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Space and time regularity for degenerate evolution equations

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Introduction.

Very recently in [9] the authors, by means of multivalued linear operators, have developped a theory which extends the classical results by T. Kato-H. Tanabe, H. Tanabe and P. E. Sobolevskii [19] to degenerate evolution problems of parabolic type, like

$$(P) \qquad \begin{cases} \dfrac{d}{dt}(M(t)u(t))+L(t)u(t)=f(t),\ 0<t\leqq T \\ M(0)u(0)=v_0(=M(0)u_0), \end{cases}$$

where $L(t)$, $M(t)$ are closed linear operators from the complex Banach space X into itself, f is a continuous function from $[0, T]$ into X, v_0 is a given element of X and u is the unknown function.

In spite of the large number of papers and books written on this subject, the approach to (P) via multivalued operators under *parabolicity* assumptions for the operator pencil $zM(t)+L(t)$ seems in fact new to us.

It is shown in [9] that under some hypotheses that one expects from the theory relative to non degenerate equations, (P) has a strict solution on $]0, T]$, (someone should prefer to call it a classical solution) in the sense that $t\rightarrow L(t)u(t)$, $t\rightarrow d(M(t)u(t))/dt$ are continuous from $]0, T]$ into X, the equation in (P) holds, and provided that the initial condition is read as convergence of $M(t)u(t)$ to v_0, with respect to the seminorm $\|M(0)L(0)^{-1};X\|$.

Some examples of application to partial differential equations were given, in particular, when $M(t)$ is the multiplication operator by a non negative function $m(t)$ or $m(t, x)$.

The present paper contains some extensions of results obtained by the first author [5, 6, 7, 8] on regular solutions for (P), where regularity means that if the data have suitable properties in space or in time, then the corresponding solution, or to be more precise, $t\rightarrow d(M(t)u(t))/dt$ has analogous properties; if

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it has the same properties, we also shall say that u has the maximal regularity. Just for this need of smoothness, our results are obtained by means of methods more classic and traditional than the ones in [**9**].

We can subdivide the paper into three parts.

The first one concerns problem (P) when the operators $L(t)$ and $M(t)$ are in fact independent of t and take their values in a reflexive Banach space. Then we are able to extend [**1**] and [**15**] to degenerate equations, obtaining maximal regularity in time and in space when the estimate $\|L(zM+L)^{-1};\mathcal{L}(X)\|$ $\leqq$const holds in the halfplane $\mathrm{Re}\,z\geqq 0$.

If we ask for strict solutions with some other Hölder regularity on the closed interval $[0, T]$, it is useful to introduce, (this is a well known fact in the regular parabolic case [**1, 11**]), some interpolation spaces between the range of T, $R(T)$, $T=ML^{-1}$, and its closure $\overline{R(T)}$ in X, precisely the real interpolation spaces $(\overline{R(T)}, R(T))_{\theta,\infty}$, $0<\theta<1$.

Using these spaces, that we can also characterize with the help of $L(tM+L)^{-1}$, $t>0$, we obtain both strict solutions +time/space regularity *provided that* some assumptions are satisfied on $(1-P)f$ and/or $(1-P)f$ and Lu_0, where $v_0=Mu_0$; here P denotes the projection operator onto the null space $N(T)$ of T.

We can dispense with such a projection if we content us with strict or else classical solutions to (P) on $(0, T]$, because then it is enough that f is Hölder continuous and $v_0\in\overline{R(T)}$, in accordance with [**9**], see Theorem 3. Notice that now the initial condition in (P) is always read in the norm of X.

The extension of these results to the general case deservers some difficulties, since even if we make the rather restrictive hypothesis that $\overline{R(T(t))}\equiv R$, $0\leqq t\leqq T$, where $T(t)=M(t)L(t)^{-1}$, the corresponding projections $P(t)$ depend on t and we should be compelled to make some differentiability assumptions on $P(t)$.

These problems are already pointed out for algebraic-differential equations in finite-dimensional spaces (we refer to the treatment given in [**17**]) and depend upon the fact that the singularities of $T(t)$ may arise either in $t=0$ only or in all a neighbourhood of $t=0$.

Further, the corresponding interpolation spaces $(\overline{R(T(t))}, R(T(t)))_{\theta,\infty}$ may change suddenly from $\overline{R(T(t_0))}$ to $\{0\}$, where $t_0\geqq 0$, for example and this does not allow to use the well-known results on maximal space regularity described in [**1**] or elsewhere. These remarks account for the second part of the paper, where problem (P) is faced and only time regularity is discussed.

However, since we now need no hypothesis of abstract potential type on the operators $T(t)$ for a projection argument, it is possible to weaken the assumptions on $\|L(t)(zM(t)+L(t))^{-1};\mathcal{L}(X)\|$ to an estimate as

$$\|L(t)(zM(t)+L(t))^{-1};\ \mathcal{L}(X)\|\leqq C(1+|z|)^{\alpha},\qquad 0\leqq t\leqq T, \tag{1}$$

with $0\leqq\alpha<1$, for all z's in a sector containing $\mathrm{Re}\,z\geqq 0$.

To handle our problem we shall improve some results of Favini-Plazzi's papers [**7**-**8**], removing, in particular, the restrictive and important condition on the exponent s_2 in Assumption (C) of [**7**, Theorem 3, p. 1023], that had limited the application of those results either to weak degenerations or to problems in Banach spaces less natural than the expected ones.

The last part contains a number of examples and applications. The first 5 examples generalize some equations considered in [**9**] and also establish the corresponding space and time regularity for their solutions according to paragraphs 1 and 2. Such equations have the form

$$\frac{d}{dt}(m(t)u(t)) = -A(t)u(t)+f(t), \qquad m(t) \geqq 0,$$

or

$$\frac{\partial}{\partial t}(m(t, x)u(t, x)) = -A(t, x, D)u(t, x)+f(t, x), \qquad m(t, x) \geqq 0$$

and they were previously considered in a lot of papers. We only quote [**2**, **4**, **10**, **11**, **12**, **13**, **16**], where some types of weak solutions are studied.

These results seem to us new at all. In particular, Example 5 permits to treat degenerate problems in L^p spaces, $p \neq 2$.

In Example 6 we suppose L, M two non negative self adjoint operators in the Hilbert space H satisfying $\langle Lu, Mu\rangle \geqq 0$, $u \in D(L) \subseteqq D(M)$, and we invoke some results from [8] in order to have the maximal regularity estimates for the modified resolvent. The hypothesis on $\langle Lu, Mu\rangle$ entailing parabolicity is to be compared with the general angle condition $\mathrm{Re}\langle Lu, Mu\rangle \geqq -\delta\|Mu\,;\, H\|^2$, $\delta \geqq 0$, or analogous ones given in [**12**, **13**], also for nonlinear equations.

Example 7 points out that one can obtain the maximal regularity estimate and need less regularity on the data if the concrete degenerate partial differential equations are studied in some suitable weight spaces or in Sobolev spaces $H^{-m}(\Omega)$, m a positive integer, instead of L^2, as in [**5**, **6**].

We remark that to our knowledge the type of applications that we have given was previously considered in the context of H^{-m} spaces; on the other hand, this has happened just because only the best exponent $\alpha=0$ in (1) has been looked for and used.

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NOTATION. If E is a Banach space, $\|\,;\, E\|$ denotes its norm. If E, F are two Banach space, $\mathcal{L}(E\,;\, F)$ is the Banach space of bounded linear operators from E into F, with the usual norm; we shall use $\mathcal{L}(E)$ for $\mathcal{L}(E\,;\, E)$.

If $T>0$, $C[0, T\,;\, E]$, $C^\delta[0, T\,;\, E]$, $0<\delta<1$, $C^1[0, T\,;\, E]$ denote the spaces of all E-valued strongly continuous, resp. Hölder-continuous with exponent δ, strongly continuously differentiable functions on $[0, T]$. The norm in

$C^{\delta}[0, T; E]$ is given by

$$\sup_{0\leq t\leq T}\|u(t); E\|+\|u\|_{\delta}, \qquad \|u\|_{\delta}=\sup_{\substack{0\leq t,s\leq T\\ t\neq s}}\|u(t)-u(s); E\||t-s|^{-\delta}.$$

$B[0, T; E]$ denotes the space of bounded functions from $(0, T]$ into E, with the supnorm. u belongs to $C(0, T; E]$ if u is continuous from $(0, T]$ into E and $u\in C^{1}(0, T; E]$ if u is strongly differentiable on $(0, T]$ and $u'\in C(0, T; E]$.

