J. Math. Soc. Japan
Vol. 44, No. 2, 1992

On unimodal Lévy processes on the nonnegative integers

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(Received Oct. 1, 1990)
(Revised April 5, 1991)

1. Introduction and main results.

Let $\boldsymbol{R}=(-\infty, \infty)$, $\boldsymbol{R}_+=[0, \infty)$, $\boldsymbol{Z}=\{0, \pm 1, \pm 2, \cdots\}$ and $\boldsymbol{Z}_+=\{0, 1, 2, \cdots\}$. A measure $\mu(dx)$ on $\boldsymbol{R}$ is said to be *unimodal* with mode a if $\mu(dx)=c\delta_a(dx)+f(x)dx$, where $-\infty<a<\infty$, $c\geqq 0$, $\delta_a(dx)$ is the delta measure at a and $f(x)$ is non-decreasing for $x<a$ and non-increasing for $x>a$. A measure $\mu(dx)=\sum_{n=-\infty}^{\infty} p_n\delta_n(dx)$ on $\boldsymbol{Z}$ is said to be *discrete unimodal* with mode a ($a\in\boldsymbol{Z}$) if p_n is non-decreasing for $n\leqq a$ and non-increasing for $n\geqq a$. A probability measure $\mu(dx)$ is said to be *strongly unimodal* (resp. *discrete strongly unimodal*) if, for every unimodal (resp. discrete unimodal) probability measure $\eta(dx)$, the convolution $\mu*\eta(dx)$ is unimodal (resp. discrete unimodal). Let X_t $(0\leqq t<\infty)$ be a Lévy process (that is, a process with stationary independent increments starting at the origin) on $\boldsymbol{R}$ or $\boldsymbol{Z}$ with the Lévy measure $\nu(dx)$. The process X_t on $\boldsymbol{R}$ (resp. on $\boldsymbol{Z}$) is said to be unimodal (resp. discrete unimodal) if the distribution of X_t is unimodal (resp. discrete unimodal) for every $t>0$. The process X_t on $\boldsymbol{R}$ (resp. on $\boldsymbol{Z}_+$) is said to be of class L (resp. discrete class L) if the distribution of X_t is of class L (resp. discrete class L). A necessary and sufficient condition for the process X_t on $\boldsymbol{R}$ (resp. on $\boldsymbol{Z}_+$) to be of class L (resp. discrete class L) is that $|x|\nu(dx)$ is unimodal with mode 0 (resp. discrete unimodal with mode 1 on $\boldsymbol{Z}_+$).

The following theorem is our main result.

THEOREM 1.1. *Let X_t be a Lévy process on $\boldsymbol{Z}_+$ with the Lévy measure $\nu(dx)=\sum_{n=1}^{3} n^{-1}k_n\delta_n(dx)$ satisfying $0<2k_1\leqq 3k_2$. Then X_t is discrete unimodal if and only if*

$$k_1\geqq k_2 \quad \text{and} \quad k_1k_3\leqq k_1^2-k_1k_2+k_2^2. \tag{1.1}$$

REMARK 1.1. In Theorem 1.1, we can choose k_n $(1\leqq n\leqq 3)$ in such a way that $k_1, k_3<k_2$. In this case, X_t is discrete unimodal but not of discrete class L.

REMARK 1.2. Let $X_t^{(1)}$ and $X_t^{(2)}$ be independent and discrete unimodal Lévy processes in Theorem 1.1 such that $k_1^{(1)}=k_2^{(1)}=k_3^{(1)}=1$ and $k_1^{(2)}=1$, $k_2^{(2)}=2/3$, $k_3^{(2)}=7/9$. Then $X_t=X_t^{(1)}+X_t^{(2)}$ satisfies the conditions $3k_2 \geqq 2k_1$ and $k_1 \geqq k_2$ but does not satisfy the condition $k_1k_3 \leqq k_1^2-k_1k_2+k_2^2$. Hence X_t is not discrete unimodal.

Many results on the unimodality of Lévy processes are obtained by Medgyessy [6], Sato [7, 8], Sato-Yamazato [9], Steutel-van Harn [11], Watanabe [12, 13], Wolfe [14, 15], and Yamazato [16, 17]. But a necessary and sufficient condition for the unimodality of Lévy processes in terms of their Lévy measures is not known. Among these works, main related results are the following. Wolfe [15] proves that if a Lévy process on $\boldsymbol{R}$ (resp. on $\boldsymbol{Z}$) is unimodal (resp. discrete unimodal), then the Lévy measure $\nu(dx)$ (resp. $\nu(dx)+c\delta_0(dx)$ for some $c>0$) is unimodal (resp. discrete unimodal) with mode 0, and that the converse does not hold. As a big advancement, Yamazato [16] shows that Lévy processes of class L are unimodal. Steutel-van Harn [11] proves the discrete unimodality of Lévy processes of discrete class L on $\boldsymbol{Z}_+$. Watanabe [12] constructs unimodal Lévy processes on $\boldsymbol{R}_+$ and $\boldsymbol{R}$ that are not of class L. Sato [8] proves that the mode $a(t)$ of the distribution of any unimodal Lévy process X_t on $\boldsymbol{R}_+$ is non-decreasing for $t>0$.

