-b_{\lambda}(W)\in\Gamma\}]\\ &= \boldsymbol{E}^{Q}[\hat{\boldsymbol{P}}_{W}^{\lambda}\{\tilde{\omega}-b^{\lambda}\in\Psi_{\lambda}^{-1}(\Gamma)\}]+o(1)\\ &= I_{\lambda}+I_{\lambda}'+o(1),\end{aligned}$$

where

$$(5.8)\qquad I_{\lambda}=\boldsymbol{E}^{Q}[\hat{\boldsymbol{P}}_{W}^{\lambda}\{\tilde{\omega}-b^{\lambda}\in\Psi_{\lambda}^{-1}(\Gamma)\},\ b^{\lambda}=b^{+,\lambda}],$$

$$(5.9)\qquad I_{\lambda}'=\boldsymbol{E}^{Q}[\hat{\boldsymbol{P}}_{W}^{\lambda}\{\tilde{\omega}-b^{\lambda}\in\Psi_{\lambda}^{-1}(\Gamma)\},\ b^{\lambda}=b^{-,\lambda}],$$

$b^{-,\lambda}$ is defined in a way similar to $b^{+,\lambda}$ and $o(1)$ is a term which tends to 0 uniformly in $\Gamma\in\tilde{\mathscr{B}}_{h}$ as $\lambda\to\infty$ (and which may depend on h). Let ε be an arbitrary small positive number (which we let tend to zero later). Then, by Lemma 4.4 I_{λ} can be written as

$$I_{\lambda}=\boldsymbol{E}^{Q}[\boldsymbol{P}_{\lambda\Phi_{\lambda}W[u_{\varepsilon}^{+,\lambda},v_{1/2}^{+,\lambda}]}\{\tilde{\omega}-b^{+,\lambda}\in\Psi_{\lambda}^{-1}(\Gamma)\},\ b^{\lambda}=b^{+,\lambda}]+o(1),$$

and by Lemma 5.2 and Lemma 5.3 the right hand of the above is equal to

$$\boldsymbol{E}^{Q}[\boldsymbol{P}_{I}\{\Gamma\},\ b_{\lambda}=b_{\lambda}^{+}]+o(1),$$

where $\boldsymbol{P}_{I}$ stands for $\boldsymbol{P}_{\tilde{W}[a,b]}$, $a=\tau^{+}(N_{\varepsilon\lambda}^{+}(\tilde{W})+1,\tilde{W})$, $b=\sigma^{+}(\tilde{N}_{\lambda/2}^{+}(\tilde{W}),\tilde{W})$ and $o(1)$ is a term which tends to 0 uniformly in $\Gamma\in\tilde{\mathscr{B}}_{h}$ as $\lambda\to\infty$ (and which may depend on h and ε). Applying Lemma 3.3 and Lemma 5.1 we obtain

$$\boldsymbol{E}^{Q}[\boldsymbol{P}_{I}\{\Gamma\},\ b_{\lambda}=b_{\lambda}^{+}]=\boldsymbol{E}_{\lambda}[\boldsymbol{P}_{II}\{\Gamma\},\ \overline{M}_{\lambda}^{+}\vee\overline{V}_{\lambda}^{+}<\overline{M}_{\lambda}^{-}\vee\overline{V}_{\lambda}^{-}],$$

where $\boldsymbol{E}_{\lambda}$ is the expectation with respect to the product probability measure

$$(5.10)\qquad \boldsymbol{Q}_{\lambda}=\boldsymbol{Q}_{\lambda}^{+}\otimes\boldsymbol{Q}_{\lambda}^{-}\qquad\text{on } \mathscr{W}^{+}\times\mathscr{W}^{-},$$

and $\boldsymbol{P}_{II}$ stands for $\boldsymbol{P}_{W[a,b]}$ with $W=W^{+}\in\mathscr{W}^{+}$, $a=\tau^{+}(N_{\varepsilon\lambda}^{+}+1)$ and $b=\sigma^{+}(\tilde{N}_{\lambda/2}^{+})$.

