J. Math. Soc. Japan
Vol. 44, No. 3, 1992

Blow-up sets and asymptotic behavior of interfaces for quasilinear degenerate parabolic equations in R^N

Dedicated to Professor Takeshi Kotake on his 60th birthday

By Kiyoshi MOCHIZUKI and Ryuichi SUZUKI

(Received Nov. 6, 1990)
(Revised Aug. 13, 1991)

Introduction.

In this paper we shall consider the Cauchy problem

$$\partial_t\beta(u)=\Delta u+f(u) \quad \text{in} \quad (x, t)\in \boldsymbol{R}^N\times(0, T), \tag{0.1}$$

$$u(x, 0)=u_0(x) \quad \text{in} \quad x\in \boldsymbol{R}^N, \tag{0.2}$$

where $\partial_t=\partial/\partial t$, Δ is the N-dimensional Laplacian and $\beta(v)$, $f(v)$ with $v\geqq 0$ and $u_0(x)$ are nonnegative functions.

Equation (0.1) describes the combustion process in a stationary medium, in which the thermal conductivity $\beta'(u)^{-1}$ and the volume heat source $f(u)$ are depending in a nonlinear way on the temperature $\beta(u)=\beta(u(x, t))$ of the medium.

Throughout this paper we assume

(A1) $\beta(v)$, $f(v)\in C^{\infty}(\boldsymbol{R}_+)\cap C(\bar{\boldsymbol{R}}_+)$, where $\boldsymbol{R}_+=(0, \infty)$ and $\bar{\boldsymbol{R}}_+=[0, \infty)$; $\beta(v)>0$, $\beta'(v)>0$, $\beta''(v)\leqq 0$ and $f(v)>0$ for $v>0$; $\lim_{v\to\infty}\beta(v)=\infty$; $f\circ\beta^{-1}$ is locally Lipschitz continuous in $[\beta(0), \infty)$.

(A2) $u_0(x)\geqq 0$, $\not\equiv 0$ and $\in B(\boldsymbol{R}^N)$ (bounded continuous in $\boldsymbol{R}^N$).

With these conditions the above Cauchy problem has a unique local solution $u(x, t)$ (in time) which satisfies (0.1) in $\boldsymbol{R}^N\times(0, T)$ in the following weak sense (see e.g., Oleinik et al. [**17**]), where $T>0$ is assumed sufficiently small.

DEFINITION 1. Let G be a domain in $\boldsymbol{R}^N$. By a solution of equation (0.1) in $G\times(0, T)$ we mean a function $u(x, t)$ such that

1) $u(x, t)\geqq 0$ in $\bar{G}\times[0, T)$, and $\in B(\bar{G}\times[0, \tau])$ for each $0<\tau<T$.

2) For any bounded domain $\Omega\subset G$, $0<\tau<T$ and nonnegative $\varphi(x, t)\in C^2(\bar{\Omega}\times[0, T))$ which vanishes on the boundary $\partial\Omega$,

$$\begin{aligned}\int_{\Omega}\beta(u(x, \tau))\varphi(x, \tau)dx-\int_{\Omega}\beta(u(x, 0))\varphi(x, 0)dx \\ =\int_0^{\tau}\int_{\Omega}\{\beta(u)\partial_t\varphi+u\Delta\varphi+f(u)\varphi\}dxdt-\int_0^{\tau}\int_{\partial\Omega}u\partial_n\varphi dSdt,\end{aligned} \tag{0.3}$$

where n denotes the outer unit normal to the boundary.

If $u(x, t)$ does not exist globally in time, its existence time T is defined by

(0.4) $$T=\sup\{\tau>0;\ u(x, t) \text{ is bounded in } \boldsymbol{R}^N\times[0, \tau]\}.$$

In this case we say that u is a *blow-up solution* and T is the *blow-up time*.

The main purpose of the present paper is the study of blow-up solutions near the blow-up time. Especially, we are interested in the shape of the *blow-up set* which locates the "hot-spots" at the blow-up time. In addition, since our quasilinear equation (0.1) has a property of finite propagation, there are some interesting subjects such as the regularity of the *interface* and its asymptotic behavior near the blow-up time. These problems have been studied by one of the authors, Suzuki [**18**], in the case $N=1$. This paper will extend some of his results to higher dimensional problems.

To deal with the finite propagation of solutions and the regularity of interfaces, we require the additional conditions

(A3) $\beta(0)=f(0)=0;\ \int_0^1\frac{dv}{\beta(v)}<\infty.$

(A4) Let $D=\{x\in\boldsymbol{R}^N;\ u_0(x)>0\}$. Then D forms in $\boldsymbol{R}^N$ a bounded convex set with smooth boundary ∂D.

We put

(0.5) $$\Omega(t)=\{x\in\boldsymbol{R}^N;\ u(x, t)>0\},\qquad \Gamma(t)=\partial\Omega(t)$$

for $t\in(0, T)$. Then under (A1)$\sim$(A4) we can show that $\Omega(t)$ is bounded and nondecreasing in $t\in[0, T)$ (Theorem 2.4). Moreover, $\Gamma(t)$ is represented by a continuous function ϑ: $\partial D\times[0, T)\to\boldsymbol{R}^N$ like

(0.6) $$\Gamma(t)=\{x=\vartheta(y, t);\ y\in\partial D\},$$

and if $\vartheta(y_0, t_0)\notin D$ for some $(y_0, t_0)\in\partial D\times(0, T)$, $\vartheta(y, t_0)$ is Lipschitz continuous in $y\in\partial D$ in a neighborhood of y_0 (Theorem 2.6).

Note that in the case of the porous medium equation

(0.7) $$\partial_t(u^{1/m})=\Delta u\ (m>1)\quad\text{in}\quad(x, t)\in\boldsymbol{R}^N\times(0, \infty),$$

there are many works studying the interfaces. Among them Caffarelli et al. [**2**] proved that $\vartheta(y, t)$ is Lipschitz continuous in $(y, t)\in\partial D\times(0, \infty)$ in a neighborhood of (y_0, t_0). So the above continuity of $\vartheta(x, t)$ in t is insufficient. However, to obtain a stronger regularity in t, as the Barenblatt solutions of (0.7) have played an important role in [**2**], it seems necessary to know suitable exact solutions of (0.1) whose space-time structure reflects the most important properties of general solutions.

Next, we restrict ourselves to the blow-up solution of (0.1), (0.2) assuming that u_0 satisfies the additional condition (A4)′ which will be given in the next section as a condition slightly weaker than (A4).

The determination of blow-up solutions has been discussed in Galaktionov et al. [**10**] for equation (0.1) with power nonlinearities

$$\partial_t(u^{1/m}) = \Delta u + u^{p/m} \quad \text{in} \quad (x, t) \in \boldsymbol{R}^N \times (0, T), \tag{0.8}$$

where $m>1$. It has been shown that for $1<p<m+2/N$ any nontrivial solution of (0.8), (0.2) blows-up in finite time, and for $p>m+2/N$ we may find global solutions. These correspond to Fujita's classical results ([**6**]) concerning the semilinear equation (0.8) with $m=1$ (cf., also Levine et al. [**16**]). Blow-up conditions have been studied in Itaya [**13**], [**14**] and Imai-Mochizuki [**11**] for general nonlinear equation (0.1) in a bounded domain, and the following condition is given in [**11**] as a "necessary" condition to raise a blow-up.

(A5) $\displaystyle\int_1^\infty \frac{\beta'(v)}{f(v)} dv < \infty.$

In this paper we require also (A5) and classify the blow-up solutions by the following three conditions.

(A6) (sublinear case) $f(v) = o(v)$ as $v \to \infty$.

(A7) (asymptotic linear case) There exist γ, $C>0$ such that

$$f(v) \leqq \gamma v + C \qquad \text{for each} \quad v > 0.$$

(A8) (superlinear case) There exists a function $\Phi(v)$ such that

(i) $\Phi(v) > 0$, $\Phi'(v) > 0$ and $\Phi''(v) \geqq 0$ for $v > 0$;

(ii) $\displaystyle\int_1^\infty \frac{dv}{\Phi(v)} < \infty$;

(iii) there are constants $c>0$ and $v_0>0$ such that

$$f'(v)\Phi(v) - f(v)\Phi'(v) \geqq c\Phi(v)\Phi'(v) \qquad \text{for} \quad v > v_0.$$

Remark 1. (0.8) satisfies (A1), (A3) and (A6) if $m>1$ and $p>1$, and satisfies (A6) (or (A7)) if $1<p<m$ (or $1<p\leqq m$). (A8) is originally introduced in Friedman-McLeod [**5**] to study the shape of blow-up set for semilinear parabolic equations. (0.8) satisfies (A8) if $p>m$. In this case we can choose $\Phi(v)=v^{\delta p/m}$, where δ is any constant satifying $0<\delta<1$ and $\delta p/m>1$.

