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The invariant differential forms on the Teichmüller space under the Fenchel-Nielsen flows

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Introduction.

We begin by giving an explanation of our motivation. Given a C^{∞} manifold M, $\Omega^*(M)$ denotes the differential graded algebra of all C^{∞} differential forms on M. Let G be a semi-simple Lie group, K its maximal compact subgroup, and Γ a discrete subgroup of G. The inclusion of invariant algebras

$$\Omega^*(G/K)^G \subset \Omega^*(G/K)^{\Gamma} = \Omega^*(\Gamma\backslash G/K)$$

induces the homomorphism

$$H^*(\Omega^*(G/K)^G) \longrightarrow H^*(\Gamma\backslash G/K).$$

In the case Γ is cocompact, Matsushima [**Ma**] established that it is isomorphic for sufficiently small $*$. Using this homomorphism Borel [**B**] determined the stable real cohomology of arithmetic groups.

Our purpose is to find an analogue of this Matsushima's theory for the moduli space of compact Riemann surfaces of genus g, $\mathbf{M}_g$. More precisely we want to find out a group which transitively acts on the Teichmüller space of compact Riemann surfaces of genus g, T_g, and includes the mapping class group $\mathcal{M}_g$ as a "discrete" subgroup. Then we want to investigate the invariant differential forms on T_g under the group and its relation to the real cohomology of $\mathbf{M}_g$.

On the other hand, by Dehn-Lickorish [**L**], the mapping class group $\mathcal{M}_g$ is generated by the Dehn twists associated to the simple closed curves in the oriented closed surface of genus g, Σ_g. The twist associated to a simple closed curve α is described as follows; cut Σ_g along α, rotate one boundary relative to the other in the angle 2π, and reglue in this new position. Varying the rotation angle over the real numbers, Fenchel-Nielsen defined the flow on the Teichmüller space T_g, called the Fenchel-Nielsen flow associated to α [**W1**][**G2**]. When we define FN the subgroup of $Diff(T_g)$ generated by the Fenchel-Nielsen flows associated to all simple closed geodesics, FN includes $\mathcal{M}_g$ as a "discrete" subgroup. Furthermore the group FN acts on T_g transitively in view of a theorem of Abikoff [Ab, Ch. 3, §4]. Thus we employ the group FN as the required analogue of G. Indeed, for $g=1$, we have $\mathcal{M}_1=SL(2, \boldsymbol{Z})$ and $FN=SL(2, \boldsymbol{R})$.

In the present paper, our main result is the following

THEOREM.

$$H^*(\Omega^*(T_g)^{FN})=\Omega^*(T_g)^{FN}=\boldsymbol{R}[\omega] \qquad \text{for} \quad * \leqq 3g-5.$$

Here $\omega\in\Omega^2(T_g)$ denotes the Weil-Petersson Kähler form.

Combining it with various known results, we obtain

COROLLARY. *The homomorphism*

$$H^*(\Omega^*(T_g)^{FN}) \longrightarrow H^*(\boldsymbol{M}_g)$$

induced by the inclusion

$$\Omega^*(T_g)^{FN} \subset \Omega^*(T_g)^{\mathcal{M}_g}=\Omega^*(\mathbf{M}_g)$$

is stably injective (Miller [**Mi**], *Morita* [**Mo**]), *(even stably) not surjective for* $*\geqq 4$ *(Miller* [**Mi**], *Morita* [**Mo**]), *stably isomorphic for* $*=0, 1, 2$ *(Harer* [**H**], *Powell* [**P**]).

The non-surjectivity follows from that the image does *not* contain the Morita-Mumford characteristic class e_i for $i\geqq 2$. Therefore, unfortunately, the group FN turned out to be unable to determine the stable real cohomology of the moduli space of compact Riemann surfaces.

Our proof of Theorem involves a tensor calculus with respect to the Fenchel-Nielsen coordinates associated to a pantalon decomposition. Following Wolpert [**W3**], we use an orientation-reversing involution ρ of Σ_g fixing the pantalon decomposition. $\Omega^*(T_g)^{FN}$ splits into the ± 1 eigenspaces of the action of ρ (§2), which enables us to compute $\Omega^*(T_g)^{FN}$. Our calculation for $*\geqq 2$ (§4) is complicated but elementary, while for $*=1$ (§3) we need the positivity of the Weil-Petersson Hessian of the geodesic length functions [**W4**].

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1. Definitions, Results, and Basic properties.

Fix an integer $g\geqq 2$. Let

Σ_g : the closed oriented C^∞ 2-manifold of genus g,

T_g : the Teichmüller space of compact Riemann surfaces of genus g,

$l_\alpha: T_g \to \boldsymbol{R}_{>0}$: the geodesic length function of α, where α is (the isotopy class of) a simple closed curve in Σ_g [**W4**],

$\omega \in \Omega^2(T_g)$: the Weil-Petersson Kähler form [**Ah**][**G1**],

Hh: the Hamiltonian vector field with respect to the symplectic structure ω whose potential is $h \in C^\infty(T_g)$.

The vector field Hh is characterized by the formula

$$\iota(Hh)\omega = -dh,$$

where ι denotes the interior product. The flow generated by $H((1/4\pi)l_\alpha{}^2)$, $\exp tH((1/4\pi)l_\alpha{}^2)$, coincides with the Fenchel-Nielsen flow associated to α that we mentioned in Introduction [**W1**].

Now fix an arbitrary real analytic function $f: \boldsymbol{R}_{>0} \to \boldsymbol{R}$, which satisfies, for all $l \in \boldsymbol{R}_{>0}$,

$$f'(l) > 0 \tag{1.1}$$

and

$$f''(l) \geqq 0 \tag{1.2}$$

(e.g., $f(l)=l$, $(1/4\pi)l^2$, $2\cosh(l/2)$.). We define the (f-) Fenchel-Nielsen group $FN = FN_f$ as the subgroup in $Diff(T_g)$ generated by $\exp(tHf(l_\alpha))$, where t (resp. α) runs over all reals (resp. all simple closed geodesics). The group denoted by FN in Introduction is $FN_{(1/4\pi)l^2}$ in the present notation.

Since FN acts on T_g preserving the 2-form ω, we obtain

$$\boldsymbol{R}[\omega] \subset \Omega^*(T_g)^{FN}. \tag{1.3}$$

Here, given a C^∞ manifold M, $\Omega^*(M)$ denotes the differential graded algebra of all C^∞ differential forms on M. Our main result is

THEOREM. *Under the above assumptions,*

$$\Omega^*(T_g)^{FN} = \boldsymbol{R}[\omega] \quad \textit{for} \quad * \leqq 3g-5.$$

The rest of this paper is devoted to the proof of this Theorem. Clearly we have

LEMMA 1. *$\theta \in \Omega^*(T_g)$ is FN-invariant if and only if*

$$L(Hf(l_\alpha))\theta = 0$$

for any simple closed curve α in Σ_g, where $L(\cdot)$ denotes the Lie derivative.