Existence of a unimodal Lévy process on $\boldsymbol{Z}_+$ which is not of class L (Remark 1.1) is a discrete version of a result of Watanabe [12]. But our method of the proof is different from the continuous case. In order to prove Theorem 1.1, we give a necessary and sufficient condition for Lévy processes on $\boldsymbol{Z}_+$, to be discrete unimodal in terms of a zero of the polynomial $Q_n(t)$, defined in (2.3), in Section 2. A discrete analogue of Sato's result [8] plays an essential role in the proof. General results on discrete unimodality and discrete strong unimodality given in Section 2 will be of interest in themselves. In Section 3, we prove Theorem 1.1. In Section 4, we apply our results in Section 2 to unimodal Lévy processes on $\boldsymbol{R}_+$, and give a necessary and sufficient condition for the unimodality of Lévy processes on $\boldsymbol{R}_+$.

2. Discrete unimodal Lévy processes on $\boldsymbol{Z}_+$.

In this section, let X_t be a Lévy process on $\boldsymbol{Z}_+$, not identically zero. Then we have

$$
\begin{aligned}
&E\exp(izX_t)=\exp(t\psi(z)), \\
(2.1)\qquad &\psi(z)=\sum_{n=1}^{\infty}(e^{izn}-1)n^{-1}k_n,
\end{aligned}
$$

with the Lévy measure $\nu(dx)=\sum_{n=1}^{\infty}n^{-1}k_n\delta_n(dx)$ satisfying $\sum_{n=1}^{\infty}n^{-1}k_n<\infty$. Let $\mu_t(dx)=\sum_{n=0}^{\infty}p_n(t)\delta_n(dx)$ be the distribution of X_t. Then we have a relation by

Katti [**4**] or Steutel [**10**]:

$$nP_n(t) = t\sum_{j=1}^{n} k_j P_{n-j}(t) \tag{2.2}$$

for $n\geqq 1$, where $P_n(t)=p_n(t)/p_0(t)$ for $n\geqq 0$. Define $P_{-1}(t)=0$ and $Q_n(t)=P_n(t)-P_{n-1}(t)$ for $n\geqq 0$. Then we obtain from (2.2) that

$$nQ_n(t) = \sum_{j=1}^{n}(tk_j-1)Q_{n-j}(t) \tag{2.3}$$

for $n\geqq 1$. From (2.2) and (2.3), we find that if $k_1>0$, then $P_n(t)$ and $Q_n(t)$ are polynomials of degree n and the highest coefficients are positive. Also the equation (2.2) implies that if $k_1=0$, then $P_1(t)=0$ for every $t>0$ and hence X_t is not discrete unimodal. Therefore we assume, from now on, that $k_1>0$.

LEMMA 2.1. *If X_t is discrete unimodal, then, for every $n\geqq 1$, there exists $t_n>0$ such that $Q_n(t)<0$ for $0<t<t_n$.*

PROOF. Suppose that X_t is discrete unimodal. We have by (2.3)

$$Q_1(t) = (k_1t-1)Q_0(t) = k_1t-1<0 \tag{2.4}$$

for $0<t<k_1^{-1}$. Hence 0 is the unique mode of $\mu_t(dx)$ for $0<t<k_1^{-1}$. It follows that $Q_n(t)\leqq 0$ for all $n\geqq 1$ and for $0<t<k_1^{-1}$. Since $Q_n(t)$ is a polynomial, it has only a finite number of zeros. Therefore there exists $t_n>0$ such that $0<t_n\leqq k_1^{-1}$ and $Q_n(t)<0$ for $0<t<t_n$.

LEMMA 2.2. *Let $T\geqq 0$. If $\mu_t(dx)$ is discrete unimodal for $t>T$, then the largest mode $a(t)$ of $\mu_t(dx)$ is non-decreasing for $t>T$.*

REMARK 2.1. If a distribution is unimodal, then either its mode is unique or the set of its modes is a closed interval. We mean by the largest mode the largest one in the set of modes of a distribution. This lemma is a discrete analogue of Theorem 2.1 of Sato [8].

PROOF. Suppose that $\mu_t(dx)$ is discrete unimodal for $t>T$. We have

$$P_{a(t)}(t+s) = \sum_{j=0}^{a(t)} P_{a(t)-j}(t)p_j(s) \tag{2.5}$$

and

$$P_{a(t)-1}(t+s) = \sum_{j=0}^{a(t)-1} P_{a(t)-j-1}(t)p_j(s)$$

for $s>0$ and for $t>T$. Hence we get

$$Q_{a(t)}(t+s) = P_0(t)p_{a(t)}(s)+\sum_{j=0}^{a(t)-1} Q_{a(t)-j}(t)p_j(s)>0 \tag{2.6}$$

for $s>0$ and for $t>T$, noting that $Q_{a(t)-j}(t)\geqq 0$ for $0\leqq j\leqq a(t)-1$. We obtain from (2.6) that $a(t)\leqq a(t+s)$ for $s>0$ and for $t>T$. This proves Lemma 2.2.