Definition 2. The blow-up set of u is defined as

$$S = \{x \in \boldsymbol{R}^N ;\ \text{there is a sequence } (x_i, t_i) \in \boldsymbol{R}^N \times (0, T) \text{ such that } x_i \to x,\ t_i \to T \text{ and } u(x_i, t_i) \to \infty \text{ as } i \to \infty\},$$

and each $x \in S$ is called a blow-up point of u.

The shape of the blow-up set will be very different to each other in the above three cases. Roughly speaking, under (A6), $S=\boldsymbol{R}^N$ (Theorem 3.2), and under (A7), it includes some ball whose radius depends only on γ (Theorem 3.3). On the other hand, under (A8), S is included in a domain depending only on the shape of the initial data $u_0(x)$ (Theorem 4.3). Especially, if $u_0=u_0(r)$, $r=|x|$, and is decreasing in r, then the so called single point blow-up occurs, i.e., $S=\{0\}$ (Corollary 4.4).

The above conditions also distinguish the asymptotic behavior of the interface near the blow-up time. We shall show that under (A6), $\Omega(t)$ grows to the whole $\boldsymbol{R}^N$ (Theorem 3.2), and under (A8), it remains bounded (Theorem 4.3) as $t\uparrow T$.

In the case of (A7), we have no results on the last problem. A very special equation (0.8) with $N=1$ and $m=p$ has been studied in Galaktionov [**9**], and the boundedness of $\Omega(t)$ is shown there by use of an exact solution to (0.8). A corresponding result to Theorem 3.2 has been proved also by Galaktionov [**8**] for the case $N=1$, where each blow-up solution is compared with a family of steady-state solutions to (0.1). It has been shown in [**18**] that S forms a finite set under (A8) if $u_0(x)$ is "sufficiently smooth" and $N=1$. In our higher dimensional problem, it remains unsolved to determine S more strictly to the case of (A8). Radially symmetric solutions are exceptional as will be shown in Corollary 4.4.

The methods used in this paper are essentially the same to those of [**18**]. Namely, we are based on three (smoothness, comparison and reflection) principles, and the main proof is done by reduction to absurdity. The three principles are summarized in the next §1. In §2 we first prepare two lemmas to show Theorem 2.4. These lemmas are also used in showing Theorems 3.2, 3.3 and 4.3 in §§3 and 4, respectively. To show Theorem 2.6 we shall use some results which follow from the reflection principle. The main parts of Theorems 3.2, 3.3 and 4.3 are proved by contradiction. To do so, for the first two theorems, a nonblow-up result for the Dirichlet problem in a bounded domain plays a key role. On the other hand, for the last theorem, we can use the argument of Friedman-McLeod [**5**] (cf., also Chen-Matano [**3**], Chen [**4**] and Fujita-Chen [**7**]). Note that in this case we need some stronger restrictions on u_0.

§1. A comparison principle and monotonicity of solutions.

In this section we begin with two propositions which will be fundamental tools in our study of the interfaces and the blow-up sets.

PROPOSITION 1.1. (*Smoothness principle*). *Let u be a solution of* (0.1) *in $G\times(0, T)$ in the sense of Definition* 1. *If $u(\bar{x}, \bar{t})>0$ for some $(\bar{x}, \bar{t})\in G\times(0, T)$,* *then u is a classical solution in a neighborhood W of $(\bar{x}, \bar{t})$, and hence $u \in C^\infty(W)$.*

PROOF. Note that $\beta(v)$, $f(v) \in C^\infty(\boldsymbol{R}_+)$ and $\beta(v)>0$ for $v>0$. Then the above proposition follows from the usual parabolic regularization method (see e.g. Ladyzenskaja et al. [**15**]). $\square$

For each domain $G \subset \boldsymbol{R}^N$, a supersolution (or subsolution) of (0.1) in $G \times (0, T)$ is defined by 1), 2) of Definition 1 with equality (0.3) replaced by $\geqq$(or $\leqq$).

PROPOSITION 1.2 (*Comparison principle*). *Let u (or v) be a supersolution (or subsolution) of* (0.1) *in $G \times (0, T)$. If $u \geqq v$ on the parabolic boundary of $G \times (0, T)$, then we have $u \geqq v$ in the whole $\bar{G} \times [0, T)$.*

PROOF. See e.g., Appendix of Bertsch et al. [**1**]. $\square$

In the rest of this section, based on these principles, we shall show some monotonicity properties of solutions to (0.1), (0.2).

For $z \in \boldsymbol{R}^N$ and $\nu \in S^{N-1}$, we put

$$A = A(z, \nu) = \{x \in \boldsymbol{R}^N ;\ \nu\cdot(x-z) = 0\}, \tag{1.1}$$

where "$\cdot$" means the innerproduct in $\boldsymbol{R}^N$. A forms a hyperplane in $\boldsymbol{R}^N$. The upper [or lower] half space of $\boldsymbol{R}^N$ with respect to A is defined as

$$\boldsymbol{R}^N_{A,+} = \{x ;\ \nu\cdot(x-z) > 0\} \ [\text{or } \boldsymbol{R}^N_{A,-} = \{x ;\ \nu\cdot(x-z) < 0\}]. \tag{1.2}$$

For any $x \notin A$, the reflection of x in A is denoted by $\sigma_A x$. Thus, we have for each $\zeta \in A$,

$$\zeta\cdot(\sigma_A x - x) = \frac{1}{2}(\sigma_A x + x)\cdot(\sigma_A x - x). \tag{1.3}$$

For any function v in $\boldsymbol{R}^N$, we define the reflection of v in A as

$$\sigma_A v(x) = v(\sigma_A x), \qquad x \in \boldsymbol{R}^N. \tag{1.4}$$

Let $u = u(x, t)$ be the solution of (0.1), (0.2).

PROPOSITION 1.3 (*Reflection principle*). *For some $A = A(z, \nu)$ suppose that*

$$\sigma_A u_0(x) \geqq u_0(x) \qquad \text{in} \quad \boldsymbol{R}^N_{A,+}. \tag{1.5}$$

Then we have

$$\sigma_A u(x, t) \geqq u(x, t) \text{ in } \boldsymbol{R}^N_{A,+} \times [0, T). \tag{1.6}$$

Moreover, if $u(z, \bar{t}) > 0$ for some $\bar{t} \in (0, T)$, then we have

$$\partial_\nu u(z, \bar{t}) \equiv \nu\cdot\nabla u(z, \bar{t}) < 0. \tag{1.7}$$

PROOF. Since Δ is invariant under the reflection of x in A, we see that

$\sigma_A u$ also satisfies (0.1) in $\bar{\boldsymbol{R}}^N \times (0, T)$. Thus, noting (1.5) and the equality

$$\sigma_A u(x, t) = u(x, t) \quad \text{in} \quad (x, t) \in A \times (0, T),$$

we can use Proposition 1.2 with $G = \boldsymbol{R}^N_{A,+}$ to obtain (1.6).