In view of a theorem of Abikoff [**Ab**, Ch. 3, §4], the group FN acts on T_g transitively, which implies

LEMMA 2.

(1) *For any* $x \in T_g$, *the evaluation map*

$$ev_x : \Omega^*(T_g)^{FN} \longrightarrow \Lambda^* T_x T_g^*$$

is injective.

(2)
$$\dim \Omega^n(T_g)^{FN} \leqq \binom{6g-6}{n}.$$

(3)
$$\Omega^0(T_g)^{FN} = \boldsymbol{R} \quad (\textit{the constant functions}).$$

(4)
$$\Omega^{6g-6}(T_g)^{FN} = \boldsymbol{R}(\omega)^{3g-3}.$$

(2) and (3) follow from (1), and (4) from (1) and (1.3).

For the rest of this paper, we fix a pantalon decomposition $\mathcal{P} = \{\alpha_i\}_{i=1}^{3g-3}$ of Σ_g. By definition, each α_i, $1 \leqq i \leqq 3g-3$, is a simple closed curve in Σ_g, no two α_i's intersect each other, and the complement $\Sigma_g - \bigcup_{i=1}^{3g-3} \alpha_i$ is diffeomorphic to the disjoint union of $2g-2$ copies of pantalons, where pantalon means the complement of three disjoint discs in S^2. For example we may fix the decomposition shown in Figure 1 associated to the trivalent graph in Figure 2.

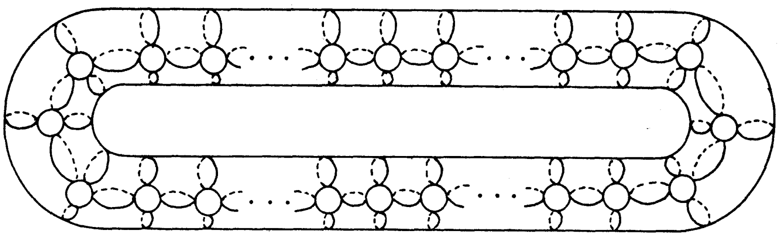


Figure 1.

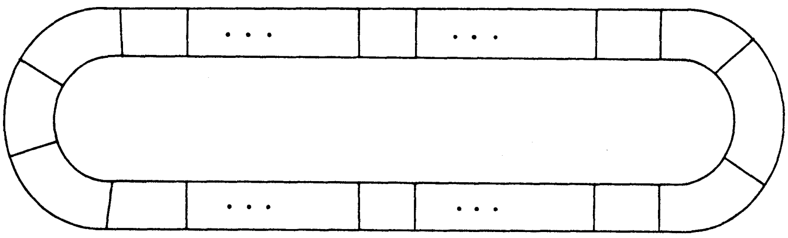


Figure 2.

Let $\{l_i, t_i\}_{i=1}^{3g-3}$ be the Fenchel-Nielsen coordinates associated to the decomposition $\mathcal{P}$. l_i is the geodesic length of α_i, l_{α_i}, and t_i measures the hyperbolic displacement between the canonical points on both sides of α_i $(1 \leqq i \leqq 3g-3)$ [**Ab**][**W3**]. Wolpert [**W3**, Theorem 1.3] shows the formula

$$\omega = \sum_{i=1}^{3g-3} dl_i \wedge dt_i. \tag{1.4}$$

We define the modified Fenchel-Nielsen coordinates $\{k_i, s_i\}_{i=1}^{3g-3}$ associated to the decomposition $\mathcal{P}$ as follows;

$$k_i = f(l_i) \tag{1.5}$$
$$s_i = \frac{t_i}{f'(l_i)}.$$

The coordinates $\{k_i, s_i\}_{i=1}^{3g-3}$ are real analytic on T_g. From (1.4) follows

$$\omega = \sum_{i=1}^{3g-3} dk_i \wedge ds_i. \tag{1.6}$$

Especially, for $1 \leqq i \leqq 3g-3$,

$$Hf(l_{\alpha_i}) = Hk_i = \frac{\partial}{\partial s_i} \tag{1.7}$$
$$Hs_i = -\frac{\partial}{\partial k_i}.$$

We introduce the following notation to carry out some tensor calculus with respect to the modified Fenchel-Nielsen coordinates.

NOTATION.

$$\theta^{(q)}_{i_1,\cdots,i_p,j_1,\cdots,j_q} := \frac{1}{p!q!}\theta\left(\frac{\partial}{\partial k_{i_1}}, \cdots, \frac{\partial}{\partial k_{i_p}}, \frac{\partial}{\partial s_{j_1}}, \cdots, \frac{\partial}{\partial s_{j_q}}\right) \in C^\infty(T_g) \tag{1.8}$$

$$\theta^{(q)} := \sum_{i_1,\cdots,i_p,j_1,\cdots,j_q} \theta^{(q)}_{i_1,\cdots,i_p,j_1,\cdots,j_q} dk_{i_1} \cdots dk_{i_p} ds_{j_1} \cdots ds_{j_q} \in \Omega^n(T_g) \tag{1.9}$$

for $\theta \in \Omega^n(T_g)$, $p+q=n$, *and* $1 \leqq i_1, \cdots, i_p, j_1, \cdots, j_q \leqq 3g-3$.

Then immediately

$$\theta = \sum_{q=0}^{n} \theta^{(q)}.$$

LEMMA 3.

(1) *The function* $\theta^{(q)}_{i_1,\cdots,i_p,j_1,\cdots,j_q}$ *does not depend on the* s_j's, $1 \leqq j \leqq 3g-3$, *when* $\theta \in \Omega^n(T_g)^{FN}$, $p+q=n$, *and* $1 \leqq i_1, \cdots, i_p, j_1, \cdots, j_q \leqq 3g-3$.

(2) *For* $\theta \in \Omega^n(T_g)^{FN}$ *and* $0 \leqq q \leqq n$,

$$d(\theta^{(q)}) = (d\theta)^{(q)}.$$

PROOF. (1) By the fact

$$L\left(\frac{\partial}{\partial s_{j_0}}\right)dk_i = L\left(\frac{\partial}{\partial s_{j_0}}\right)ds_j = 0, \qquad \text{for } 1 \leqq i, j, j_0 \leqq 3g-3$$

and Lemma 1, we obtain

$$
\begin{aligned}
0 &= L(Hf(l_{\alpha_{j0}}))\theta \\
&= L\Big(\frac{\partial}{\partial s_{j0}}\Big)\theta \\
&= \sum_{p+q=n}\sum_{i_1,\cdots,i_p,j_i,\cdots,j_q}\frac{\partial\theta^{(q)}_{i_1,\cdots,i_p,j_i,\cdots,j_q}}{\partial s_{j0}}dk_{i_1}\cdots dk_{i_p}ds_{j_1}\cdots ds_{j_q}.
\end{aligned}
$$

Hence, for $i\leqq i_1, \cdots, i_p, j_i, \cdots, j_q, j_0\leqq 3g-3$,

$$\frac{\partial\theta^{(q)}_{i_1,\cdots,i_p,j_i,\cdots,j_q}}{\partial s_{j0}}=0.$$

(2) follows from (1) immediately.