Next, denote by $F^{+}(x)$ (resp. $F^{-}(x)$) the probability that the process $\{W(t), t\geqq 0, \mathbf{Q}\}$ (resp. $\{W(-t), t\geqq 0, \mathbf{Q}\}$) hits $(-\infty, x-1]$ before it hits $[x,\infty)$ and put

$$(5.11)\qquad \mathfrak{p}=\int_{0}^{1}F^{+}(x)dF^{-}(x).$$

We are going to prove

$$(5.12)\qquad \boldsymbol{E}_{\lambda}[\boldsymbol{P}_{II}\{\Gamma\},\ \overline{M}_{\lambda}^{+}\vee\overline{V}_{\lambda}^{+}<\overline{M}_{\lambda}^{-}\vee\overline{V}_{\lambda}^{-}]=\mathfrak{p}\boldsymbol{E}_{\lambda}^{+}[\boldsymbol{P}_{II}\{\Gamma\}]+\Delta_{1}(\Gamma,\varepsilon,\lambda),$$

where $\boldsymbol{E}_{\lambda}^{+}$ is the expectation with respect to $\boldsymbol{Q}_{\lambda}^{+}$ and $\Delta_{1}(\Gamma,\varepsilon,\lambda)$ (and also $\Delta_{2}(\Gamma,\varepsilon,\lambda)$, appearing below) represents a term which is dominated in modulus by some $\Delta(\varepsilon, \lambda)$ satisfying $\lim_{\varepsilon\downarrow 0}\overline{\lim}_{\lambda\to\infty}|\Delta(\varepsilon, \lambda)|=0$. Let $W'(x)=W^+(\tau^+(N_{\varepsilon\lambda}^+ +1)+x) - W^+(\tau^+(N_{\varepsilon\lambda}^+ +1))$ and define $\overline{M}_\lambda'$ and $\overline{V}_\lambda'$ by (5.2) and (5.3), respectively, but with the replacement of W^+ by W' (it can happen that $W'(x)<0$ for some $x>0$ so that $W'\notin\mathcal{W}^+$; but even in this case $\overline{M}_\lambda'$ and $\overline{V}_\lambda'$ are well-defined). Moreover, let $\overline{m}_\lambda^+$ be the smallest $n\geqq\tau^+(N_\lambda^+)$ such that $W^+(n)=\overline{M}_\lambda^+$. Then

$$Q_\lambda\{\overline{M}_\lambda^+\vee\overline{V}_\lambda^+\neq\overline{M}_\lambda'\vee(\overline{V}_\lambda'-W^+(\tau^+(N_{\varepsilon\lambda}^+ +1)))\}\leqq Q_\lambda\{\overline{m}_\lambda^+\geqq\tau^+(N_{\varepsilon\lambda}^+ +1)\}.$$

Using Lemma 5.1 (after a suitable scaling) and noting (5.6a), we see that

$$Q_\lambda\{\overline{m}_\lambda^+\geqq\tau^+(N_{\varepsilon\lambda}^+ +1)\}\leqq Q\{m^{+,\lambda}\geqq u_\varepsilon^{+,\lambda}\},$$

where $m^{+,\lambda}=m^+(\Phi_\lambda W)$, m^+ being the smallest $x\geqq 0$ such that $W(x)=M_1^+$. Lemma 4.3(ii) then implies

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\overline{\lim_{\lambda\to\infty}}\, Q_\lambda\{\overline{M}_\lambda^+\vee\overline{V}_\lambda^+\neq\overline{M}_\lambda'\vee(\overline{V}_\lambda'-W^+(\tau^+(N_{\varepsilon\lambda}^+ +1)))\}\\ &\leqq \lim_{\varepsilon\downarrow 0}\overline{\lim_{\lambda\to\infty}}\, Q\{m^{+,\lambda}\geqq u_\varepsilon^{+,\lambda}\}=0.\end{aligned}$$

Therefore,

$$\begin{aligned}&E_\lambda[P_{\mathrm{II}}\{\Gamma\},\ \overline{M}_\lambda^+\vee\overline{V}_\lambda^+<\overline{M}_\lambda^-\vee\overline{V}_\lambda^-]\\ &=E_\lambda[P_{\mathrm{II}}\{\Gamma\},\ \overline{M}_\lambda'\vee(\overline{V}_\lambda'-W^+(\tau^+(N_{\varepsilon\lambda}^+ +1)))<\overline{M}_\lambda^-\vee\overline{V}_\lambda^-]+\Delta_2(\Gamma, \varepsilon, \lambda).\end{aligned}$$

