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Density properties for solenoidal vector fields, with applications to the Navier-Stokes equations in exterior domains

Dedicated to Professor Dr. Reimund Rautmann on his 60th birthday

By Hideo KOZONO and Hermann SOHR

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Introduction.

In this paper we give some density properties for the space of solenoidal vector fields in *exterior domains*. Then we shall apply such results to the *stationary* problem for the Navier-Stokes equations. Let Ω be an exterior domain in $\boldsymbol{R}^n$ ($n\geqq 2$), i.e., a domain having a compact complement $\boldsymbol{R}^n/\Omega$, and assume that the boundary $\partial\Omega$ is of class $C^{2+\mu}$ with $0<\mu<1$. Consider the following boundary value problem for the Navier-Stokes equations in Ω:

$$\text{(N-S)}\qquad \begin{aligned} -\Delta u+u\cdot\nabla u+\nabla p &= f \quad \text{in } \Omega\,, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned}$$

where $u=(u_1(x), \cdots, u_n(x))$ and $p=p(x)$ denote the unknown velocity and pressure, respectively; $f=(f_1(x), \cdots, f_n(x))$ denotes the given external force.

Density properties for solenoidal vector fields, i.e., vector fields u with $\operatorname{div} u=0$, are essentially important for the Navier-Stokes equations in *exterior domain*. The reason is that, compared with the interior problems, the possible space of test functions for weak solutions of (N-S) is "too small" in *unbounded domains*. So, one needs as a wider space as possible for the test functions. Therefore Masuda [**17**], Heywood [**14**] and Giga [**12**] gave certain types of density theorems for such vector fields; Masuda [**17**] applied them to the uniqueness problem of weak solutions for the *non-stationary* Navier-Stokes equations. Heywood [**14**] proved a similar result to ours in L^2-space and applied it to the uniqueness problem for the *stationary Stokes equations*.

The purpose of this paper is to give more general density theorems for solenoidal vector fields in L^q-spaces on *exterior domains* and to apply them to the *stationary* Navier-Stokes equations. In particular, we will prove global L^q-bounds and a uniqueness criterion for weak solutions of (N-S). On account of the nonlinear term $u \cdot \nabla u$, we need such density properties not only in L^q but also in the intersection $L^q \cap L^r$ for $1<q, r<\infty$, because we have the different behavior at infinity of Δu and $u \cdot \nabla u$ for weak solutions u of (N-S) in *unbounded* domains.

First we shall apply our result to a problem on regularity at infinity of weak solutions u and its associated pressure p of (N-S). To this end, the same problem on the linearlized equations of (N-S), i.e., the Stokes equations will be also investigated. For bounded domains, Cattabriga [**7**] showed the most general result in L^q on the Stokes equations. Our result (Lemma 2.5) clarifies a typical difference between interior and exterior problems. When $n=3$, Fujita [**9**] gave an explicit representation formula of weak solutions of (N-S) for smooth f decreasing rapidly at infinity, which seems to give a similar application to ours. However, our method enables us to treat a much wider class of f. Our second application is a uniqueness criterion for weak solutions of (N-S). Our criterion for the *stationary* problem is closely related to that of Serrin's [**20**] for the *non-stationary* case.

In this paper we avoid such a complicated tool as the hydrodynamical potential theory; our method is based on a cut-off procedure. We shall first prove the corresponding results in bounded domains and the whole space $\boldsymbol{R}^n$. These results are also interesting in itself. Then the exterior problem in question can be treated as perturbation of both cases. To this end, we shall make fully use of the result on the boundary-value problem $\operatorname{div} u=g$ in Ω, $u=0$ on $\partial\Omega$, which was given by Bogovski [**4, 5**] and Borchers-Sohr [**6**].

1. Results.

Before stating our results we introduce some notations. For $1<q<\infty$, $q'=q/(q-1)$, $\| \ \|_q$ and $(\cdot, \cdot)$ denote the usual norm of $L^q(\Omega)$ and the inner product between $L^q(\Omega)$ and $L^{q'}(\Omega)$, respectively. $\hat{H}_0^{1,q}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla u\|_q$. Since Ω is an exterior domain, $\hat{H}_0^{1,q}(\Omega)$ is larger than $H_0^{1,q}(\Omega)$. Let $\hat{H}^{-1,q}(\Omega):=\hat{H}_0^{1,q'}(\Omega)^*$($X^*$; dual space of X). $\| \ \|_{-1,q}$ denotes the norm of $\hat{H}^{-1,q}(\Omega)$ defined by $\|f\|_{-1,q}:=\sup\{|\langle f, \phi\rangle|/\|\nabla\phi\|_{q'}\,;\ \phi\in C_0^\infty(\Omega),\ \phi\neq 0\}$, where $\langle\cdot,\cdot\rangle$ is the duality paring of $\hat{H}^{-1,q}(\Omega)$ and $\hat{H}_0^{1,q'}(\Omega)^*$. $C_0^\infty(\Omega)^n$, $L^q(\Omega)^n, \cdots$, and $C_0^\infty(\Omega)^{n^2}$, $L^p(\Omega)^{n^2}, \cdots$ denote the corresponding spaces for the vector-valued and the matrix-valued functions, respectively. In such spaces, we shall also use the same notations $\| \ \|_q$ and $(\cdot, \cdot)$. $C_{0,\sigma}^\infty(\Omega)$ is the set of all C^∞-vector functions $\phi=(\phi_1, \cdots, \phi_n)$ such that $\operatorname{div}\phi=0$.

Our results now read:

THEOREM 1. *Let* $\hat{X}_\sigma^q(\Omega)\equiv\{u\in\hat{H}_0^{1,q}(\Omega)^n;\ \operatorname{div} u=0\}$. *Then for all* $1<q<\infty$, $1<r<\infty$, $C_{0,\sigma}^\infty(\Omega)$ *is dense in* $\hat{X}_\sigma^q(\Omega)\cap\hat{X}_\sigma^q(\Omega)$ *under the norm* $\|\nabla u\|_q+\|\nabla u\|_r$.

THEOREM 2. *Let q and r satisfy one of the following cases* (i) *or* (ii):

(i) $1<q<n$ *and* $1<r<\infty$;

(ii) $n\leqq q<r<\infty$.

Then $C_{0,\sigma}^{\infty}(\Omega)$ *is dense* $\hat{X}_{\sigma}^{q}(\Omega)\cap L^{r}(\Omega)^{n}$ *under the norm* $\|\nabla u\|_{q}+\|u\|_{r}$.

REMARKS 1. In case $q=r=2$, Heywood [**14**] showed the same result of Theorem 1.

2. When Ω is the whole space $\boldsymbol{R}^{n}$ or a bounded domain, Masuda [**17**] and Giga [**12**] proved that $C_{0,\sigma}^{\infty}(\Omega)$ is dense in $H_{0,\sigma}^{1,2}(\Omega)\cap L^{r}(\Omega)^{n}$, where $H_{0,\sigma}^{1,2}(\Omega)$ is the closure of $C_{0,\sigma}^{\infty}(\Omega)$ in $H^{1,2}(\Omega)$. In Remark after his proof, Giga [**12**, p. 210] conjectured that one can prove the same result even in unbounded domains.

We next apply the above results to (N-S). Our definition of a weak solution of (N-S) is as follows.

DEFINITION. Let $f\in\hat{H}^{-1,2}(\Omega)^{n}$. Then a measurable function u on Ω is called a weak solution of (N-S) if

(i) $u\in\hat{X}_{\sigma}^{2}(\Omega)$;

(ii) $(\nabla u, \nabla\phi)+(u\cdot\nabla u, \phi)=\langle f, \phi\rangle$ for all $\phi\in C_{0,\sigma}^{\infty}(\Omega)$.

Concerning the existence of weak solutions, see, e.g., Temam [**24**, p. 169, Theorem 1.4]. For every weak solution u of (N-S) there is a scalar function $p\in L_{\mathrm{loc}}^{1}(\bar{\Omega})$, unique up to an additive constant, such that

$$(\nabla u, \nabla\psi)+(u\cdot\nabla u, \psi)-(p, \operatorname{div}\psi)=\langle f, \psi\rangle$$

holds for all $\psi\in C_{0}^{\infty}(\Omega)^{n}$. This means that the pair $\{u, p\}$ satisfies (N-S) in the sense of distributions. We call such p the pressure associated with u (see Fujita [**9**, Definition 2.3]).

Our result on regularity of weak solutions of (N-S) reads:

THEOREM 3. (1) (*associated pressure*) *Let* $n\geqq 3$ *and* $f\in\hat{H}^{-1,2}(\Omega)^{n}$. *Suppose that u is a weak solution of* (N-S). *Then the pressure p associated with u can be chosen in the class* $p\in L^{2}(\Omega)+L^{n/(n-2)}(\Omega)$.

(2) (*more regularity*) (i) *Let* $n=3$ *and* $f\in\hat{H}^{-1,2}(\Omega)^{3}\cap\hat{H}^{-1,q}(\Omega)^{3}$ *for* $3\leqq q<\infty$. *Suppose that u is a weak solution of* (N-S) *and that p is the pressure associated with u. Then we have*

$$\nabla u\in L^{r}(\Omega)^{3^{2}} \quad \text{for } 2\leqq r\leqq q, \qquad u\in L^{s}(\Omega)^{3} \quad \text{for } 6\leqq s<\infty,$$
$$p\in L^{q}(\Omega).$$

In particular, if $q>3$, *we have also* $u\in L^{\infty}(\Omega)^{3}$.

(ii) *Let* $n\geqq 5$ *and* $f\in\hat{H}^{-1,2}(\Omega)^{n}\cap\hat{H}^{-1,q}(\Omega)^{n}$ *for* $n/(n-1)<q\leqq n/(n-2)$. *Let u and p be as above. Then it holds*

$$\nabla u \in L^r(\Omega)^{n^2} \quad for\ q \leqq r \leqq 2\,,$$

$$u \in L^s(\Omega)^n \quad for\ nq/(n-q) \leqq s \leqq 2n/(n-2),\ p \in L^q(\Omega)\,.$$

Next we shall proceed to the uniqueness criterion for the weak solutions of (N-S).

THEOREM 4. *Let* $n \geqq 3$ *and* $f \in \hat{H}^{-1,2}(\Omega)^n$. *Let* u *and* v *be weak solutions of* (N-S). *Suppose also that* u *satisfies the energy inequality*

$$\text{(E. I.)} \qquad \|\nabla u\|_2^2 \leqq \langle f,\, u\rangle$$

and that $v \in L^n(\Omega)^n$. *Then there is a positive constant* λ *such that if* $\|v\|_n \leqq \lambda$, *we have* $u \equiv v$ *in* Ω.

REMARKS 1. If Ω is a *bounded* domain in $\boldsymbol{R}^n$ with $n \leqq 4$, then every weak solution u belongs to $L^n(\Omega)^n$ and satisfies the *energy equality* $\|\nabla u\|_2^2 = \langle f,\, u\rangle$. Hence in such a case, we have $u \equiv v$ under the assumption that $\|f\|_{-1,2}$ is sufficiently small (see Temam [**24**, p. 167, Theorem 1.3]).

2. In the *non-stationary* Navier-Stokes equations, Wahl [**23**] and Masuda [**17**] improved Serrin's uniqueness criterion [**20**] for the weak solutions on $\Omega \times (0, T)$ in the spaces $C([0, T];\, L^n(\Omega)^n)$ and $L^\infty(0, T;\, L^n(\Omega)^n)$, respectively. So Theorem 4 yields a uniqueness criterion for the *stationary problem* which is similar to that of Serrin's [**20**] for the non-stationary problem.

2. Preliminaries.

Let us recall the spaces $\hat{H}_0^{1,q}(\Omega)$ and $\hat{H}^{-1,q}(\Omega)$. Since the norms $\|\nabla u\|_q$ and $\|\nabla u\|_r (1<q,\ r<\infty)$ are consistent on $C_0^\infty(\Omega)$, we can define the two Banach spaces $\hat{H}_0^{1,q}(\Omega) + \hat{H}_0^{1,r}(\Omega)$ and $\hat{H}_0^{1,q}(\Omega) \cap \hat{H}_0^{1,r}(\Omega)$ as usual (see, e.g., Reed-Simon [**19**, p. 35]). In the same way, we denote by $\hat{H}_0^{1,q}(\boldsymbol{R}^n)$ the closure of $C_0^\infty(\boldsymbol{R}^n)$ with respect to the norm $\|\nabla u\|_{q,\boldsymbol{R}^n}$; $\|\ \|_{q,\boldsymbol{R}^n}$ is the L^q-norm over $\boldsymbol{R}^n$. Note that $\hat{H}_0^{1,q}(\boldsymbol{R}^n)$ consists of equivalent classes of all measurable functions whose differences are only constants in $\boldsymbol{R}^n$. $\hat{H}^{-1,q}(\boldsymbol{R}^n)$ is the dual space of $\hat{H}_0^{1,q'}(\boldsymbol{R}^n)$ whose norm we denote by $\|\ \|_{-1,q,\boldsymbol{R}^n}$. For simplicity, we shall abbreviate the above norms and the duality on $\boldsymbol{R}^n$ as $\|\ \|_q$, $\|\ \|_{-1,q}$, $(\cdot,\, \cdot)$ and $\langle\cdot,\, \cdot\rangle$, unless it causes confusions between Ω and $\boldsymbol{R}^n$. In what follows C denotes a positive constant which may change from line to line. In particular, $C = C(\cdot,\, \cdot,\, \cdots,\, \cdot)$ denotes a constant depending only on the quantities appearing in the parentheses.