1. The autonomous equation.

Let X be a reflexive Banach space over the complex plane, and suppose T a closed linear operator in X such that the inverse $(sT+1)^{-1}\in\mathcal{L}(X)$ exists for all $s>0$ and

$$\|(sT+1)^{-1}; \mathcal{L}(X)\|\leqq M, \qquad s>0. \tag{2}$$

Then it is well known (see [**20**, p. 218]) that the representation $X=N(T)\oplus\overline{R(T)}$ holds. It suffies, to this end, to define $J_s=T(sT+1)^{-1}$, $s>0$, and to apply Corollary 1' in [**20**, p. 218].

Let X, Y be now two complex Banach spaces and L, M be two closed linear operators from Y into X such that $zM+L$ has a bounded inverse $\forall z$, $\mathrm{Re}\,z\geqq 0$, $D(L)\subseteqq D(M)$, (of course $D(L)$ denotes the domain of L) and

$$\|L(zM+L)^{-1}; \mathcal{L}(X)\|\leqq C, \qquad \mathrm{Re}\,z\geqq 0. \tag{3}$$

It easily follows that an estimate analogous to (3) is satisfied in a sector Σ: $\mathrm{Re}\,z\geqq -a_0-b_0|\mathrm{Im}\,z|$, $a_0, b_0>0$.

If X is reflexive, we immediately deduce that the bounded linear operator $T=ML^{-1}$ satisfies (2).

Denote by P the projection operator onto $N(T)$ defined by

$$Pu=\lim_{n\to\infty}(nT+1)^{-1}u, \qquad u\in X.$$

In the paper [**5**, p. 220] it has been shown that the restriction $\tilde{T}$ of T to $\overline{R(T)}$ has an inverse whose opposite generates an analytic semigroup in $\overline{R(T)}$.

Notice that $D(\tilde{T}^{-1})=R(T)$ is everywhere dense in $\overline{R(T)}$.

We now consider the initial value problem

$$(P_1) \qquad \begin{cases} \dfrac{d}{dt}(Mu(t))=-Lu(t)+f(t), & 0\leqq t\leqq T \\ Mu(0)=Mu_0. \end{cases}$$

Analogously to the definition for the regular situation $M=I$, we introduce

DEFINITION 1. A function $u: [0, T]\to Y$ is a strict solution for (P_1) if (P_1) holds, with $u\in C[0, T; D(L)]$ and $Mu(\cdot)\in C^{1}[0, T; X]$.

In order to treat the maximal regularity property of (P_1), to which we are mainly interested, we define

$$X_\theta=(\overline{D(\tilde{T}^{-1})},\ D(\tilde{T}^{-1}))_{\theta,\infty}=(\overline{R(T)},\ R(T))_{\theta,\infty},\ 0<\theta<1,\ T=ML^{-1}.$$

Known characterizations of these interpolation spaces [**11**] say that $x\in X_\theta$ if and only if $x\in R(T)$ and

$$\sup_{t>0} t^\theta \|\tilde{T}^{-1}(t+\tilde{T}^{-1})^{-1}x\,;\ X\|<+\infty .$$

On the other hand, clearly, the preceding sup coincides with

$$\begin{aligned}\sup_{t>0} t^\theta\|(tT+1)^{-1}x\,;\ X\|&=\sup_{t>0} t^\theta\|(tT+1)^{-1}(1-P)x\,;\ X\|\\ &=\sup_{t>0} t^\theta\|L(tM+L)^{-1}(1-P)x\,;\ X\|.\end{aligned}$$

We establish our first result, extending [**15**] to (P_1).

THEOREM 1. *Assume that* (3) *is satisfied in a reflexive Banach space* X. *Suppose that* $f\in C^\theta[0, T;\ X]$, $0<\theta<1$ *and* $u_0\in D(L)$, *with*

$$(1-P)[f(0)-Lu_0]\in X_\theta .$$

Then there exists a unique strict solution u *of* (P_1) *such that*

$$\frac{d}{dt}(Mu(\cdot)),\quad Lu(\cdot)\in C^\theta[0, T;\ X]\quad and\quad \frac{d}{dt}(Mu(\cdot))\in B[0, T;\ X_\theta].$$

PROOF. Define $u(t)=u_1(t)+u_2(t)$, $0\leqq t\leqq T$, where

$$u_1(t)=u_0-(2\pi i)^{-1}\int_\Gamma z^{-1}e^{zt}(zM+L)^{-1}[Lu_0-f(0)]dz,$$

and u_2 shall be precised in the sequel.

Here Γ is the union of two rays originating from $a_1>a_0$ and angles $\pm\varphi$, $\varphi>\pi/2$, customary in analytic semigroup theory.

It easily follows that

$$\begin{aligned}u_1(t)=&\ u_0-L^{-1}P[Lu_0-f(0)]\\ &-(2\pi i)^{-1}L^{-1}\int_\Gamma z^{-1}e^{zt}(zT+1)^{-1}(1-P)[Lu_0-f(0)]dz,\quad 0\leqq t\leqq T.\end{aligned}$$

Notice that the integral converges also for $t=0$ in view of the regularity imposed on $Lu_0-f(0)$. We also have:

$$\begin{aligned}Mu_1(t)&=Mu_0-(2\pi i)^{-1}\int_\Gamma z^{-2}e^{zt}(zT+1-1)(zT+1)^{-1}(1-P)[Lu_0-f(0)]dz\\ &=Mu_0-t(1-P)[Lu_0-f(0)]+(2\pi i)^{-1}\int_\Gamma z^{-2}e^{zt}(zT+1)^{-1}(1-P)[Lu_0-f(0)]dz,\end{aligned}$$

and

$$d(Mu_1(t))/dt = -(1-P)[Lu_0-f(0)] + (2\pi i)^{-1}\int_\Gamma z^{-1}e^{zt}(zT+1)^{-1}(1-P)[Lu_0-f(0)]dz,$$

$$Lu_1(t) = Lu_0 - P[Lu_0-f(0)] - (2\pi i)^{-1}\int_\Gamma z^{-1}e^{zt}(zT+1)^{-1}(1-P)[Lu_0-f(0)]dz.$$

It follows that

$$d(Mu_1(t))/dt + Lu_1(t) = f(0), \qquad 0 \leqq t \leqq T;$$

further,

$$\begin{aligned} d(Mu_1(t))/dt &= (P-1)[Lu_0-f(0)] \\ &\quad + (2\pi i)^{-1}\int_\Gamma z^{-1}e^{zt}\tilde{T}^{-1}(z+\tilde{T}^{-1})^{-1}(1-P)[Lu_0-f(0)]dz \\ &= -(2\pi i)^{-1}\int_\Gamma e^{zt}(z+\tilde{T}^{-1})^{-1}(1-P)[Lu_0-f(0)]dz; \end{aligned}$$

in view of Sinestrari's results [**15**, Theorem 4.5], we deduce that

$$\sup_{0\leq t\leq T} \|d(Mu_1(t))/dt;\ X_\theta\| < +\infty.$$

Let us define now $u_2(t)$. Precisely

$$u_2(t) = (2\pi i)^{-1}\int_0^t\int_\Gamma e^{z(t-s)}(zM+L)^{-1}[f(s)-f(0)]dz\,ds, \qquad 0 \leqq t \leqq T.$$

In virtue of the meaning of P and the properties of Laplace transform, we have

$$u_2(t) = L^{-1}P[f(t)-f(0)] + L^{-1}\int_0^t (2\pi i)^{-1}\int_\Gamma e^{z(t-s)}\tilde{T}^{-1}(z+\tilde{T}^{-1})^{-1}(1-P)[f(s)-f(0)]dz\,ds;$$

in fact,

$$L^{-1}P[f(t)-f(0)] = (2\pi i)^{-1}\int_0^t\int_\Gamma e^{z(t-s)}L^{-1}(zT+1)^{-1}P[f(s)-f(0)]dz\,ds,$$

and $(zT+1)^{-1}(1-P) = \tilde{T}^{-1}(z+\tilde{T}^{-1})^{-1}(1-P)$.