Next suppose that $u(z, \bar{t}) > 0$. Then by Proposition 1.1, u becomes a classical solution of (0.1) in a neighborhood of $(z, \bar{t})$. Thus, (1.7) follows from the strong maximal principle applied to the function $u(x, t) - \sigma_A u(x, t)$. □

For $R>0$, $0<\gamma<\pi/2$, $z \in \boldsymbol{R}^N$ and $\nu \in S^{N-1}$, we put

$$B(R, z) = \{x \in \boldsymbol{R}^N;\ |x-z| < R\}, \qquad B(R) = B(R, 0), \tag{1.8}$$

$$U(\gamma, \nu) = \{\mu \in S^{N-1};\ \langle \mu, \nu \rangle < \gamma\}, \tag{1.9}$$

where $\langle \mu, \nu \rangle \in [0, \pi]$ means the angle between μ and ν, and

$$V(\gamma, z, \nu) = \left\{x \in \boldsymbol{R}^N;\ \frac{x-z}{|x-z|} \in U(\gamma, \nu)\right\}, \tag{1.10}$$

$$V^*(\gamma, z, \nu) = \{x \in \boldsymbol{R}^N;\ \mu \cdot (x-z) > 0 \text{ for any } \mu \in \bar{U}(\gamma, \nu)\}. \tag{1.11}$$

By definition

$$V^*(\gamma, z, \nu) = V\left(\frac{\pi}{2} - \gamma, z, \nu\right). \tag{1.12}$$

In the following of this section we require

(A4)′ There exists a bounded convex domain $D \subset \boldsymbol{R}^N$ with smooth boundary ∂D such that we have

$$\sigma_A u_0(x) \geqq u_0(x) \qquad \text{in} \quad \boldsymbol{R}^N_{A,+}$$

for any hyperplane A satisfying $\bar{D} \subset \boldsymbol{R}^N_{A,-}$.

Let $x \notin \bar{D}$. Then there exists a unique $y(x) \in \partial D$ such that

$$|x - y(x)| = \operatorname{dis}(x, \partial D). \tag{1.13}$$

We denote by $\mu(x)$ the direction of the vector $x - y(x)$:

$$\mu(x) = \frac{x - y(x)}{|x - y(x)|}. \tag{1.14}$$

Obviously $\mu(x) = n(y(x))$, where n means the outer unit normal to the boundary. Moreover, we can define $\gamma(x) \in (0, \pi/2)$ to satisfy

$$D \subset V(\gamma(x), x, -\mu(x)), \qquad \partial D \cap \partial V(\gamma(x), x, -\mu(x)) \neq \emptyset. \tag{1.15}$$

LEMMA 1.4. (1) *$y(x)$, $\mu(x)$ and $\gamma(x)$ are all continuous in $x \in \boldsymbol{R}^N \setminus \bar{D}$. $y(x)$ and $\mu(x)$ are extended continuously to $x \in \partial D$ as $y(x) = x$ and $\mu(x) = n(x)$, and $\gamma(x)$ depends only on* $\operatorname{dis}(x, \partial D)$.

(2) *Let $y\in\partial D$. Put $\xi=y+sn(y)$, $\eta=y+s'n(y)$ for some $0<s<s'$, and $\gamma=\min\{\gamma(\xi), \pi/2-\gamma(\xi)\}$, and define*

$$K_-(\gamma, \xi, \eta)=V(\gamma, \xi, n(y))\cap V(\gamma, \eta, -n(y)), \tag{1.16$_-$}$$

$$K_+(\gamma, \xi, \eta)=V(\gamma, \eta, n(y)). \tag{1.16$_+$}$$

Then for any $x\in K_-(\gamma, \xi, \eta)$, we have

$$V^*(\gamma(x), x, \mu(x))\supset K_+(\gamma, \xi, \eta). \tag{1.17}$$

PROOF. (1) is obvious from the definition of these functions.

(2) Since $V(\gamma(x), x, -\mu(x))\subset V(\gamma(\xi), x, -n(y))$, it follows from (1.12) that $V^*(\gamma(x), x, \mu(x))\supset V^*(\gamma(\xi), x, n(y))$. Noting $\gamma\leqq\gamma(\xi)$ and $\eta\in V^*(\gamma(\xi), x, n(y))$, we obtain (1.17). $\square$

LEMMA 1.5. *Let $y\in\partial D$. Then $u(x, t)$, $t\in(0, T)$, is nonincreasing along the ray $\ell(y)=\{x=y+sn(y);\ s>0\}$.*

PROOF. For $0\leqq s<s'$, we put $\xi=y+sn(y)$ and $\eta=y+s'n(y)$. Let $A=A((\xi+\eta)/2, n(y))$. Then obviously $D\subset\boldsymbol{R}^N_{A,-}$, and hence we have $\sigma_A u_0(x)\geqq 0=u_0(x)$ in $\boldsymbol{R}^N_{A,+}$. Since $\eta=\sigma_A\xi\in\boldsymbol{R}^N_{A,+}$, it follows from Proposition 1.3 that

$$\begin{aligned} u(\eta, t) &\leqq \sigma_A u(\eta, t)\\ &= u(\xi, t). \quad \square \end{aligned}$$

LEMMA 1.6. *Let $R>0$ be chosen to satisfy $D\subset B(R)$. Then we have for any $t\in[0, T)$,*

$$\inf_{x\in B(R)} u(x, t)\geqq \sup_{x\notin B(3R)} u(x, t). \tag{1.18}$$

PROOF. Let $\xi\in\bar{B}(R)$ a point to attain

$$u(\xi, t)=\inf\{u(x, t);\ x\in B(R)\}.$$

Let $\eta\notin B(3R)$ and $A=A((\eta+\xi)/2, (\eta-\xi)/|\eta-\xi|)$. Then noting $|\eta+\xi|/2\geqq R$, we have $D\subset\boldsymbol{R}^N_{A,-}$. Thus, it follows from Proposition 1.3 that

$$u(\eta, t)\leqq\sigma_A u(\eta, t)=u(\xi, t). \quad \square$$

LEMMA 1.7. *Let $x\notin\bar{D}$. Then for any $\nu\in U(\pi/2-\gamma(x), \mu(x))$ and $t\in[0, T)$, $u(x+s\nu, t)$ is nonincreasing in $s>0$. Thus, we have*

$$u(x, t)\geqq u(z, t) \qquad \text{for any} \quad z\in V^*(\gamma(x), x, \mu(x)). \tag{1.19}$$

PROOF. For $\nu\in U(\pi/2-\gamma(x), \mu(x))$ put $\xi=x+s\nu$ and $\eta=x+s'\nu$, where $0\leqq s<s'$, and let $A=A((\xi+\eta)/2, \nu)$. Then since $\nu\cdot v<0$ for any $v\in U(\gamma(x), -\mu(x))$, we have $V(\gamma(x), x, -\mu(x))\subset\boldsymbol{R}^N_{A,-}$. Thus, this A divides $\{\xi, D\}$ and η

as in the above proof. Combining this and (1.12), we conclude the assertions. $\square$

LEMMA 1.8. *Let* $\xi \notin \bar{D}$. *If* $u(x, t)>0$ *for some* $(x, t) \in V(\gamma(\xi), \xi, \mu(\xi)) \times (0, T)$, *then*

$$\mu(\xi)\cdot\nabla u(x, t) \leqq -\cos\{\gamma(\xi)\}\,|\nabla u(x, t)|\,. \tag{1.20}$$

PROOF. Note that $A=A(x, \nu)$ with $\nu \in U(\pi/2-\gamma(\xi), \mu(\xi))$ also satisfies the conditions of Proposition 1.3. Then we have $\nu\cdot\nabla u(x, t)<0$ for any $\nu \in U(\pi/2-\gamma(\xi), \mu(\xi))$. Thus, it follows that $\nabla u(x, t)/|\nabla u(x, t)| \in U(\gamma(\xi), -\mu(\xi))$, which implies

$$-\mu(\xi)\cdot\frac{\nabla u(x, t)}{|\nabla u(x, t)|} = \cos\left\langle -\mu(\xi), \frac{\nabla u(x, t)}{|\nabla u(x, t)|}\right\rangle \geqq \cos\gamma(\xi)\,. \quad \square$$

§ 2. Existence and regularity of interfaces.

We shall first show three lemmas which describe a finite propagation of solutions to (0.1), (0.2). All the lemmas are based on the comparison principle.