2.1. First we consider the boundary-value problem of the equation:

$$\text{(2.1)} \qquad \operatorname{div} u = f \quad \text{in } \Omega\,, \qquad u = 0 \quad \text{on } \partial\Omega\,.$$

The following lemma is essentially due to Bogovski [**4, 5**]; for the special formulation and extension, see Borchers-Sohr [**6**].

LEMMA 2.1. *Let* $1<q<\infty$. (i) *There is a bounded operator* $f\rightarrow u$ *from* $L^q(\Omega)$ *to* $\hat{H}_0^{1,q}(\Omega)^n$ *such that* $\operatorname{div} u=f$.

(ii) *Let* D *be a bounded Lipschitz domain in* $\boldsymbol{R}^n$. *For* $0\neq\tau\in\boldsymbol{R}$ *we set* $D_\tau\equiv\{\tau x;\, x\in D\}$. *Then there is a linear operator* $R_\tau: f\rightarrow R_\tau f$ *from* $\left\{f\in C_0^\infty(D_\tau);\, \int_{D_\tau} f(x)dx=0\right\}$ *to* $C_0^\infty(D_\tau)^n$ *such that*

(a) $\operatorname{div} R_\tau f=f$ *in* D_τ;

(b) $\|\nabla R_\tau f\|_{L^q(D_\tau)}\leqq C\|f\|_{L^q(D_\tau)}$ *for* $1<q<\infty$.

Here $C=C(D, q)$ *is a constant independent of* τ *and* f.

Using the well-known closed range theorem and the Sobolev inequality, we obtain immediately from this lemma the following result.

COROLLARY 2.2. (i) *Let* $1<q<\infty$ *and let* $\hat{X}_\sigma^q(\Omega)=\{u\in\hat{H}_0^{1,q}(\Omega)^n;\, \operatorname{div} u=0\}$. *Suppose that* $f\in\hat{H}^{-1,q'}(\Omega)^n$ *satisfies* $\langle f, u\rangle=0$ *for all* $u\in\hat{X}_\sigma^q(\Omega)$. *Then there is a unique* $p\in L^{q'}(\Omega)$ *such that* $f=\nabla p$, i.e., $\langle f, \phi\rangle=-(p, \operatorname{div}\phi)$ *for all* $\phi\in\hat{H}_0^{1,q}(\Omega)^n$ *and that* $\|p\|_{q'}\leqq C\|f\|_{-1,q'}$ *with* C *independent of* f.

(ii) *Let* $1<q<n$ *and let* $u\in L_{\mathrm{loc}}^1(\bar{\Omega})$ *with* $\nabla u\in L^q(\Omega)^n$. *Then there is a constant* K_u *such that* $u+K_u\in L^{q^*}(\Omega)$ *with* $1/q^*=1/q-1/n$ *and* $\|u+K_u\|_{q^*}\leqq C\|\nabla u\|_q$ *with* C *independent of* u. *Here* $\bar{\Omega}$ *is the closure of* Ω *and* $u\in L_{\mathrm{loc}}^1(\bar{\Omega})$ *means that* $u\in L^1(\Omega\cap B)$ *for all balls* $B\subset\boldsymbol{R}^n$ *with* $\Omega\cap B\neq\emptyset$.

For the proof, see Giga-Sohr [**13**, Corollary 2.2].

2.2. Next we shall characterize the space $\hat{H}_0^{1,q}(\boldsymbol{R}^n)$. The following variational inequality in L^q is simple but plays an important role for our purpose; see also Simader-Sohr [**22**].

Let $1<q<\infty$. Then there is a constant $C=C(n, q)>0$ such that

$$\|\nabla u\|_q\leqq C\sup\{|(\nabla u, \nabla\phi)|/\|\nabla\phi\|_{q'};\, 0\neq\phi\in C_0^\infty(\boldsymbol{R}^n)\} \tag{2.2}$$

holds for all $u\in L_{\mathrm{loc}}^q(\boldsymbol{R}^n)$ with $\nabla u\in L^q(\boldsymbol{R}^n)^n$.

Indeed, note that the space $H\equiv\{\Delta\psi;\, \psi\in C_0^\infty(\boldsymbol{R}^n)\}$ is dense in $L^{q'}(\boldsymbol{R}^n)$. Then using the Calderon-Zygmund inequality $\|\nabla\nabla\psi\|_{q'}\leqq C\|\Delta\psi\|_{q'}$ ($\psi\in C_0^\infty(\boldsymbol{R}^n)$), we have for each $i=1, \cdots, n$

$$\begin{aligned}
&\sup\{|(\nabla u, \nabla\phi)|/\|\nabla\phi\|_{q'};\, \phi\in C_0^\infty(\boldsymbol{R}^n), \phi\neq0\}\\
&\geqq\sup\{|(\nabla u, \nabla(\partial_i\psi))|/\|\nabla(\partial_i\psi)\|_{q'};\, \psi\in C_0^\infty(\boldsymbol{R}^n), \psi\neq0\}\\
&\geqq C\sup\{|(\partial_i u, \Delta\psi)|/\|\Delta\psi\|_{q'};\, \psi\in C_0^\infty(\boldsymbol{R}^n), \psi\neq0\}
\end{aligned}$$

$$= C \sup\{|(\partial_i u, g)|/\|g\|_{q'};\ g\in L^{q'}(\boldsymbol{R}^n),\ g\neq 0\}$$
$$= C\|\partial_i u\|_q$$

with $C=C(n, q)$ and (2.2) follows.

Let $L^{1,q}\equiv\{u\in L^q_{\mathrm{loc}}(\boldsymbol{R}^n);\ \nabla u\in L^q(\boldsymbol{R}^n)^n\}$. For $u\in L^{1,q}$ we denote by $[u]$ the set of all $v\in L^{1,q}$ such that $u-v$ is constant in $\boldsymbol{R}^n$ and define the space $L^{1,q}/\boldsymbol{R}=\{[u];\ u\in L^{1,q}\}$ with norm $\|[u]\|_{L^{1,q}/\boldsymbol{R}}:=\|\nabla u\|_q$. Clearly $L^{1,q}/\boldsymbol{R}$ is isometric to the space $G_q:=\{\nabla u;\ [u]\in L^{1,q}/\boldsymbol{R}\}(\subset L^q(\boldsymbol{R}^n)^n)$. By the theory of Helmholtz decomposition (see Simader-Sohr [22] and Miyakawa [18]), G_q is a *closed subspace* in $L^q(\boldsymbol{R}^n)^n$. Therefore, G_q is a reflexive Banach space. Moreover, we have the following relation:

LEMMA 2.3. *Let $1<q<\infty$. Then the spaces $\hat{H}_0^{1,q}(\boldsymbol{R}^n)$, $L^{1,q}/\boldsymbol{R}$ and G_q are isometric as Banach spaces; each element of $\hat{H}_0^{1,q}(\boldsymbol{R}^n)$ can be identified with some $[u]\in L^{1,q}/\boldsymbol{R}$ such that $\nabla u\in G_q$.*

PROOF. Let $\phi_j\in C_0^\infty(\boldsymbol{R}^n)$ $(j=1, 2, \cdots)$ be a Cauchy sequence with respect to the norm $\|\nabla u\|_q$. There is function $w\in L^q(\boldsymbol{R}^n)^n$ such that $\nabla\phi_j\to w$ in $L^q(\boldsymbol{R}^n)^n$. Since $(w, \psi)=\lim_{j\to\infty}(\nabla\phi_j, \psi)=-\lim_{j\to\infty}(\phi_j, \operatorname{div}\psi)=0$ for all $\psi\in C_{0,\sigma}^\infty(\boldsymbol{R}^n)$, it follows from the Helmholtz decomposition that w has the form $w=\nabla p$ with some $p\in L^{1,q}$. Clearly, such p is uniquely determined up to an additive constant and hence we see that $\hat{H}_0^{1,q}(\boldsymbol{R}^n)$ is isometrically embedded into G_q. To prove the assertion, it remains to show that the space $W\equiv\{\nabla\phi;\ \phi\in C_0^\infty(\boldsymbol{R}^n)\}$ is dense in G_q. To this end, let us consider a linear operator $B_q:\nabla u\in G_q\to B_q(\nabla u)\in G_{q'}^*$, defined by

$$\langle B_q(\nabla u), \nabla v\rangle = (\nabla u, \nabla v) \qquad \text{for } \nabla v\in G_{q'},$$

where $\langle\cdot, \cdot\rangle$ denotes the duality between $G_{q'}^*$ and $G_{q'}$. Then by (2.2) we see that B_q is injective and that its range is closed in $G_{q'}^*$. Since $B_q^*=B_{q'}$ (T^*; adjoint operator of T), it follows from the closed range theorem that B_q is surjective and hence bijective. Now, suppose that $F\in G_q^*$ satisfies $\langle F, \nabla\phi\rangle=0$ for all $\phi\in C_0^\infty(\boldsymbol{R}^n)$. Since $B_{q'}$ is also bijective, there is a unique $\nabla u\in G_{q'}$ such that

$$\langle F, \nabla v\rangle = \langle B_{q'}(\nabla u), \nabla v\rangle = (\nabla u, \nabla v)$$

holds for all $\nabla v\in G_q$. Then by the assumption and (2.2) we get $\nabla u=0$ and hence $F=0$, which implies that W is dense in G_q. This completes the proof. ∎

REMARKS. 1. By the proof of this lemma we see that for every $u\in L^{1,q}$, there is a sequence $u_j\in C_0^\infty(\boldsymbol{R}^n)$ $(j=1, 2, \cdots)$ such that $\nabla u_j\to\nabla u$ in $L^q(\boldsymbol{R}^n)^n$. This holds also with $\boldsymbol{R}^n$ replaced by $\bar{\Omega}$. Simader [21] gave another proof for the latter convergence by using the Poincaré inequality on annulus domains and a scaling argument.

2. The above approximation and the application of the Sobolev inequality yield the following concrete characterization of the space $\hat{H}_0^{1,q}(\Omega)$.

(i) *For* $1<q<n$, *we have*

$$\hat{H}_0^{1,q}(\Omega)=\{u\in L^{nq/(n-q)}(\Omega);\ \nabla u\in L^q(\Omega)^n,\ u|_{\partial\Omega}=0\}.$$

(ii) *For* $n\leqq q<\infty$, *we have*

$$\hat{H}_0^{1,q}(\Omega)=\{u\in L_{\mathrm{loc}}^q(\bar{\Omega});\ \nabla u\in L^q(\Omega)^n,\ u|_{\partial\Omega}=0\}.$$

2.3. Let us consider the Stokes equations:

$$-\Delta u+\nabla p=f\,,\qquad \operatorname{div} u=0\quad \text{in } \boldsymbol{R}^n \tag{2.3}$$

Recall that $\hat{X}_\sigma^q(\boldsymbol{R}^n)=\{u\in\hat{H}_0^{1,q}(\boldsymbol{R}^n)^n;\ \operatorname{div}u=0\}$. Then we have

LEMMA 2.4. *Let* $1<q<\infty$, $1<r<\infty$. *For every* $f\in\hat{H}^{-1,q}(\boldsymbol{R}^n)^n\cap\hat{H}^{-1,r}(\boldsymbol{R}^n)^n$, *there is a unique pair* $\{u, p\}$ *with* $u\in\hat{X}_\sigma^q(\boldsymbol{R}^n)\cap\hat{X}_\sigma^r(\boldsymbol{R}^n)$ *and* $p\in L^q(\boldsymbol{R}^n)\cap L^r(\boldsymbol{R}^n)$ *such that*

$$(\nabla u, \nabla\phi)-(p, \operatorname{div}\phi)=\langle f, \phi\rangle \tag{2.4}$$

for all $\phi\in C_0^\infty(\boldsymbol{R}^n)^n$. *Such* $\{u, p\}$ *is subject to the inequality*

$$\|\nabla u\|_q+\|\nabla u\|_r+\|p\|_q+\|p\|_r=C(\|f\|_{-1,q}+\|f\|_{-1,r}), \tag{2.5}$$

where $C=C(n, q, r)$.