THEOREM 2.1. *A process X_t is discrete unimodal if and only if $Q_n(t)$ has a unique positive zero α_n of odd order for every $n\geqq 1$ and α_n is non-decreasing in n.*

PROOF OF THE "IF" PART OF THEOREM 2.1. The polynomial $Q_n(t)$ is non-positive for $0\leqq t\leqq \alpha_n$ and non-negative for $t\geqq \alpha_n$. It follows from (2.4) that $\alpha_1=k_1^{-1}$. Since α_n is non-decreasing, $Q_n(t)\leqq 0$ for all $n\geqq 1$ and for $0<t<k_1^{-1}$. Hence $\mu_t(dx)$ is discrete unimodal with mode 0 for $0<t<\alpha_1$. For $\alpha_n\leqq t\leqq \alpha_{n+1}$, we have $Q_j(t)\geqq 0$ for $1\leqq j\leqq n$ and $Q_j(t)\leqq 0$ for $j\geqq n+1$. Therefore, $\mu_t(dx)$ is discrete unimodal with mode n when $\alpha_n\leqq t\leqq \alpha_{n+1}$. We shall prove that $T=\sup_{n\geqq 1}\alpha_n=\infty$, which will complete the proof of the "if" part. Suppose that $T<\infty$. Then we get $Q_n(t)\geqq 0$ for $t>T$ and for all $n\geqq 1$. But this implies that $\sum_{n=0}^{\infty}P_n(t)=\infty$ for $t>T$. This is a contradiction.

PROOF OF THE "ONLY IF" PART OF THEOREM 2.1. Suppose that X_t is discrete unimodal. We find from Lemma 2.1 and from $Q_n(t)\to\infty$ as $t\to\infty$ that $Q_n(t)$ has at least one positive zero of odd order. Suppose that $Q_n(t)$ has distinct positive zeros of odd orders. Let β_n and γ_n be, respectively, the smallest and the largest such zero. Then we can choose $\varepsilon>0$ such that $Q_n(\beta_n+\varepsilon)>0$, $Q_n(\gamma_n-\varepsilon)<0$ and $\beta_n+\varepsilon<\gamma_n-\varepsilon$. But this contradicts Lemma 2.2. Hence $Q_n(t)$ has a unique positive zero of odd order. Suppose that $\alpha_m>\alpha_{m+1}$ for some $m\geqq 1$. Then we can find $\varepsilon>0$ such that $Q_{m+1}(\alpha_{m+1}+\varepsilon)>0$, $Q_m(\alpha_m-\varepsilon)<0$ and $\alpha_{m+1}+\varepsilon<\alpha_m-\varepsilon$. But this contradicts Lemma 2.2. Therefore, α_n is non-decreasing in $n\geqq 1$. The proof is complete.

COROLLARY 2.1. *If X_t is discrete unimodal, then* (1.1) *holds.*

PROOF. The polynomial $Q_1(t)$ has a unique positive zero $\alpha_1=k_1^{-1}$. We obtain from (2.3) that

$$2Q_2(t)=k_1^2t^2+(k_2-2k_1)t \tag{2.7}$$

and

$$6Q_3(t)=k_1^3t^3+3(k_2-k_1)k_1t^2+(2k_3-3k_2)t\,. \tag{2.8}$$

Hence $Q_2(t)$ has a unique positive zero $\alpha_2=-k_1^{-2}k_2+2k_1^{-1}$, if $2k_1>k_2$. The inequality $\alpha_1\leqq\alpha_2$ holds if and only if $k_1\geqq k_2$. From (2.8), $Q_3(t)$ has a unique positive zero α_3 if and only if either $2k_3=3k_2$ and $k_1>k_2$ or $2k_3<3k_2$. And α_3 is given by

$$\alpha_3=2^{-1}k_1^{-2}[3(k_1-k_2)+\{9(k_1-k_2)^2-4k_1(2k_3-3k_2)\}^{1/2}]\,. \tag{2.9}$$

The inequality $\alpha_2 \leqq \alpha_3$ holds if and only if $k_1 k_3 \leqq k_1^2 - k_1 k_2 + k_2^2$. Hence (1.1) holds by Theorem 2.1.

COROLLARY 2.2. *Suppose that $\lambda(dx)=\sum_{n=0}^{\infty} k_{n+1}\delta_n(dx)$ is discrete unimodal. Then X_t is discrete unimodal if and only if X_t is of discrete class L, that is, k_n is non-increasing for $n \geqq 1$.*

PROOF. If X_t is of discrete class L on $\boldsymbol{Z}_+$, then X_t is discrete unimodal by Steutel-van Harn [**11**]. Conversely, suppose that X_t and $\lambda(dx)$ are discrete unimodal on $\boldsymbol{Z}_+$. From Corollary 2.1, the inequality $k_1 \geqq k_2$ holds. Hence there are two cases.

Case 1. $k_1 > k_2$ or $k_1 = k_2 = \cdots = k_m > k_{m+1}$ for some $m \geqq 2$. Then, since $\lambda(dx)$ is discrete unimodal, k_n is non-increasing for $n \geqq 1$.