LEMMA 2.1. *If* $u(\bar{x}, \bar{t})>0$ *for some* $(\bar{x}, \bar{t}) \in \boldsymbol{R}^N \times [0, T)$, *then*

$$u(\bar{x}, t) > \rho(t) > 0 \quad in \quad t \in [\bar{t}, T), \tag{2.1}$$

where $\rho(t)$, $t \geqq \bar{t}$, *solves the initial value problem*

$$\rho' = -\frac{\lambda\rho}{\beta'(\rho)} \quad in\ (\bar{t}, \infty)\ with\ \rho(\bar{t}) \in (0, u(\bar{x}, \bar{t}))\,. \tag{2.2}$$

PROOF. Cf., Lemma 2.3 of [**18**] or [**11**]. Since u is continuous in $\boldsymbol{R}^N \times (0, T)$, for given $\rho(\bar{t})$, there exists a $\delta>0$ such that

$$u(x, t) \geqq \rho(\bar{t}) \quad \text{in} \quad B(\delta, \bar{x}) \times [\bar{t}, \bar{t}+\delta)\,.$$

Let $(\varphi(x), \lambda)$ be the principal eigensolution of $-\Delta$ in $B(\delta, \bar{x})$ with zero Dirichlet condition. We normalize φ to satisfy $0<\varphi(x)\leqq 1$ and $\varphi(\bar{x})=1$. Integrating (2.2), we have

$$\rho(t) = W^{-1}(W(a)-\lambda(t-\bar{t})), \ \text{where}\ W(s) = \int_1^s \frac{\beta'(v)}{v}\,dv\,. \tag{2.3}$$

Note that $\beta'(v)>0$ and $\beta''(v)\leqq 0$ in $v>0$. Then as is easily seen, $W(s)$ is increasing in $s>0$ and $W(s)\to -\infty$ as $s \downarrow 0$. Thus, $\rho(t)>0$ for each $t>\bar{t}$.

Now we put

$$v(x, t) = \varphi(x)\rho(t)\,.$$

Then since $\beta'(\rho)\leqq\beta'(v)$, we see that this v forms a subsolution of (0.1) in $B(\delta, \bar{x}) \times (\bar{t}, T)$. Moreover,

and
$$v(x, \bar{t}) \leqq \rho(\bar{t}) \leqq u(x, \bar{t}) \quad \text{in} \quad B(\delta, \bar{x})$$
$$v(x, t) = 0 \leqq u(x, t) \quad \text{in} \quad \partial B(\delta, \bar{x}) \times (\bar{t}, T).$$

Thus, Proposition 1.2 shows that $u(x, t) \geqq v(x, t)$ in $B(\delta, \bar{x}) \times (\bar{t}, T)$. Especially, $u(\bar{x}, t) \geqq \rho(t) > 0$ in $(\bar{t}, T)$. □

In the following of this section we require (A3) and (A4).

LEMMA 2.2. *Suppose that there exist $R>0$, $\bar{t} \in [0, T)$ and $M>0$ such that*

$$u(x, \bar{t}) = 0 \quad in \quad \boldsymbol{R}^N \setminus B(R), \tag{2.4}$$

$$u(x, t) \leqq M \quad in \quad \partial B(R) \times (\bar{t}, T). \tag{2.5}$$

Then there exist $\ell>0$ and $h \in (0, T-\bar{t})$ depending only on M such that

$$u(x, t) = 0 \quad in \quad [\boldsymbol{R}^N \setminus B(R+\ell)] \times [\bar{t}, \bar{t}+h]. \tag{2.6}$$

PROOF. Cf., Lemma 6.4 of [**18**]. We construct a supersolution of (0.1) in the following form.

$$w(x, t) = \phi^{-1}([\tau(t) - |x| + R]_+), \tag{2.7}$$

where $[g]_+ = \max\{g, 0\}$ and

$$\phi(u) = \int_0^u \frac{dv}{\beta(v)}, \tag{2.8}$$

$$\tau(t) = C(M)(t-\bar{t}) + \phi(M); \; C(M) = 1 + \sup_{0<v\leqq 2M} \frac{f(v)}{\beta(v)\beta'(v)}. \tag{2.9}$$

Note that (2.8) is well defined by (A3). On the other hand, $C(M)<\infty$ by (A1) and (A3). In fact, $\beta'(v)$ is decreasing in $(0, 2M]$ and $f(v)/\beta(v) = f \circ \beta^{-1}(\xi)/\xi$, $\xi = \beta(v)$, is bounded in $(0, 2M]$ since we have $\beta(0)=0$ and $f \circ \beta^{-1}(0)=0$.

Now, in the domain $\{[\boldsymbol{R}^N \setminus B(R)] \times [\bar{t}, \bar{t}+k]\} \cap \{\tau(t) \geqq |x| - R\}$, where $k = C(M)^{-1}\{\phi(2M) - \phi(M)\}$, we have $w = \phi^{-1}(\tau(t) - |x| + R) \leqq \phi^{-1}(\tau(\bar{t}+k)) = 2M$, and hence,

$$\begin{aligned} &\frac{1}{\beta'(w)} \Delta\phi(w) + |\nabla\phi(w)|^2 + \frac{f(w)}{\beta(w)\beta'(w)} \\ &\quad = \frac{-(N-1)}{\beta'(w)|x|} + 1 + \frac{f(w)}{\beta(w)\beta'(w)} \leqq C(M) = \partial_t \phi(w). \end{aligned} \tag{2.10}$$

This w is extended by 0 to the whole $[\boldsymbol{R}^N \setminus B(R)] \times [\bar{t}, \bar{t}+k]$ as a supersolution of (0.1). For this aim we have only to note that $f(0)=0$ and $\partial_{|x|} w(\tau(t)+R, t) = -(\phi^{-1})'(0) = -\beta(0) = 0$.

Moreover, we have

$$w(x, \bar{t}) \geqq 0 = u(x, \bar{t}) \quad \text{on} \quad \boldsymbol{R}^N \setminus B(R),$$
$$w(x, t) = \phi^{-1}(\tau(t))$$
$$\geqq \phi^{-1}(\phi(M)) = M \geqq u(x, t) \quad \text{on} \quad \partial B(R) \times (\bar{t}, T).$$

Thus, Proposition 1.2 implies that

$$w(x, t) \geqq u(x, t) \quad \text{in} \quad [\boldsymbol{R}^N \setminus B(R)] \times [\bar{t}, \bar{t}+k'), \tag{2.11}$$

where $k'=\min\{k, T-\bar{t}\}$.

Choosing $h=k'/2$ and $\ell=\tau(\bar{t}+h)$, we conclude the assertion of the lemma from (2.7) and (2.11). □

LEMMA 2.3. *Suppose that there exist* $(\bar{x}, \bar{t}) \in \boldsymbol{R}^N \times [0, T)$ *and* $R>0$ *such that*

$$u(x, \bar{t}) = 0 \quad \textit{in} \quad B(R, \bar{x}). \tag{2.12}$$

Then for any $\varepsilon>0$ *there exists a* $\delta>0$ *such that*

$$u(x, t) = 0 \quad \textit{in} \quad B(R-\varepsilon, \bar{x}) \times [\bar{t}, \bar{t}+\delta). \tag{2.13}$$

PROOF. Without loss of generality we can assume $\bar{x}=0$. We define $w(x, t)$ as

$$w(x, t) = \phi^{-1}([\tau(t)+|x|-R]_+), \tag{2.7$'$}$$

where ϕ is as given in (2.8) and

$$\tau(t) = c(\varepsilon)(t-\bar{t})+\frac{\varepsilon}{2}\,; \tag{2.9$'$}$$

$$c(\varepsilon) = \frac{N-1}{\beta'(\phi^{-1}(\varepsilon))(R-\varepsilon)}+1+\sup_{0<v\leq\phi^{-1}(\varepsilon)} \frac{f(v)}{\beta(v)\beta'(v)}.$$

Choose $0<\delta<\varepsilon/2c(\varepsilon)$ so small that

$$0 \leqq u(x, t) \leqq \phi^{-1}(\varepsilon/2) \quad \text{on} \quad \partial B(R) \times [\bar{t}, \bar{t}+\delta). \tag{2.14}$$

Then the above w forms a supersolution of (0.1) in $B(R) \times (\bar{t}, \bar{t}+\delta)$. In fact, corresponding to (2.10), we have

$$\frac{1}{\beta'(w)}\Delta\phi(w)+|\nabla\phi(w)|^2+\frac{f(w)}{\beta(w)\beta'(w)}$$
$$=\frac{N-1}{\beta'(w)|x|}+1+\frac{f(w)}{\beta(w)\beta'(w)} \leqq c(\varepsilon) = \partial_t\phi(w).$$

For the above inequality we have used the fact $0 \leqq \phi(w) \leqq \rho(t) \leqq \varepsilon$.