PROOF. By the definition of the space $\hat{H}_0^{1,q'}(\boldsymbol{R}^n)$, we see that the operator $-\nabla:\hat{H}_0^{1,q'}(\boldsymbol{R}^n)\to L^{q'}(\boldsymbol{R}^n)^n$ is injective and has a closed range. Hence by the closed range theorem, the adjoint operator $\operatorname{div}=(-\nabla)^*: L^q(\boldsymbol{R}^n)^n\to\hat{H}^{-1,q}(\boldsymbol{R}^n)$ is surjective. Since the null space Ker (div) of div is a closed subspace in $L^q(\boldsymbol{R}^n)^n$, for each $h\in\hat{H}^{-1,q}(\boldsymbol{R}^n)$, there is at least one $u\in L^q(\boldsymbol{R}^n)^n$ such that

$$-(u, \nabla\phi)=\langle h, \phi\rangle \quad\text{for all } \phi\in C_0^\infty(\boldsymbol{R}^n) \text{ and that } \|u\|_q\leqq C\|h\|_{-1,q} \tag{2.6}$$

with C independent of h. Let us recall the space G_q and the bijective operator $B_q: G_q\to G_{q'}^*$ in the proof of Lemma 2.3. Since $u\in L^q(\boldsymbol{R}^n)^n$, the map $\nabla\phi\in G_{q'}\to-(u, \nabla\phi)\in\boldsymbol{R}$ is an element in $G_{q'}^*$, so we can choose $\pi\in\hat{H}_0^{1,q}(\boldsymbol{R}^n)$ so that

$$(\nabla\pi, \nabla\phi)=\langle B_q(\nabla\pi), \nabla\phi\rangle=-(u, \nabla\phi)=\langle h, \phi\rangle \tag{2.7}$$

for all $\phi\in C_0^\infty(\boldsymbol{R}^n)$. By (2.2) such π is uniquely determined by h and so we can define a bounded linear operator $S_q: h\in\hat{H}^{-1,q}(\boldsymbol{R}^n)\to\pi\in\hat{H}_0^{1,q}(\boldsymbol{R}^n)$ by the relation (2.7). If in addition, $h\in\hat{H}^{-1,r}(\boldsymbol{R}^n)$, we have also $\pi\in\hat{H}_0^{1,r}(\boldsymbol{R}^n)$. Indeed, with q replaced by r, we see by the above argument that there is a unique $\eta\in\hat{H}_0^{1,r}(\boldsymbol{R}^n)=L^{1,r}/\boldsymbol{R}$ such that $(\nabla\eta, \nabla\phi)=\langle h, \phi\rangle$ for all $\phi\in C_0^\infty(\boldsymbol{R}^n)$ and $\|\nabla\eta\|_r\leqq C\|h\|_{-1,r}$ with $C=C(n, r)$ independent of h. Thus we get $\pi\in L^{1,q}$, $\eta\in L^{1,r}$

and $\pi-\eta\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ satisfies $\Delta(\pi-\eta)=0$ in the sense of distributions. Then it follows from Weyl's lemma that $\pi-\eta$ is of class C^∞ and harmonic in $\boldsymbol{R}^n$; so is $\nabla\pi-\nabla\eta$. Applying the mean value property and the Hölder inequality, we get

$$(2.8)\qquad |\nabla\pi(x)-\nabla\eta(x)| \leqq C(\|\nabla\pi\|_q|x|^{-n/q}+\|\nabla\eta\|_r|x|^{-n/r})$$

for all $x(\neq 0)\in\boldsymbol{R}^n$ with C independent of x. Then the classical Liouville theorem yields that $\nabla\pi-\nabla\eta\equiv 0$ and hence $\pi-\eta\equiv$const. in $\boldsymbol{R}^n$. This shows that $\pi\in L^{1,r}$ and hence $\pi\in\hat{H}_0^{1,q}(\boldsymbol{R}^n)\cap\hat{H}_0^{1,r}(\boldsymbol{R}^n)$ for π in (2.7). From this we conclude now that $S: h\to\pi$ is a bounded operator from $\hat{H}^{-1,q}(\boldsymbol{R}^n)\cap\hat{H}^{-1,r}(\boldsymbol{R}^n)$ to $\hat{H}_0^{1,q}(\boldsymbol{R}^n)\cap\hat{H}_0^{1,r}(\boldsymbol{R}^n)$ with

$$\|\nabla\pi\|_q+\|\nabla\pi\|_r \leqq C(\|h\|_{-1,q}+\|h\|_{-1,r}),$$

where $C=C(n, q, r)$ is independent of h.

Using S, we give an explicit formula for the pair $\{u, p\}$ of solution in (2.3). For each $f\in\hat{H}^{-1,q}(\boldsymbol{R}^n)^n\cap\hat{H}^{-1,r}(\boldsymbol{R}^n)^n$, we define $\{u, p\}$ by

$$u = Sf+S(\nabla\operatorname{div} Sf), \qquad p = -\operatorname{div} Sf .$$

Here $Sf=S(f_1, \cdots, f_n)\equiv(Sf_1, \cdots, Sf_n)$ and correspondingly for $S(\nabla\operatorname{div} Sf)$. Now it is easy to see that such $\{u, p\}$ satisfies (2.4). To show that $\operatorname{div} u=0$, we observe that

$$(\operatorname{div} S(\nabla\psi)+\psi, \Delta\phi) = 0 \qquad \text{for all } \psi\in L^q(\boldsymbol{R}^n),\ \phi\in C_0^\infty(\boldsymbol{R}^n).$$

Since the space $H=\{\Delta\phi;\ \phi\in C_0^\infty(\boldsymbol{R}^n)\}$ is dense in $L^{q'}(\boldsymbol{R}^n)$, the above identity yields that $\operatorname{div} S(\nabla\psi)=-\psi$ for all $\psi\in L^q(\boldsymbol{R}^n)$. Then we get $\operatorname{div} u=0$ and see that the above pair $\{u, p\}$ has the desired properties.

Now it remains to show the uniqueness. Let $\{u', p'\}$ with $u'\in\hat{X}_\sigma^q(\boldsymbol{R}^n)\cap\hat{X}_\sigma^r(\boldsymbol{R}^n)$ and $p'\in L^q(\boldsymbol{R}^n)\cap L^r(\boldsymbol{R}^n)$ satisfy (2.4). Then $\bar{u}\equiv u-u'$, $\bar{p}\equiv p-p'$ satisfies (2.4) with $f=0$. Applying the operator div to both sides of the first equation, we get $\Delta\bar{p}=0$ in the sense of distributions in $\boldsymbol{R}^n$. Hence $\bar{p}$ is of class C^∞ and harmonic. Since $\bar{p}\in L^q(\boldsymbol{R}^n)\cap L^r(\boldsymbol{R}^n)$, it follows from the Liouville theorem that $\bar{p}\equiv 0$ in $\boldsymbol{R}^n$. Therefore $(\nabla\bar{u}, \nabla\psi)=0$ for all $\psi\in C_0^\infty(\boldsymbol{R}^n)^n$. From (2.2) we obtain $\bar{u}=0$. This completes the proof. ■

2.4. In this subsection we show a regularity property at infinity for solutions of the Stokes equations in Ω:

$$(2.9)\qquad \begin{aligned} -\Delta u+\nabla p &= f, \qquad \operatorname{div} u = 0 \quad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega . \end{aligned}$$

Compared with the case when $\Omega=\boldsymbol{R}^n$, we have a restriction on r in Lemma

2.4. Let us recall the trace theorem for vector functions. Take $R>0$ so that $B_R\equiv\{x\in\boldsymbol{R}^n;\ |x|<R\}\supset\partial\Omega$ and set $\Omega_R\equiv\Omega\cap B_R$, $E^q(\Omega_R)\equiv\{u\in L^q(\Omega_R)^n;\ \operatorname{div} u\in L^q(\Omega_R)\}$ $(1<q<\infty)$. Then it follows from Fujiwara-Morimoto [**10**, Lemma 1] that the boundary value $u\cdot\nu$ of the normal component to $\partial\Omega_R=\partial\Omega\cup\{|x|=R\}$ exists as element belonging to $W^{-1/q,q}(\partial\Omega_R)\equiv W^{1/q,q'}(\partial\Omega_R)^*$ and that the following generalized Stokes formula holds:

$$
\begin{aligned}
(2.10)\qquad &(\operatorname{div} u,\ \phi)_{\Omega_R}+(u,\ \nabla\phi)_{\Omega_R}\\
&=-\langle u\cdot\nu,\ \phi|_{\partial\Omega}\rangle_{\partial\Omega}+\langle u\cdot\nu_R,\ \phi|_{\partial B_R}\rangle_{\partial B_R}\qquad\text{for } \phi\in W^{1,q'}(\Omega_R).
\end{aligned}
$$

Here ν and ν_R denote unit outer normals to $\partial\Omega$ and $\partial B_R\equiv\{|x|=R\}$, respectively; $\langle\cdot,\ \cdot\rangle_{\partial D}$ denotes the duality paring of $W^{-1/q,q}(\partial D)$ and $W^{1/q,q'}(\partial D)$. Moreover, the map $u\in E^q(\Omega_R)\to u\cdot\nu\in W^{-1/q,q}(\partial\Omega_R)$ is surjective. Our regularity result now reads:

LEMMA 2.5. *Let $1<q<\infty$, and $n'(=n/(n-1))<r<\infty$. Let $f\in\hat{H}^{-1,q}(\Omega)^n\cap\hat{H}^{-1,r}(\Omega)^n$. Suppose that $\{u,\ p\}\in\hat{X}_\sigma^q(\Omega)\times L^q(\Omega)$ and satisfies* (2.9) *in the sense of distributions in Ω. Then we have $\nabla u\in L^r(\Omega)^{n^2}$ and $p\in L^r(\Omega)$. In case $1<q<n$, we have in particular, $u\in\hat{X}_\sigma^r(\Omega)$. In case $n\leqq r$ for $n\geqq3$ and in case $2<r$ for $n=2$, we have also $u\in\hat{X}_\sigma^r(\Omega)$.*

PROOF. *Step* 1. We shall first show the local regularity

$$
(2.11)\qquad u\in H^{1,r}(\Omega_R)^n,\qquad p\in L^r(\Omega_R).
$$

This is trivial if $n'<r\leqq q$. Suppose that $q<r<\infty$. Let us first assume that $1/q-1/n\leqq1/r$. Choose $N(>R)$ sufficiently large and take $\chi\in C_0^\infty(\boldsymbol{R}^n)$ with $0\leqq\chi\leqq1$, $\chi(x)\equiv1$ for $|x|\leqq N$, $\chi(x)\equiv0$ for $|x|\geqq N+1$. From (2.9) we get the following equation on $\Omega_{N+1}=\Omega\cap B_{N+1}$:

$$
\begin{aligned}
(2.12)\qquad &-\Delta(\chi u)+\nabla(\chi p)=\hat{f},\qquad \operatorname{div}(\chi u)=\hat{g}\quad\text{in } \Omega_{N+1},\\
&\chi u=0\qquad\text{on } \partial\Omega_{N+1},
\end{aligned}
$$

where $\hat{f}=\chi f-2\nabla\chi\cdot\nabla u-\Delta\chi\cdot u+\nabla\chi\cdot p$, $\hat{g}=\nabla\chi\cdot u$. Since $1/q-1/n\leqq1/r$, by the Sobolev inequality we have the continuous embeddings $L^q(\Omega_{N+1})\subset H^{-1,r}(\Omega_{N+1})$ $(\equiv H_0^{1,r'}(\Omega_{N+1})^*)$, $H^{1,q}(\Omega_{N+1})\subset L^r(\Omega_{N+1})$. Hence from the assumption, $\hat{f}\in H^{-1,r}(\Omega_{N+1})^n$ and $\hat{g}\in L^r(\Omega_{N+1})$. Since $\int_{\Omega_{N+1}}\hat{g}\,dx=-\int_{\partial\Omega}u\cdot\nu\,dS+\int_{\partial B_{N+1}}\chi u\cdot\nu\,dS=0$, it follows from Cattabriga [7] and Kozono-Sohr [**15**, Proposition 2.10] that $\chi u\in H^{1,r}(\Omega_{N+1})^n$ and $\chi p\in L^r(\Omega_{N+1})$. Since $\chi\equiv1$ on Ω_N, we obtain (2.11).

We next consider the case $1/q-2/n\leqq1/r<1/q-1/n$. From the above argument we have $u\in H^{1,q^*}(\Omega_N)^n$ and $p\in L^{q^*}(\Omega_N)$ with $1/q^*=1/q-1/n$. Taking q^* instead of q and then using the same argument as above, we get $u\in H^{1,r}(\Omega_{N-1})^n$

and $p\in L^{r}(\Omega_{N-1})$ for $r>n'$ with $1/r\geqq 1/q-2/n$. Proceeding in the same way to the case $1/r>1/q-2/n$, by the bootstrap argument with finite steps, we get (2.11) for all $r<n'$.

Step 2. Since $f\in\hat{H}^{-1,q}(\Omega)^{n}\cap\hat{H}^{-1,r}(\Omega)^{n}$, there is a function $F\in L^{q}(\Omega)^{n^2}\cap L^{r}(\Omega)^{n^2}$ such that $f=\operatorname{div}F$, i.e., $\langle f, \phi\rangle=-(F, \nabla\phi)$ holds for all $\phi\in C_0^{\infty}(\Omega)^{n}$.