This implies that

$$Mu_2(t) = (2\pi i)^{-1}\int_0^t\int_\Gamma e^{z(t-s)}(z+\tilde{T}^{-1})^{-1}(1-P)[f(s)-f(0)]dz\,ds.$$

Let $Mu_2(t)=z(t)$. Since f is Hölder-continuous, we know by [**15**, Theorem 4.5] that $t \to d((Mu_2(t))/dt \in C^\theta[0, T;\ \overline{R(T)}] \cap B[0, T;\ X_\theta]$. Further,

$$\frac{d}{dt}(Mu_2(t)) = -\tilde{T}^{-1}z(t) + (1-P)[f(t)-f(0)], \qquad 0 \leqq t \leqq T.$$

On the other hand,

$$Lu_2(t) = P[f(t)-f(0)] + (2\pi i)^{-1}\int_0^t\int_\Gamma e^{z(t-s)}\tilde{T}^{-1}(z+\tilde{T}^{-1})^{-1}(1-P)[f(s)-f(0)]dz\,ds = P[f(t)-f(0)]+\tilde{T}^{-1}z(t), \quad 0 \leqq t \leqq T.$$

Hence

$$\frac{d}{dt}(Mu_2(t)) = -Lu_2(t)+f(t)-f(0), \quad 0 \leqq t \leqq T.$$

The statement is completely proved. #

The results in [**15**] give also existence (and uniqueness) of both strict and classical solutions. Accordingly, we define

DEFINITION 2. A classical solution u of (P_1) is a function u from $(0, T]$ into Y, such that $u \in C(0, T; D(L)]$, $Mu \in C^1(0, T; X]$ and (P_2) holds, where $f \in C[0, T; X]$, $z_0 \in X$ and

$$(P_2) \qquad \begin{cases} \dfrac{d}{dt}(Mu(t)) = -Lu(t)+f(t), & 0 < t \leqq T, \\ \|Mu(t)-z_0;\ X\| \longrightarrow 0 \quad \text{as} \quad t \downarrow 0. \end{cases}$$

Let us describe how it is possible to use the theorems in [**15**] to have the strict solution for (P_1) and the classical solution for (P_2), of course without maximal regularity.

We begin with (P_1). To this end, we transform it into

$$\begin{cases} \dfrac{d}{dt}(Tv(t))+v(t) = f(t), & 0 \leqq t \leqq T, \\ (Tv)(0) = TLu_0 \end{cases} \tag{4}$$

by means of the change of variable $Lu=v$.

Consider then the Cauchy problem in $\overline{R(T)} = Z$,

$$\begin{cases} z'(t)+\tilde{T}^{-1}z(t) = (1-P)f(t), & 0 \leqq t \leqq T, \\ z(0) = \tilde{T}(1-P)Lu_0. \end{cases} \tag{5}$$

If $f \in C^\theta[0, T; X]$, $0<\theta<1$, since $(1-P)f(0)-\tilde{T}^{-1}(\tilde{T}(1-P)Lu_0)=(1-P)[f(0)-Lu_0] \in \overline{D(\tilde{T}^{-1})}=\overline{R(\tilde{T})}=\overline{R(T)}$, (5) has a unique strict solution z [**15**, Theorem 4.5].

Thus we know that $\tilde{T}^{-1}z(\cdot) \in C[0, T; Z]$, $z \in C^1[0, T; Z]$. Define $v(t)$ by means of $(1-P)v(t)=\tilde{T}^{-1}z(t)$, $Pv(t)=Pf(t)$, $0 \leqq t \leqq T$. Then $v(\cdot)$ solves (4) since $z(t)=\tilde{T}(1-P)v(t)=Tv(t)$ and also

$$\frac{d}{dt}(Tv(t)) = z'(t) = -(1-P)v(t)+(1-P)f(t) = -v(t)+f(t), \quad 0 \leqq t \leqq T.$$

Hence, $u(t)=L^{-1}v(t)$ solves the equation in (P_1) and $Mu(0)=Tv(0)=\tilde{T}(1-P)v(0) =\tilde{T}(1-P)Lu_0=TLu_0=Mu_0$.

We conclude

THEOREM 2. *Let X be a reflexive Banach space. Assume* (3) *and* $f\in C^{\theta}[0, T; X]$, $0<\theta<1$, $u_0\in D(L)$. *Then* (P_1) *has a unique strict solution.*

The same reasoning as in the preceding proof, this time by using [**15**, Theorem 4.4] leads to

THEOREM 3. *If X is a reflexive Banach space, the linear operators L, M satisfy* (3), $f\in C^{\theta}[0, T; X]$, $0<\theta<1$, *then* (P_2) *has a unique classical solution u for any* $z_0\in\overline{R(T)}$.

PROOF. The Cauchy problem in the space Z

$$\begin{cases} z'(t)+\tilde{T}^{-1}z(t)=(1-P)f(t), & 0<t\leqq T, \\ \|z(t)-z_0;\ X\|\longrightarrow 0 \quad \text{as} \quad t\downarrow 0, \end{cases} \tag{6}$$

has a unique classical solution z. Let $v(t)=\tilde{T}^{-1}z(t)+Pf(t)$.

Then this $v(\cdot)$ satisfies the equation $d(Tv(t))/dt+v(t)=f(t)$, $0<t\leqq T$, and $Tv(t)=z(t)$ converges to z_0 as $t\to 0$. #

THEOREMS 2 and 3 are interesting since they give existence results without any knowledge of the projection P, which can be difficult to characterize. On the contrary, this cannot be avoided if we seek space regularity. In fact, we have

THEOREM 4. *Assume X a reflexive Banach space; if* (3) *holds for L and M and* $f\in C[0, T; X]$, $(1-P)f\in B[0, T; X_\theta]$, $0<\theta<1$, $z_0\in\overline{R(T)}$, *then* (P_2) *has a unique classical solution.*

PROOF. It suffices to observe that $z_0\in\overline{D(\tilde{T}^{-1})}$ and hence we can apply [**15**, Theorem 5.4] to problem (6). #

THEOREM 5. *Let X be a reflexive Banach space, with* (3) *satisfied by L, M and* $f\in C[0, T; X]$, $(1-P)f\in B[0, T; X_\theta]$, $0<\theta<1$.

If $u_0\in D(L)$, *then* (P_1) *has a unique strict solution u.*

If $(1-P)Lu_0\in X_\theta$ *too, then this solution u satisfies*

$$\frac{d}{dt}(Mu)(\cdot)\in B[0, T; X_\theta],\quad (1-P)Lu\in C^{\theta}[0, T; X]\cap B[0, T; X_\theta].$$

PROOF. We want to apply [**15**, Theorem 5.5]. For this, we observe that in equation (5)

$$\tilde{T}(1-P)Lu_0=z_0\in R(T)=D(\tilde{T}^{-1})$$

and

$$\tilde{T}^{-1}z_0=(1-P)Lu_0\in\overline{R(T)}=\overline{D(\tilde{T}^{-1})}.\quad \#$$

2. The general equation.

In this section we consider problem (P) in its generality, also when the range of $T(0)=M(0)L(0)^{-1}$ may be very small, and then interpolation spaces fail to be useful. In view of this circumstance, we shall use some improvements of [7, Theorems 1-3].

In that paper the equation $BMu+Lu=h$ was studied under the assumptions which we now recall for convenience of the reader.

$(H1)$ B is a closed linear operator from the complex Banach space E into itself such that $\forall z\in \boldsymbol{C}$, $|\pi-\arg z|\leqq\varphi<\pi/2$ or $z=0$, $B-z$ has a bounded inverse $(B-z)^{-1}$ such that $\|(B-z)^{-1};\mathcal{L}(E)\|\leqq C(1+|z|)^{-1}$, for that z's, $D(B)$ being everywhere dense in E.

$(H2)$ L and M are closed linear operators from the complex Banach space F into E, $D(L)\subseteqq D(M)$, L has a bounded inverse, $zM+L$ has a bounded inverse for $|\arg z|\leqq\pi-\varphi+\varepsilon$, $\varepsilon>0$, φ as in $(H1)$, with $\|L(zM+L)^{-1};\mathcal{L}(E)\|\leqq C(1+|z|)^{\alpha}$, where $\alpha\in[0, 1)$.