If we note (1.14), the inequality

$$w(x, t) \geqq u(x, t) \tag{2.15}$$

is easily verified on the parabolic boundary, and hence, it holds in the whole domain. Since $w=0$ in $\rho(t)\leqq R-|x|$ and $\varepsilon/2\leqq\rho(t)\leqq\varepsilon$ in $(\bar{t}, \bar{t}+\delta)$, the lemma follows from (2.7)′ and (2.15). □

Now we put

$$\Omega(t)=\{x\in\boldsymbol{R}^N;\ u(x, t)>0\}\quad\text{and}\quad \Gamma(t)=\partial\Omega(t) \tag{2.16}$$

for $t\in(0, T)$. The interface Γ is then given by

$$\Gamma=\bigcup_{0\leq t\leq T}\Gamma(t)\times\{t\}. \tag{2.17}$$

THEOREM 2.4. *Assume* (A1)~(A4). *Let u be any weak solution to* (0.1), (0.2). *Then $\Omega(t)$ forms a bounded set in $\boldsymbol{R}^N$ and is nondecreasing in t:*

$$\Omega(t_1)\subset\Omega(t_2)\qquad if\quad t_1<t_2. \tag{2.18}$$

PROOF. (2.18) directly follows from Lemma 2.1. So, we shall show that $\Omega(t')$ is bounded for any $0<t'<T$. Choose $R>0$ so large that $u(x, 0)=0$ in $\boldsymbol{R}^N\setminus B(R)$ (see (A4)). Let $t'<T'<T$ and

$$M=\sup_{(x,t)\in R^N\times(0,T')}u(x, t)<\infty.$$

Then applying Lemma 2.2 with $\bar{t}=0$ and $T=T'$, we see that there exist $\ell>0$ and $0<h<T'-t'$ such that $u(x, t)=0$ in $[\boldsymbol{R}^N\setminus B(R+\ell)]\times[0, h]$. Note that ℓ and h depend only on M. Thus, repeating the argument n times, where n is chosen to satisfy $(n-1)h<t'\leqq nh$, we conclude that $u(x, t')=0$ in $\boldsymbol{R}^N\setminus B(R+nh)$. □

To show the regularity of the interface we put

$$r=r(y, t)=\sup\{s\in\boldsymbol{R};\ u(y+sn(y), t)>0\} \tag{2.19}$$

for $(y, t)\in\partial D\times(0, T)$. Since $\Omega(t)$ is bounded in $\boldsymbol{R}^N$, it follows that $0\leqq r<\infty$. With this r we define the function $\mathcal{S}$: $\partial D\times[0, T)\to\boldsymbol{R}^N$ as follows;

$$\mathcal{S}(y, t)=y+r(y, t)n(y). \tag{2.20}$$

LEMMA 2.5. *Let $z=y+sn(y)$ for some $0\leqq s<r(y, t)$ (or $s>r(y, t)\geqq 0$). Then $u(x, t)>0$ (or $=0$) in a neighborhood of $x=z$.*

PROOF. If $0\leqq s<r(y, t)$, then $u(z, t)>0$ by Lemma 1.5 and (2.19). Hence the assertion follows from the continuity of u. If $s>r(y, t)\geqq 0$, then for any $\xi=y+\tilde{s}n(y)$, where $r(y, t)<\tilde{s}<s$, we have $z\in V^*(\gamma(\xi), \xi, n(y))$. Since $u(\xi, t)=0$, we have the assertion from Lemma 1.7. □

THEOREM 2.6. *Assume* (A1)~(A4), *and let u be any weak solution to* (0.1), (0.2). (1) *Then $\mathcal{S}$: $\partial D\times[0, T)\to\boldsymbol{R}^N$ is continuous, and $\Gamma(t)$ is represented as*

$$\Gamma(t) = \{x = \mathcal{J}(y, t);\ y \in \partial D\}. \tag{2.21}$$

(2) *For each* $t \geqq 0$, $\mathcal{J}(\cdot, t)\colon \partial D \to \Gamma(t)$ *is bicontinuous.*

(3) *If* $\mathcal{J}(y_0, t_0) \notin \bar{D}$ *for some* $(y_0, t_0) \in \partial D \times (0, T)$, *then* $\mathcal{J}(y, t_0)$ *is Lipschitz continuous in* $y \in \partial D$ *in a neighborhood of* y_0.

PROOF. (1) Firstly, we shall show (2.21). By definition (2.19), $\Gamma(t) \supset \{\mathcal{J}(y, t);\ y \in \partial D\}$. On the other hand, let $z \in \Gamma(t)$. We can write $z = y(z) + s\mu(z)$ for some $s \geqq 0$. Then by means of the above lemma, we see that $s = r(y(z), t)$. Hence, $z = \mathcal{J}(y(z), t)$ and the converse inclusion $\Gamma(t) \subset \{\mathcal{J}(y, t);\ y \in \partial D\}$ holds.

Secondly, we shall show the continuity in $y \in \partial D$ of $\mathcal{J}(y, t)$. Let $y_0 \in \partial D$ and let $\{y_i;\ i = 1, 2, \cdots\} \subset \partial D$ be any sequence which converges to y_0 as $i \to \infty$. Since $\{\mathcal{J}(y_i, t)\}$ is bounded in $\boldsymbol{R}^N$, we can choose a convergent subsequence. For the sake of simplicity, we write this also as $\{\mathcal{J}(y_i, t)\}$. Let

$$z = \lim_{i \to \infty} \mathcal{J}(y_i, t). \tag{2.22}$$

Then from the continuity of $y(x)$ (lemma 1.4 (1)) and the fact that $y(y_i + sn(y_i)) = y_i$ for any $s \geqq 0$, we have

$$y(z) = \lim_{i \to \infty} y(\mathcal{J}(y_i, t)) = \lim_{i \to \infty} y_i = y_0. \tag{2.23}$$

This shows that z lies on the closed ray $\bar{\ell}(y_0)$, i.e., $z = y_0 + sn(y_0)$ for some $s \geqq 0$. Since $z \in \Gamma(t)$ by (2.22), Lemma 2.5 shows that $s = r(y_0, t)$ as above. Hence, $z = \mathcal{J}(y_0, t)$, and we see that any convergent subsequence has the same limit $\mathcal{J}(y_0, t)$. This implies the desired continuity of $\mathcal{J}(y, t)$.

Thirdly, we shall show the continuity of $\mathcal{J}(y, t)$ in $t \in (0, T)$. Note that $r(y, t)$ is nondecreasing in t. If $r(y, t_0) = 0$ for some $t_0 \in (0, T)$, then $r(y, t) = 0$ for any $t \in (0, t_0)$. Thus, to show the left continuity of $r(y, t)$ at $t = t_0$, we can restrict ourselves to the case $r(y, t_0) > 0$. Choose any $0 < s < r(y, t_0)$. Then $u(y + sn(y), t)$ is continuous in t and by Lemma 1.5, $u(y + sn(y), t_0) > 0$. So, if $t < t_0$ is chosen very close to t_0, we have $u(y + sn(y), t) > 0$. Hence,

$$s < r(y, t) \leqq r(y, t_0), \tag{2.24}$$

and the left continuity follows. Next let us show the right continuity of $r(y, t)$ at $t = t_0$. Lemma 2.5 shows that for any $s > r(y, t_0)$ we can choose $\gamma > 0$ such that $u(x, t_0) = 0$ in $x \in B(\gamma, y + sn(y))$. We use here Lemma 2.3. Then for any $\varepsilon \in (0, \gamma)$ there exists a δ such that

$$u(x, t) = 0 \quad \text{in} \quad (x, t) \in B(\gamma - \varepsilon, y + sn(y)) \times (t_0, t_0 + \delta). \tag{2.25}$$

This shows that

$$r(y, t_0) \leqq r(y, t) < s \tag{2.26}$$

if $t>t_0$ is chosen very close to t_0, and the right continuity follows.

Summarizing the second and third results, we can say that $\mathcal{G}(y, t)$ is continuous in $(y, t)\in\partial D\times(0, T)$. In fact, since $r(y, t)$ is nondecreasing in t, we can use the Dini theorem to see its uniform continuity in t. The continuity in (y, t) is then easily follows.