Case 2. $k_1 = k_2 = \cdots = k_m < k_{m+1}$ for some $m \geqq 2$. We shall show that absurdity occurs in this case. We obtain from (2.2) that

$$
\begin{aligned}
(m+1)Q_{m+1}(t) &= (k_1 t - 1)P_m(t) + t\sum_{j=1}^{m}(k_{j+1}-k_j)P_{m-j}(t) \\
&= (k_1 t - 1)P_m(t) + (k_{m+1}-k_m)t\,.
\end{aligned}
\tag{2.10}
$$

Letting $t = \alpha_1 = k_1^{-1}$, we get

$$
(m+1)Q_{m+1}(\alpha_1) = (k_{m+1}-k_m)\alpha_1 > 0\,. \tag{2.11}
$$

But this contradicts $\alpha_1 \leqq \alpha_{m+1}$. This proves Corollary 2.2.

We can prove the following theorem by argument similar to Theorem 2.1.

THEOREM 2.2. *Fix $T>0$. The distribution $\mu_t(dx)$ is discrete unimodal for every $t>T$ if and only if there exists an integer $A \geqq 0$ such that, for $1 \leqq n \leqq A$, $Q_n(t)$ has no zero of odd order on (T, ∞) and, for $n \geqq A+1$, $Q_n(t)$ has a unique zero β_n of odd order on (T, ∞) and β_n is non-decreasing in $n \geqq A+1$.*

PROOF OF THE "IF" PART OF THEOREM 2.2. For $1 \leqq n \leqq A$, the polynomial $Q_n(t)$ is non-negative for $t>T$. For every $n \geqq A+1$, $Q_n(t)$ is non-positive for $T<t\leqq \beta_n$ and non-negative for $t \geqq \beta_n$. It follows that $Q_n(t) \leqq 0$ for every $n \geqq A+1$ and for $T<t<\beta_{A+1}$. Hence $\mu_t(dx)$ is discrete unimodal with mode A for $T<t<\beta_{A+1}$. By argument similar to Theorem 2.1, we can show that $\mu_t(dx)$ is discrete unimodal with mode n when $\beta_n \leqq t \leqq \beta_{n+1}$ $(n \geqq A+1)$. Also we can prove that $\sup_{n\geqq A+1}\beta_n = \infty$, which completes the proof of the "if" part.

PROOF OF THE "ONLY IF" PART OF THEOREM 2.2. Suppose that $\mu_t(dx)$ is discrete unimodal for every $t>T$. Then $\mu_T(dx)$ is discrete unimodal, because $\mu_t(dx)$ converges weakly to $\mu_T(dx)$ as $t \to T$. Let A be the largest mode of

$\mu_T(dx)$. We prove that, for $1 \leqq n \leqq A$, $Q_n(t)$ does not have a zero β_n of odd order satisfying $\beta_n > T$. In fact, if such a zero β_m exists for some m $(1 \leqq m \leqq A)$, then we can find $\varepsilon > 0$ such that $\beta_m - \varepsilon > T$ and $Q_m(\beta_m - \varepsilon) < 0$. But this contradicts Lemma 2.2. Next we show that, for every $n \geqq A+1$, there exists $t_n > T$ such that $Q_n(t) < 0$ for $T < t < t_n$. Suppose that, for some $m \geqq A+1$, there exists a sequence s_n such that $T < s_n$, $Q_m(s_n) \geqq 0$ and $s_n \to T$ as $n \to \infty$. Since $Q_m(t)$ has only a finite number of zeros, we can assume $Q_m(s_n) > 0$. This implies that $m \leqq a_n$, where a_n is a mode of $\mu_{s_n}(dx)$. Because a_n converges to a mode a of $\mu_T(dx)$ as $n \to \infty$, we have $A+1 \leqq m \leqq a \leqq A$, which is a contradiction. It follows from this and from $Q_n(t) \to \infty$ as $t \to \infty$ that, for every $n \geqq A+1$, $Q_n(t)$ has at least one zero β_n of odd order satisfying $\beta_n > T$. By argument similar to Theorem 2.1, we can prove that such a zero β_n is unique and non-decreasing in $n \geqq A+1$. Thus we have proved Theorem 2.2.

We consider the following condition. Let N be a positive integer.

(H) $$k_n > 0 \text{ for } 1 \leqq n \leqq N \quad \text{and} \quad k_n = 0 \text{ for } n \geqq N+1\,.$$

LEMMA 2.3. (Hansen [**2**]) *Suppose that* $k_n^2 \geqq k_{n+1} k_{n-1}$ *for all* $n \geqq 2$. *Then* $\mu_t(dx)$ *is discrete strongly unimodal if and only if* $t \geqq k_1^{-2} k_2$.

LEMMA 2.4. *Suppose that* X_t *satisfies the condition* (H). *Then there exists* $T \geqq 0$ *such that* $\mu_t(dx)$ *is discrete strongly unimodal for every* $t \geqq T$.

The smallest T *satisfying the above condition is denoted by* T_N. *This* T_N *depends not only on* N *but also on* k_n $(1 \leqq n \leqq N)$ *in general.*

PROOF OF LEMMA 2.4. We shall prove by induction in N.

(i) Suppose that $N=1$. Then $\mu_t(dx)$ is a Poisson distribution and hence discrete strongly unimodal by Keilson-Gerber [**5**]. This means $T_1 = 0$. (In case $N=2$, the assertion is a direct consequence of Lemma 2.3. Thus $T_2 = k_1^{-2} k_2$.)