2. Wolpert's involution.

Following Wolpert [**W3**] we introduce an orientation-reversing involution ρ of Σ_g fixing the pantalon decomposition $\mathcal{P}=\{\alpha_i\}_{i=1}^{3g-3}$.

To define ρ, consider a pantalon with a hyperbolic structure with geodesic boundary. Then the pantalon has a unique isometric reflection ρ_0 fixing each boundary component. On each boundary component ρ_0 has two fixed points, which are the canonical points mentioned in the preceding section. Let $\{l_i, t_i\}_{i=1}^{3g-3}$ be the Fenchel-Nielsen coordinates associated to the decomposition $\mathcal{P}$, and $R\in T_g$ a marked Riemann surface which satisfies

$$t_i=0 \qquad (1\leqq i\leqq 3g-3). \tag{2.1}$$

Represent each α_i as a unique geodesic in R. Then each component of the complement $R-\cup\alpha_i$ is a pantalon with a hyperbolic structure with geodesic boundary, which has a unique isometric reflection ρ_0 fixing each boundary component. From (2.1), assembling the reflections ρ_0 of each hyperbolic pantalon, we can construct a reflection ρ of R. ρ is an orientation-reversing involution of Σ_g fixing each α_i setwise, and determines an element of the extended mapping class group $E\mathcal{M}_g=\pi_0 Diff(\Sigma_g)$. Recall the extended mapping class group $E\mathcal{M}_g$ acts on the Teichmüller space T_g naturally [**W3**].

LEMMA 4.

(1) $$\rho^*l_i=l_i \qquad (1\leqq i\leqq 3g-3).$$

(2) $$\rho^*t_i=-t_i \qquad (1\leqq i\leqq 3g-3).$$

(3) $$\rho^*\omega=-\omega.$$

PROOFS. (1) and (3) are contained in [**W3**, Lemma 1.1].

(2) From [**W3**, Lemma 1.1] follows

$$\rho^* t_i = -t_i + \frac{n_i}{2} l_i \qquad (n_i \in \boldsymbol{Z}).$$

But, from the construction of ρ, $\rho(R)=R$, and, from (2.1), $t_i(R)=0$. Hence we obtain $n_i=0$.

Using (1.5) we define the modified Fenchel-Nielsen coordinates $\{k_i, s_i\}_{i=1}^{3g-3}$ from the coordinates $\{l_i, t_i\}_{i=1}^{3g-3}$. Lemma 4 implies the following.

LEMMA 5.

(1) $$\rho^* k_i = k_i \qquad (1 \leqq i \leqq 3g-3).$$

(2) $$\rho^* ds_i = -ds_i \qquad (1 \leqq i \leqq 3g-3).$$

(3) $$\rho_* Hf(l_\alpha) = -Hf(l_{\rho(\alpha)}) \qquad \alpha: \textit{a simple closed curve.}$$

(3) follows from Lemma 4(3) and the fact

$$\rho^* l_\alpha = l_{\rho(\alpha)}.$$

Especially, from (3), the action of ρ preserves $\Omega^*(T_g)^{FN}$. Since ρ is involutive, $\Omega^*(T_g)^{FN}$ splits into the ± 1 eigenspaces of the action of ρ. Thus we define

$$\Omega^*(T_g)^{FN}{}_+ := \text{the } +1 \text{ eigenspace of } \rho \text{ in } \Omega^*(T_g)^{FN}$$

$$\Omega^*(T_g)^{FN}{}_- := \text{the } -1 \text{ eigenspace of } \rho \text{ in } \Omega^*(T_g)^{FN}.$$

Then we have

$$\Omega^*(T_g)^{FN} = \Omega^*(T_g)^{FN}{}_+ \oplus \Omega^*(T_g)^{FN}{}_-.$$

The following enables us to compute $\Omega^*(T_g)^{FN}$.

LEMMA 6. *If we set, for* $\theta \in \Omega^*(T_g)^{FN}$,

$$\theta_+ := \sum_{q:\ \text{even}} \theta^{(q)}$$

$$\theta_- := \sum_{q:\ \text{odd}} \theta^{(q)},$$

then we have

$$\theta_+ \in \Omega^*(T_g)^{FN}{}_+$$

$$\theta_- \in \Omega^*(T_g)^{FN}{}_-.$$

PROOF. By Lemma 3(1) each $\theta^{(q)}_{i_1,\dots,i_p,j_1,\dots,j_q}$ does not depend on the s_j's, which implies

$$\rho^* \theta^{(q)}_{i_1,\dots,i_p,j_1,\dots,j_q} = \theta^{(q)}_{i_1,\dots,i_p,j_1,\dots,j_q}.$$

Hence, by Lemma 5(2),

$$\rho^*\theta^{(q)} = (-1)^q\theta^{(q)}$$

This proves Lemma 6.

COROLLARY. *Let $\theta \in \Omega^n(T_g)^{FN}$ and $0 \leqq q \leqq n$. Then we have*

$$(L(Hf(l_\alpha))\theta^{(q)})^{(q)} = 0.$$

PROOF. By Lemma 6, we may assume

$$\theta = \sum_{q' \equiv q(2)} \theta^{(q')}.$$

Since, for a vector field X on T_g,

$$(L(X)\theta^{(q)})^{(r)} = 0 \qquad \text{if} \quad |q-r| > 1,$$

we have

$$(L(X)\theta)^{(q)} = (L(X)\theta^{(q)})^{(q)},$$

which proves Corollary.

3. $\Omega^1(\boldsymbol{T}_g)^{FN}=0$.

In this section we prove $\Omega^1(T_g)^{FN}=0$ using the positivity of the Weil-Petersson Hessian of geodesic length functions [**W4**]. Let $\{\gamma_a\}_{a=1}^A$ be a collection of simple closed curves filling up Σ_g. We set

$$k_\gamma := \sum_{a=1}^{A} f(l_{\gamma a}).$$

Then Kerckhoff [**K.** Lemma 3.1] shows

LEMMA (KERCKHOFF). *The function $\sum_{a=1}^A l_{\gamma a} : T_g \to \boldsymbol{R}$ attains a minimum in T_g.*

We remark the same holds for the function $k_\gamma = \sum f(l_{\gamma a}) : T_g \to \boldsymbol{R}$, since f satisfies the condition (1.1).