(2) As is already discussed, $y(\cdot)$: $\Gamma(t)\to\partial D$ gives, for fixed any $t\in(0, T)$, the inverse of $\mathcal{G}(\cdot, t)$: $\partial D\to\Gamma(t)$. Since $y(\cdot)$ is continuous, this shows the bicontinuity of $\mathcal{G}(\cdot, t)$.

(3) We put $\eta=\mathcal{G}(y_0, t_0)$, $\xi=y(\eta)+s\mu(\eta)$ for $0<s<|\eta-y(\eta)|$ and $\gamma=\min\{\gamma(\xi), \pi/2-\gamma(\xi)\}$ as given in Lemma 1.4 (2). Since $\mathcal{G}(y, t)$ is continuous in y, the Lipschitz continuity at $y=y_0$ follows if we can prove

$$\mathcal{G}(y, t)\in \boldsymbol{R}^N\setminus\{K_+(\gamma, \xi, \eta)\cup K_-(\gamma, \xi, \eta)\}, \tag{2.28}$$

where $K_\pm(\gamma, \eta)$ are as given by $(1.16)_\pm$, when $y\in\partial D$ moves in a neighborhood of y_0. From Lemmas 1.4 (2) and 1.7 and by reduction to absurdity, it follows that $u(x, t)>0$ in $K_-(\gamma, \eta)$. On the other hand, since $u(\eta, t)=0$, it also follows from Lemma 1.7 that $u(x, t)=0$ in $K_+(\gamma, \xi, \eta)$. Thus, (2.28) is proved. □

§ 3. Blow-up sets and asymptotic behavior of interfaces 1.

In this sections we treat the case of (A4)′, (A5) and (A6), or (A4)′, (A5) and (A7). In this case a nonblow-up result to the corresponding Dirichlet boundary value problem in a bounded domain will play a key role to investigate the blow-up set.

Let $\Omega\subset\boldsymbol{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the initial-boundary value problem

$$\partial_t\beta(u)=\Delta u+f(u) \quad \text{in} \quad (x, t)\in\Omega\times(0, \infty), \tag{3.1}$$

$$u(x, t)=b(x, t) \quad \text{on} \quad (x, t)\in\partial\Omega\times(0, \infty), \tag{3.2}$$

$$u(x, 0)=u_0(x) \quad \text{in} \quad x\in\Omega. \tag{3.3}$$

Here $\beta(v)$, $f(v)$ and $u_0(x)$ satisfy (A1) and (A2), and equation (3.1) is considered in the weak sense as in Definition 1.

Assume further the following

(A6)′ There exists a $\gamma\in(0, \lambda)$ and $C>0$ such that

$$f(v)\leqq\gamma v+C \quad \text{for} \quad v\geqq 0,$$

where $\lambda>0$ is the principal eigenvalue of $-\Delta$ in Ω with zero Dirichlet condition.

(A9) $g(x, t)\in C(\partial\Omega\times(0, T))$; $0\leqq g(x, t)\leqq M$; $g(x, 0)=u_0(x)$.

LEMMA 3.1. *Under these conditions the above problem* (3.1), (3.2), (3.3) *has a unique global solution* $u=u(x, t)$, *which is uniformly bounded in* $\Omega\times(0, \infty)$.

REMARK 3.1. Cf. [11] and [12]. In [11] an energy method has been used to obtain similar results. In [12] the above result has been applied to blow-up problems for radially symmetric solutions in a ball of $\boldsymbol{R}^N$.

PROOF. We choose $\tilde{\Omega} \supset\supset \Omega$ so that the principal eigenvalue of $-\Delta$ in $\tilde{\Omega}$ equals to γ. Let $v(x)>0$ be a corresponding eigenfunction, and put

$$v_h(x) = hv(x) - \frac{C}{\gamma} \qquad \text{for} \quad h > 0 . \tag{3.4}$$

If h is sufficiently large, we have

$$v_h(x) \geqq M \quad \text{in} \quad \partial\Omega, \quad \text{and} \quad \geqq u_0(x) \quad \text{in} \quad \Omega . \tag{3.5}$$

Furthermore, it follows from (A6)′ that

$$-\Delta v_h = -h\Delta v = \gamma\left(hv - \frac{C}{\gamma}\right) + C = \gamma v_h + C \geqq f(v_h).$$

Thus, v_h is a supersolution of (3.1). Noting (3.5) and Proposition 1.2, we have the a-priori estimate

$$v_h(x) \geqq u(x, t) \qquad \text{for any} \quad t \geqq 0 ,$$

which ensures simultaneously the global existence and the boundedness of solutions. □

THEOREM 3.2. *Assume* (A1), (A2), (A4)′, (A5) *and* (A6). *Let u be a blow-up solution of* (0.1), (0.2). (1) *Then*

$$S = \boldsymbol{R}^N , \tag{3.6}$$

and u blows up uniformly in each compact set K of $\boldsymbol{R}^N$:

$$\lim_{t \uparrow T} \inf_{x \in K} u(x, t) = \infty . \tag{3.7}$$

(2) *Assume further* (A3) *and* (A4). *Then the support $\bar{\Omega}(t)$ of $u(t)$ grows to $\boldsymbol{R}^N$ as $t \uparrow T$, in other words,*

$$\lim_{t \uparrow T} \inf_{y \in \partial D} |\mathcal{J}(y, t)| = \infty . \tag{3.8}$$

PROOF. (1) First we shall show that $S=\boldsymbol{R}^N$. Assume contrary that there exists a point $z\in \boldsymbol{R}^N$ and $M>0$ such that

$$u(z, t) \leqq M \qquad \text{in} \quad t \in (0, T). \tag{3.9}$$

We choose $R>0$ so large that both D in (A4)′ and this z are contained in the ball $B(R)$. By Lemma 1.6

$$\sup_{x\notin B(3R)} u(x, t) \leqq \inf_{x\in B(R)} u(x, t) \leqq u(z, t) \leqq M, \tag{3.10}$$

Thus, if we consider this u as a solution of (3.1), (3.2), (3.3) with $\Omega=B(4R)$, then $g(x, t)\equiv u(x, t)$ on $\partial\Omega\times(0, \infty)$ satisfies (A9). Since (A6) is a stronger condition than (A6)$'$, we can apply Lemma 3.1 to show the boundedness of u in $B(4R)\times(0, T)$. This and (3.10) imply that u is nonblow-up, and a contradiction occurs.

Let us next prove (3.7). Choose $R>0$ so large that $K\subset B(R)$. Let $z\in \boldsymbol{R}^N\setminus B(3R)$. Then since $z\in S$, there exists a sequence $(x_i, t_i)\in\{\boldsymbol{R}^N\setminus B(3R)\}\times(0, T)$ such that

$$x_i \longrightarrow z,\ t_i\uparrow T \quad \text{and} \quad a_i\equiv u(x_i, t_i)\longrightarrow\infty \quad \text{as} \quad i\longrightarrow\infty.$$

Then by means of Lemma 1.6 we have

$$\inf_{x\in B(R)} u(x, t_i) \geqq a_i \longrightarrow \infty \quad \text{as} \quad i\longrightarrow\infty. \tag{3.11}$$

Let $\rho_i(t)$ be the solution of (2.2) with $\rho_i(t_i)=a_i$. Then by (2.3)

$$\rho_i(t) \geqq W^{-1}(W(a_i)-\lambda(T-t_i)) \qquad \text{for} \quad t\in[t_i, T). \tag{3.12}$$

Since $W(a_i)-\lambda(T-t_i)\uparrow\int_1^\infty \frac{\beta'(v)}{v}dv$ as $i\to\infty$, from this it follows that $\rho_i(t)\to\infty$ as $i\to\infty$. Thus, by (2.1) and (3.11) we have (3.7).

(2) As is proved in Theorem 2.4, for each $t\in(0, T)$, $\Omega(t)$ determines a bounded set in $\boldsymbol{R}^N$ which is nondecreasing in t. Since $S=\boldsymbol{R}^N$, the assertion is obvious. □

THEOREM 3.3. *Assume* (A1), (A2), (A4)$'$, (A5) *and* (A7). *Let u be a blow-up solution of* (0.1), (0.2). *We choose $R_\gamma>0$ so that γ is the principal eigenvalue of $-\Delta$ in $B(3R_\gamma)$ with zero Dirichlet condition. Suppose that D in* (A4)$'$ *is included in $B(R_\gamma)$. Then we have*

$$S\supset B(R_\gamma), \tag{3.13}$$

and u blows up uniformly in each compact set of $B(R_\gamma)$.