$(H3)$ Let $T=ML^{-1}$ and $V_{\theta}=(E, D(B))_{\theta,\infty}$, $0<\theta<1$. Let us denote by γ the path in $\boldsymbol{C}$ parametrized by $z=r\exp(\pm i\Phi)$, $r\geqq a_0>0$, $\Phi=\pi-\varphi+\varepsilon/2$, and $z=a_0\exp(i\omega)$, $|\omega|\leqq\Phi$, oriented by $(+\infty)e^{-i\Phi}$ to $(+\infty)e^{i\Phi}$, and by $[B;(zT+1)^{-1}]$ the commutator $B(zT+1)^{-1}-(zT+1)^{-1}B$ on $D(B)$. Then $[B;(zT+1)^{-1}]$ has a bounded extension in $\mathcal{L}(E)$, transforms continuously $D(B)$ into itself, and there are $\sigma\in[0, 1)$, $k\geqq1$ such that

i) $\|[B;(zT+1)^{-1}];\mathcal{L}(E)\|\leqq C(1+|z|)^{\sigma}, \qquad z\in\gamma,$

ii) $\|[B;(zT+1)^{-1}];\mathcal{L}(D(B))\|\leqq C(1+|z|)^{k}, \qquad z\in\gamma.$

We remark that if $[B;[B;(zT+1)^{-1}]]$ exists and has a bounded extension to E for all $z\in\gamma$, with $\|[B;[B;(zT+1)^{-1}]];\mathcal{L}(E)\|\leqq C(1+|z|)^{k}$, $z\in\gamma$, then ii) in $(H3)$ is satisfied. The novelty with respect to [7] is that the exponent k is allowed to be $\geqq1$. This is a crucial fact in order to treat strongly degenerate or singular problems, because then $(H3)$ ii) holds only for k large. We have

THEOREM 6. *Assume $(H1\text{-}3)$, with $0\leqq\alpha<1$, $\alpha<(1-\sigma)(k-\sigma)^{-1}$. Given $\omega\in(\alpha,(1-\sigma)(k-\sigma)^{-1})$, for all $h\in V_{\omega}$ and for every $s>0$ sufficiently large, the equation $(B+s)Mu+Lu=h$ has exactly one solution u such that Lu, $BMu\in V_{\omega-\alpha}$.*

PROOF. Take $\omega>0$ such that $\omega<(1-\sigma)(k-\sigma)^{-1}$. In view of $(H3$, i-ii$)$ and the interpolation property of the spaces V_{θ}, $\|[B;(zT+1)^{-1}];\mathcal{L}(V_{\omega})\|\leqq C(1+|z|)^{\sigma_1}$, where $\sigma_1=\sigma+\omega(k-\sigma)$.

If ω is chosen sufficiently small so that $\sigma_1<1$, we can apply Theorems 1/2 in [7] and deduce the result. #

Hypothesis $H3$ ii) can be weakened to

ii)′ $\|[B;\ (zT+1)^{-1}];\ \mathcal{L}(V_{\varphi})\| \leqq C(1+|z|)^{\delta}, \qquad z\in\gamma,$

where $\varphi\in(0, 1)$ and $\delta\geqq 1$.

In fact, all what we need in proving THEOREM 6 [see also Theorems 1-3 in [7]] is that there are $0<\sigma_1<1$, $0<\theta_0<1$ such that

$$\|[B;\ (zT+1)^{-1}];\ \mathcal{L}(V_{\theta_0})\| \leqq C(1+|z|)^{\sigma_1}, \qquad z\in\gamma.$$

On the other hand, if ii)′ is satisfied, then $[B;\ (zT+1)^{-1}]\in\mathcal{L}(V_{\varphi\rho})$, $0<\rho<1$, and $\|[B;\ (zT+1)^{-1}];\ \mathcal{L}(V_{\varphi\rho})\|\leqq C(1+|z|)^{\sigma+\rho(\delta-\sigma)}$, by interpolation and the reiteration theorem for the real method. If ρ is chosen $<(1-\sigma)(\delta-\sigma)^{-1}$ and accordingly $\theta_0=\varphi\rho<\varphi(1-\sigma)(\delta-\sigma)^{-1}$, $\sigma_1=\sigma+\rho(\delta-\sigma)$, we have what we wanted. Hence,

THEOREM 7. *Assume* (H1-3), *with* ii)′ *instead of* ii), *and suppose* $\alpha<\varphi(1-\sigma)(\delta-\sigma)^{-1}=\alpha^*$. *Then for all* $\omega\in(\alpha, \alpha^*)$, *all* $h\in V_{\omega}$ *and s sufficiently large, the equation* $(B+s)Mu+Lu=h$ *has one and only one solution u such that* Lu, $BMu\in V_{\omega-\alpha}$.

Now let us clarify how THEOREMS 6 and 7 work when applied to abstract differential equations.

We also observe that then, by a change of independent variable argument, it is not restrictive to take $s=0$ in these theorems.

APPLICATION. Let $L(t)$, $M(t)$, $0\leqq t\leqq T$, be two families of closed linear operators from the Banach space X into itself, such that

(K1) $T(t)=M(t)L(t)^{-1}$ is continuous from $[0, T]$ into $\mathcal{L}(X)$ and $L(t)^{-1}$ is continuous from $[0, T]$ into $\mathcal{L}(X)$.

(K2) There is $0\leqq\alpha<1$ such that

$$\|L(t)(zM(t)+L(t))^{-1};\ \mathcal{L}(X)\| \leqq C(1+|z|)^{\alpha},$$
$$z\in\Sigma:\ \mathrm{Re}\, z\geqq -a_0-b_0|\mathrm{Im}\, z|, \qquad a_0, b_0>0.$$

(K3) $D(L(t))=D$ is independent of $t\in[0, T]$.

(K4) For all $f\in D$, $t\to L(t)f$ is twice strongly continuously differentiable on $[0, T]$, and hence $\|L^{(k)}(t)L(t)^{-1};\ \mathcal{L}(X)\|\leqq\mathrm{Const}$, $k=1, 2$, $0\leqq t\leqq T$.

(K5) For all $t\in[0, T]$ $M(t)$ is a non negative bounded linear operator in X, $M(t)f$ is twice strongly continuously differentiable on $[0, T]$ for any $f\in D$, and there are $C>0$, $0<\beta\leqq 1$ such that

$$\|M'(t)f;\ X\| \leqq C\|M(t)^{\beta}f;\ X\|, \qquad f\in D, \qquad 0\leqq t\leqq T.$$

To apply THEOREM 6 we use also [7, Theorem 3] and we choose $E=F=C_0[0, T;X]=\{u\in C[0, T;X];\ u(0)=0\}$, $D(B)=\{u\in C_0^1[0, T;\ X],\ u'(0)=0\}$,

$C_0^1[0, T, X]=\{u\in C^1[0, T, X]\,;\, u(0)=0\}$, $Bu=u'$, $(Tu)(t)=T(t)u(t)=M(t)L(t)^{-1}u(t)$. It is well known that $[B;\,(zT+1)^{-1}]$ and $[B;\,[B;\,(zT+1)^{-1}]]$ are the multiplication operators by $\partial/\partial t(zT(t)+1)^{-1}$ and $\partial^2/\partial t^2(zT(t)+1)^{-1}$, respectively.

With $f\in X$, we calculate

$$\frac{\partial}{\partial t}(zT(t)+1)^{-1}f=-z(zT(t)+1)^{-1}M'(t)L(t)^{-1}(zT(t)+1)^{-1}f$$
$$+L'(t)L(t)^{-1}(zT(t)+1)^{-1}f-(zT(t)+1)^{-1}L'(t)L(t)^{-1}(zT(t)+1)^{-1}f\,.$$

In virtue of $(K5)$ and the moment inequality,

$$\|zM'(t)L(t)^{-1}(zT(t)+1)^{-1}f\,;\,X\|$$
$$\leqq C|z|\,\|T(t)(zT(t)+1)^{-1}f\,;\,X\|^{\beta}\|L(t)^{-1}(zT(t)+1)^{-1}f\,;\,X\|^{1-\beta}$$
$$\leqq C|z|^{1-\beta}(1+|z|)^{\alpha}\|f\,;\,X\|\leqq C(1+|z|)^{1-\beta+\alpha}\|f\,;\,X\|\,.$$

It follows that

$$\left\|\frac{\partial}{\partial t}(zT(t)+1)^{-1}f\,;\,X\right\|\leqq C[(1+|z|)^{1-\beta+2\alpha}+(1+|z|)^{2\alpha}]\|f\,;\,X\|$$
$$\leqq C'(1+|z|)^{1-\beta+2\alpha}\|f\,;\,X\|\,.$$

We also have for all $f\in X$,

$$\frac{\partial^2}{\partial t^2}(zT(t)+1)^{-1}f$$
$$=-z\left\{\frac{\partial}{\partial t}R(z,\,t)\right\}T'(t)R(z,\,t)f-zR(z,\,t)T''(t)R(z,\,t)f-zR(z,\,t)^{-1}T'(t)$$
$$\cdot\left\{\frac{\partial}{\partial t}R(z,\,t)\right\}f\,,$$

where we have used $R(z,\,t)$ for $(zT(t)+1)^{-1}$.