Indeed, $\hat{H}_0^{1,q'}(\Omega)\cap\hat{H}_0^{1,r'}(\Omega)$ is dense in $\hat{H}_0^{1,q'}(\Omega)$ and in $\hat{H}_0^{1,r'}(\Omega)$. Hence $(\hat{H}_0^{1,q'}(\Omega)+\hat{H}_0^{1,r'}(\Omega))^{*}=\hat{H}^{-1,q}(\Omega)\cap\hat{H}^{-1,r}(\Omega)$ (see Aronszajn-Gagliardo [**2**, Theorem 8.3]). Consider the bounded operator $-\nabla:\hat{H}_0^{1,q'}(\Omega)+\hat{H}_0^{1,r'}(\Omega)\rightarrow L^{q'}(\Omega)^{n}+L^{r'}(\Omega)^{n}$. Using the closed range theorem for the adjoint operator $\operatorname{div}=(-\nabla)^{*}: L^{q}(\Omega)^{n}\cap L^{r}(\Omega)^{n}\rightarrow\hat{H}^{-1,q}(\Omega)\cap\hat{H}^{-1,r}(\Omega)$ in a similar manner as in (2.6), we get a function $F\in L^{q}(\Omega)^{n^2}\cap L^{r}(\Omega)^{n^2}$ with $f=\operatorname{div}F$.

Now the first equation of (2.9) can be rewritten in the following divergence form:

$$\operatorname{div}(T(u, p)+F)=0 \quad \text{in } \Omega, \tag{2.13}$$

where $T(u, p)=\{T_{ij}(u, p)\}_{1\leq i,j\leq n}$; $T_{ij}(u, p)=-\delta_{ij}p+(\partial_i u_j+\partial_j u_i)$. From the assumption and the argument in Step 1, we see $T(u, p)+F\in E^{q}(\Omega_R)^{n}\cap E^{r}(\Omega_R)^{n}$ and hence we can take $H\in E^{q}(\Omega_R)^{n}\cap E^{r}(\Omega_R)^{n}$ such that

$$H\cdot\nu|_{\partial\Omega}=(T(u, p)+F)\cdot\nu|_{\partial\Omega}, \qquad H\cdot\nu|_{|x|=R}=0. \tag{2.14}$$

Set $\tilde{H}(x)=H(x)$ for $x\in\Omega_R$, $\tilde{H}(x)=0$ for $|x|>R$. Then we have $\tilde{H}\in L^{q}(\Omega)^{n^2}\cap L^{r}(\Omega)^{n^2}$ with $\operatorname{div}\tilde{H}\in L^{q}(\Omega)^{n}\cap L^{r}(\Omega)^{n}$. Take $s\in(1, \infty)$ so that $1/s=1/r+1/n$. Then we have also $\tilde{H}\in L^{s}(\Omega)^{n^2}$ with $\operatorname{div}\tilde{H}\in L^{s}(\Omega)^{n}$, since $s<r$ and since $\tilde{H}$ has a compact support. Now it follows from Lemma 2.1(i) that there exists $G\in\hat{H}_0^{1,s}(\Omega)^{n^2}$ such that

$$\operatorname{div}G=\operatorname{div}\tilde{H} \quad \text{in } \Omega. \tag{2.15}$$

By the Sobolev inequality we have also $G\in L^{r}(\Omega)^{n^2}$. Set $V\equiv F-\tilde{H}+G$. Then $V\in L^{r}(\Omega)^{n^2}$ and from (2.13)-(2.15) we obtain that

$$\begin{aligned}&\operatorname{div}(T(u, p)+V)=0 \quad \text{in the sense of distributions on } \Omega,\\ &(T(u, p)+V)\cdot\nu|_{\partial\Omega}=0 \quad \text{in } W^{-1/r,r}(\partial\Omega)^{n}.\end{aligned} \tag{2.16}$$

Let us define the function $\tilde{u}$ on $\boldsymbol{R}^{n}$ by $\tilde{u}(x)=u(x)$ for $x\in\Omega$, $\tilde{u}(x)=0$ for $x\in\boldsymbol{R}^{n}/\Omega$. In the same way, we define also $\tilde{p}$ and $\tilde{V}$ on $\boldsymbol{R}^{n}$. Clearly $\tilde{u}\in\hat{X}_0^{q}(\boldsymbol{R}^{n})$, $\tilde{p}\in L^{q}(\boldsymbol{R}^{n})$ and $\tilde{V}\in L^{r}(\boldsymbol{R}^{n})^{n^2}$. Moreover, it holds

$$\operatorname{div}(T(\tilde{u}, \tilde{p})+\tilde{V})=0 \tag{2.17}$$

in the sense of distributions on $\boldsymbol{R}^{n}$. To see this, we take a function $\eta\in C^{\infty}(\boldsymbol{R}^{n})$ with $0\leqq\eta\leqq 1$ so that $\eta(x)\equiv 0$ near $\boldsymbol{R}^{n}/\Omega$, $\eta(x)\equiv 1$ for $|x|\geqq R$. By the generalized Stokes formula (2.10) and (2.16), we have

$$(T(\tilde{u}, \tilde{p})+\tilde{V}, \nabla\Phi)_{R^n} = (T(u, p)+V, \nabla(\eta\Phi))_\Omega+(T(u, p)+V, \nabla((1-\eta)\Phi))_{\Omega_R}$$
$$= -\langle (T(u, p)+V)\cdot\nu|_{\partial\Omega}, \Phi|_{\partial\Omega}\rangle = 0$$

for all $\Phi\in C_0^\infty(\boldsymbol{R}^n)^n$. This implies (2.17).

On the other hand, since $\tilde{V}\in L^r(\boldsymbol{R}^n)^{n^2}$, it follows from Lemma 2.4 that there is a pair $\{u', p'\}$ with $u'\in\hat{X}_\sigma^r(\boldsymbol{R}^n)$ and $p'\in L^r(\boldsymbol{R}^n)$ satisfying $\operatorname{div}(T(u', p')+\tilde{V})=0$ in the sense of distributions on $\boldsymbol{R}^n$. Applying the theory of harmonic functions for $\bar{u}\equiv\tilde{u}-u'$ and $\bar{p}=\tilde{p}-p'$ with such an aid of inequality as (2.8), we get as in Lemma 2.4 $\tilde{u}=u'$ and $\tilde{p}=p'$. From this it follows that $\nabla u\in L^r(\Omega)^{n^2}$ and $p\in L^r(\Omega)$.

Now it remains to show that $u\in\hat{H}_0^{1,r}(\Omega)^n$ in case $r\geqq n$ $(n\geqq 3)$, $r>2$ $(n=2)$ and in case $1<q<n$. For the former case, by Remark 2(ii) to Lemma 2.3, we get $u\in\hat{H}_0^{1,r}(\Omega)^n$. Suppose the latter case $1<q<n$ and $n'<r<n$ $(n\geqq 3)$. By the Sobolev inequality, we have $u\in L^{q^*}(\Omega)^n$ for $1/q^*=1/q-1/n$. Moreover it follows from Corollary 2.2(ii) that there is a constant vector $M\in\boldsymbol{R}^n$ such that $u+M\in L^{r^*}(\Omega)^n$ for $1/r^*=1/r-1/n$. Since $u\in L^{q^*}(\Omega)^n$, we see $M=0$ and hence $u\in L^{r^*}(\Omega)^n$. Then again by Remark 2(i) to Lemma 2.3, we get $u\in\hat{H}_0^{1,r}(\Omega)^n$. This completes the proof of Lemma 2.5. ■

3. Proof of Theorem 1.

3.1. We shall first show the corresponding result to Theorem 1 in the whole space $\boldsymbol{R}^n$.

LEMMA 3.1. *Let* $1<q<\infty$, $1<r<\infty$. *Then* $C_{0,\sigma}^\infty(\boldsymbol{R}^n)$ *is dense in* $\hat{X}_\sigma^q(\boldsymbol{R}^n)\cap\hat{X}_\sigma^r(\boldsymbol{R}^n)$ *with respect to the norm* $\|\nabla u\|_q+\|\nabla u\|_r$.

PROOF. Let us recall some basic properties of interpolation couples. From Lemma 2.3 we conclude that $\hat{H}_0^{1,q}(\boldsymbol{R}^n)$ and $\hat{H}_0^{1,r}(\boldsymbol{R}^n)$ are reflexive Banach spaces, because G_q and G_r are closed subspaces in $L^q(\boldsymbol{R}^n)^n$ and $L^r(\boldsymbol{R}^n)^n$, respectively. Since $C_0^\infty(\boldsymbol{R}^n)\subset\hat{H}_0^{1,q}(\boldsymbol{R}^n)\cap\hat{H}_0^{1,r}(\boldsymbol{R}^n)$ is dense in $\hat{H}_0^{1,q}(\boldsymbol{R}^n)$ and $\hat{H}_0^{1,r}(\boldsymbol{R}^n)$, if follows from Aroszajn-Gagliardo [2, Corollary 8.4] that

$$(\hat{H}^{-1,q}(\boldsymbol{R}^n)\cap\hat{H}^{-1,r}(\boldsymbol{R}^n))^* = \hat{H}_0^{1,q'}(\boldsymbol{R}^n)+\hat{H}_0^{1,r'}(\boldsymbol{R}^n). \tag{3.1}$$

We shall first show that

$$Y \equiv \{\operatorname{div}(\nabla\Delta\phi);\ \phi\in C_0^\infty(\boldsymbol{R}^n)\} \text{ is dense in } \hat{H}^{-1,q}(\boldsymbol{R}^n)\cap\hat{H}^{-1,r}(\boldsymbol{R}^n). \tag{3.2}$$

Suppose the contrary. Then by (3.1) there exists $0\neq h=f+g$ with $f\in\hat{H}_0^{1,q'}(\boldsymbol{R}^n)$, $g\in\hat{H}_0^{1,r'}(\boldsymbol{R}^n)$ such that $\langle\operatorname{div}(\nabla\Delta\phi), h\rangle=0$ for all $\phi\in C_0^\infty(\boldsymbol{R}^n)$. Moreover, by Lemma 2.3 we can choose $\bar{f}\in L^{1,q'}$, $\bar{g}\in L^{1,r'}$ such that

(3.3) $$\|h\|_{\hat{H}_0^{1,q'}+\hat{H}_0^{1,r'}} = \|\nabla\bar{f}+\nabla\bar{g}\|_{L^{q'}+L^{r'}};$$

$$(\nabla\bar{f}+\nabla\bar{g}, \nabla(\Delta\phi)) = 0 \qquad \text{for all } \phi\in C_0^\infty(\boldsymbol{R}^n).$$

Let $J_\varepsilon*(\varepsilon>0)$ denote the well-known Friedrichs mollifier. Then we have by the Hausdorff-Yang inequality that

$$\|\partial_i J_\varepsilon*\nabla\bar{f}\|_{q'} \leqq \|\partial_i J_\varepsilon\|_1\|\nabla\bar{f}\|_{q'}, \qquad \|\partial_i J_\varepsilon*\nabla\bar{g}\|_{r'} \leqq \|\partial_i J_\varepsilon\|_1\|\nabla\bar{g}\|_{r'}$$

for $i=1, 2, \cdots, n$; this leads to

(3.4) $$\operatorname{div}(J_\varepsilon*\nabla\bar{f}) \in L^{q'}(\boldsymbol{R}^n), \qquad \operatorname{div}(J_\varepsilon*\nabla\bar{g}) \in L^{r'}(\boldsymbol{R}^n).$$

Taking $\phi=J_\varepsilon*\psi$, $\psi\in C_0^\infty(\boldsymbol{R}^n)$ in (3.3), we have $(\operatorname{div}(J_\varepsilon*(\nabla\bar{f}+\nabla\bar{g})), \Delta\psi)=0$, which implies by Weyl's lemma that the function $\operatorname{div}(J_\varepsilon*(\nabla\bar{f}+\nabla\bar{g}))$ is of class C^∞ and harmonic in $\boldsymbol{R}^n$. Because of (3.4) we can apply the same argument as in (2.8) and conclude that $\operatorname{div}(J_\varepsilon*(\nabla\bar{f}+\nabla\bar{g}))\equiv 0$ in $\boldsymbol{R}^n$. Hence it holds

$$(\nabla\bar{f}+\nabla\bar{g}, \nabla\phi) = \lim_{\varepsilon\downarrow 0}(J_\varepsilon*(\nabla\bar{f}+\nabla\bar{g}), \nabla\phi)$$

$$= \lim_{\varepsilon\downarrow 0}(\operatorname{div}(J_\varepsilon*(\nabla\bar{f}+\nabla\bar{g})), \phi) = 0 \qquad \text{for all } \phi\in C_0^\infty(\boldsymbol{R}^n).$$

Again by Weyl's lemma we conclude that $\bar{f}+\bar{g}$ is harmonic in $\boldsymbol{R}^n$, so is $\nabla\bar{f}+\nabla\bar{g}$. Since $\nabla\bar{f}+\nabla\bar{g}\in L^{q'}(\boldsymbol{R}^n)+L^{r'}(\boldsymbol{R}^n)$, we can use the same argument as in (2.8) and hence $\nabla\bar{f}+\nabla\bar{g}\equiv 0$ in $\boldsymbol{R}^n$. Then from (3.3) we get that $h=0$, which causes a contradiction.