PROOF. By (A4)$'$ and Lemma 1.5 we see that D includes blow-up points. Suppose that some $z\in B(R_\gamma)$ does not belong to S. Let $R=|z|<R_\gamma$ (without loss of generality we can assume $z\neq0$). Then by Lemma 1.6 we have

$$\sup_{x\notin B(3R)} u(x, t)\leqq u(z, t)\leqq C<\infty \qquad \text{for any} \quad t\in(0, T).$$

Thus, if u is considered as the solution of (3.1), (3.2), (3.3) with $\Omega=B(3R)$, it satisfies (A9). Moreover, since $R<R_\gamma$, (A6)$'$ also follows from (A7). Then Lemma 3.1 shows that this u remains bounded in $\Omega\times(0, T)$, which contradicts the fact that $D\cap S\neq\emptyset$.

The remainder of the theorem can be proved by the same argument of Theorem 3.2. □

§4. Blow-up sets and asymptotic behavior of interfaces 2.

In this section we treat the case (A4)′, (A5) and (A8), requiring the following additional condition on u_0.

(A10) $\Delta u_0(x)+f(u_0(x))\geqq 0$ in the distribution sense in $\boldsymbol{R}^N$.

We begin with a lemma, which extends the corresponding results for semilinear equations due to [**5**] (see also [**3**], [**4**] and [**7**]). In one dimensional quasilinear problems, similar results have already been proved in [**11**] and [**18**].

LEMMA 4.1. *Assume* (A4)′, (A5) *and* (A8). *Let* $\Omega\subset\boldsymbol{R}^N$ *be a domain and let* $u>0$ *be a solution of* (0.1) *in* $\Omega\times(0, T)$. *If*

$$\partial_t u(x, t)\geqq 0 \qquad in \quad \Omega\times(0, T), \tag{4.1}$$

and if there exists $\nu\in S^{N-1}$ *and* $\delta>0$ *such that*

$$\nu\cdot\nabla u(x, t)\leqq -\delta|\nabla u(x, t)|<0 \qquad in \quad \Omega\times(0, T), \tag{4.2}$$

then u *does not uniformly blow-up in* Ω:

$$\inf_{x\in\Omega} u(x, t)\leqq M<\infty \qquad in \quad t\in(0, T). \tag{4.3}$$

PROOF. We shall show this lemma also by contradiction assuming

$$\lim_{t\uparrow T}\inf_{x\in\Omega} u(x, t)=\infty. \tag{4.4}$$

For $a=(a_1, \cdots, a_N)\in\Omega$ we choose $\alpha>0$ so small that the rectangular domain

$$\omega(\alpha)=\{x=(x_1, \cdots, x_N)\in\boldsymbol{R}^N;\ \alpha_j<x_j<\alpha_j+\alpha,\ j=1, \cdots, N\} \tag{4.5}$$

is included in Ω. If we choose $\tau\in(0, T)$ very close to T, then since u is assumed to blow-up uniformly in Ω, we see that it is a C^∞-classical solution of (0.1) in $\omega(a)\times(\tau, T)$.

Without loss of generality we can choose $\nu=(-1, 0, \cdots, 0)$ in (4.2). It then follows that

$$0<|\nabla u|\leqq\delta^{-1}\partial_1 u, \quad \text{where} \quad \partial_1=\partial/\partial x_1, \qquad \text{in} \quad \overline{\omega(a)}\times(0, T). \tag{4.6}$$

We put

$$J(x, t)=\partial_1 u(x, t)-\rho(x)\Phi(u(x, t)), \tag{4.7}$$

where $\rho(x)=\varepsilon\prod_{k=1}^N\sin\pi(x_k-a_k)/\alpha$ and Φ is as given in (A8). By a direct calculation J is shown to satisfy the equation

$$\partial_t(\beta' J)-\Delta J=\rho\{f'\Phi-\Phi' f-N(\pi/\alpha)^2\Phi\}$$
$$+2\Phi'\nabla\rho\cdot\nabla u-\rho\Phi\beta''\partial_t u+\rho\Phi''|\nabla u|^2+f'J \tag{4.8}$$

in $\omega(a)\times(\tau, T)$. Here $\beta''\partial_t u\leqq 0$ by (A1) and (4.1), and

$$|\nabla\rho\cdot\nabla u|\leqq\frac{\varepsilon N\pi}{\alpha}|\nabla u|\leqq\frac{2\varepsilon N\pi}{\alpha\delta}\partial_1 u$$

by (4.6). Thus, we have

$$\partial_t(\beta' J)-\Delta J-\left(f'-\frac{2\varepsilon N\pi}{\alpha\delta}\Phi'\right)J \tag{4.9}$$
$$\geqq\rho\left\{f'\Phi-\Phi' f-N(\pi/\alpha)^2\Phi+\frac{2\varepsilon N\pi}{\alpha\delta}\Phi'\Phi\right\}.$$

Since $\Phi'(v)\to\infty$ as $v\to\infty$ by (A8, ii), choosing τ very close to T, we have $N(\pi/\alpha)^2\Phi'^{-1}\leqq c/2$. We fix such a τ and take $\varepsilon>0$ very small. Then we see that the right side of (4.9) becomes nonnegative:

$$\partial_t(\beta' J)-\Delta J-\left(f'-\frac{2\varepsilon N\pi}{\alpha\delta}\Phi'\right)J\geqq 0 \quad \text{in} \quad \omega(a)\times(\tau, T). \tag{4.10}$$

We can make use of the maximal principle to (4.10) to obtain

$$J(x, t)>0 \quad \text{in} \quad \omega(a)\times(\tau, T). \tag{4.11}$$

In fact, the inequality

$$J(x, \tau)=\partial_1 u(x, \tau)-\rho(x)\Phi(u(x, \tau))>0 \quad \text{on} \quad \omega(a)$$

follows from (4.6) since $\varepsilon>0$ has been chosen sufficiently small. Moreover, since $\rho(x)=0$ on $\partial\omega(a)$, we have

$$J(x, t)=\partial_1 u(x, t)>0 \quad \text{on} \quad \partial\omega(a)\times(\tau, T).$$

We rewrite (4.11) as

$$\Phi^{-1}\partial_1 u>\rho(x) \quad \text{in} \quad \omega(a)\times(\tau, T), \tag{4.12}$$

and integrate both sides by x_1 over the interval $(a_1, a_1+\alpha)$. Then we have for any $t\in(\tau, T)$,

$$\int_{u(a_1, x', t)}^{u(a_1+\alpha, x', t)}\Phi(u)^{-1}du>\varepsilon\prod_{k=2}^{N}\sin\frac{\pi}{\alpha}(x_k-a_k)\int_0^\alpha\sin\frac{\pi}{\alpha}x_1 dx_1 \tag{4.13}$$
$$=\frac{2\varepsilon\alpha}{\pi}\prod_{k=2}^{N}\sin\frac{\pi}{\alpha}(x_k-a_k)>0,$$

where $x'=(x_2, \cdots, x_N)$. By (A8, ii) and (4.4), the left side decays 0 as $t\uparrow T$. Hence, a contradiction occurs and the lemma is proved. □

We need one more lemma which ensures (4.1).