(ii) Assume that Lemma 2.4 is true when $N=j$. Consider the case $N=j+1$. We can choose $k_n^{(1)}$ such that $(k_n^{(1)})^2 \geqq k_{n+1}^{(1)} k_{n-1}^{(1)}$ for $2 \leqq n \leqq j$, $k_n^{(1)} < k_n$ for $1 \leqq n \leqq j$ and $k_n^{(1)} = k_n$ for $n \geqq j+1$. Let $k_n^{(2)} = k_n - k_n^{(1)}$. Then we have $\mu_t(dx) = \mu_t^{(1)} * \mu_t^{(2)}(dx)$, where $\mu_t^{(i)}(dx)$ $(i=1, 2)$ is the distribution of the process $X_t^{(i)}$ whose Lévy measure is given by $\nu^{(i)}(dx) = \sum_{n=1}^{\infty} n^{-1} k_n^{(i)} \delta_n(dx)$. The distribution $\mu_t^{(2)}(dx)$ is discrete strongly unimodal for $t \geqq T_j$ by the assumption. And, by Lemma 2.3, $\mu_t^{(1)}(dx)$ is discrete strongly unimodal for $t \geqq (k_1^{(1)})^{-2} k_2^{(1)} = T$. Hence $\mu_t(dx)$ is discrete strongly unimodal for $t \geqq T' = \max(T, T_j)$.

Let us denote by $[x]$ the largest integer not exceeding x.

THEOREM 2.3. *Suppose that* X_t *satisfies the condition* (H). *Then* X_t *is discrete unimodal if and only if* $Q_n(t)$ *has a unique positive zero* α_n *of odd order for* $1 \leqq n \leqq M+N$ *and* α_n *is non-decreasing in* $1 \leqq n \leqq M$, *where* $M = [T_N \sum_{j=1}^{N} k_j]$.

Proof of the "only if" part of Theorem 2.3 is trivial by Theorem 2.1.

PROOF OF THE "IF" PART OF THEOREM 2.3. Suppose that $Q_n(t)$ has α_n for $1\leqq n\leqq M+N$ and α_n is non-decreasing in $1\leqq n\leqq M$. We shall prove that

$$\alpha_M \leqq T_N < \alpha_{M+1} \quad \text{or} \quad T_N < \alpha_M . \tag{2.12}$$

Suppose that $T_N\geqq\alpha_M$. Then we have $Q_j(T_N)\geqq 0$ for $1\leqq j\leqq M$. Hence we get $P_j(T_N)\leqq P_M(T_N)$ for $0\leqq j\leqq M$. We obtain from (2.2) that

$$\begin{aligned}(M+1)P_{M+1}(T_N) &= T_N \sum_{j=1}^{N} k_j P_{M-j+1}(T_N) \\ &\leqq P_M(T_N)T_N \sum_{j=1}^{N} k_j < (M+1)P_M(T_N) .\end{aligned} \tag{2.13}$$

Hence we have $Q_{M+1}(T_N)<0$ and $T_N<\alpha_{M+1}$. Thus we have proved (2.12). Recalling Lemma 2.4, (2.12), and the proof of the "only if" part of Theorem 2.2, we find that there exists a non-negative integer $A\leqq M$ ($A=M$ if $\alpha_M\leqq T_N<\alpha_{M+1}$ and $A\leqq M-1$ if $T_N<\alpha_M$) such that, for every $n\geqq A+1$, $Q_n(t)$ has a unique zero β_n of odd order satisfying $\beta_n>T_N$ and β_n is non-decreasing in $n\geqq A+1$. This implies that α_n is non-decreasing in $1\leqq n\leqq M+N$ and that

$$T_N < \alpha_{M+1} \leqq \cdots \leqq \alpha_{M+N} \leqq \beta_{M+N+1} \leqq \beta_{M+N+2} \leqq \cdots , \tag{2.14}$$

noting that $\alpha_n=\beta_n$ for $A+1\leqq n\leqq M+N$. From (2.14), there exists $\varepsilon>0$ such that $Q_{M+j}(t)\leqq 0$ for $1\leqq j\leqq N$ and for $0<t<T_N+\varepsilon$. Therefore we have by (2.3)

$$(M+N+1)Q_{M+N+1}(t) = t\sum_{j=1}^{N} k_j Q_{M+N+1-j}(t) - P_{M+N}(t) < 0 \tag{2.15}$$

for $0<t<T_N+\varepsilon$. By induction in j, we get $Q_{M+N+j}(t)<0$ for $0<t<T_N+\varepsilon$ and for all $j\geqq 1$. Hence the unique zero β_n of odd order satisfying $\beta_n>T_N$ is a unique positive zero of odd order for every $n\geqq M+N+1$. It follows from (2.14) that $Q_n(t)$ has a unique positive zero α_n of odd order for every $n\geqq 1$ and α_n is non-decreasing in $n\geqq 1$. Therefore, X_t is discrete unimodal by Theorem 2.1. The proof of Theorem 2.3 is complete.

REMARK 2.2. Suppose that X_t satisfies the condition (H) with $N=2$. Then X_t is discrete unimodal if and only if $k_1\geqq k_2$.