Next we consider the operator A with the domain $D(A)=(\hat{X}_\sigma^q(\boldsymbol{R}^n)\cap\hat{X}_\sigma^r(\boldsymbol{R}^n))\times(L^q(\boldsymbol{R}^n)\cap L^r(\boldsymbol{R}^n))$ defined by

$$\langle A(u, p), \phi\rangle = (\nabla u, \nabla\phi)-(p, \operatorname{div}\phi)$$

for $\{u, p\}\in D(A)$ and $\phi\in\hat{H}_0^{1,q'}(\boldsymbol{R}^n)^n+\hat{H}_0^{1,r'}(\boldsymbol{R}^n)^n$. By Lemma 2.4 we see that A is a *bijective* bounded operator from $D(A)$ onto $\hat{H}^{-1,q}(\boldsymbol{R}^n)^n\cap\hat{H}^{-1,r}(\boldsymbol{R}^n)^n$. We define now the spaces V_σ, W and Y by $V_\sigma=\{-\Delta\phi+\nabla(\operatorname{div}\phi);\ \phi\in C_0^\infty(\boldsymbol{R}^n)^n\}$, $W=\{\operatorname{div}\Delta\phi;\ \phi\in C_0^\infty(\boldsymbol{R}^n)^n\}$ and $\boldsymbol{Y}=\{\operatorname{div}\nabla(\Delta\phi);\ \phi\in C_0^\infty(\boldsymbol{R}^n)^n\}$, respectively. A direct calculation yields that $A(V_\sigma\times W)=\boldsymbol{Y}$. Now from (3.2) together with the bijectivity of A, we can conclude that V_σ is dense in $\hat{X}_\sigma^q(\boldsymbol{R}^n)\cap\hat{X}_\sigma^r(\boldsymbol{R}^n)$. Since $V_\sigma\subset C_{0,\sigma}^\infty(\boldsymbol{R}^n)$, we get the desired result. This completes the proof. ■

In the next step we go over to the exterior domain Ω by using cut-off procedures. Recall that $\hat{X}_\sigma^q(\Omega)=\{u\in\hat{H}_0^{1,q}(\Omega)^n;\ \operatorname{div}u=0\}$. For a concrete characterization of the space $\hat{H}_0^{1,q}(\Omega)$, see Remark to Lemma 2.3. The following lemma gives us an approximation property.

LEMMA 3.2. *Let* $1<q<\infty$, $1<r<\infty$ *and* $u\in\hat{X}_\sigma^q(\Omega)\cap\hat{X}_\sigma^r(\Omega)$. *Suppose that* $\tilde{u}(x)=u(x)$ *for* $x\in\Omega$, $\tilde{u}(x)=0$ *for* $x\in\boldsymbol{R}^n/\Omega$. *Then there is a sequence* $v_j\in$

$C_{0,\sigma}^{\infty}(\boldsymbol{R}^n)$ $(j=1, 2, \cdots)$ *such that*

(3.5) $$\nabla v_j \longrightarrow \nabla \tilde{u} \text{ in both } L^q(\boldsymbol{R}^n)^{n^2} \text{ and } L^r(\boldsymbol{R}^n)^{n^2};$$

(3.6) $$v_j \longrightarrow \tilde{u} \text{ in both } L^q_{\mathrm{loc}}(\boldsymbol{R}^n)^n \text{ and } L^r_{\mathrm{loc}}(\boldsymbol{R}^n)^n.$$

PROOF. Let us assume that $1<q\leqq r<\infty$. Since $\tilde{u}\in L^{1,q}\cap L^{1,r}$ with div $\tilde{u}=0$ in $\boldsymbol{R}^n$, we have by Lemma 3.1 that there is a sequence $v_j\in C_{0,\sigma}^{\infty}(\boldsymbol{R}^n)$ $(j=1, \cdots)$ satisfying (3.5). Since $q\leqq r$, it is enough to show that we can choose such a sequence $\{v_j\}^{\infty}{}_{=1}$ as

(3.7) $$v_j \longrightarrow \tilde{u} \quad \text{in } L^r_{\mathrm{loc}}(\boldsymbol{R}^n)^n.$$

(i) Let $1<q\leqq r<n$. By Remark 2(ii) to Lemma 2.3, $\tilde{u}\in L^{r*}(\boldsymbol{R}^n)$ for $1/r^*=1/r-1/n$. Moreover by the Sobolev inequality, we have $\|v\|_{r*}\leqq C\|\nabla v\|_r$ for all $v\in\hat{H}_0^{1,r}(\boldsymbol{R}^n)$ and hence (3.5) yields $v_j\rightarrow\tilde{u}$ in $L^{r*}(\boldsymbol{R}^n)^n$, which leads to (3.7).

(ii) Let $1<q<n\leqq r<\infty$. By the interpolation inequality we see that (3.5) holds even in $L^s(\boldsymbol{R}^n)^{n^2}$ for all s with $q\leqq s\leqq r<\infty$. So it follows from the Sobolev inequality as above that $v_j\rightarrow\tilde{u}$ in $L^{s*}(\boldsymbol{R}^n)$ for all $s^*\geqq nq/(n-q)$. In particular, we get (3.7).

(iii) Let $n\leqq q\leqq r<\infty$. First we conclude from (3.5) that there is a sequence $c_j\in\boldsymbol{R}^n$ $(j=1, 2, \cdots)$ such that

(3.8) $$\|v_j+c_j-\tilde{u}\|_{L^r(B)} \longrightarrow 0 \quad \text{as } j\longrightarrow\infty \qquad \text{for all ball } B\subset\boldsymbol{R}^n.$$

To see this take a sequence $B_1\subset B_2\subset\cdots\subset B_N\subset\cdots$ of balls in $\boldsymbol{R}^n$, where $B_j=\{x\in\boldsymbol{R}^n;\ |x|<j\}$. By making use of the Poincaré inequality $\inf_{c\in\boldsymbol{R}^n}\|v+c\|_{L^r(B_N)}\leqq K_N\|\nabla v\|_{L^r(B_N)}$ holding for $v\in H^{1,r}(B_N)$, we can choose for each N a sequence $\{c_j^{(N)}\}_{j=1}^{\infty}$ in $\boldsymbol{R}^n$ and a function $v^{(N)}\in H^{1,r}(B_N)$ such that $\|v_j+c_j^{(N)}-v^{(N)}\|_{L^r(B_N)}\rightarrow 0$ as $j\rightarrow\infty$. Since $\{C_j^{(N)}\}_{j=1}^{\infty}$ and $\{C_j^{(N')}\}_{j=1}^{\infty}$ $(N<N')$ can differ by a constant in the limit as $j\rightarrow\infty$ and since $v^{(N)}$ and $v^{(N')}$ can differ at most by a constant in B_N, it is possible to redefine the sequence $\{c_j^{(N)}, v^{(N)}\}_{N=1}^{\infty}$ so that they are all equal in common regions of definition, thereby, determining a sequence $\{c_j\}_{j=1}^{\infty}$ in $\boldsymbol{R}^n$ and a function $v\in L^{1,r}$ with property $\|v_j+c_j-v\|_{L^r(B)}\rightarrow 0$ for all ball $B\subset\boldsymbol{R}^n$. Clearly $\tilde{u}-v\equiv$const. in $\boldsymbol{R}^n$ and hence (3.8) follows.

Let us first assume that $n<q$. Take a function $\zeta\in C_0^{\infty}(\boldsymbol{R}^n)$ satisfying $0\leqq\zeta\leqq 1$, $\zeta(x)=1$ for $|x|\leqq 1+\delta$ and $\zeta(x)=0$ for $|x|\geqq 2-\delta$, where $0<\delta<1/4$ and define $\zeta_j(x)\equiv\zeta(x/j)$ for $j=1, 2, \cdots$. $\{\zeta_j\}_{j=1}^{\infty}$ will be called a sequence of n-dimensional cut-off functions. Then we get $\|\nabla\zeta_j\|_q\leqq Cj^{-1+n/q}$ for all $j=1, 2, \cdots$, so there is a subsequence $\{\zeta_{j(k)}\}_{k=1}^{\infty}$ of $\{\zeta_j\}_{j=1}^{\infty}$ such that

(3.9) $$\lim_{k\rightarrow\infty}|c_k|\|\nabla\zeta_{j(k)}\|_q=0, \qquad \lim_{k\rightarrow\infty}|c_k|\|\nabla\zeta_{j(k)}\|_r=0.$$

Now let us consider the equations:

$$\text{(3.10)}\qquad \operatorname{div} w_k = c_k\cdot\nabla\zeta_{j(k)} \quad \text{in } D_{j(k)}, \qquad w_k = 0 \quad \text{on } \partial D_{j(k)},$$

where $D_{j(k)}=\{x\in \boldsymbol{R}^n;\ j(k)<|x|<2j(k)\}$ and $\partial D_{j(k)}=\{x\in \boldsymbol{R}^n;\ |x|=j(k), 2j(k)\}$. We see that $c_k\cdot\nabla\zeta_{j(k)}\in C_0^\infty(D_{j(k)})$ and that $\int_{D_{j(k)}} c_k\cdot\nabla\zeta_{j(k)}dx=0$. It follows from Lemma 2.1(ii) that there exists $w_k\in C_0^\infty(D_{j(k)})^n$ such that (3.10) holds and that

$$\text{(3.11)}\qquad \begin{aligned} \|\nabla w_k\|_{q,D_{j(k)}} &\leqq M|c_k|\,\|\nabla\zeta_{j(k)}\|_q, \\ \|\nabla w_k\|_{r,D_{j(k)}} &\leqq M|c_k|\,\|\nabla\zeta_{j(k)}\|_r \end{aligned}$$

with $M>0$ independent of k.

Now we set $u_k:=v_k+c_k\zeta_{j(k)}-\tilde{w}_k$ $(k=1,\cdots)$, where $\tilde{w}_k$ denotes the zero extension of w_k to $\boldsymbol{R}^n$. Then from (3.10), $u_k\in C_{0,\sigma}^\infty(\boldsymbol{R}^n)$ and by (3.11) we get

$$\|\nabla u_k-\nabla\tilde{u}\|_q \leqq \|\nabla v_k-\nabla\tilde{u}\|_q+(M+1)|c_k|\,\|\nabla\zeta_{j(k)}\|_q$$

and the same inequality with q replaced by r. From (3.5) and (3.9) we obtain

$$\nabla u_k \longrightarrow \nabla\tilde{u} \ \text{in both}\ L^q(\boldsymbol{R}^n)^{n^2} \ \text{and}\ L^r(\boldsymbol{R}^n)^{n^2}.$$

Moreover we have

$$\|u_k-\tilde{u}\|_{L^r(B)} \leqq \|v_k+c_k-\tilde{u}\|_{L^r(B)}+|c_k|\,\|\zeta_{j(k)}-1\|_{L^r(B)}+\|\tilde{w}_k\|_{L^r(B)}$$

for each fixed ball $B\subset\boldsymbol{R}^n$ and $k=1, 2, \cdots$. Since $\zeta_{j(k)}=1$, $\tilde{w}_k=0$ on B for large k, we obtain from the above and (3.8)

$$u_k \longrightarrow \tilde{u} \qquad \text{in } L_{\mathrm{loc}}^r(\boldsymbol{R}^n)^n.$$

Hence we get the desired sequence in case $n<q$.