Since $(0, W^+(-1), \cdots, W^+(\tau^+(N_{\varepsilon\lambda}^+ +1)))$ and

$$(0, W^+(\tau^+(N_{\varepsilon\lambda}^+ +1)-1)-W^+(\tau^+(N_{\varepsilon\lambda}^+ +1)), \cdots, W^+(\tau^+(N_\lambda^+))-W^+(\tau^+(N_{\varepsilon\lambda}^+ +1)))$$

are independent under Q_λ^+, we have

$$\begin{aligned}(5.13)\qquad &E_\lambda[P_{\mathrm{II}}\{\Gamma\},\ \overline{M}_\lambda'\vee(\overline{V}_\lambda'-W^+(\tau^+(N_{\varepsilon\lambda}^+ +1)))<\overline{M}_\lambda^-\vee\overline{V}_\lambda^-]\\ &=E_\lambda^+[P_{\mathrm{II}}\{\Gamma\}\phi_\lambda(W^+(\tau^+(N_{\varepsilon\lambda}^+ +1)))],\end{aligned}$$

where

$$(5.14)\qquad \phi_\lambda(x)=Q_\lambda\{\overline{M}_\lambda^+\vee(\overline{V}_\lambda^+-x)<\overline{M}_\lambda^-\vee\overline{V}_\lambda^-\}.$$

Using Lemma 5.1 and a scaling we have

$$\begin{aligned}(5.15)\qquad \phi_\lambda(\lambda x)&=Q_\lambda\{\overline{M}_\lambda^+\vee(\overline{V}_\lambda^+-\lambda x)<\overline{M}_\lambda^-\vee\overline{V}_\lambda^-\}\\ &=Q\{M_1^{+,\lambda}\vee(V_1^{+,\lambda}-x)<M_1^{-,\lambda}\vee V_1^{-,\lambda}\},\end{aligned}$$

where $M_1^{+,\lambda}=M_1^+(\Phi_\lambda W)$, $M_1^{-,\lambda}=M_1^-(\Phi_\lambda W)$, $V_1^{+,\lambda}=V_1^+(\Phi_\lambda W)$ and $V_1^{-,\lambda}=V_1^-(\Phi_\lambda W)$. Therefore from the assumption (A) we see that

$$(5.16)\qquad \lim_{\lambda\to\infty}\phi_\lambda(\lambda x)=\phi(x)\equiv \mathbf{Q}\{M_1^+\vee(V_1^+-x)<M_1^-\vee V_1^-\}\qquad \text{for } x\geqq 0.$$

The identity $\phi(0)=\int_0^1 F^+(x)dF^-(x)$ can be verified by the fact that $M_1^+\vee V_1^+$ and

$M_{\mathrm{I}}^{-}\vee V_{\mathrm{I}}^{-}$ are independent under $\mathbf{Q}$ and distributed according to dF^{+} and dF^{-}, respectively. Thus we have

$$(5.17)\qquad \psi_{\lambda}(0)\boldsymbol{E}_{\lambda}^{+}[\boldsymbol{P}_{\mathrm{II}}\{\Gamma\}] \leqq \boldsymbol{E}_{\lambda}^{+}[\boldsymbol{P}_{\mathrm{II}}\{\Gamma\}\psi_{\lambda}(W^{+}(\tau^{+}(N_{\varepsilon\lambda}^{+}+1)))]$$
$$\leqq \boldsymbol{E}_{\lambda}^{+}[\boldsymbol{P}_{\mathrm{II}}\{\Gamma\}\psi_{\lambda}(\lambda\delta),\ W^{+}(\tau^{+}(N_{\varepsilon\lambda}^{+}+1))/\lambda\leqq\delta]+\boldsymbol{Q}_{\lambda}^{+}\{W^{+}(\tau^{+}(N_{\varepsilon\lambda}^{+}+1))/\lambda>\delta\}.$$

Using Lemma 4.3 it is easy to see that for any $\delta>0$

$$\lim_{\varepsilon\downarrow 0}\varlimsup_{\lambda\to\infty}\boldsymbol{Q}_{\lambda}^{+}\{W^{+}(\tau^{+}(N_{\varepsilon\lambda}^{+}+1))/\lambda>\delta\}$$
$$=\lim_{\varepsilon\downarrow 0}\varlimsup_{\lambda\to\infty}\boldsymbol{Q}\{(\Phi_{\lambda}W)(u_{\varepsilon}^{+,\lambda})>\delta\}=0,$$

which combined with (5.13) and (5.17) proves (5.12).