PROOF. The "only if" part of the proof is clear from Corollary 2.1. Conversely, if $k_1\geqq k_2$, then X_t is of discrete class L and, by Steutel-van Harn [**11**], discrete unimodal.

3. Proof of Theorem 1.1.

In this section, we prove Theorem 1.1, by using Corollary 2.1, Theorem 2.3, and the following lemma.

LEMMA 3.1. *Let* $A_n(t)=\sum_{j=0}^{n} a_j t^j$ *be a polynomial of degree* n $(n\geqq 1)$, *Suppose that there exists an integer* m $(0\leqq m\leqq n-1)$ *such that* $a_j\leqq 0$ *for* $0\leqq j\leqq m-1$, $a_m<0$, $a_j\geqq 0$ *for* $m+1\leqq j\leqq n-1$, *and* $a_n>0$. *Then* $A_n(t)$ *has a unique positive zero, which is of order one.*

PROOF. We shall prove by induction in m.

(i) Suppose that $m=0$. Then the derivative $A_n'(t)>0$ for each $t>0$ and $A_n(0)=a_0<0$. Hence, for every $n\geqq 1$, $A_n(t)$ has a unique positive zero, which is of order one.

(ii) Assume that, for every $n\geqq 1$, Lemma 3.1 is true when $m=j$ $(0\leqq j\leqq n-1)$. Consider the case $m=j+1$. Since the derivative $A_n'(t)$ satisfies the conditions of Lemma 3.1 with $m=j$, it has a unique positive zero θ, which is of order one by the assumption. Hence $A_n'(t)<0$ for $0<t<\theta$, $A_n'(\theta)=0$, and $A_n'(t)>0$ for $t>\theta$. Because $A_n(0)=a_0\leqq 0$, $A_n(t)$ has a unique positive zero, which is of order one.

Proof of the "only if" part of Theorem 1.1 is clear from Corollary 2.1.

Conversely, suppose that $0<2k_1\leqq 3k_2$ and (1.1) hold. If $k_2\geqq k_3$, then X_t is of discrete class L and, by Steutel-van Harn [11], discrete unimodal. Therefore we can assume $k_2<k_3$. Then we have $T_3\leqq k_2^{-3}k_3^2$ (see the proof of Lemma 2.4). Let $a=k_1^{-1}k_2$, $b=k_1^{-1}k_3$, and $c=k_2^{-1}k_3$. Then we obtain from (1.1), $2k_1\leqq 3k_2$, and $k_2<k_3$ that

$$2/3\leqq a<b\leqq a^2-a+1\leqq 1, \qquad c\leqq a+a^{-1}-1\leqq 7/6\,. \tag{3.1}$$

Hence we have

$$M=[T_3(k_1+k_2+k_3)]\leqq [c^2(a^{-1}+1+c)]\leqq [539/108]=4\,. \tag{3.2}$$

From (3.2) and Theorem 2.3, we have only to prove the unique existence of α_n for $1\leqq n\leqq 7$ and the inequality $\alpha_1\leqq\alpha_2\leqq\alpha_3\leqq\alpha_4$. Define $A_n(t)=n!Q_n(k_1^{-1}t)=\sum_{j=0}^{n} a_{nj}t^j$ for $n\geqq 1$. Then we get by (2.3) that

$$\begin{aligned}
A_1(t)&=t-1\,,\\
A_2(t)&=t^2+(a-2)t\,,\\
A_3(t)&=t^3+(3a-3)t^2+(2b-3a)t\,,\\
A_4(t)&=t^4+(6a-4)t^3+(3a^2-12a+8b)t^2-8bt\,,\\
A_5(t)&=t^5+(10a-5)t^4+(15a^2-30a+20b)t^3+(-15a^2+20ab-40b)t^2,
\end{aligned} \tag{3.3}$$

$$A_6(t)=t^6+(15a-6)t^5+(45a^2-60a+40b)t^4$$
$$+(15a^3-90a^2+120ab-120b)t^3+(40b^2-120ab)t^2\,,$$

and

$$A_7(t)=t^7+(21a-7)t^6+(105a^2-105a+70b)t^5$$
$$+(105a^3-315a^2+420ab-280b)t^4$$
$$+(-105a^3+210a^2b-840ab+280b^2)t^3-280b^2t^2\,.$$

Hence we obtain from (3.1) and (3.3) that $a_{nn}=1$ for all $n\geqq 1$ and $a_{10}<0$, $a_{20}=0$, $a_{21}<0$, $a_{30}=0$, $a_{31}<0$, $a_{32}<0$, $a_{40}=0$, $a_{41}<0$, $a_{42}<0$, $a_{43}\geqq 0$, $a_{50}=a_{51}=0$, $a_{52}<0$, $a_{53}>0$, $a_{54}>0$, $a_{60}=a_{61}=0$, $a_{62}<0$, $a_{63}<0$, $a_{64}>0$, $a_{65}>0$, $a_{70}=a_{71}=0$, $a_{72}<0$, $a_{73}<0$, $a_{74}<0$, $a_{75}>0$, and $a_{76}>0$. Thus $A_n(t)$ satisfies the condition in Lemma 3.1 and hence, for $1\leqq n\leqq 7$, $Q_n(t)$ has a unique positive zero α_n, which is of order one. The proof of Corollary 2.1 shows that $\alpha_1\leqq\alpha_2\leqq\alpha_3$. We shall show that $\alpha_3\leqq\alpha_4$, which will complete the proof of Theorem 1.1. We have