Now it remains to show the case $q=n$. In this case we have only $\|\nabla\zeta_j\|_n \leqq$const. for all $j=1, 2, \cdots$. For each fixed k, let us consider the following equation in $D_j=\{x\in\boldsymbol{R}^n;\ j<|x|<2j\}$:

$$\text{(3.12)}\qquad \begin{aligned} &\operatorname{div} w_j^k = c_k\cdot\nabla\zeta_j \qquad \text{in } D_j, \\ &w_j^k = 0 \qquad \text{on } \partial D_j = \{x\in\boldsymbol{R}^n;\ |x|=j, 2j\}, \end{aligned}$$

where c_k is the same constant in (3.8). Then by Lemma 2.1(ii) we can choose $w_j^k\in C_0^\infty(D_j)^n$ satisfying (3.12) with

$$\|\nabla w_j^k\|_n \leqq K|c_k|\,\|\nabla\zeta_j\|_n, \qquad \|\nabla w_j^k\|_r \leqq K|c_k|\,\|\nabla\zeta_j\|_r,$$

where $K=K(n, r)$ is a constant independent of k and j. It is easy to see that

$$\text{(3.13)}\qquad \begin{aligned} &\nabla w_j^k \longrightarrow 0 \ \textit{weakly in}\ L^n(\boldsymbol{R}^n)^{n^2} \qquad \text{as } j\to\infty; \\ &\nabla w_j^k \longrightarrow 0 \ \textit{strongly in}\ L^r(\boldsymbol{R}^n)^{n^2} \qquad \text{as } j\to\infty \end{aligned}$$

holds for each fixed k. We use now the Mazur theorem [**26**, p. 120 Theorem 2]; for each fixed k, there are sequences $\{m_j^k\}_{j=1}^\infty$ and $\{\hat{m}_j^k\}_{j=1}^\infty$ of positive integers with $m_j^k<\hat{m}_j^k$, $\lim_{j\to\infty} m_j^k=\infty$ and $\hat{m}_j^k-m_j^k+1$ real numbers $\beta_i^k\geqq 0$ $(i=m_j^k, m_j^k+1, \cdots, \hat{m}_j^k)$ with $\sum_{i=m_j^k}^{\hat{m}_j^k}\beta_i^k=1$ such that the functions $\bar{\zeta}_j^k\equiv\sum_{i=m_j^k}^{\hat{m}_j^k}\beta_i^k\zeta_i$, $\bar{w}_j^k\equiv\sum_{i=m_j^k}^{\hat{m}_j^k}\beta_i^k w_i$ satisfy $c_k\bar{\zeta}_j^k\to 0$, $\nabla\bar{w}_j^k\to 0$ *strongly* in $L^n(\boldsymbol{R}^n)^{n^2}\cap L^r(\boldsymbol{R}^n)^{n^2}$ as $j\to\infty$ for each fixed k. Hence we can choose subsequences $\{\bar{\zeta}_{j(k)}^k\}_{k=1}^\infty$ of $\{\bar{\zeta}_j^k\}_{j,k=1}^\infty$ and $\{\bar{w}_{j(k)}^k\}_{k=1}^\infty$ of $\{\bar{w}_j^k\}_{j,k=1}^\infty$ so that

$$c_k\nabla\bar{\zeta}_{j(k)}^k \longrightarrow 0, \qquad \nabla\bar{w}_{j(k)}^k \longrightarrow 0 \tag{3.14}$$

$$\textit{strongly in } L^n(\boldsymbol{R}^n)^{n^2}\cap L^r(\boldsymbol{R}^n)^{n^2} \qquad \text{as } k\to\infty\,.$$

Now the desired sequence $\{u_k\}_{k=1}^\infty$ is defined by

$$u_k \equiv v_k + c_k\bar{\zeta}_{j(k)}^k - \bar{w}_{j(k)}^k\,.$$

Indeed, we have by (3.12) that $u_k\in C_{0,\sigma}^\infty(\boldsymbol{R}^n)$ for all k and it follows from (3.5) and (3.14) that

$$\|\nabla u_k-\nabla\tilde{u}\|_{L^n\cap L^r} \leqq \|\nabla v_k-\nabla\tilde{u}\|_{L^n\cap L^r}+\|c_k\nabla\bar{\zeta}_{j(k)}^k\|_{L^n\cap L^r}+\|\nabla\bar{w}_{j(k)}^k\|_{L^n\cap L^r} \tag{3.15}$$

$$\longrightarrow 0 \qquad \text{as } k\to 0\,.$$

In the same way, we have

$$\|u_k-\tilde{u}\|_{L^r(B)} \leqq \|v_k+c_k-\tilde{u}\|_{L^r(B)}$$

$$+|c_k|\sum_{i=m_{j(k)}^k}^{\hat{m}_{j(k)}^k}\beta_i^k\|\zeta_i-1\|_{L^r(B)}+\sum_{i=m_{j(k)}^k}^{\hat{m}_{j(k)}^k}\beta_i^k\|w_i^k\|_{L^r(B)}$$

for each fixed ball $B\subset\boldsymbol{R}^n$. Since we may assume $m_{j(k)}^k\geqq k$, we see that $\zeta_i\equiv 1$, $w_i^k\equiv 0$ $(i=m_{j(k)}^k, \cdots, \hat{m}_{j(k)}^k)$ on B for sufficiently large k. Hence from the above inequality and (3.8), it follows that

$$\limsup_{k\to\infty}\|u_k-\tilde{u}\|_{L^r(B)} = 0\,. \tag{3.16}$$

Now (3.15-16) shows that $\{u_k\}_{k=1}^\infty$ has the desired property. ■

3.2. *Completion of the Proof of Theorem* 1.

Let $1<q<\infty$, $1<r<\infty$ and $u\in\hat{X}_\sigma^q(\Omega)\cap\hat{X}_\sigma^r(\Omega)$. Then we have to show that there is a sequence $u_j\in C_{0,\sigma}^\infty(\Omega)$ $(j=1, 2, \cdots)$ such that

$$\nabla u_j \longrightarrow \nabla u \text{ in both } L^q(\Omega)^{n^2} \text{ and } L^r(\Omega)^{n^2}\,. \tag{3.17}$$

Let $\tilde{u}(x)=u(x)$ for $x\in\Omega$, $\tilde{u}(x)=0$ for $x\in\boldsymbol{R}^n/\Omega$. Then by Lemma 3.2 there is a sequence $v_j\in C_{0,\sigma}^\infty(\boldsymbol{R}^n)$ $(j=1, 2, \cdots)$ such that (3.5) and (3.6) hold. Take a function $\eta\in C^\infty(\boldsymbol{R}^n)$ satisfying $0\leqq\eta\leqq 1$, $\eta(x)=0$ in a neighbourhood of $\partial\Omega$, $\eta(x)=1$ for large $|x|$ and take $R>0$ so that the subdomain $\Omega_R\equiv\Omega\cap\{x\in\boldsymbol{R}^n;$

$|x|<R\}$ contains supp $\nabla\eta$. For each $j=1, \cdots$ we consider the following equation in Ω_R:

(3.18) $$\operatorname{div} w_j = v_j\cdot\nabla\eta \quad \text{in } \Omega_R, \qquad w_j=0 \quad \text{on } \partial\Omega_R \equiv \partial\Omega\cup\{|x|=R\}.$$

Since $v_j\cdot\nabla\eta\in C_0^\infty(\Omega_R)$ and $\int_{\Omega_R} v_j\cdot\nabla\eta\, dx=0$, it follows from Lemma 2.1(ii) that there is a sequence $w_j\in C_0^\infty(\Omega_R)^n$ $(j=1, \cdots)$ such that (3.18) holds and

$$\begin{aligned}&\|\nabla w_j-\nabla w_k\|_{q,\Omega_R}+\|\nabla w_j-\nabla w_k\|_{r,\Omega_R}\\ &\leqq C(\|(v_j-v_k)\cdot\nabla\eta\|_{q,\Omega_R}+\|(v_j-v_k)\cdot\nabla\eta\|_{r,\Omega_R})\end{aligned}$$

with C independent of $j, k=1, 2, \cdots$. By (3.6) we obtain a function $w\in H_0^{1,q}(\Omega_R)^n\cap H_0^{1,r}(\Omega_R)^n$ satisfying

(3.19) $$\begin{aligned}&w_j \longrightarrow w \text{ in both } H_0^{1,q}(\Omega_R)^n \text{ and } H_0^{1,r}(\Omega_R)^n;\\ &\operatorname{div} w=\tilde{u}\cdot\nabla\eta \quad \text{in } \Omega_R.\end{aligned}$$

Since $\tilde{u}|_{\partial\Omega}=u|_{\partial\Omega}=0$, we conclude from the assumption on u that $(1-\eta)\tilde{u}+w$ belongs to $H_0^{1,q}(\Omega_R)^n\cap H_0^{1,r}(\Omega_R)^n$ and that $\operatorname{div}[(1-\eta)\tilde{u}+w]=0$. Hence there is a sequence $h_j\in C_{0,\sigma}^\infty(\Omega_R)$ $(j=1, \cdots)$ such that

(3.20) $$h_j \longrightarrow (1-\eta)\tilde{u}+w \text{ in both } H_0^{1,q}(\Omega_R)^n \text{ and } H_0^{1,r}(\Omega_R)^n$$

(see, e.g., [**6**, Lemma 4.1] and note that Ω_R is bounded). Now the desired sequence $\{u_j\}_{j=1}^\infty$ is obtained by $u_j:=\tilde{h}_j+\eta v_j-\tilde{w}_j$, where $\tilde{h}_j$ and $\tilde{w}_j$ denote zero extensions of h_j and w_j to Ω, respectively. Indeed, by (3.18) we see that $u_j\in C_{0,\sigma}^\infty(\Omega)$. Denoting by $\tilde{w}$ as the zero extension of w, we obtain from (3.5-6), (3.19-20).

$$\begin{aligned}&\|\nabla u_j-\nabla u\|_{L^q(\Omega)\cap L^r(\Omega)}\\ &=\|\nabla(\tilde{h}_j+\eta v_j-\tilde{u}_j)-\nabla[(1-\eta)\tilde{u}+\tilde{w}+\eta\tilde{u}-\tilde{w}]\|_{L^q(\Omega)\cap L^r(\Omega)}\\ &\leqq \|\nabla h_j-\nabla[(1-\eta)\tilde{u}+w]\|_{L^q(\Omega_R)\cap L^r(\Omega_R)}+C\|v_j-\tilde{u}\|_{L^q(\Omega_R)\cap L^r(\Omega_R)}\\ &\qquad+\|\nabla v_j-\nabla\tilde{u}\|_{L^q(\Omega)\cap L^r(\Omega)}+\|\nabla w_j-\nabla w\|_{L^q(\Omega_R)\cap L^r(\Omega_R)}\\ &\longrightarrow 0 \quad \text{as } j\to\infty.\end{aligned}$$

This completes the proof of Theorem 1. ∎

REMARK 3.3. (i) Taking $q=r$ in Theorem 1, we get in particular that

(3.21) $$C_{0,\sigma}^\infty(\Omega) \text{ is dense in } \hat{X}_\sigma^q(\Omega).$$

In case $n=3$ and $q=2$, (3.21) has been proved by Heywood [**14**, Theorem 8]. See also [**6**, Lemma 4.1].

(ii) Using Corollary 2.2(i), we conclude from (i) above that for each $f\in \hat{H}^{-1,q}(\Omega)^n$ there is a unique $p\in L^q(\Omega)$ satisfying $f=\nabla p$ if and only if $\langle f, \phi\rangle=0$ for all $\phi\in C_{0,\sigma}^{\infty}(\Omega)$.

4. Proof of Theorem 2.

4.1. We shall first show the corresponding results to Theorem 2 in bounded domains and in the whole space $\boldsymbol{R}^n$. The assertion in the former case has been proved by Giga [**12**, Proposition in Appendix]:

LEMMA 4.1. (*Giga*) *Let D be a bounded domain in $\boldsymbol{R}^n$ with $C^{2+\mu}$-boundary* $(0<\mu<1)$ *and* $1<q<\infty$, $1<r<\infty$. *Let $H_{0;\sigma}^{1;q}(D)$ be the completion of the space $C_{0,\sigma}^{\infty}(D)$ in $H^{1,q}(D)^n$, i.e., with respect to the norm $\|\nabla u\|_{q,D}+\|u\|_{q,D}$. Then $C_{0,\sigma}^{\infty}(D)$ is dense in $H_{0;\sigma}^{1;q}(D)\cap L^r(D)^n$.*

Giga [**12**] gave the proof by using his result on the concrete characterization of fractional powers of the Stokes operator [**11**]. Although he proved only the case $q=2$, we see easily that the parallel argument works even for $q\in(1, \infty)$, so we may omit the detail.

REMARK. Giga [**12**] conjectured in Remark that the same proof works even for unbounded domains.

In the whole space $\boldsymbol{R}^n$, we have restriction on q and r. The following lemma is essentially due to Masuda [**17**, Proposition 1].

LEMMA 4.2. *Let* $1<q<\infty$, $1<r<\infty$ *satisfy the following cases* (i) *or* (ii):

$$\text{(i)}\ 1<q<n,\quad 1<r<\infty\,;\qquad \text{(ii)}\ n\leqq q<r<\infty\,.$$

Then $C_{0,\sigma}^{\infty}(\boldsymbol{R}^n)$ is dense in $\hat{X}_{\sigma}^{q}(\boldsymbol{R}^n)\cap L^r(\boldsymbol{R}^n)^n$.