On the other hand Wolpert [**W4**, Theorem 4.6] proves

THEOREM (WOLPERT). *For an arbitrary closed curve α in Σ_g, the Weil-Petersson Hessian of the function $l_\alpha : T_g \to \boldsymbol{R}$ is positive definite.*

By the convexity of the function f (1.2), we have

LEMMA 7. *For an arbitrary α, the Weil-Petersson Hessian of the function $f(l_\alpha) : T_g \to \boldsymbol{R}$ is positive definite.*

REMARK. Only this lemma requires the condition (1.2).

Combining the above lemmata, we obtain

LEMMA 8. *For almost every $x \in T_g$, the matrices*

$$\left(\frac{\partial^2 k_\gamma}{\partial k_i \partial k_j}(x)\right)_{1 \leq i, j \leq 3g-3} \quad \text{and} \quad \left(\frac{\partial^2 k_\gamma}{\partial s_i \partial s_j}(x)\right)_{1 \leq i, j \leq 3g-3}$$

are non-degenerate.

PROOF. By Lemma (Kerckhoff) and the succeeding remark, there is a point $x_0 \in T_g$ where the function k_γ attains a minimum. At x_0 the Hessian of k_γ with respect to the coordinates $\{k_i, s_i\}_{i=1}^{3g-3}$ coincides with the Weil-Petersson Hessian of k_γ and is positive definite by Lemma 7. Hence the given matrices are positive definite and non degenerate at x_0. Since k_γ, k_i, and s_i are all real analytic on T_g, the matrices are nondegenerate almost everywhere on T_g, which proves Lemma 8.

Now we can prove

LEMMA 9.

$$\Omega^1(T_g)^{FN} = 0.$$

PROOF. Take an arbitrary $\theta \in \Omega^1(T_g)^{FN}$. Lemma 6 implies $\theta^{(0)}$, $\theta^{(1)} \in \Omega^1(T_g)^{FN}$. Let $\{\gamma_a\}_{a=0}^{A}$ be the collection mentioned above. Using $(\partial/\partial k_j) = -Hs_j$ (1.7), we obtain

$$\begin{aligned} 0 &= \left(\sum_a L(Hf(l_{\gamma_a}))\theta^{(1)}\right)^{(0)} = \left(\sum_a \sum_j L(Hf(l_{\gamma_a}))(\theta_j^{(1)} ds_j)\right)^{(0)} \\ &= \sum_a \sum_j \theta_j^{(1)} (L(Hf(l_{\gamma_a}))ds_j)^{(0)} = \sum_a \sum_j \sum_i \theta_j^{(1)} \frac{\partial (Hf(l_{\gamma_a}))s_j}{\partial k_i} dk_i \\ &= \sum_a \sum_j \sum_i \theta_j^{(1)} \frac{\partial^2 f(l_{\gamma_a})}{\partial k_i \partial k_j} dk_i = \sum_j \sum_i \theta_j^{(1)} \frac{\partial^2 k_\gamma}{\partial k_i \partial k_j} dk_i. \end{aligned}$$

Hence, for each i,

$$0 = \sum_j \theta_j^{(1)} \frac{\partial^2 k_\gamma}{\partial k_i \partial k_j}.$$

From Lemma 8 follows

$$\theta_j^{(1)} = 0 \qquad (1 \leq j \leq 3g-3),$$

that is

$$\theta^{(1)} = 0.$$

The same holds for $\theta^{(0)}$.

4. Fenchel-Nielsen tensor calculus.

Fix a pantalon decomposition $\mathcal{P}=\{\alpha_i\}_{i=1}^{3g-3}$ of Σ_g. We impose upon $\mathcal{P}$ the condition that each α_i is a boundary component of two different pantalons. For example the decomposition shown in Figure 1 satisfies the condition. In this section we prove our Theorem by means of a tensor calculus with respect to the modified Fenchel-Nielsen coordinates $\{k_i, s_i\}_{i=1}^{3g-3}$ associated to the decomposition $\mathcal{P}$.

The restriction $*\leqq 3g-5$ comes from the following.

LEMMA 10. *For $n\leqq 3g-4$, the map*

$$\omega\wedge : \Omega^n(T_g) \longrightarrow \Omega^{n+2}(T_g)$$

is injective.

PROOF. Let

$V_n=\boldsymbol{R}^{2n}$,

$x_1, \cdots, x_n, x_{n+1}, \cdots, x_{2n}$: a basis of V_n,

$u_1, \cdots, u_n, u_{n+1}, \cdots, u_{2n}$: the basis of $V_n{}^*$ (the dual of V_n) satisfying

$$\langle u_i, x_j\rangle=\delta_{ij} \qquad (1\leqq i, j\leqq 2n),$$

$\omega_i = u_i\wedge u_{n+i} \in \Lambda^2 V_n{}^* \qquad (1\leqq i\leqq n)$,

$\omega = \sum_{i=1}^{n} \omega_i = \sum_{i=1}^{n} u_i\wedge u_{n+i}$.

Lemma 10 follows from the following

(1) The map $\omega\wedge : \Lambda^{n-1}V_n{}^* \to \Lambda^{n+1}V_n{}^*$ is isomorphic.

(2) The map $\omega\wedge : \Lambda^{*}V_n{}^* \to \Lambda^{*+2}V_n{}^*$ is injective for $*\leqq n-1$.

PROOF OF (1). Let

$$\{1, \cdots, n\} = \{i_1, \cdots, i_p\}\amalg\{j_1, \cdots, j_q\}\amalg\{a_1, \cdots, a_r\}\amalg\{b_1, \cdots, b_s\}.$$

$$p+q+2r=n+1$$

Then the map $\omega\wedge$ maps

$$u_{i_1}\wedge \cdots \wedge u_{i_p}\wedge u_{n+j_1}\wedge \cdots \wedge u_{n+j_q} \wedge\frac{1}{r!}\sum_{t=0}^{s}(-1)^t(\omega_{a_1}+\cdots+\omega_{a_r})^{s-t}\wedge(\omega_{b_1}+\cdots+\omega_{b_s})^t$$

into

$$u_{i_1}\wedge\cdots\wedge u_{i_p}\wedge u_{n+j_1}\wedge\cdots\wedge u_{n+j_q}\wedge\omega_{a_1}\wedge\cdots\wedge\omega_{a_r},$$

which means its surjectivity. Since the dimension of the range is equal to that of the domain, the map $\omega\wedge: \Lambda^{n-1}V_n^* \to \Lambda^{n+1}V_n^*$ is isomorphic.