Finally we prove that for any fixed $\varepsilon>0$

$$(5.18)\qquad \boldsymbol{E}_{\lambda}^{+}[\boldsymbol{P}_{\mathrm{II}}\{\Gamma\}]\longrightarrow\tilde{\boldsymbol{E}}^{+}[\tilde{\boldsymbol{P}}_{W}\{\Gamma\}]\qquad\text{as }\lambda\to\infty,$$

where $\tilde{\boldsymbol{P}}_{W}$ and $\tilde{\boldsymbol{Q}}$ are defined in § 1 and $\tilde{\boldsymbol{E}}^{+}$ denotes the expectation with respect to $\tilde{\boldsymbol{Q}}^{+}$. First we show (5.18) in the case where $\Gamma=\{\tilde{\omega}\in\tilde{\Omega}:\tilde{\omega}(0)\in A\}$, $A\in\mathcal{B}(\boldsymbol{R})$. In this case (5.18) is expressed as

$$(5.19)\qquad \boldsymbol{E}_{\lambda}^{+}\Big(\int_{J(\lambda)\cap A}\mathrm{e}^{-W(x)}dx\Big/\int_{J(\lambda)}\mathrm{e}^{-W(x)}dx\Big)$$
$$\longrightarrow\tilde{\boldsymbol{E}}^{+}\Big(\int_{A}\mathrm{e}^{-W(x)}dx\Big/\int_{-\infty}^{\infty}\mathrm{e}^{-W(x)}dx\Big),\qquad\text{as }\lambda\to\infty,$$

where $J(\lambda)=[\tau^{+}(N_{\varepsilon\lambda}^{+}+1),\ \sigma^{+}(\tilde{N}_{\lambda/2}^{+})]$. From Proposition 2.3, we know that

$$(5.20)\qquad \boldsymbol{Q}_{\lambda}^{+}\text{-distribution of }\int_{J(\lambda)\cap[0,\infty]}\mathrm{e}^{-W(x)}dx$$
$$\longrightarrow\tilde{\boldsymbol{Q}}^{+}\text{-distribution of }\int_{0}^{\infty}\mathrm{e}^{-W(x)}dx,\qquad\lambda\to\infty.$$

Since $\{W(x),\ x\geqq 0\}$ and $\{W(x),\ x\leqq 0\}$ are independent under $\boldsymbol{Q}_{\lambda}^{+}$ and the law of $\{W(x),\ x\leqq 0,\ \boldsymbol{Q}_{\lambda}^{+}\}$ does not depend on λ, (5.20) implies

$$(5.21)\qquad \boldsymbol{Q}_{\lambda}^{+}\text{-distribution of }\int_{J(\lambda)}\mathrm{e}^{-W(x)}dx$$
$$\longrightarrow\tilde{\boldsymbol{Q}}^{+}\text{-distribution of }\int_{-\infty}^{\infty}\mathrm{e}^{-W(x)}dx,\qquad\lambda\to\infty.$$

Taking an arbitrary positive sequence $\{\lambda_{n}\}$ tending to infinity, we now prove that (5.19) holds as $\lambda\to\infty$ via $\{\lambda_{n}\}$. We make use of Skorohod's theorem of almost sure convergence to construct, on a suitable probability space $(\bar{\Omega},\bar{\boldsymbol{Q}})$, $\mathcal{W}^{+}$-valued random variables $\overline{W}$ and $\overline{W}_{\lambda}$, $\lambda=\lambda_{1},\lambda_{2},\cdots$, such that (i) $\{\overline{W},\bar{\boldsymbol{Q}}\}\overset{d}{=}\{W,\tilde{\boldsymbol{Q}}^{+}\}$ and $\{\overline{W}_{\lambda},\bar{\boldsymbol{Q}}\}\overset{d}{=}\{W,\boldsymbol{Q}_{\lambda}^{+}\}$ for each $\lambda=\lambda_{n}$ and (ii) $\overline{W}_{\lambda}\to\overline{W}$ ($\bar{\boldsymbol{Q}}$-a.s.) as $\lambda\to\infty$