PROOF. We make use of the argument of Masuda [**17**, Proposition 1]. Let $1<p<\infty$ and $B=B_p=-\Delta$ with domain $D(B_p)=H^{2,p}(\boldsymbol{R}^n)^n$. Since the range of $-\Delta$ is dense in $L^p(\boldsymbol{R}^n)$, we have

$$\lim_{\lambda\downarrow 0} B_p(\lambda+B_p)^{-1}f=f \ \text{in}\ L^p(\boldsymbol{R}^n)^n \qquad \text{for all}\ f\in L^p(\boldsymbol{R}^n)^n\,. \tag{4.1}$$

(i) *Case* $1<q<n$, $1<r<\infty$. Let $u\in\hat{X}_{\sigma}^{q}(\boldsymbol{R}^n)\cap L^r(\boldsymbol{R}^n)^n$. By the Sobolev inequality, we have $u\in L^{q*}(\boldsymbol{R}^n)^n\cap L^r(\boldsymbol{R}^n)^n$ for $1/q*=1/q-1/n$ with $\nabla u\in L^q(\boldsymbol{R}^n)^{n^2}$. The approximation $\{u_{\varepsilon,\lambda,j}\}_{\varepsilon,\lambda>0}^{j=1,\cdots}$ of u is defined by

$$u_{\varepsilon,\lambda,j}:=(-\Delta+\nabla\,\mathrm{div})\zeta_j(\lambda+B_{q*})^{-1}J_\varepsilon * u\,,$$

where $J_\varepsilon *$ and $\{\zeta_j\}_{j=1}^{\infty}$ denote the Friedrichs mollifier and the sequence of n-dimensional cut-off functions, respectively (see Lemma 3.2). Clearly $u_{\varepsilon,\lambda,j}\in$

$C_{0,\sigma}^{\infty}(\boldsymbol{R}^n)$ for all $\varepsilon, \lambda>0$, $j=1, 2, \cdots$. Let us first show that

$$\nabla u_{\varepsilon,\lambda,j} \longrightarrow \nabla u_{\varepsilon,\lambda} \quad \text{in } L^q(\boldsymbol{R}^n)^{n^2} \text{ as } j\to\infty; \tag{4.2}$$

$$u_{\varepsilon,\lambda,j} \longrightarrow u_{\varepsilon,\lambda} \quad \text{in } L^r(\boldsymbol{R}^n)^n \text{ as } j\to\infty, \tag{4.3}$$

where $u_{\varepsilon,\lambda}:=(-\Delta+\nabla \operatorname{div})(\lambda+B)^{-1}J_\varepsilon*u(=B(\lambda+B)^{-1}J_\varepsilon*u$, since $\operatorname{div} u=0)$. Indeed, using $\|\nabla^k\zeta_j\|_\infty\leqq Cj^{-1}$ for $k=1, 2$ and $\|\nabla^3\zeta_j\|_p\leqq Cj^{n/p-3}$ $(\nabla^k=(\partial/\partial x^1)^{\alpha_1}\cdots(\partial/\partial x^n)^{\alpha_n}$, $\alpha_1+\cdots+\alpha_n=k)$ for $1<p<\infty$ with a constant C independent of $j=1, \cdots$, we have by the Hölder inequality

$$\begin{aligned}
&\|\nabla u_{\varepsilon,\lambda,j}-\nabla u_{\varepsilon,\lambda}\|_q\\
&\quad\leqq C(\|(\zeta_j-1)\nabla^2(\lambda+B)^{-1}\nabla J_\varepsilon*u\|_q+\|(\nabla\zeta_j)(\lambda+B)^{-1}\nabla J_\varepsilon*u\|_q\\
&\qquad+\|(\nabla^2\zeta_j)(\lambda+B)^{-1}\nabla J_\varepsilon*u\|_q+\|(\nabla^3\zeta_j)(\lambda+B)^{-1}J_\varepsilon*u\|_q)\\
&\quad\leqq C\{\|\nabla^2(\lambda+B)^{-1}\nabla J_\varepsilon*u\|_{L^q(|x|>j)}\\
&\qquad+(\|\nabla\zeta_j\|_\infty+\|\nabla^2\zeta_j\|_\infty)\|(\lambda+B)^{-1}\nabla J_\varepsilon*u\|_{H^{1,q}}+\|\nabla^3\zeta_j\|_n\cdot\|(\lambda+B)^{-1}J_\varepsilon*u\|_{q*}\}\\
&\quad\leqq C\{\|\nabla^2(\lambda+B)^{-1}\nabla J_\varepsilon*u\|_{L^q(|x|>j)}+j^{-1}\cdot\|(\lambda+B)^{-1}\nabla J_\varepsilon*u\|_{H^{1,q}}\\
&\qquad+j^{-2}\cdot\|(\lambda+B)^{-1}J_\varepsilon*u\|_{q*}\},
\end{aligned} \tag{4.4}$$

Since $\nabla u\in L^q(\boldsymbol{R}^n)^{n^2}$ and $u\in L^{q*}(\boldsymbol{R}^n)$, (4.2) follows from the above inequality. Similarly since $u\in L^r(\boldsymbol{R}^n)^n$, we have

$$\begin{aligned}
\|u_{\varepsilon,\lambda,j}-u_{\varepsilon,\lambda}\|_r &\leqq C\{\|\nabla^2(\lambda+B)^{-1}J_\varepsilon*u\|_{L^r(|x|>j)}\\
&\qquad+(\|\nabla\zeta_j\|_\infty+\|\nabla^2\zeta_j\|_\infty)\|(\lambda+B)^{-1}J_\varepsilon*u\|_{H^{1,r}}\}\\
&\leqq C\{\|\nabla^2(\lambda+B)^{-1}J_\varepsilon*u\|_{L^r(|x|>j)}+j^{-1}\cdot\|(\lambda+B)^{-1}J_\varepsilon*u\|_{H^{1,r}}\\
&\longrightarrow 0 \quad \text{as } j\to\infty.
\end{aligned}$$

This implies (4.3). Now, letting $\lambda\to 0$ and then $\varepsilon\to 0$, we have by (4.1) that

$$\nabla u_{\varepsilon,\lambda} \longrightarrow \nabla u \quad \text{in } L^q(\boldsymbol{R}^n)^{n^2}, \qquad u_{\varepsilon,\lambda}\longrightarrow u \quad \text{in } L^r(\boldsymbol{R}^n)^n. \tag{4.5}$$

From (4.2-3) and (4.5) we get the desired result.

(ii) *Case* $n\leqq q<r<\infty$. Compared with the case (i), we may only show

$$\lim_{j\to\infty}\|(\nabla^3\zeta_j)\cdot(\lambda+B)^{-1}J_\varepsilon*u\|_q=0 \qquad \text{for each fixed } \varepsilon, \lambda>0, \tag{4.6}$$

(see (4.4)). Set $1/p=1/q-1/r$. Then $1<p<\infty$ and we have by the Hölder inequality

$$\|\nabla^3\zeta_j\cdot(\lambda+B)^{-1}J_\varepsilon*u\|_q\leqq C\|\nabla^3\zeta_j\|_p\cdot\|(\lambda+B)J_\varepsilon*u\|_r\leqq Cj^{n/p-3}\|(\lambda+B)^{-1}J_\varepsilon*u\|_r.$$

Since $n\leqq q<r$, we have $n/p-3=n/q-n/r-3<-2$ and hence from the above inequality (4.6) follows. The proof for (4.5) is quite similar as in case (i), so we may omit it. This completes the proof. ∎

4.2. *Completion of the Proof of Theorem* 2.

Let (i) $1<q<n$, $1<r<\infty$ or (ii) $n\leqq q<r<\infty$. Let $u\in \hat{X}_\sigma^q(\Omega)\cap L^r(\Omega)^n$. Then we have to prove that there is a sequence $u_j\in C_{0,\sigma}^\infty(\Omega)$ $(j=1, 2, \cdots)$ such that

(4.7) $$\nabla u_j \longrightarrow \nabla u \text{ in } L^q(\Omega)^{n^2} \text{ and } u_j \longrightarrow u \text{ in } L^r(\Omega)^n .$$

Our argument is similar to that of the proof of Theorem 1. Set $\tilde{u}(x)=u(x)$ for $x\in\Omega$, $\tilde{u}(x)=0$ for $x\in \boldsymbol{R}^n/\Omega$. Then it follows from Lemma 4.2 that there is a sequence $v_j\in C_{0,\sigma}^\infty(\boldsymbol{R}^n)$ $(j=1, 2, \cdots)$ satisfying

(4.8) $$\nabla v_j \longrightarrow \nabla \tilde{u} \text{ in } L^q(\Omega)^{n^2} \text{ and } v_j \longrightarrow \tilde{u} \text{ in } L^r(\Omega)^n .$$

Together with our assumption on q and r and the Sobolev inequality, (4.8) yields that

(4.9) $$v_j \longrightarrow \tilde{u} \quad \text{in } L_{\text{loc}}^q(\boldsymbol{R}^n)^n .$$

Now, take $R>0$, the function $\eta\in C^\infty(\boldsymbol{R}^n)$ and the subdomain Ω_R as in subsection 3.2. Since $v_j\cdot\nabla\eta\in C_0^\infty(\Omega_R)$ and $\int_{\Omega_R} v_j\cdot\nabla\eta dx=0$, it follows from Lemma 2.1 (ii) that there exist $w_j\in C_0^\infty(\Omega_R)^n$ $(j=1, \cdots)$ satisfying

(4.10) $$\operatorname{div} w_j = v_j\cdot\nabla\eta \quad \text{in } \Omega_R, \qquad w_j=0 \quad \text{on } \partial\Omega_R .$$

Moreover, such $\{w_j\}_{j=1}^\infty$ is subject to the inequality

$$\|\nabla w_j-\nabla w_k\|_{q,\Omega_R}+\|w_j-w_k\|_{r,\Omega_R}$$
$$\leqq C(\|(v_j-v_k)\cdot\nabla\eta\|_{q,\Omega_R}+\|(v_j-v_k)\cdot\nabla\eta\|_{r,\Omega_R})$$

with C independent of j, $k=1, 2, \cdots$. Hence from (4.9) we see that there is a function $w\in H_0^{1,q}(\Omega_R)^n\cap L^r(\Omega_R)^n$ such that

(4.11) $$\nabla w_j \longrightarrow \nabla w \quad \text{in } L^q(\Omega_R)^{n^2}, \qquad w_j \longrightarrow w \quad \text{in } L^r(\Omega_R)^n ,$$
$$\operatorname{div} w = \tilde{u}\cdot\nabla\eta \quad \text{in } \Omega_R .$$

Set $h\equiv(1-\eta)\tilde{u}+w$. Then we have $h\in H_{0,\sigma}^{1,q}(\Omega_R)\cap L^r(\Omega_R)^n$. Hence it follows from Lemma 4.1 that there is a sequence $h_j\in C_{0,\sigma}^\infty(\Omega_R)$ $(j=1, \cdots)$ such that

(4.12) $$\nabla h_j \longrightarrow \nabla h \quad \text{in } L^q(\Omega_R)^{n^2}, \qquad h_j \longrightarrow h \quad \text{in } L^r(\Omega_R)^n .$$

Denoting by $\tilde{h}_j$ and $\tilde{w}_j$ the zero-extensions of h_j and w_j to Ω, respectively, we define the desired sequence $\{u_j\}_{j=1}^\infty$ with (4.7) as $u_j:=\eta v_j+\tilde{h}_j-\tilde{w}_j$. Clearly by (4.10) $u_j\in C_{0,\sigma}^\infty(\Omega)$ and it follows from (4.8-9) and (4.11-12) that

$$\begin{aligned}&\|\nabla u_j-\nabla u\|_{L^q(\Omega)}+\|u_j-u\|_{L^r(\Omega)}\\&\quad\leqq \|\nabla\eta(v_j-\tilde u)\|_{L^q(\Omega)}+\|\eta(\nabla v_j-\nabla\tilde u)\|_{L^q(\Omega)}+\|\eta(v_j-\tilde u)\|_{L^r(\Omega)}\\&\qquad+\|\nabla h_j-\nabla h\|_{L^q(\Omega_R)}+\|h_j-h\|_{L^r(\Omega_R)}\\&\qquad+\|\nabla w_j-\nabla w\|_{L^q(\Omega_R)}+\|w_j-w\|_{L^r(\Omega_R)}\\&\quad\longrightarrow 0 \quad \text{as } j\to\infty\,.\end{aligned}$$

This implies (4.7) and the proof is complete. ■

REMARK. Our cut-off procedure enables us to prove that $C_{0,\sigma}^{\infty}(\Omega)$ is also dense in $H_{0,\sigma}^{1,2}(\Omega)\cap L^n(\Omega)^n$, which has an application to the weak solutions of the *non-stationary* Navier-Stokes equations constructed by Masuda [**17**, Proposition 1].

5. L^q-gradient bounds for the Navier-Stokes equations; Proof of Theorem 3.

5.1. Let us first recall some fundamental facts for interpolation couples. For a closed subspace X of a Banach space E we denote by $X^{\perp}$ the annihilator of X, i.e., the set of all continuous linear functionals on E vanishing on X. By Corollary 2.2 (i), we have

$$\hat X_\sigma^q(\Omega)^{\perp}=\{f\in\hat H^{-1,q'}(\Omega)^n;\ f=\nabla p \text{ with } p\in L^{q'}(\Omega)\} \tag{5.1}$$

for $1<q<\infty(q'=q/(1-q))$. Moreover by Theorem 1, $\hat X_\sigma^q(\Omega)\cap\hat X_\sigma^r(\Omega)$ is dense in $\hat X_\sigma^q(\Omega)$ and $\hat X_\sigma^r(\Omega)\,(1<q,\,r<\infty)$. Hence it follows from Aronszajn-Gagliardo [**2**, Theorem 8.3] that

$$(\hat X_\sigma^q(\Omega)\cap\hat X_\sigma^r(\Omega))^{\perp}=\hat X_\sigma^q(\Omega)^{\perp}+\hat X_\sigma^r(\Omega)^{\perp}\,. \tag{5.2}$$

For L^q-gradient bounds of weak solutions of (N-S), we need the following variational inequality.