LEMMA 4.2. *Assume* (A10), *and Let u be the solution of* (0.1), (0.2). *Then $u(x, t)$ is nondecreasing in t, and in the domain where $u>0$, we have $\partial_t u(x, t)\geqq 0$.*

PROOF. Let $u_i (i=1, 2, \cdots)$ be the solution of (0.1) in $\boldsymbol{R}^N\times(0, T_i)$, where $T_i\uparrow T$ as $i\to\infty$, with initial data $u_{0i}(x)=u_0(x)+(1/i)$. Then by Lemma 2.1 each $u_i>0$ in the whole space-time, and hence by Proposition 1.1 it is a C^∞-classical solution. Under (A10), an argument by the maximal principle shows that $\partial_t u_i>0$ in $\boldsymbol{R}^N\times(0, T_i)$. Take $i\to\infty$. Then u_i converges to u locally uniformly in $\boldsymbol{R}^N\times(0, T)$, and the assertion of the lemma holds. □

THEOREM 4.3. *Assume* (A1), (A2), (A4)′, (A5), (A8) *and* (A10). *Let u be a blow-up solution of* (0.1), (0.2). (1) *Then*

$$S\subset\bar{D}. \tag{4.14}$$

(2) *Assume further* (A3) *and* (A4). *Then the support $\bar{\Omega}(t)$ of $u(t)$ remains bounded as $t\uparrow T$, in other words,*

$$\limsup_{t\uparrow T}\sup_{y\in\partial D}|\vartheta(y, t)|<\infty. \tag{4.15}$$

PROOF. (1) We shall first show (4.14). Suppose the contrary that there exists a point $\zeta\in S\cap\{\boldsymbol{R}^N\setminus\bar{D}\}$. By definition we can choose a sequence $(\zeta_i, t_i)\in\boldsymbol{R}^N\times(0, T)$ such that

$$\zeta_i\longrightarrow\zeta,\ t_i\uparrow T \quad\text{and}\quad u(\zeta_i, t_i)\longrightarrow\infty \quad\text{as}\quad i\longrightarrow\infty. \tag{4.16}$$

Let $\xi=y(\zeta)+s\mu(\zeta)$, $\eta=y(\zeta)+s'\mu(\zeta)$, where $0<s<s'<|\zeta-y(\zeta)|$, and $\gamma=\min\{\gamma(\xi), \pi/2-\gamma(\xi)\}$. Then by Lemmas 1.4 (2) and 1.7, we have for any $x\in K_-(\gamma, \xi, \eta)$ and $z\in K_+(\gamma, \xi, \eta)$, where $K_\pm(\gamma, \xi, \eta)$ are defined by $(1.16)_\pm$,

$$u(x, t)\geqq u(z, t), \quad\text{where}\quad t\in(0, T). \tag{4.17}$$

Since $\zeta\in K_+(\gamma, \xi, \eta)$, combining this and (4.16), we have for any $x\in K_-(\gamma, \xi, \eta)$,

$$u(x, t_i)\geqq u(\zeta_i, t_i)\longrightarrow\infty \quad\text{as}\quad i\longrightarrow\infty. \tag{4.18}$$

By means of Lemma 4.2, u is nondecreasing in t. Thus, (4.18) shows that u blows up uniformly in $K_-(\gamma, \xi, \eta)$.

On the other hand, this result shows that $u(x, t)>0$ in $(x, t)\in\bar{K}_-(\gamma, \xi, \eta)\times[\tau, T)$ if $\tau\in(0, T)$ is sufficiently close to T. Then (4.1) holds from Lemma 4.2 and (4.2) with $\nu=\mu(\xi)$, $\delta=\cos\gamma(\xi)$ and $\Omega=K_-(\gamma, \xi, \eta)$ holds by Lemma 1.8. Thus, Lemma 4.1 shows that $u(x, t)$ never blows up uniformly in $K_-(\gamma, \xi, \eta)$.

This is a contradiction.

(2) We choose $R>0$ so large that $\bar{D}\subset B(R)$. Then since $S\subset\bar{D}$, u is uniformly bounded, say $u\leqq M$, in $\{\boldsymbol{R}^N\setminus B(R)\}\times(0, T)$ (cf. e.g., Lemma 1.5). Noting that $u_0(x)=0$ in $\boldsymbol{R}^N\setminus B(R)$, we see that the conditions of Lemma 2.2 with $\bar{t}=0$

are satisfied by this u. Since $\ell>0$ and $0<h<T$ in (2.6) depend only on M, we can repeat the same argument n times, where n is the least integer satisfying $nh>T$, to conclude $u(x, t)=0$ in $\{\boldsymbol{R}^N\setminus B(R+n\ell)\}\times(0, T)$. Hence, (4.15) holds true. $\square$

COROLLARY 4.4. *Assume* (A1), (A2), (A5), (A8), (A10) *and*

(A4)$''$ $u_0=u_0(r)$, *where* $r=|x|$; $u_0(r)$ *is monotone decreasing in* $r>0$.

Let $u=u(r, t)$ *be a blow-up solution of* (0.1), (0.2). *Then*

$$S=\{0\}. \tag{4.19}$$

PROOF. We fix any $R>0$. For $z\notin B(R)$ put $A=A(z, z/|z|)$. Then by (A4)$''$ we have

$$\sigma_A u_0(r)\geqq u_0(r) \quad \text{in } \boldsymbol{R}^N_{A,+}.$$

Thus, we can apply Theorem 4.3 (1) with $D=B(R)$ to obtain

$$S\subset \bar{D}=\bar{B}(R).$$

$R>0$ being arbitrary, this concludes (4.19). $\square$

References

[1] M. Bertsch, R. Kersner and L. A. Peletie, Positivity versus localization in degenerate diffusion equations, Nonlinear Anal., **9** (1985), 987-1008.

[2] L. A. Caffareli, J. L. Vaquez and N. I. Wolanski, Lipschitz continuity of solutions and interfaces of the N-dimensional porous medium equation, Indiana Univ. Math. J., **36** (1987), 373-400.

[3] X.-Y. Chen and H. Matano, Convergence, asymptotic periodicity and finite-point blow-up in one-dimensional semilinear heat equations, J. Differential Equations, **78** (1989), 160-190.

[4] Y.-G. Chen, Blow-up solutions of a semilinear parabolic equation with Neumann and Robin boundary conditions, Hokkaido Univ. Preprint series in Math., **65** (1989).

[5] A. Friedman and B. McLeod, Blow-up of positive solutions of semilinear heat equations, Indiana Univ. Math. J., **34** (1985), 425-447.

[6] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t=\Delta u+u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo, **13** (1966), 109-124.

[7] H. Fujita and Y.-G. Chen, On the set of blow-up points and asymptotic behaviours of blow-up solutions to a semilinear parabolic equation, Preprint 87-13, Univ. Tokyo.

[8] V. A. Galaktionov, On localization conditions for unbounded solutions of quasilinear parabolic equations, Soviet Math. Dokl., **23** (1982), 775-780.

[9] V. A. Galaktionov, Asymptotic behavior of unbounded solutions of the nonlinear parabolic equation $u_t=(u^\sigma u_x)_x+u^{\sigma+1}$, Differents. Urab., **21** (1985), 1126-1134.

[10] V. A. Galaktionov, S. P. Kurdyumov, A. P. Mikhailov and A. A. Samarskii, Unbounded solutions of the Cauchy problem for the parabolic equation $u_t=\nabla(u^\sigma\nabla u)+u^\beta$, Soviet Phys. Dokl., **25** (1980), 458-459.

[11] T. Imai and K. Mochizuki, On blow-up of solutions for quasilinear degenerate pa-

rabolic equations, Publ. RIMS, Kyoto Univ., to appear.
[12] I. Imai, K. Mochizuki and R. Suzuki, On blow-up sets for the parabolic equation $\partial_t\beta(u)=\Delta u+f(u)$ in a ball, J. Fac. Sci. Shinshu Univ., **25** (1990), 51-58.
[13] N. Itaya, A note on the blowup-nonblowup problems in nonlinear parabolic equations, Proc. Japan Acad., **55** (1979), 241-244.
[14] N. Itaya, On some subjects related to the blowing-up problem in nonlinear parabolic equations, Lecture Notes Numer. Appl. Anal., **2** (1980), 27-38.
[15] O. A. Ladyzenskaja, V. A. Solonikov and N. N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, Transl. Math. Monographs, **23** AMS Providence, R. I., 1968.
[16] H. A. Levine, G. M. Lieberman and P. Meier, On critical exponents for some quasilinear parabolic equations, Math. Mech. Anal. Sci., to appear.
[17] O. A. Oleinik, A. S. Kalashnikov and Chzou Yui-Lin, The Cauchy problem and boundary problems for equations of the type of nonlinear filtration, Izv. Akad. Nauk. SSSR Ser. Math., **22** (1958), 667-704.
[18] R. Suzuki, On blow-up sets and asymptotic behavior of interfaces of one dimensional quasilinear degenerate parabolic equations, Publ. RIMS, Kyoto Univ., to appear.

Kiyoshi MOCHIZUKI
Department of Mathematics
Faculty of Science
Shinshu University
Matsumoto 390
Japan

Ryuichi SUZUKI
Department of Mathematics
Tokyo College of Aertonautic
Engineering
Tokyo 116
Japan