A trivial calculation shows that

$$\|\partial^2R(z,\,t)f/\partial t^2\,;\,X\|\leqq C(1+|z|)^{2+3\alpha-\beta}\|f\,;\,X\|\,.$$

Take $k=2+3\alpha-\beta$ and suppose $0\leqq\sigma=1-\beta+2\alpha<1$, that is, $2\alpha<\beta\leqq 1$. The condition $\alpha<(1-\sigma)(k-\sigma)^{-1}$ in THEOREM 6 reads $\beta>3\alpha+\alpha^2(\geqq 2\alpha)$. Hence we obtain the following result on problem (P), where a strict solution for (P) is defined according

DEFINITION 2. A function $u:\,[0,\,T]\to X$ is a strict solution to (P) if $u(t)\in D(L(t))$ for $t\in[0,\,T]$, $L(\cdot)u(\cdot)\in C[0,\,T\,;\,X]$, $M(\cdot)u(\cdot)\in C^1[0,\,T,\,X]$ and (P) holds.

THEOREM 8. *Assume* $(K1/5)$ *and* $3\alpha+\alpha^2<\beta\leqq 1$. *Then for all* $f\in C^{\theta}[0,\,T,\,X]$, $\alpha<\theta<(\beta-2\alpha)(1+\alpha)^{-1}$, *and* $u_0\in D$, *with*

$$f(0)-(1+T'(0))L(0)u_0 \in M(0)(D) = \{M(0)y\,;\ y\in D\}, \tag{7}$$

there is a unique strict solution u of (P) such that $L(\cdot)u(\cdot)$ *and* $t\to d(M(t)u(t))/dt \in C^{\theta-\alpha}[0, T, X]$.

As a simple example, suppose $-L(t)$ to be for all $0\leqq t\leqq T$ the infinitesimal generator of an analytic semigroup in the Banach space X with a common domain D and satisfying the Tanabe-Sobolevskii conditions [**19**, p. 118] with also the further regularity assumptions in (K4). Let $M(t)$ be the multiplication operator by t^m, where $m\geqq 2$. Then (K2) holds with $\alpha=0$. Since

$$(M'(t)f)(t) = mt^{m-1}f(t), \qquad f\in C[0, T\,;\ X],$$

(K5) is verified if we take $\beta=1-1/m$.

On the other hand, $T'(0)=0$, and thus condition (7) reduces to $f(0)=L(0)u_0$, which says how u_0 has to be chosen so that $t^m u(t)$ vanishes in $t=0$, our actual initial condition in (P).

The assumption $m\geqq 2$ seems to be too restrictive and is due to (H3 ii). In fact it can be refined to $m>1$ if we make use of the following general

THEOREM 9. *Assume* (K1, 2) *and* $T(t)=M(t)L(t)^{-1}$ *have the properties*

$$(K6)\quad \begin{cases} \left\|\dfrac{\partial}{\partial t}(zT(t)+1)^{-1};\ \mathcal{L}(X)\right\| \leqq C(1+|z|)^{\sigma},\ 0\leqq\sigma<1, \\ \qquad z\in\Sigma:\ |\arg z|\leqq \pi-\varphi+\varepsilon,\ \varepsilon>0,\ 0<\varphi<\pi/2,\ 0\leqq t\leqq T \\ \left\|\dfrac{\partial}{\partial t}(zT(t)+1)^{-1}-\dfrac{\partial}{\partial s}(zT(s)+1)^{-1};\ \mathcal{L}(X)\right\| \\ \qquad \leqq C(1+|z|)^{\delta}|t-s|^{\varphi},\ \text{where}\ \ \delta\geqq 1,\ 0<\varphi<1,\ z\in\Sigma. \end{cases}$$

If $\alpha<\varphi(1-\sigma)(\delta-\sigma)^{-1}=\alpha^*$, *then for any* $\omega\in(\alpha, \alpha^*)$, *all* $f\in C^{\omega}[0, T\,;\ X]$, $u_0\in D(L(0))$ *such that* $f(0)-(1+T'(0))L(0)u_0\in R(T(0))$, *there is a unique strict solution u to (P) for which* $L(\cdot)u(\cdot)$ *and* $d(M(\cdot)u(\cdot))/dt\in C^{\omega-\alpha}[0, T\,;\ X]$.

PROOF. In view of [7] and THEOREM 7, it suffices to observe that if $f\in C_0^{\varphi}[0, T, X]=V_{\varphi}$, then by ($K$6)

$$\left\|\left\{\frac{\partial}{\partial t}R(z, t)\right\}f(t)-\left\{\frac{\partial}{\partial t}R(z, s)\right\}f(s);\ X\right\||t-s|^{-\varphi}$$
$$\leqq C(1+|z|)^{\delta}\|f\,;\ V_{\varphi}\|+C'(1+|z|)^{\sigma}\|f\,;\ V_{\varphi}\|\leqq C''(1+|z|)^{\delta}\|f\,;\ V_{\varphi}\|. \quad \#$$

We shall finish to discuss the preceding example in next section.

3. Applications and Examples.

1. Let us consider problem (P) under Hypothesis (K1/4) and, for sake of simplicity, suppose $L(t)=L$. Concerning $M(t)$, we now shall assume that $M(t)$

is positive, once strongly continuously differentiable from $[0, T]$ into $\mathcal{L}(X)$ and

$$(K7)\quad \begin{cases} \|M'(t)-M'(s);\ \mathcal{L}(X)\| \leqq C|t-s|^{\varphi}, & 0<\varphi\leqq 1,\quad 0\leqq t,\ s\leqq T, \\ \|M'(t)f;\ X\| \leqq C\|M(t)^{\beta}f;\ X\|, & f\in D(L),\quad 0\leqq t\leqq T,\quad 0<\beta\leqq 1. \end{cases}$$

From the expression $-zR(z, t)M'(t)L^{-1}R(z, t)$ for $\partial R(z, t)/\partial t$, $R(z, t)=(zT(t)+1)^{-1}$, it follows that

$$\|R(z, t)-R(z, s);\ \mathcal{L}(X)\| \leqq C(1+|z|)^{1-\beta+2\alpha}|t-s|,$$

$$\|\partial R(z, t)/\partial t-\partial R(z, s)/\partial s;\ \mathcal{L}(X)\| \leqq C(1+|z|)^{2-\beta+3\alpha}|t-s|^{\varphi},\quad 0\leqq t,\ s\leqq T,$$

and hence we can apply THEOREM 9 if $2\alpha<\beta\leqq 1$, with $\sigma=1-\beta+2\alpha$ and $\delta=2-\beta+3\alpha$. On the other hand, $(1-\sigma)(\delta-\sigma)^{-1}=(\beta-2\alpha)(1+\alpha)^{-1}$ and hence, if $(K2)$ and $(K7)$ are satisfied, $0\leqq 2\alpha\leqq\beta\leqq 1$, then for any $f\in C^{\omega}[0, T;\ X]$, $\alpha<\omega<\varphi\,(\beta-2\alpha)(1+\alpha)^{-1}$, and any $u_0\in D(L)$ such that $f(0)-(1+M'(0)L^{-1})Lu_0=f(0)-(L+M'(0))u_0$ belongs to the range of $M(0)L^{-1}$, there is a unique strict solution u of (P) with $Lu(\cdot)\in C^{\omega-\alpha}[0, T;\ X]$.

2. Let us consider (P) under the assumptions that $-L(t)$, $0\leqq t\leqq T$, generate holomorphic semigroup (even with non dense domain) in the Banach space X, have a common domain D, for simplicity, and are strongly differentiable on $[0, T]$, with

$$\|L'(t)-L'(s);\ \mathcal{L}(D;\ X)\| \leqq C|t-s|^{\varepsilon},\qquad 0<\varepsilon\leqq 1.$$

Further, there exists $M>0$ such that $\|(L(t)+z)^{-1};\ \mathcal{L}(X)\|\leqq M(1+|z|)^{-1}$, $\operatorname{Re} z\geqq 0$. Let $m(t)\geqq 0$ on $[0, T]$ be a C^1 function satisfying

$$|m'(t)| \leqq Cm(t)^{\eta},\qquad 0<\eta\leqq 1,\qquad C>0,$$

$$|m'(t)-m'(s)| \leqq K|t-s|^{\nu},\qquad 0<\nu\leqq 1,\qquad K>0.$$

These hypotheses imply that in a sector Σ containing $\operatorname{Re} z\geqq 0$ and for any $0\leqq t\leqq T$, the estimates $\|L(t)(zm(t)+L(t))^{-1};\ \mathcal{L}(X)\|\leqq C_1$ hold.