LEMMA 5.1. *Let $u\in\hat X_\sigma^q(\Omega)$ for $1<q<\infty$. Suppose that*

$$\sup\{|(\nabla u,\nabla\phi)|/\|\nabla\phi\|_{r'};\ 0\neq\phi\in C_{0,\sigma}^{\infty}(\Omega)\}<\infty$$

for some $r>n'(=n/(n-1))$. Then it follows $\nabla u\in L^r(\Omega)^{n^2}$. If in addition $1<q<n$, we have also $u\in\hat X_\sigma^r(\Omega)$.

PROOF. Since $C_{0,\sigma}^{\infty}(\Omega)$ is dense in $\hat X_\sigma^{r'}(\Omega)$ and since $\hat X_\sigma^{r'}(\Omega)$ is a closed subspace of $\hat H_0^{1,r'}(\Omega)^n$, it follows from the assumption and the Hahn-Banach theorem that there is a functional $f\in\hat H^{-1,r}(\Omega)^n$ such that $(\nabla u,\nabla\phi)=\langle f,\phi\rangle$ holds for all $\phi\in C_{0,\sigma}^{\infty}(\Omega)$. Now by Theorem 1, $C_{0,\sigma}^{\infty}(\Omega)$ is dense in $\hat X_\sigma^{q'}(\Omega)\cap\hat X_\sigma^{r'}(\Omega)$ and therefore, from the above identity, we get $\langle -\Delta u-f, v\rangle=0$ for all $v\in\hat{X}_\sigma^{q'}(\Omega)\cap\hat{X}_\sigma^{r'}(\Omega)$. Then by (5.1) and (5.2) there are functions $p_1\in L^q(\Omega)$ and $p_2\in L^r(\Omega)$ such that $-\Delta u+\nabla p_1=f+\nabla p_2$ in the sense of distributions on Ω. Since $f+\nabla p_2\in\hat{H}^{-1,r}(\Omega)^n$ with $r>n'$, by Lemma 2.5 we get the desired result. ■

We next consider the complex interpolation space $[X, Y]_\theta$, $0\leqq\theta\leqq 1$. Note that the norms $\|\nabla u\|_q$ and $\|\nabla u\|_r$ are consistent on $C_0^\infty(\Omega)$ and that the pair $\{\hat{H}_0^{1,q}(\Omega), \hat{H}_0^{1,r}(\Omega)\}$ is an interpolation couple. Moreover from Remark 2 to Lemma 2.3, we get the following concrete characterization (see, e.g., Triebel [**25**, 1.9]):

If $1<q<n$, $1<r<n$ or if $n\leqq q<\infty$, $n\leqq r<\infty$,

$$[\hat{H}_0^{1,q}(\Omega), \hat{H}_0^{1,r}(\Omega)]_\theta = \hat{H}_0^{1,s}(\Omega),$$

where $1/s=(1-\theta)/q+\theta/r$, $0\leqq\theta\leqq 1$. Applying duality argument [**25**, 1.11.2], we get

$$[\hat{H}^{-1,q}(\Omega), \hat{H}^{-1,r}(\Omega)]_\theta = \hat{H}^{-1,s}(\Omega) \tag{5.3}$$

for $n'<q<\infty$, $n'<r<\infty$, where $1/s=(1-\theta)/q+\theta/r$, $0\leqq\theta\leqq 1$.

5.2. *Completion of the Proof of Theorem* 3.

(1) *Associated Pressure.* Since $u\in\hat{X}_\sigma^2(\Omega)$, we get $-\Delta u-f\in\hat{H}^{-1,2}(\Omega)^n$. By the Sobolev inequality we have the continuous embeddings $\hat{H}_0^{1,2}(\Omega)\subset L^{2n/(n-2)}(\Omega)$, $\hat{H}_0^{1,n/2}(\Omega)\subset L^n(\Omega)$, so it follows from the Hölder inequality that

$$|(u\cdot\nabla u, \phi)| \leqq \|u\|_{2n/(n-2)}\|\nabla u\|_2\|\phi\|_n \leqq C\|\nabla u\|_2^2\|\nabla\phi\|_{n/2}$$

for all $\phi\in\hat{H}_0^{1,n/2}(\Omega)^n$ with C independent of u and ϕ. This implies that $u\cdot\nabla u\in\hat{H}^{-1,n/(n-2)}(\Omega)^n$ and hence we get

$$-\Delta u+u\cdot\nabla u-f \in \hat{H}^{-1,2}(\Omega)^n+\hat{H}^{-1,n/(n-2)}(\Omega)^n. \tag{5.4}$$

On the other hand, by Theorem 1, $C_{0,\sigma}^\infty(\Omega)$ is dense in $\hat{X}_\sigma^2(\Omega)\cap\hat{X}_\sigma^{n/2}(\Omega)$. Now by (5.4) and the definition of the weak solution of (N-S), we get $-\Delta u+u\cdot\nabla u-f\in(\hat{X}_\sigma^2(\Omega)\cap\hat{X}_\sigma^{n/2}(\Omega))^\perp$. Then it follows from (5.1) and (5.2) that there exist scalar functions $p_1\in L^2(\Omega)$ and $p_2\in L^{n/(n-2)}(\Omega)$ such that $-\Delta u+u\cdot\nabla u-f=-\nabla p_1-\nabla p_2$, which means that

$$(\nabla u, \nabla\phi)+(u\cdot\nabla u, \phi)-(p_1+p_2, \mathrm{div}\phi) = \langle f, \phi\rangle$$

for all $\phi\in C_0^\infty(\Omega)^n$. Now we see that $p_1+p_2\in L^2(\Omega)+L^{n/(n-2)}(\Omega)$ is the pressure associated with u.

(2) *More Regularity.* (i) Since $n=3$, we have by (5.4) that $u\cdot\nabla u\in\hat{H}^{-1,3}(\Omega)^3$ and hence from the assumption on f with the aid of (5.3) it follows that $u\cdot\nabla u$

$-f\in\hat{H}^{-1,3}(\Omega)^3$. Now applying Lemma 5.1, we get $\nabla u\in L^3(\Omega)^{3^2}$. By interpolation, $\nabla u\in L^r(\Omega)^{3^2}$ for $2\leqq r\leqq 3$. Since $u\in L^6(\Omega)^3$, it follows from Corollary 2.2 (ii) that $u\in L^s(\Omega)^3$ for all s with $6\leqq s<\infty$. Since $2q\geqq 6$, we obtain by integration by parts and the Hölder inequality $|(u\cdot\nabla u, \phi)|=|(u\cdot\nabla\phi, u)|\leqq\|u\|_{2q}^2\|\nabla\phi\|_{q'}$ for all $\phi\in C_0^\infty(\Omega)^3$, which implies that $u\cdot\nabla u\in\hat{H}^{-1,q}(\Omega)^3$. By assumption $u\cdot\nabla u-f\in\hat{H}^{-1,q}(\Omega)^3$ and Lemma 5.1 yields, together with interpolation, $\nabla u\in L^r(\Omega)^{3^2}$ for $2\leqq r\leqq q$. Now $-\Delta u+u\cdot\nabla u-f$ belongs to $\hat{H}^{-1,q}(\Omega)^3$ and vanishes on $C_{0,\sigma}^\infty(\Omega)$. By Remark 3.3 (ii) the pressure p associated with u can be chosen in the class that $p\in L^q(\Omega)$.

Suppose in particular that $q>3$. By interpolation we have $u\in\hat{X}_\sigma^{\tilde{q}}(\Omega)\cap L^6(\Omega)^3$ for $3<\tilde{q}<6$. Then we have $u\in L^\infty(\Omega)^3$ because it holds

$$\|\phi\|_\infty\leqq C\|\nabla\phi\|_{\tilde{q}}^\alpha\|\phi\|_6^{1-\alpha}\qquad\text{for all}\quad \phi\in\hat{X}_\sigma^{\tilde{q}}(\Omega)\cap L^6(\Omega)^3, \tag{5.5}$$

where $\alpha=\tilde{q}/3(\tilde{q}-2)$. Indeed, from Gagliardo-Nirenberg inequality (see, e.g., Friedman [8, p. 24 Theorem 9.4]), we see that (5.5) holds for all $\phi\in C_{0,\sigma}^\infty(\Omega)$. Now since $C_{0,\sigma}^\infty(\Omega)$ is dense in $\hat{X}_\sigma^{\tilde{q}}(\Omega)\cap L^6(\Omega)^3$ (by Theorem 2), we get (5.5) by passage to the limit.

(ii) By (5.3) and the assumption on f, we see as in case (1) that $u\cdot\nabla u-f\in\hat{H}^{-1,n/(n-2)}(\Omega)^n$. It follows from Lemma 5.1 and interpolation that $\nabla u\in L^r(\Omega)^{n^2}$ for $n/(n-2)\leqq r\leqq 2$. Since $u\in L^{2n/(n-2)}(\Omega)^n$, we have by Corollary 2.2 (ii) that $u\in L^y(\Omega)^n$ for $n/(n-3)\leqq y\leqq 2n/(n-2)$, which yields $u\cdot\nabla u\in\hat{H}^{-1,\delta}(\Omega)^n$ for $n/2(n-3)\leqq\delta\leqq n/(n-2)$. Since $n/2(n-3)\leqq n'<q\leqq n/(n-2)$, we have in particular $u\cdot\nabla u\in\hat{H}^{-1,q}(\Omega)^n$. Then in the same way as above we have by Lemma 5.1 and Remark 3.3 (ii) that $\nabla u\in L^q(\Omega)^{n^2}$ and $p\in L^q(\Omega)$. Now, the assertion follows from interpolation and Corollary 2.2 (ii). This completes the proof. ■

6. Uniqueness for the weak solutions of the stationary Navier-Stokes equations; Proof of Theorem 4.

As we have seen in the proof of Theorem 3 (1), it holds $|(u\cdot\nabla u, \phi)|\leqq C\|\nabla u\|_2^2\|\phi\|_n$ for all $\phi\in C_{0,\sigma}^\infty(\Omega)$. By Theorem 2, $C_{0,\sigma}^\infty(\Omega)$ is dense in $\hat{X}_\sigma^2(\Omega)\cap L^n(\Omega)^n$ and hence by passage to the limit we can insert $v\in\hat{X}_\sigma^2(\Omega)\cap L^n(\Omega)^n$ as a test function ϕ in the definition of weak solution of (N-S). Since $(v\cdot\nabla v, v)=0$, we obtain

$$(\nabla u, \nabla v)+(u\cdot\nabla u, v)=\langle f, v\rangle, \tag{6.1}$$

$$\|\nabla v\|_2^2=\langle f, v\rangle. \tag{6.2}$$

Moreover, we have by the Hölder and the Sobolev inequalities that

$$|(v\cdot\nabla v, \phi)|\leqq\|v\|_n\|\nabla v\|_2\|\phi\|_{2n/(n-2)}\leqq C\|v\|_n\|\nabla v\|_2\|\nabla\phi\|_2$$

for all $\phi\in C_{0,\sigma}^{\infty}(\Omega)$. By Theorem 1, $C_{0,\sigma}^{\infty}(\Omega)$ is dense in $\hat{X}_{\sigma}^{2}(\Omega)$ and hence from the above inequality we can insert $u\in\hat{X}_{\sigma}^{2}(\Omega)$ as a test function defining the weak solution $v\in\hat{X}_{\sigma}^{2}(\Omega)\cap L^{n}(\Omega)^{n}$;

$$(6.3)\qquad (\nabla v, \nabla u)+(v\cdot\nabla v, u)=\langle f, u\rangle.$$

Adding (6.1-3) and (E.I.), we get by integration by parts

$$\begin{aligned}\|\nabla u-\nabla v\|_2^2 &\leqq (u\cdot\nabla u, v)+(v\cdot\nabla v, u)=(u\cdot\nabla u, v)-(v\cdot\nabla u, v)\\ &=((u-v)\cdot\nabla(u-v), v).\end{aligned}$$

Here we used $((u-v)\cdot\nabla v, v)=0$. Letting $w\equiv u-v$ and then using the Hölder inequality and the Sobolev inequality $\|\phi\|_{2n/(n-2)}\leqq C_*\|\nabla\phi\|_2(\phi\in\hat{H}_0^{1,2}(\Omega))$, we have from above

$$\|\nabla w\|_2^2\leqq\|w\|_{2n/(n-2)}\|\nabla w\|_2\|v\|_n\leqq C_*\|\nabla w\|_2^2\|v\|_n.$$

Take $0<\lambda<C_*^{-1}$. Then under the assumption that $\|v\|_n\leqq\lambda$, we conclude $\|\nabla w\|_2^2\leqq 0$, which implies $u\equiv v$ on Ω. This completes the proof. ■

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Hideo KOZONO
Department of Mathematics
College of General Education
Kyushu University
Fukuoka 810
Japan

Hermann SOHR
Fachbereich Mathematik
Universität Paderborn
D-4790 Paderborn
Bundesrepublik
Deutschland