via $\{\lambda_n\}$. Then, (5.21) implies

$$(5.22) \qquad \int_{\bar{J}(\lambda)} \exp\{-\overline{W}_\lambda(x)\}dx \longrightarrow \int_{-\infty}^{\infty} e^{-\overline{W}(x)}dx \text{ in probability,}$$

as $\lambda\to\infty$ via $\{\lambda_n\}$, where $\bar{J}(\lambda)$ corresponds to $J(\lambda)$. From the above (ii) and (5.22) we can easily show that

$$\int_{\bar{J}(\lambda)\cap A} \exp\{-\overline{W}_\lambda(x)\}dx \longrightarrow \int_A e^{-\overline{W}(x)}dx \text{ in probability, } \lambda\to\infty \text{ via } \{\lambda_n\},$$

for any Borel set A in $\boldsymbol{R}$. This clearly implies

$$\int_{\bar{J}(\lambda)\cap A} \exp\{-\overline{W}_\lambda(x)\}dx \Big/ \int_{\bar{J}(\lambda)} \exp\{-\overline{W}_\lambda(x)\}dx$$

$$\longrightarrow \int_A e^{-\overline{W}(x)}dx \Big/ \int_{-\infty}^{\infty} e^{-\overline{W}(x)}dx \text{ in probability, } \quad \text{as } \lambda\to\infty \text{ via } \{\lambda_n\}.$$

Since $\{\lambda_n\}$ is arbitrary, (5.19) holds.

Next we prove (5.18) for general $\Gamma\in\tilde{\mathscr{B}}_h$. For positive integers k and l with $k<l$ we put

$$A_{\lambda,l} = \{W\in\mathscr{W}^+ : J(\lambda)\supset[-l, l]\},$$

$$B_{\rho,l} = \{W\in\mathscr{W}^+ : 0\leqq W(x)\leqq\rho \text{ (for all } 0\leqq x\leqq l)\},$$

$$T_l = \text{the first exit time of } \tilde{\omega}(t) \text{ from } (-l, l),$$

$$\Gamma_{k,l} = \{\tilde{\omega}\in\Gamma : \tilde{\omega}(0)\in[-k, k],\ T_l\geqq h\}.$$

We are going to estimate

$$\tilde{\Delta} = |\boldsymbol{E}_\lambda^\dagger[\boldsymbol{P}_{\mathrm{II}}\{\Gamma\}]-\tilde{\boldsymbol{E}}^+[\tilde{\boldsymbol{P}}_W\{\Gamma\}]|.$$

We can write

$$\tilde{\Delta} \leqq \tilde{\Delta}_1+\tilde{\Delta}_2+\tilde{\Delta}_3+\tilde{\Delta}_4+\tilde{\Delta}_5,$$

where

$$\tilde{\Delta}_1 = \boldsymbol{E}_\lambda^\dagger[\boldsymbol{P}_{\mathrm{II}}\{\Gamma\}]-\boldsymbol{E}_\lambda^\dagger[\boldsymbol{P}_{\mathrm{II}}\{\Gamma_{k,l}\};\ A_{\lambda,l}\cap B_{\rho,l}],$$

$$\tilde{\Delta}_2 = |\boldsymbol{E}_\lambda^\dagger[\boldsymbol{P}_{\mathrm{II}}\{\Gamma_{k,l}\}-\boldsymbol{P}_{W[-l,l]}\{\Gamma_{k,l}\};\ A_{\lambda,l}\cap B_{\rho,l}]|,$$

$$\tilde{\Delta}_3 = \boldsymbol{E}_\lambda^\dagger[\boldsymbol{P}_{W[-l,l]}\{\Gamma_{k,l}\};\ A_{\lambda,l}^c\cap B_{\rho,l}],$$

$$\tilde{\Delta}_4 = |\boldsymbol{E}_\lambda^\dagger[\boldsymbol{P}_{W[-l,l]}\{\Gamma_{k,l}\};\ B_{\rho,l}]-\tilde{\boldsymbol{E}}^+[\boldsymbol{P}_{W[-l,l]}\{\Gamma_{k,l}\};\ B_{\rho,l}]|,$$

$$\tilde{\Delta}_5 = |\tilde{\boldsymbol{E}}^+[\boldsymbol{P}_{W[-l,l]}\{\Gamma_{k,l}\};\ B_{\rho,l}]-\tilde{\boldsymbol{E}}^+[\tilde{\boldsymbol{P}}_W\{\Gamma\}]|.$$