PROOF OF (2). We prove (2) inductively on n. Clearly it holds for $n=0, 1$. Suppose $n \geqq 2$. In view of (1), we may assume $*=k \leqq n-2$. Let

$$\begin{aligned} \iota: V_{n-1} &\longrightarrow V_n \\ x_i &\longmapsto x_i, \qquad x_{n-1+i} \longmapsto x_{n+i} \qquad (1 \leqq i \leqq n-1) \end{aligned}$$

be the inclusion. Consider the morphism of the exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \iota^* & \longrightarrow & \Lambda^k V_n^* & \stackrel{\iota^*}{\longrightarrow} & \Lambda^k V_{n-1}^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker \iota^* & \longrightarrow & \Lambda^{k+2} V_n^* & \stackrel{\iota^*}{\longrightarrow} & \Lambda^{k+2} V_{n-1}^* & \longrightarrow & 0, \end{array}$$

where the vertical arrows denote $\omega\wedge$. Because of the assumption $k \leqq n-2$ and the inductive assumption the right vertical arrow is injective. Hence it suffices to show that the left arrow is injective. We have the direct sum decomposition

$$\ker \iota^* = x_n \wedge \Lambda^{k-1} V_{n-1}^* \oplus x_{2n} \wedge \Lambda^{k-1} V_{n-1}^* \oplus \omega_n \wedge \Lambda^{k-2} V_{n-1}^*.$$

The map $\omega\wedge$ preserves the decomposition and is injective on each component by the inductive assumption. So the left arrow is injective, which completes the proof of Lemma 10.

We prove our Theorem by assembling the following five lemmas

LEMMA 11. *Let* $2 \leqq n \leqq 3g-4$. *Suppose* $\theta \in \Omega^n(T_g)$ *satisfies*

$$d\theta^{(n)} = 0$$

$$(L(Hf(l_\alpha))\theta^{(n)})^{(n)} = 0 \qquad \textit{for an arbitrary } \alpha.$$

Then $\theta^{(n)} = 0$.

LEMMA 12. *Let* $0 \leqq q < n$, $p = n-q$, *and*

$$\theta = \sum_{q' \leqq q} \theta^{(q')} \in \Omega^n(T_g)^{FN}.$$

Then we have

$$\theta^{(q)}_{i_1,\cdots,i_p,j_1,\cdots,j_q} = 0, \qquad \textit{if} \quad \{i_1, \cdots, i_p\} \not\subset \{j_1, \cdots, j_q\}.$$

LEMMA 13. *Let*

$$\theta = \sum_{q < n-q} \theta^{(q)} \in \Omega^n(T_g)^{FN}.$$

Then $\theta = 0$.

LEMMA 14. *Let* $n = 2m+1$, $\varphi = \varphi^{(1)} \in \Omega^1(T_g)$, *and*

$$\theta=\omega^{m}\wedge\varphi+\sum_{q\leq m}\theta^{(q)}\in\Omega^{n}(T_{g})^{FN}.$$

Then $\theta=0$.

LEMMA 15. *Let* $p+q=n$, *and* $\theta\in\Omega^{n}(T_{g})^{FN}$. *Suppose*

$$\theta_{i_{1},\cdots,i_{p},j_{1},\cdots,j_{q}}^{(q)}=0, \qquad if \ \{i_{1},\cdots,i_{p}\}\not\subset\{j_{1},\cdots,j_{q}\}.$$

Then there exists $\varphi=\varphi^{(q-p)}\in\Omega^{(q-p)}(T_{g})$ *such that*

$$\theta^{(q)}=\omega^{p}\wedge\varphi.$$

The proofs are delayed until that of the following theorem finishes.

THEOREM.

$$\Omega^{*}(T_{g})^{FN}=\boldsymbol{R}[\omega] \qquad for \quad *\leqq 3g-5.$$

PROOF. The cases $*=0$, 1 are proved in Lemmas 2(3) and 9, respectively. For $*\geqq 2$, we claim as follows.

(4.1) $2\leqq n\leqq 3g-4$, n : even, $\theta\in\Omega^{n}(T_{g})^{FN}$, and $d\theta=0\Rightarrow\theta\in\boldsymbol{R}\omega^{n/2}$

(4.2) $2\leqq n\leqq 3g-4$, n : odd, $\theta\in\Omega^{n}(T_{g})^{FN}$, and $d\theta=0\Rightarrow\theta=0$

(4.3) $2\leqq n\leqq 3g-5$, n : even, and $\theta\in\Omega^{n}(T_{g})^{FN}\Rightarrow\theta\in\boldsymbol{R}\omega^{n/2}$

(4.4) $2\leqq n\leqq 3g-5$, n : odd, and $\theta\in\Omega^{n}(T_{g})^{FN}\Rightarrow\theta=0$

PROOF OF (4.1). From the assumption $d\theta=0$ and Lemma 3(2), $d\theta^{(n)}=0$. Corollary of Lemma 6 implies $(L(Hf(l_{\alpha}))\theta^{(q)})^{(q)}=0$ for an arbitrary α. Hence, by Lemma 11, we have $\theta=\sum_{q\leq n-1}\theta^{(q)}$. Applying Lemmas 12 and 15 to $\theta^{(n-1)}$, we obtain $\varphi=\varphi^{(n-2)}\in\Omega^{n-2}(T_{g})$ such that

$$\theta^{(n-1)}=\omega\wedge\varphi$$

Lemma 10 and the assumption $2\leqq n\leqq 3g-4$ show φ satisfies the hypothesis of Lemma 11. Hence we have $\varphi=0$, and $\theta^{(n-1)}=0$. Iterating these considerations, we conclude

$$\theta=\sum_{q<n-q}\theta^{(q)}+\lambda\omega^{n/2} \qquad (\lambda\in\boldsymbol{R}).$$

Since $\lambda\omega^{n/2}\in\Omega^{n}(T_{g})^{FN}$, $\theta-\lambda\omega^{n/2}\in\Omega^{n}(T_{g})^{FN}$, which implies $\theta=\lambda\omega^{n/2}$ by Lemma 13. This proves (4.1).

PROOF OF (4.2). In a similar manner to (4.1), we conclude

$$\theta=\varphi\wedge\omega^{(n-1)/2}+\sum_{q<n-q}\theta^{(q)},$$

where $\varphi=\varphi^{(1)}\in\Omega^{1}(T_{g})$. Lemma 14 implies $\theta=0$.

PROOF OF (4.3). Since $dd=0$, we can apply (4.2) to $d\theta$. Hence $d\theta=0$.

Thus, by (4.1), $\theta\in\boldsymbol{R}\omega^{n/2}$.