Now, since

$$\begin{aligned}\frac{\partial}{\partial t}R(z, t) = &-zR(z, t)m'(t)L(t)^{-1}R(z, t)+L'(t)L(t)^{-1}R(z, t)\\ &-R(z, t)L'(t)L(t)^{-1}R(z, t),\end{aligned}$$

$$\left\|\frac{\partial}{\partial t}R(z, t);\ \mathcal{L}(X)\right\| \leqq C|z|^{1-\eta},\quad 0\leqq t\leqq T,\quad z\in\Sigma,\quad |z|\ \text{large}.$$

It is not too difficult to see that also the estimates

$$\left\|\frac{\partial}{\partial t}R(z, t)-\frac{\partial}{\partial s}R(z, s);\ \mathcal{L}(X)\right\| \leqq C(1+|z|)^{2-\eta}|t-s|^{\varphi},\quad \varphi=\min\{\varepsilon, \nu\},$$

hold and hence we can apply THEOREM 9 with $\alpha=0$, $\sigma=1-\eta$, $\delta=2-\eta$, $\varphi=\min\{\varepsilon, \nu\}$, so that $\alpha^*=\eta\varphi$.

REMARK. If $m(t)=t^m$, $m>1$, then all preceding assumptions are true, with $\eta=1-1/m$, $\nu=\min\{m-1, 1\}$, because

$$|m'(t)-m'(s)| = m(m-1)\left|\int_s^t y^{m-2}dy\right| = m(m-1)\left|\int_s^t r^{m-2}(t-s)^{m-1}dr\right| \leqq K|t-s|^{m-1}.$$

3. Let $V\subseteqq H\subseteqq V^*$ be given complex Hilbert spaces, with V densely embedded in H. Let $a(u, v)$, $u, v\in V$, $b(x, y)$, $x, y\in H$, two bounded sesquilinear forms such that

$$b(x, y)=\overline{b(y, x)},\ x, y\in H, \qquad b(x, x)\geqq 0 \quad \text{for all} \quad x\in H,$$
$$a(u, u)\geqq \delta\|u\,;\,V\|^2, \qquad u\in V, \qquad \delta>0.$$

Denote by $M\in\mathcal{L}(H)$, $\tilde{L}\in\mathcal{L}(V, V^*)$ the operators induced by these sesquilinear forms; by L we intend the operator in H given by $D(L)=\{u\in V\,;\,\tilde{L}u\in H\}$, $Lu=\tilde{L}u$, $u\in D(L)$.

Let $\langle\ ,\ \rangle$ be the inner product in H. If $(\lambda M+L)u=f$, from

$$\lambda\langle Mu, u\rangle+\langle Lu, u\rangle=\langle f, u\rangle$$

and $|\langle f, u\rangle|\leqq\|u\,;\,H\|\|f\,;\,H\|$, we deduce that there are two positive constants C_0, C_1 such that

$$\operatorname{Re}\lambda\langle Mu, u\rangle+C_0\|u\,;\,V\|^2\leqq\operatorname{Re}\langle f, u\rangle, \tag{8}$$
$$\operatorname{Re}\lambda\langle Mu, u\rangle+C_1\|M^{1/2}u\,;\,H\|^2\leqq\operatorname{Re}\langle f, u\rangle \tag{9}$$
$$|\operatorname{Im}\lambda|\,\|M^{1/2}u\,;\,H\|^2\leqq\|f\,;\,H\|\|u\,;\,H\|.$$

Hence,

$$(2\operatorname{Re}\lambda+|\operatorname{Im}\lambda|+C_1)\|M^{1/2}u\,;\,H\|^2+C_0\|u\,;\,V\|^2\leqq 3\|f\,;\,H\|\|u\,;\,H\|.$$

Take $2\operatorname{Re}\lambda+|\operatorname{Im}\lambda|+C_1\geqq C_2>0$. Then we conclude that for these λ's, $\|u\,;\,V\|\leqq C'\|f\,;\,H\|$.

By (8), (9) one then obtains

$$|\operatorname{Re}\lambda|\,\|M^{1/2}u\,;\,H\|^2\leqq\|f\,;\,H\|\|u\,;\,H\|+C_3\|u\,;\,V\|^2\leqq C_4\|f\,;\,H\|^2.$$

Therefore the estimate

$$|z|\,\|M^{1/2}u\,;\,H\|^2\leqq C\|f\,;\,H\|^2$$

is satisfied in a sector Σ containing $\operatorname{Re}z\geqq 0$ and, since M is bounded, $\|zM(zM+L)^{-1}\,;\,\mathcal{L}(H)\|\leqq C(1+|z|)^{1/2}$, $z\in\Sigma$. THEOREM 9 immediately applies and ensures that for any $f\in C^\omega[0, T\,;\,H]$, $1/2<\omega<1$, $u_0\in D(L)$, with $f(0)-Lu_0=Mu_1$, where $u_1 \in D(L)$, problem (P) has a unique strict solution u such that $Lu(\cdot) \in C^{\omega-1/2}[0, T; H]$.

4. This is a concrete example of initial boundary value problem. Suppose that $m(t, x) \geqq 0$ is a $C^{(1)}$-function on $[0, T] \times [0, 1]$ such that

$$\left|\frac{\partial}{\partial x} m(t, x)\right| \leqq Cm(t, x)^{\rho}, \quad 0 \leqq t \leqq T, \quad 0 \leqq x \leqq 1, \quad C > 0, \quad 0 < \rho \leqq 1, \tag{10}$$

$$\left|\frac{\partial}{\partial t} m(t, x)\right| \leqq Cm(t, x)^{\beta}, \quad 0 \leqq t \leqq T, \quad 0 \leqq x \leqq 1, \quad C > 0, \quad 0 < \beta \leqq 1, \tag{11}$$

$$\left|\frac{\partial}{\partial t} m(t, x) - \frac{\partial}{\partial s} m(s, x)\right| \leqq C|t-s|^{\varphi}, \quad 0 \leqq s, t \leqq T, \quad 0 \leqq x \leqq 1, \quad C > 0, \quad 0 < \varphi \leqq 1. \tag{12}$$

Let L be the operator defined on $L^2(0, 1) = X$ by

$$D(L) = H_0^1(0, 1) \cap H^2(0, 1), \qquad Lu = -u'' = \partial^2 u/\partial x^2.$$

If $M(t)$ denotes the multiplication operator by $m(t, x)$, from $\lambda M(t)u + Lu = f \in L^2(0, 1)$, $\lambda \in \boldsymbol{C}$, we deduce, by taking the inner product of f by $M(t)u$,

$$\lambda \int_{]0,1[} m(t, x)^2 |u(x)|^2 dx + \int_{]0,1[} \frac{\partial m}{\partial x}(t, x) u'(x) \bar{u}(x) dx + \int_{]0,1[} m(t, x) |u'(x)|^2 dx = \int_{]0,1[} m(t, x) f(x) \bar{u}(x) dx$$

and hence, by (10),

$$\operatorname{Re}\lambda \|M(t)u ; X\|^2 + \int_{]0,1[} m(t, x) |u'(x)|^2 dx \leqq \|f ; X\| \|M(t)u ; X\| + C' \|M(t)u ; X\|^{\rho} \|u ; X\|^{1-\rho} \|u' ; X\|,$$

$$|\operatorname{Im}\lambda| \, \|M(t)u ; X\|^2 \leqq \|f ; X\| \|M(t)u ; X\| + C'' \|M(t)u ; X\|^{\rho} \|u ; X\|^{1-\rho} \|u' ; X\|.$$

Further, as in Example 3, $\|u' ; X\| \leqq C\|f ; X\|$.