Now we prove

$$(5.23) \qquad \lim_{k\to\infty}\overline{\lim_{l\to\infty}}\ \overline{\lim_{\rho\to\infty}}\ \overline{\lim_{\lambda\to\infty}}\ \sup_{\Gamma\in\tilde{\mathscr{B}}_h}\tilde{\Delta}_i = 0, \qquad 1\leqq i\leqq 5.$$

To prove (5.23) for $i=1$, we write $\tilde{\Delta}_1$ as the sum

(5.24) $$\boldsymbol{E}_{\lambda}^{\dagger}[\boldsymbol{P}_{\mathrm{II}}\{\Gamma\cap\Gamma_{k,l}^{c}\}\,;\,A_{\lambda,l}\cap B_{\rho,l}]+\boldsymbol{E}_{\lambda}^{\dagger}[\boldsymbol{P}_{\mathrm{II}}\{\Gamma\}\,;\,A_{\lambda,l}^{c}\cup B_{\rho,l}^{c}]=\tilde{\Delta}_{11}+\tilde{\Delta}_{12},$$

and then note that the first term $\tilde{\Delta}_{11}$ is dominated by

$$\boldsymbol{E}_{\lambda}^{\dagger}[\boldsymbol{P}_{\mathrm{II}}\{\tilde{\omega}(0)\in[-k,\,k]^{c}\}+\boldsymbol{P}_{\mathrm{II}}\{\tilde{\omega}(0)\in[-k,\,k],\ T_{l}<h\}\,;\,A_{\lambda,l}\cap B_{\rho,l}]$$

$$\leqq \boldsymbol{E}_{\lambda}^{\dagger}\left[\left\{\int_{J(\lambda)}e^{-W(y)}dy\right\}^{-1}\cdot\int_{J(\lambda)\cap[-k,\,k]^{c}}e^{-W(x)}dx\right]$$

$$+\boldsymbol{E}_{\lambda}^{\dagger}\left[\left\{\int_{[-l,\,l]}e^{-W(y)}dy\right\}^{-1}\cdot\int_{[-k,\,k]}e^{-W(x)}\boldsymbol{P}_{W}^{(x)}\{T_{l}<h\}dx\,;\,B_{\rho,l}\right]$$

$$=\tilde{\Delta}_{111}+\tilde{\Delta}_{112},$$

where $\boldsymbol{P}_{W}^{(x)}$ is the probability law of the diffusion process with generator $\mathcal{L}_{W}$ and starting at x. $\tilde{\Delta}_{111}$ depends only on k and λ and it follows from (5.19) that

$$\lim_{\lambda\to\infty}\tilde{\Delta}_{111}=\tilde{\boldsymbol{E}}^{+}\left[\left\{\int_{-\infty}^{\infty}e^{-W(y)}dy\right\}^{-1}\cdot\int_{[-k,\,k]^{c}}e^{-W(x)}dx\right]\longrightarrow 0,\qquad k\to\infty.$$

As for $\tilde{\Delta}_{112}$, recalling the definition of $q_{\lambda}(\mathfrak{S})$- and $q(\mathfrak{S})$-chains with $\mathfrak{S}=\mathfrak{W}^{+}$ we note that

$$d\boldsymbol{Q}_{\lambda}^{\dagger}/d\tilde{\boldsymbol{Q}}^{+}=R(0)\eta_{\lambda}(0)^{-1}R(W(l))^{-1}\eta_{\lambda}(W(l))\qquad\text{on } B_{\rho,l},$$

and that this tends to 1 uniformly on $B_{\rho,l}$ as $\lambda\to\infty$ by Theorem 2.3 and Remark 2.1. Therefore $\tilde{\Delta}_{112}$ tends to Δ_{112} as $\lambda\to\infty$, where

$$\Delta_{112}=\tilde{\boldsymbol{E}}^{+}\left[\left\{\int_{[-l,\,l]}e^{-W(y)}dy\right\}^{-1}\cdot\int_{[-k,\,k]}e^{-W(x)}\boldsymbol{P}_{W}^{(x)}\{T_{l}<h\}dx\,;\,B_{\rho,l}\right]$$

$$\leqq\tilde{\boldsymbol{E}}^{+}\left[\left\{\int_{[-l,\,l]}e^{-W(y)}dy\right\}^{-1}\cdot\int_{[-k,\,k]}e^{-W(x)}\boldsymbol{P}_{W}^{(x)}\{T_{l}<h\}dx\right]\equiv\Delta_{112}'.$$