PROOF OF (4.4). Applying (4.1) to $d\theta$,

$$d\theta=\lambda\omega^{(n+1)/2}\qquad(\lambda\in\boldsymbol{R}).$$

From Lemma 3(2) follows

$$d\theta^{(q)}=0\qquad\left(\frac{n}{2}+1\leqq q\right).$$

Hence Lemmas 10, 11, 12 and 15 can be applied to θ in a similar manner to (4.1), and we conclude

$$\theta=\varphi\wedge\omega^{(n-1)/2}+\sum_{q<n-q}\theta^{(q)},$$

where $\varphi=\varphi^{(1)}\in\Omega^1(T_g)$, which proves $\theta=0$ by Lemma 14.

Thus the proof of Theorem is completed except for those of Lemmas 11-15.

PROOF OF LEMMA 11. First of all we prove $\theta^{(n)}_{j_1,\cdots,j_n}$ is constant on T_g. By Lemma 3(1), we have

$$\frac{\partial\theta^{(n)}_{j_1,\cdots,j_n}}{\partial s_j}=0\qquad(1\leqq j,\ j_1,\ \cdots,\ j_n\leqq 3g-3).$$

Hence

$$\begin{aligned}0&=d\theta^{(n)}\\&=\sum_{j_1,\cdots,j_n}\frac{\partial\theta^{(n)}_{j_1,\cdots,j_n}}{\partial k_i}dk_i ds_{j_1}\cdots ds_{j_n},\end{aligned}$$

that is,

$$\frac{\partial\theta^{(n)}_{j_1,\cdots,j_n}}{\partial k_i}=0\qquad(1\leqq i,\ j_1,\ \cdots,\ j_n\leqq 3g-3).$$

Thus $\theta^{(n)}_{j_1,\cdots,j_n}(1\leqq j_1,\ \cdots,\ j_n\leqq 3g-3)$ is constant on T_g.

Fix $1\leqq i\leqq 3g-3$. Let β be a simple closed curve in Σ_g which intersects α_i and does not intersect α_j $(j\neq i)$ (see Figure 3). We notice $\partial f(l_\beta)/\partial s_j=0$ $(j\neq i)$. Since each $\theta^{(n)}_{j_1,\cdots,j_n}$ is constant,

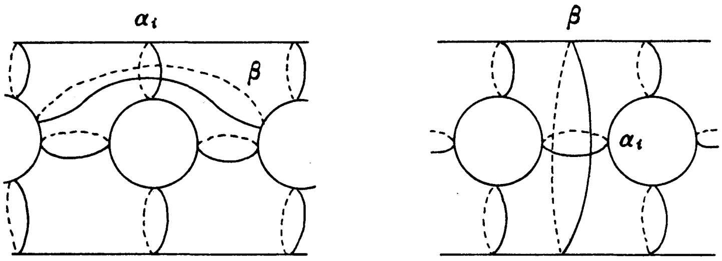


Figure 3.

$$0=(L(Hf(l_\beta))\theta^{(n)})^{(n)}$$
$$=n\sum_{j_0}\sum_{j_1,\cdots,j_n}\theta^{(n)}_{j_1,\cdots,j_n}\frac{\partial^2 f(l_\beta)}{\partial s_{j_0}\partial k_{j_1}}ds_{j_0}ds_{j_2}\cdots ds_{j_n}$$
$$=n\sum_{j_1,\cdots,j_n}\theta^{(n)}_{j_1,\cdots,j_n}\frac{\partial^2 f(l_\beta)}{\partial s_i\partial k_{j_1}}ds_i ds_{j_2}\cdots ds_{j_n}.$$

Applying the interior product $\iota(\partial/\partial s_i)$,

$$0=\sum_{j_1,\cdots,j_n, i\neq j_2,\cdots,j_n}\theta^{(n)}_{j_1,\cdots,j_n}\frac{\partial^2 f(l_\beta)}{\partial s_i\partial k_{j_1}}ds_{j_2}\cdots ds_{j_n}.$$

Hence we obtain, if $i\neq j_2,\cdots,j_n$,

$$0=\sum_{j_i}\theta^{(n)}_{j_1,\cdots,j_n}\frac{\partial^2 f(l_\beta)}{\partial s_i\partial k_{j_1}}. \tag{4.5}$$

From (4.5) we can deduce the following

$$\theta^{(n)}_{j_0,j_2,\cdots,j_n}=\theta^{(n)}_{j_1,j_2,\cdots,j_n}$$

(4.6) if $\exists i(\neq j_0, j_1, j_2,\cdots,j_n)$ such that

α_i, α_{j_0}, and α_{j_1} are boundary components of a single pantalon.

$$\theta^{(n)}_{j_1,j_2,\cdots,j_n}=0$$

(4.7) if $\exists i(\neq j_1, j_2,\cdots,j_n)$ such that

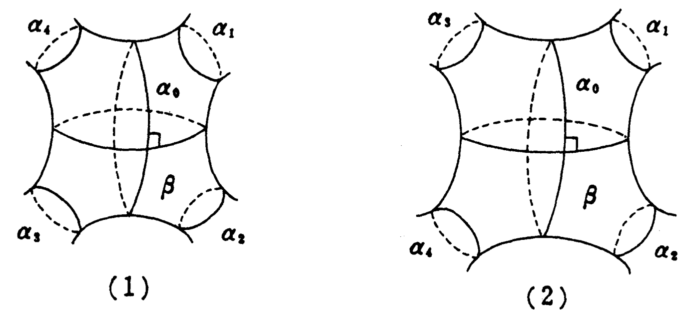
α_i, α_{j_1}, and α_{j_2} are boundary components of a single pantalon.

Lemma 11 follows from (4.6) and (4.7). In fact, for an arbitrary $\theta^{(n)}_{j_1,\cdots,j_n}$, applying (4.6) repeatedly, there is obtained a $\theta^{(n)}_{j'_1,\cdots,j'_n}$ satisfying the hypothesis of (4.7) and

$$\theta^{(n)}_{j_1,\cdots,j_n}=\theta^{(n)}_{j'_1,\cdots,j'_n}.$$

By (4.7) we conclude $\theta^{(n)}_{j_1,\cdots,j_n}=0\,(1\leqq j_1,\cdots,j_n\leqq 3g-3)$.

To prove (4.6) and (4.7), consider the situation shown in Figure 4(1). For simplicity we renumber the α_i's. We set



$t=0,\ l_1=l_2=l_3=l_4.$

Figure 4.