Thus there is a positive constant c_0 such that

$$(\operatorname{Re}\lambda + |\operatorname{Im}\lambda| + c_0) \|M(t)u ; X\|^2 \leqq \|f ; X\| \|M(t)u ; X\| + C \|M(t)u ; X\|^{\rho} \|f ; X\|^{2-\rho}.$$

Take $\operatorname{Re}\lambda + |\operatorname{Im}\lambda| + c_0 \geqq \varepsilon_0 > 0$. Then

$$|\operatorname{Re}\lambda| \, \|M(t)u ; X\|^2 \leqq \|f ; X\| \|M(t)u ; X\| + C \|M(t)u ; X\|^{\rho} \|u ; X\|^{1-\rho} \|u' ; X\|$$

and also

$$|\lambda|\,\|M(t)u,\ X\|^{2-\rho} \leqq C\|f;\ X\|^{2-\rho},$$

for any λ's in the sector Σ that we have defined.

We conclude that for each $\lambda\in\Sigma$ one has

$$\|\lambda M(t)(\lambda M(t)+L)^{-1};\ \mathcal{L}(X)\| \leqq C(1+|\lambda|)^{\alpha}, \qquad \alpha=(1-\rho)(2-\rho)^{-1}(<1/2).$$

Assumptions (11), (12) allow to estimate suitably $\|(\partial/\partial t)R(z, t);\ \mathcal{L}(X)\|$ and $\|(\partial/\partial t)R(z, t)-(\partial/\partial s)R(z, s);\ \mathcal{L}(X)\|$ as in Application 1 provided that $\rho>2(1-\beta)(2-\beta)^{-1}$. Therefore we obtain a strict solution u to (P), with $t\to(\partial/\partial t)(m(t, \cdot)u(t, \cdot))\in C^{\omega-\alpha}[0, T;\ L^2(0, 1)]$ if $f\in C^{\omega}[0, T, X]$, $\alpha<\omega<\varphi(\rho(2-\beta)-2(1-\beta))(3-2\rho)^{-1}$, $u_0\in H_0^1(0, 1)\cap H^2(0, 1)$, $f(0, x)+u_0''(x)-(\partial/\partial t)m(0, x)u_0(x)=m(0, x)u_1(x)$, $u_1\in D(L)$.

REMARK. Let $m(t, x)=t^q m(x)$, $m(x)\geqq 0$, $m\in C^1[0, 1]$, $q>1$, $0\leqq t\leqq T$. If

$$|m'(x)| \leqq Cm(x)^{\rho}, \qquad 0<\rho\leqq 1, \qquad x\in[0, 1],$$

then (10) is verified. Since

$$\frac{\partial}{\partial t}m(t, x) = qm(t, x)^{1-1/q}m(x)^{1/q},$$

(11) holds with $\beta=1-1/q$.

Also (12) is satisfied if one takes $\varphi=\min\{q-1, 1\}$. Therefore to use the preceding result it suffices to suppose $2(q+1)^{-1}<\rho\leqq 1$, since $\alpha=(1-\rho)(2-\rho)^{-1}$.

5. Example 4 can be considered also in the space $L^p(0, 1)$, $p\neq 2$. For sake of simplicity, we confine ourselves to the case in which $m(t, x)=m(x)$ is in fact independent of t and the operator L is defined by $D(L)=W^{2,p}(0, 1)\cap W_0^{1,p}(0, 1)$, $p>1$, $(Lu)(x)=-u''(x)+c_0u(x)$, $c_0>0$, $x\in(0, 1)$. This time $X=L^p(0,1)$ with the usual norm. If M denotes the multiplication operator by $m(\cdot)$, and $\lambda Mu+Lu=f\in X$, one obviously has

$$\begin{aligned}
&\operatorname{Re}\lambda\int_{]0,1[}m(x)|u(x)|^p dx-\operatorname{Re}\int_{]0,1[}u''(x)\bar{u}(x)|u(x)|^{p-2}dx+c_0\|u;\ X\|^p\\
&=\operatorname{Re}\int_{]0,1[}f(x)\bar{u}(x)|u(x)|^{p-2}dx,\\
&\operatorname{Im}\lambda\int_{[0,1[}m(x)|u(x)|^p dx-\operatorname{Im}\int_{]0,1[}u''(x)\bar{u}(x)|u(x)|^{p-2}dx\\
&=\operatorname{Im}\int_{]0,1[}f(x)\bar{u}(x)|u(x)|^{p-2}dx.
\end{aligned} \tag{13}$$

Now, it is well known [18, pp. 310-311] that

$$\begin{aligned}
&\left|\operatorname{Im}\int_{]0,1[}u''(x)\bar{u}(x)|u(x)|^{p-2}dx\right| \leqq C\int_{]0,1[}|u'(x)|^2|u(x)|^{p-2}dx\\
&\leqq C_1\operatorname{Re}\Big(-\int_{]0,1[}u''(x)\bar{u}(x)|u(x)|^{p-2}dx\Big).
\end{aligned}$$

Hence,

$$|\mathrm{Im}\lambda|\int_{]0,1[}m(x)|u(x)|^p dx \leqq \left|\int_{]1,0[}f(x)\bar{u}(x)|u(x)|^{p-2}dx\right| + C_1 \mathrm{Re}\left(-\int_{]0,1[}u''(x)\bar{u}(x)|u(x)|^{p-2}dx\right).$$

It follows that

$$((C+1)\mathrm{Re}\lambda+|\mathrm{Im}\lambda|)\|M^{1/p}u;\ X\|^p + (C+1)\mathrm{Re}\left(-\int_{]0,1[}u''(x)\bar{u}(x)|u(x)|^{p-2}dx\right)+c_0(C+1)\|u;\ X\|^p$$
$$-C\,\mathrm{Re}\left(-\int_{]0,1[}u''(x)\bar{u}(x)|u(x)|^{p-2}dx\right)$$
$$\leqq (C+1)\mathrm{Re}\int_{]0,1[}f(x)\bar{u}(x)|u(x)|^{p-2}dx+\|f;\ X\|\|u;\ X\|^{p/p'},$$

where $1/p+1/p'=1$.

Therefore

$$((C+1)\mathrm{Re}\lambda+|\mathrm{Im}\lambda|)\|M^{1/p}u;\ X\|^p+\mathrm{Re}\int_{]0,1[}u''(x)\bar{u}(x)|u(x)|^{p-2}dx+C_2\|u;\ X\|^p$$
$$\leqq (C+2)\|f;\ X\|\|u;\ X\|^{p-1}. \tag{14}$$

Let $(C+1)\mathrm{Re}\lambda+|\mathrm{Im}\lambda|\geqq\varepsilon_0>0$. We then deduce that there is $K>0$ such that

$$\int_{]0,1[}|u'(x)|^2|u(x)|^{p-2}dx\leqq K\|f;\ X\|\|u;\ X\|^{p-1}.$$

In virtue of (13), this implies

$$|\mathrm{Re}\lambda|\,\|M^{1/p}u;\ X\|^p \leqq (C+2)\|f;\ X\|\|u;\ X\|^{p-1}+\|f;\ X\|\|u;\ X\|^{p-1},$$
$$|\mathrm{Im}\lambda|\,\|M^{1/p}u;\ X\|^p \leqq C_3\|f;\ X\|\|u;\ X\|^{p-1},$$

and thus, since $\|u;\ X\|\leqq K_1\|f;\ X\|$ by (14),

$$|\lambda|\,\|M^{1/p}u;\ X\|^p \leqq C_4\|f;\ X\|\|u;\ X\|^{p-1} \leqq C_5\|f;\ X\|^p.$$

It follows that $\|\lambda Mu;\ X\|=\|\lambda M(\lambda M+L)^{-1}f;\ X\|\leqq C(1+|\lambda|)^{1/p'}\|f;\ X\|$.

We also want to refine the preceding estimate if our $m(x)$ satisfies $0\leqq m\in C^1[0,1]$ and $|m'(x)|\leqq Cm(x)^\rho$, $0<\rho\leqq 1$.