(3.4) $$24Q_4(k_1^{-1}t)=(t+3a-1)6Q_3(k_1^{-1}t)+3B(t)\,,$$

where $B(t)=(-2a^2+a+2b-1)t^2+(3a^2-2ab-a-2b)t$. Hence the inequality $\alpha_3\leqq\alpha_4$ is equivalent to

(3.5) $$B(k_1\alpha_3)\leqq 0\,.$$

Since $0<-2a^2+a+2b-1\leqq 1-a$ and $-3a^2+2ab+a+2b>-a^2+3a$ by (3.1), it is sufficient for (3.5) that

(3.6) $$k_1\alpha_3\leqq(1-a)^{-1}(3a-a^2)\,.$$

We obtain from (2.9) and (3.1) that

(3.7) $$\begin{aligned}k_1\alpha_3&=2^{-1}(3(1-a)+\{9(1-a)^2+4(3a-2b)\}^{1/2})\\&<3(1-a)+1\leqq 2\leqq(1-a)^{-1}(3a-a^2)\,,\end{aligned}$$

which implies (3.6) and hence $\alpha_3\leqq\alpha_4$. Thus the proof is complete.

4. Application to Lévy processes on $\boldsymbol{R}_+$.

Let $\mu(dx)$ be a measure on $\boldsymbol{R}_+$ for which the Laplace transform $L\mu(s)=\int_0^\infty e^{-sx}\mu(dx)$ exists for $s>0$. For $s>0$, define the measure $\eta^{(s)}(\mu, dx)$ on $\boldsymbol{Z}_+$ by

(4.1) $$\eta^{(s)}(\mu, dx)=\sum_{n=0}^{\infty}p_n^{(s)}(\mu)\delta_n(dx)\,,$$

where

$$p_n^{(s)}(\mu)=(n\,!)^{-1}\int_0^\infty e^{-sx}(sx)^n\mu(dx)\,.$$

Note that if $\mu(dx)$ is a probability measure on $\boldsymbol{R}_+$, then $\eta^{(s)}(\mu, dx)$ is a probability measure on $\boldsymbol{Z}_+$ for every $s>0$.

In this section, let X_t be a non-deterministic Lévy process on $\boldsymbol{R}_+$ without drift and let $\mu_t(dx)$ be the distribution of X_t. Then we have

$$\int_0^\infty e^{izx}\mu_t(dx) = e^{t\psi(z)},$$

$$\psi(z) = \int_0^\infty (e^{izx}-1)\nu(dx) \tag{4.2}$$

with $\nu(\{0\})=0$ and $\int_0^\infty x(1+x)^{-1}\nu(dx)<\infty$,

By argument in the proof of Forst's theorem [**1**], we find that $\eta_t^{(s)}(dx)= \eta^{(s)}(\mu_t, dx)$ is the distribution of a Lévy process $Y_t^{(s)}$ on $\boldsymbol{Z}_+$, whose Lévy measure is given by

$$\nu^{(s)}(dx) = \sum_{n=1}^\infty p_n^{(s)}(\nu)\delta_n(dx), \tag{4.3}$$

where $p_n^{(s)}(\nu)=(n\,!)^{-1}\int_0^\infty e^{-sx}(sx)^n\nu(dx)$ for $n\geqq 1$.

A measure $\mu(dx)$ on $\boldsymbol{R}$ (resp. on $\boldsymbol{Z}$) is said to be unimodal (resp. discrete unimodal) with mode ∞ if $\mu(dx)=f(x)dx$ (resp. $\mu(dx)=\sum_{n=-\infty}^{\infty}p_n\delta_n(dx)$), where $f(x)$ (resp. p_n) is non-decreasing for $-\infty<x<\infty$ (resp. $-\infty<n<\infty$). In the following lemma, $\mu(dx)$ and $\eta^{(s)}(\mu, dx)$ may have the mode ∞.

LEMMA 4.1. *Let $\mu(dx)$ be a measure on $\boldsymbol{R}_+$ for which the Laplace transform $L\mu(s)$ exists for $s>0$. Then $\mu(dx)$ is unimodal on $\boldsymbol{R}_+$ if and only if $\eta^{(s)}(\mu, dx)$ is discrete unimodal on $\boldsymbol{Z}_+$ for every $s>0$.*

PROOF. Suppose that $\mu(dx)$ is unimodal with mode a. If $a<\infty$, then we can write $\mu(dx)=c\delta_a(dx)+f(x)dx$, where $c\geqq 0$ and $f(x)$ is non-decreasing for $0<x<a$ and non-increasing for $x>a$. If $a=\infty$, then we can write $\mu(dx)= f(x)dx$ with non-decreasing $f(x)$.