$t = t_0 =$ the twist parameter along α_0

$l_i = l_{\alpha_i} \quad (0 \leqq i \leqq 4)$

$s = \dfrac{t}{f'(l_0)}$

$k_i = f(l_i) \quad (0 \leqq i \leqq 4)\,.$

From Wolpert's cosine formula [**W2**][**G2**] it follows

$$\frac{\partial l_\beta}{\partial t} = \cos\theta_1 + \cos\theta_2\,,$$

where θ_1 and θ_2 denote the angles between the geodesics β and α_0. Hence we have

$$\frac{\partial l_\beta}{\partial t} = 0$$

$$\frac{\partial^2 l_\beta}{\partial t \partial l_0} = 0\,.$$

$$\frac{\partial^2 l_\beta}{\partial t \partial l_i} = \pm \frac{\mu(\lambda - \lambda^{-1})}{2(\mu+1)(\lambda^{-1}\mu+\lambda)^{1/2}(\lambda\mu+\lambda^{-1})^{1/2}} \neq 0$$

at the point $t=0$, $l_0 = 2\log\mu$, and $l_i = 2\log\lambda\,(1 \leqq i \leqq 4)$ (see [**W1**]). Therefore, when we denote by c_i the value of $\partial^2 f(l_\beta)/\partial s \partial k_i$ at the point $s=0$, $k_1 = k_2 = k_3 = k_4$, we have

$$c_i \neq 0 \quad (1 \leqq i \leqq 4)\,.$$

It is easy to show

$$c_1 = -c_2 = c_3 = -c_4\,.$$

Furthermore, at the point,

$$\frac{\partial^2 f(l_\beta)}{\partial s \partial k_0} = 0\,.$$

Thus, evaluating (4.5) there, we obtain

$$0 = \theta^{(n)}_{1,j_2,\cdots,j_n} - \theta^{(n)}_{2,j_2,\cdots,j_n} + \theta^{(n)}_{3,j_2,\cdots,j_n} - \theta^{(n)}_{4,j_2,\cdots,j_n}\,. \tag{4.8}$$

Since the same holds for the situation shown in Figure 4(2),

$$0 = \theta^{(n)}_{1,j_2,\cdots,j_n} - \theta^{(n)}_{2,j_2,\cdots,j_n} - \theta^{(n)}_{3,j_2,\cdots,j_n} + \theta^{(n)}_{4,j_2,\cdots,j_n}\,. \tag{4.9}$$

Summing up (4.8) and (4.9), we conclude

$\theta^{(n)}_{1,j_2,\cdots,j_n} = \theta^{(n)}_{2,j_2,\cdots,j_n}$

if $0 \neq j_2, \cdots, j_n$, and

α_0, α_1, and α_2 are boundary components of a single pantalon, since each $\theta^{(n)}_{j_1,\cdots,j_n}$ is constant. (4.6) and (4.7) follows from (4.10) immediately, which completes the proof of Lemma 11.

PROOF OF LEMMA 12. Fix $1 \leqq i \leqq 3g-3$. Let β be a simple closed curve in Σ_g which intersects α_i and does not intersect α_j $(j \neq i)$ (see Figure 3). It should be observed that $\partial f(l_\beta)/\partial s_j = 0$ $(j \neq i)$.

$$\begin{aligned} 0 &= (L(Hf(l_\beta))\theta)^{(q+1)} \\ &= -p \sum_j \sum_{\substack{i_1,\cdots,i_p,\\ j_1,\cdots,j_q}} \theta^{(q)}_{i_1,\cdots,i_p,j_1,\cdots,j_q} \frac{\partial^2 f(l_\beta)}{\partial s_j \partial s_{i_1}} ds_j dk_{i_2} \cdots dk_{i_p} ds_{j_1} \cdots ds_{j_q} \\ &= -p \frac{\partial^2 f(l_\beta)}{\partial s_i^2} \sum_{i_2,\cdots,i_p,j_1,\cdots,j_q} \theta^{(q)}_{i,i_2,\cdots,i_p,j_1,\cdots,j_q} ds_i dk_{i_2} \cdots dk_{i_p} ds_{j_1} \cdots ds_{j_q}. \end{aligned}$$

Easily we have $\partial^2 f(l_\beta)/\partial s_i^2 \not\equiv 0$, which is real analytic on T_g. Hence

$$0 = \sum_{i_2,\cdots,i_p,j_1,\cdots,j_q} \theta^{(q)}_{i,i_2,\cdots,i_p,j_1,\cdots,j_q} ds_i dk_{i_2} \cdots dk_{i_p} ds_{j_1} \cdots ds_{j_q}.$$

Thus we obtain

$$\theta^{(q)}_{i,i_2,\cdots,i_p,j_1,\cdots,j_q} = 0, \qquad \text{if} \quad i \notin \{j_1, \cdots, j_q\},$$

which proves Lemma 12.

PROOF OF LEMMA 13. Let $\theta = \sum_{q' \leq q} \theta^{(q')} \in \Omega^n(T_g)^{FN}$, and $q < n-q =: p$. Applying Lemma 12 to $\theta^{(q)}$,

$$\theta^{(q)}_{i_1,\cdots,i_p,j_1,\cdots,j_q} = 0, \qquad \text{if} \quad \{i_1, \cdots, i_p\} \not\subset \{j_1, \cdots, j_q\}.$$

Since $p > q$, the hypothesis of (4.11) is always satisfied, and so $\theta^{(q)} = 0$. Iterating this procedure, we obtain $\theta = 0$.

PROOF OF LEMMA 14. From Lemma 6 follows

$$\sum_{q \equiv m (2)} \theta^{(q)} \in \Omega^n(T_g)^{FN},$$

which vanishes by Lemma 13. Hence we may assume

$$\theta = \sum_{q \not\equiv m (2)} \theta^{(q)} = \omega^m \wedge \varphi + \theta^{(m-1)} + \theta^{(m-3)} + \cdots.$$

Set

$$\varphi = \sum_{j=1}^{3g-3} \varphi_j ds_j.$$

Fix $1 \leqq i \leqq 3g-3$, and let β be as in Lemma 12. We have

$$\begin{aligned} 0 &= (L(H(f(l_\beta))\theta)^{(m)} \\ &= \omega^m \wedge \sum_{i_1,i_2} \varphi_{i_1} \frac{\partial Hf(l_\beta) s_{i_1}}{\partial k_{i_2}} dk_{i_2} \end{aligned}$$

$$+(m+2)\sum_j \sum_{\substack{i_2,\cdots,i_{m+2}\\ j_1,\cdots,j_{m-1}}} \Big\{\theta^{(m-1)}_{i_1,i_2,\cdots,i_{m+2},j_1,\cdots,j_{m-1}} \frac{\partial Hf(l_\beta)k_{i_1}}{\partial s_j}$$