If $(\lambda M+L)u=f\in X=L^p(0,1)$, and $p>2$, we have the following estimates:

$$\mathrm{Re}\lambda\|Mu;\ X\|^p+C\int_{]0,1[}m(x)^{p-1}|u'(x)|^2|u(x)|^{p-2}dx+c_0\int_{]0,1[}m(x)^{p-1}|u(x)|^p dx$$
$$\leqq \|f;\ X\|\|Mu;\ X\|^{p-1}+|p-1|\int_{]0,1[}m(x)^{p-2}|m'(x)|\,|u'(x)|\,|u(x)|^{p-1}dx,$$
$$|\mathrm{Im}\lambda|\,\|Mu;\ X\|^p-|p-2|\int_{]0,1[}m(x)^{p-1}|u'(x)|^2|u(x)|^{p-2}$$
$$\leqq |p-1|\int_{]0,1[}m(x)^{p-2}|m'(x)|\,|u'(x)|\,|u(x)|^{p-1}dx+\|f;\ X\|\|Mu;\ X\|^{p-1},$$

with a suitable positive constant C. After multiplication by $K>0$, which is sufficiently small so that $C-K|p-2|>0$, we then deduce

$$(\mathrm{Re}\lambda+K|\mathrm{Im}\lambda|)\|Mu;\ X\|^p+C\int_{]0,1[}m(x)^{p-1}|u'(x)|^2|u(x)|^{p-2}dx$$
$$+C_1\int_{]0,1[}m(x)^{p-1}|u(x)|^p dx$$
$$\leqq (K+1)\|f;\ X\|\|Mu;\ X\|^{p-1}$$
$$+(K+1)|p-1|\int_{]0,1[}m(x)^{p-2}|m'(x)||u'(x)||u'(x)|^{p-1}dx\,.$$

Last integral, in view of the hypothesis on m, is estimated by

$$(*) =\int_{]0,1[}m(x)^{p-2+\rho}|u'(x)||u(x)|^{p-1}dx \leqq \text{(by Hölder inequality)}$$
$$\leqq \left(\int_{]0,1[}m(x)^{p\rho}|u(x)|^p dx\right)^{1/2}\left(\int_{]0,1[}m(x)^{(p-2)(2-\rho)}|u'(x)|^2|u(x)|^{p-2}dx\right)^{1/2}.$$

Since $\|M^\rho u;\ X\| \leqq C\|Mu;\ X\|^\rho\|u;\ X\|^{1-\rho}$ and

$$\|u;\ X\| \leqq K_1\|f;\ X\|,\qquad \int_{]0,1[}|u'(x)|^2|u(x)|^{p-2}dx \leqq K_2\|f;\ X\|^p,$$

as already proved, we have

$$(*) \leqq C'\|Mu;\ X\|^{\rho p/2}\|f;\ X\|^{p(1-\rho/2)}.$$

If we suppose $\lambda\in\Sigma$: $\mathrm{Re}\lambda+K|\mathrm{Im}\lambda|\geqq\varepsilon_0>0$, we deduce

$$\int_{]0,1[}m(x)^{p-1}|u'(x)|^2|u(x)|^{p-2}dx$$
$$\leqq C''\|f;\ X\|\|Mu;\ X\|^{p-1}+C_1'\|Mu;\ X\|^{\rho p/2}\|f;\ X\|^{p(1-\rho/2)}=(**)$$
$$\int_{]0,1[}m(x)^{p-1}|u(x)|^p dx \leqq (**).$$

Taking into account that $\lambda Mu+Lu=f$, we have

$$|\mathrm{Re}\lambda|\,\|Mu;\ X\|^p \leqq C\|f;\ X\|\|Mu;\ X\|^{p-1}\int_{]0,1[}m(x)^{p-1}|u'(x)|^2|u(x)|^{p-2}dx$$
$$+\tilde{C}\|Mu;\ X\|^{\rho p/2}\|f;\ X\|^{p(1-\rho/2)} \leqq C_1\{\|f;\ X\|\|Mu;\ X\|^{p-1}$$
$$+\|f;\ X\|^{p(1-\rho/2)}\|Mu;\ X\|^{\rho/p/2}\}=(***),$$
$$|\mathrm{Im}\lambda|\,\|Mu;\ X\|^p \leqq (***).$$

Hence

$$|\lambda|\,\|Mu;\ X\|^{p(1-\rho/2)} \leqq C'\{\|f;\ X\|\|Mu;\ X\|^{p(1-\rho/2)-1}+\|f;\ X\|^{p(1-\rho/2)}\}$$

and, since $\|Mu;\ X\|\leqq C\|f;\ X\|$,

$$|\lambda|\,\|Mu;\ X\|^{p(1-\rho/2)} \leqq C\|f;\ X\|^{p(1-\rho/2)},\qquad \lambda\in\Sigma,$$

that is,

$$\|\lambda Mu;\ X\| \leqq |\lambda|^{\alpha}\|f;\ X\|, \qquad \lambda\in\Sigma,$$

where $\alpha=(2p-\rho p-2)(p(2-\rho))^{-1}$. We remark that $\alpha<1/p'$.

Of course, following the lines of the preceding examples, we could treat the case in which m, L depend on t, $L(t)$ being defined by a suitable differential operator of order $2k$ in x, whose coefficients have some regularity in the variables t and x, with Dirichlet boundary conditions.

6. Let A be a positive self-adjoint operator in a complex Hilbert space X, with inner product $\langle\ ,\ \rangle$, such that $\langle Au, u\rangle\geqq\delta_0\|u;\ X\|^2$, $\delta_0>0$, $u\in D=D(A)$.

Then the operator $A-\delta=M$ is non negative for any $\delta\leqq\delta_0$ and we suppose that it has a closed range. Let L be another self-adjoint positive operator in X, with $D(L)=D(M^k)=D(A^k)$, k being an integer $\geqq 2$, such that the "angle-condition" $\langle Lu, Mu\rangle\geqq 0$, $\forall u\in D(L)$ is satisfied.

In view of the moment's inequality, M is L-bounded and has L-bound equal to 0. Thus, by [8], the estimate $\|L(zM+L)^{-1};\ \mathcal{L}(X)\|\leqq$constant, is satisfied.

Hence, we can make use of all what we have obtained both on space and time regularity.

We notice that in this case $R(T)=R(A-\delta)\cap D(A^{k-1})$, since $u\in R(T)$ implies $u=(A-\delta)v$, $v\in D(A^k)$ and also $u\in D(A^{k-1})\cap R(A-\delta)$.

Conversely, $u\in D(A^{k-1})\cap R(A-\delta)$ implies $u=(A-\delta)v$, $v\in D(A)$ and also $Av=\delta v+u\in D(A)$, that is, $v\in D(A^2)$ and so on until $v\in D(A^k)$.

We also observe that $Lu=\sum_{s=0}^{k}a_sA^su$, $a_s\geqq 0$, $k\geqq 2$, is a possible choice, since

$$\langle (A-\delta_0)u,\ A^{2k}u\rangle=\langle A^ku,\ A^{k+1}u\rangle-\delta_0\|A^ku;\ X\|^2\geqq 0,$$

$$\begin{aligned}\langle (A-\delta_0)u,\ A^{2k+1}u\rangle &= \|A^{k+1}u;\ X\|^2-\delta_0\langle A^ku,\ A(A^ku)\rangle = (\text{put}\,A^ku=v)\\ &= \|Av;\ X\|^2-\delta_0\langle v,\ Av\rangle\geqq\|Av;\ X\|^2-\delta_0\|v;\ X\|\|Av;\ X\|.\end{aligned}$$

But $\|Av;\ X\|\|v;\ X\|\geqq\langle Av, v\rangle\geqq\delta_0\|v;\ X\|^2$ implies $\|Av;\ X\|\geqq\delta_0\|v, X\|$.

Hence our inner product is non negative.

If $\delta\leqq\delta_0$, clearly $\langle (A-\delta)u, A^hu\rangle\geqq(\delta_0-\delta)\langle u, A^hu\rangle\geqq 0$, for any $u\in D(A^k)$, $0\leqq h\leqq k$. By means of this example we could treat some interesting, degenerate or not degenerate, types of boundary value problems, in which A is defined by a differential operator, as in [**16**, p. 184]. We want to this purpose to quote the thesis of Brown [**3**].

7. We could treat equations of the type

$$\frac{\partial}{\partial t}(m(t, x)u(t, x))+L(t, x, D)u(t, x)=f(t, x), \quad 0\leqq t\leqq T, \quad x\in\Omega\subset\boldsymbol{R}^n$$

where $C_1m(x)\leqq m(t, x)\leqq C_2m(x)$, C_1, $C_2>0$, $m(x)\geqq 0$ vanishes only on a part of the boundary $\partial\Omega$ of Ω, by help of the weight-spaces $L^2_{\sqrt{m}}(\Omega)$ and $L^2_{1/\sqrt{m}}(\Omega)$, as it has been done in [**16**]. Now one assumes higher regularity in time to obtain some solution u for which $t \to L(t, \cdot, D)u \in C^{\omega}[0, T; L^2_{1/\sqrt{m}}(\Omega)]$.

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