Δ_{112}' depends only on k and l, and $\Delta_{112}'\to 0$ as $l\to\infty$ since $\boldsymbol{P}_{W}^{(x)}\{T_{l}<h\}\to 0$ ($\tilde{\boldsymbol{Q}}^{+}$-a. s.) as $l\to\infty$. Thus we have

(5.25) $$\lim_{k\to\infty}\overline{\lim_{l\to\infty}}\,\overline{\lim_{\rho\to\infty}}\,\overline{\lim_{\lambda\to\infty}}\,\sup_{\Gamma\in\tilde{\mathscr{B}}_{h}}\tilde{\Delta}_{11}=0.$$

The second term $\tilde{\Delta}_{12}$ in (5.24) can be controlled by

(5.26) $$\tilde{\Delta}_{12}\leqq\boldsymbol{Q}_{\lambda}^{\dagger}\{A_{\lambda,l}^{c}\cup B_{\rho,l}^{c}\},$$

(5.27) $$\lim_{\rho\to\infty}\overline{\lim_{\lambda\to\infty}}\,\boldsymbol{Q}_{\lambda}^{\dagger}\{A_{\lambda,l}^{c}\cup B_{\rho,l}^{c}\}=0\quad(l\colon\text{fixed}).$$

From (5.24), (5.25), (5.26) and (5.27) we obtain (5.23) for $i=1$. To prove (5.23) for $i=2$, it is enough to write

$$\tilde{\Delta}_2 \leqq \boldsymbol{E}_\lambda^+\Big[\Big|\Big\{\int_{J(\lambda)} e^{-W(y)}dy\Big\}^{-1}\cdot\int_{[-k,k]} e^{-W(x)}\boldsymbol{P}_W^{(x)}\{\Gamma_{k,l}\}dx$$

$$-\Big\{\int_{[-l,l]} e^{-W(y)}dy\Big\}^{-1}\cdot\int_{[-k,k]} e^{-W(x)}\boldsymbol{P}_W^{(x)}\{\Gamma_{k,l}\}dx\Big|\,;\,A_{\lambda,l}\cap B_{\rho,l}\Big]$$

$$\leqq \boldsymbol{E}_\lambda^+\Big[\Big|\Big\{\int_{J(\lambda)} e^{-W(y)}dy\Big\}^{-1}-\Big\{\int_{[-l,l]} e^{-W(y)}dy\Big\}^{-1}\Big|\cdot\int_{[-k,k]} e^{-W(x)}dx\Big]$$

and then use (5.19). The rest of the statement (5.23) can be proved by similar arguments. The statement (5.18) now follows from (5.23).

From what we proved up to now, we have

$$\lim_{\lambda\to\infty} I_\lambda = \mathfrak{p}\cdot\tilde{\boldsymbol{E}}^+[\tilde{\boldsymbol{P}}_W\{\Gamma\}].$$

Since a similar formula for I'_λ can also be obtained, we finally obtain

$$\lim_{\lambda\to\infty} \mathscr{P}\{X(e^\lambda+\cdot, W)-b_\lambda(W)\in\Gamma\} = \mathfrak{p}\cdot\tilde{\boldsymbol{E}}^+[\tilde{\boldsymbol{P}}_W\{\Gamma\}]+(1-\mathfrak{p})\tilde{\boldsymbol{E}}^-[\tilde{\boldsymbol{P}}_W\{\Gamma\}].$$

The above convergence is uniform in $\Gamma\in\tilde{\mathscr{B}}_h$ and hence the convergence also holds with respect to the total variation norm on $\tilde{\mathscr{B}}_h$. Thus our main theorem is completely proved.

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