Suppose first that $c=0$ and $\mu(dx)$ is a finite measure. Then we can prove that $\eta^{(s)}(\mu, dx)$ is discrete unimodal on $\boldsymbol{Z}_+$ for every $s>0$. In fact, by Holgate [3], $\eta^{(1)}(\mu, dx)$ is discrete unimodal. For $s\neq 1$, define $\mu_s(dx)=s^{-1}f(s^{-1}x)dx$. Then $\eta^{(s)}(\mu, dx)=\eta^{(1)}(\mu_s, dx)$ is discrete unimodal.

Secondly suppose that $c>0$ or $\mu(dx)$ is a infinite measure. Then we can make a sequence $\mu_n(dx)$ of measures on $\boldsymbol{R}_+$ such that if $a<\infty$, $\mu_n(dx)= (cnI_{[a,a+n^{-1}]}(x)+I_{[0,a+n]}(x)f(x))dx$ and if $a=\infty$, $\mu_n(dx)=I_{[0,n]}(x)f(x)dx$, where $I_E(x)$ is the indicator function of the interval E. The finite measure $\mu_n(dx)$ is unimodal and does not have a point mass. Since $\eta^{(s)}(\mu_n, dx)$ is discrete unimodal and converges vaguely to $\eta^{(s)}(\mu, dx)$ as $n\to\infty$, $\eta^{(s)}(\mu, dx)$ is discrete unimodal for every $s>0$.

Conversely suppose that $\eta^{(s)}(\mu, dx)$ is discrete unimodal for every $s>0$. Define $\zeta^{(s)}(dx)=\sum_{n=0}^{\infty}p_n^{(s)}(\mu)\delta_{n/s}(dx)$. Then $\zeta^{(s)}(dx)$ is vaguely convergent to $\mu(dx)$ as $s\to\infty$ by Forst [1] and discrete unimodal on $\{n/s: n\in \boldsymbol{Z}_+\}$ with some mode $a(s)$ for each $s>0$. We can find a sequence s_n such that $s_n\to\infty$ and $a(s_n)\to a$ $(0\leqq a\leqq\infty)$ as $n\to\infty$. It is clear that $\mu(dx)$ is unimodal with mode a. The proof of Lemma 4.1 is complete.

REMARK 4.1. Lemma 4.1 is essentially due to Forst [**1**] and Holgate [**3**]. Also Forst [**1**] proves that $\mu(dx)$ is unimodal with mode 0 if and only if $\eta^{(s)}(\mu, dx)$ is discrete unimodal with mode 0 for every $s>0$. Similarly we can prove that $\mu(dx)$ is unimodal with mode ∞ if and only if $\eta^{(s)}(\mu, dx)$ is discrete unimodal with mode ∞ for every $s>0$.

THEOREM 4.1. *A Lévy process X_t without drift is unimodal on $\boldsymbol{R}_+$ if and only if $Y_t^{(s)}$ is discrete unimodal on $\boldsymbol{Z}_+$ for every $s>0$.*

Proof is clear from Lemma 4.1.

Let $Q_n^{(s)}(t)$ be the polynomial $Q_n(t)$ in (2.3) corresponding to the Lévy process $Y_t^{(s)}$ on $\boldsymbol{Z}_+$. We obtain the following corollary from Theorems 2.1 and 4.1.

COROLLARY 4.1. *A Lévy process X_t without drift is unimodal on $\boldsymbol{R}_+$ if and only if, for every $s>0$, $Q_n^{(s)}(t)$ has a unique positive zero $\alpha_n^{(s)}$ of odd order for each $n\geqq 1$ and $\alpha_n^{(s)}$ is non-decreasing in n.*

COROLLARY 4.2. *Suppose that $x\nu(dx)$ is unimodal on $\boldsymbol{R}_+$. Then X_t without drift is unimodal on $\boldsymbol{R}_+$ if and only if X_t is of class L.*

PROOF. If X_t is of class L on $\boldsymbol{R}_+$, then X_t is unimodal by Wolfe [**14**]. Conversely suppose that X_t and $\tilde{\nu}(dx)=x\nu(dx)$ are unimodal on $\boldsymbol{R}_+$. Let $\nu^{(s)}(dx)$ $=\sum_{n=1}^{\infty}n^{-1}k_n^{(s)}\delta_n(dx)$ (see (4.3)) and define $\lambda^{(s)}(dx)$ by

$$\lambda^{(s)}(dx)=\sum_{n=1}^{\infty}k_{n+1}^{(s)}\delta_n(dx)=\sum_{n=0}^{\infty}sp_n^{(s)}(\tilde{\nu})\delta_n(dx)\,. \tag{4.4}$$

Then $\lambda^{(s)}(dx)$ is discrete unimodal on $\boldsymbol{Z}_+$ for every $s>0$ by Lemma 4.1. Since $Y_t^{(s)}$ is discrete unimodal on $\boldsymbol{Z}_+$ by Theorem 4.1, $\lambda^{(s)}(dx)$ is discrete unimodal with mode 0 for every $s>0$ by Corollary 2.2. Hence $x\nu(dx)$ is unimodal with mode 0 by Remark 4.1. It follows that X_t is of class L on $\boldsymbol{R}_+$. We have proved Corollary 4.2.

ACKNOWLEDGEMENT. The author would like to thank Prof. K. Sato for his helpful advice.

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