$$ds_j dk_{i_2}\cdots dk_{i_{m+2}} ds_{j_1}\cdots ds_{j_{m-1}}\Big\}$$

$$= \sum_{j_1,\cdots,j_m}\sum_{i_1,i_2}\varphi_{i_1}\frac{\partial^2 f(l_\beta)}{\partial k_{i_1}\partial k_{i_2}} dk_{j_1}ds_{j_1}\cdots dk_{j_m}ds_{j_m}dk_{i_2}$$

$$-(m+2)\frac{\partial^2 f(l_\beta)}{\partial s_i{}^2}\sum_{\substack{i_2,\cdots,i_{m+2}\\ j_1,\cdots,j_{m-1}}}\{\theta^{(m-1)}_{i,i_2,\cdots,i_{m+2},j_1,\cdots,j_{m-1}}$$

$$ds_i dk_{i_2}\cdots dk_{i_{m+2}} ds_{j_1}\cdots ds_{j_{m-1}}\}.$$

Here for each term in the first summation the indices of dk includes those of ds, while for each (nontrivial) term in the second the index i of ds is not contained in the indices of dk. Hence we conclude

$$\sum_{\substack{i_2,\cdots,i_{m+2}\\ j_1,\cdots,j_{m-1}}}\theta^{(m-1)}_{i,i_2,\cdots,i_{m+2},j_1,\cdots,j_{m-1}} ds_i dk_{i_2}\cdots dk_{i_{m+2}}ds_{j_1}\cdots ds_{j_{m-1}}=0.$$

In a similar manner to Lemmas 12 and 13, we obtain $\theta^{(m-1)}=0$, and so

$$\theta=\omega^m\wedge\varphi+\theta^{(m-3)}+\cdots.$$

Hence $\theta-\omega^m\wedge\varphi=\sum_{q\le m-3}\theta^{(q)}\in\Omega^{2m+1}(T_g)^{FN}$ vanishes by Lemma 13. Since, for $n\leqq 3g-2$, the map

$$\omega^n\wedge:\Omega^1(T_g)\longrightarrow\Omega^{2n+1}(T_g)$$

is injective, $\varphi\in\Omega^1(T_g)^{FN}$. Thus $\theta=\omega^m\wedge\varphi=0$ by Lemma 9, which completes the proof of Lemma 14.

PROOF OF LEMMA 15. We set

$$\Theta_{i_1,\cdots,i_p,j_1,\cdots,j_r}=\frac{1}{p!\,r!}\theta\Big(\frac{\partial}{\partial k_{i_1}},\frac{\partial}{\partial s_{i_1}},\cdots,\frac{\partial}{\partial k_{i_p}},\frac{\partial}{\partial s_{i_p}},\frac{\partial}{\partial s_{j_1}},\cdots,\frac{\partial}{\partial s_{j_r}}\Big),$$

where $r:=q-p$. By the hypothesis,

$$\theta^{(q)}=\sum_{\substack{i_1,\cdots,i_p\\ j_1,\cdots,j_r}}\Theta_{i_1,\cdots,i_p,j_1,\cdots,j_r}dk_{i_1}ds_{i_1}\cdots dk_{i_p}ds_{i_p}ds_{j_1}\cdots ds_{j_p}.$$

Our purpose is to show that the function $\Theta_{i_1,\cdots,i_p,j_1,\cdots,j_r}$ does not depend upon the indices $i_1,\cdots,i_p$.

For a simple closed curve β in Σ_g,

$$0=(L(Hf(l_\beta))\theta^{(q)})^{(q)}$$

$$=\sum_{\substack{i_1,\cdots,i_p\\ j_1,\cdots,j_r}}(Hf(l_\beta)\Theta_{i_1,\cdots,i_p,j_1,\cdots,j_r})dk_{i_1}ds_{i_1}\cdots dk_{i_p}ds_{i_p}ds_{j_1}\cdots ds_{j_p}$$

$$+p\sum_{i_0}\sum_{\substack{i_1,\cdots,i_p\\ j_1,\cdots,j_r}}\Big\{(\Theta_{i_0,i_2,\cdots,i_p,j_1,\cdots,j_r}-\Theta_{i_1,i_2,\cdots,i_p,j_1,\cdots,j_r})$$

$$\frac{\partial^2 f(l_\beta)}{\partial k_{i_0}\partial s_{i_1}} dk_{i_0}ds_{i_1}\cdots dk_{i_p}ds_{i_p}ds_{j_1}\cdots ds_{j_p}\Big\}$$

$$+r\sum_{j_0}\sum_{\substack{i_1,\cdots,i_p\\ j_1,\cdots,j_r}}\Big\{\Theta_{i_1,\cdots,i_p,j_1,\cdots,j_r}\frac{\partial^2 f(l_\beta)}{\partial s_{j_0}\partial k_{j_1}} dk_{i_1}ds_{i_1}\cdots dk_{i_p}ds_{i_p}ds_{j_0}ds_{j_2}\cdots ds_{j_r}\Big\}.$$

Collecting the terms satisfying

$$\#(\{\text{the indices of } dk\}\cap\{\text{the indices of } ds\})=p-1,$$

we have

$$0=\sum\Big\{(\Theta_{i_0,i_2,\cdots,i_p,j_1,\cdots,j_r}-\Theta_{i_1,i_2,\cdots,i_p,j_1,\cdots,j_r})\frac{\partial^2 f(l_\beta)}{\partial k_{i_0}\partial s_{i_1}} dk_{i_0}ds_{i_1}\cdots dk_{i_p}ds_{i_p}ds_{j_1}\cdots ds_{j_p}\Big\}, \tag{4.11}$$

where the sum runs over the indices $i\leqq i_0, i_1, i_2, \cdots, i_p, j_1, \cdots, j_r\leqq 3g-3$ under the condition $i_0\neq i_1, j_1, \cdots, j_r$. Applying the interior product $\iota(\partial/\partial k_{i_2})\iota(\partial/\partial s_{i_2})\cdots \iota(\partial/\partial k_{i_p})\iota(\partial/\partial s_{i_p})$ to (4.11), we obtain, for $1\leqq i_2, \cdots, i_p\leqq 3g-3$,

$$0=\sum_{i_0\neq i_1,j_1,\cdots,j_r}\Big\{(\Theta_{i_0,i_2,\cdots,i_p,j_1,\cdots,j_r}-\Theta_{i_1,i_2,\cdots,i_p,j_1,\cdots,j_r})\frac{\partial^2 f(l_\beta)}{\partial k_{i_0}\partial s_{i_1}} dk_{i_0}ds_{i_1}ds_{j_1}\cdots ds_{j_r}\Big\}. \tag{4.12}$$

To prove Lemma 15 by means of (4.12), we define the notion of a $\mathcal{P}$-path: $P=(a_0, \cdots, a_A)$ is a $\mathcal{P}$-path, if

$a_i\in\{1, \cdots, 3g-3\}$ $(0\leqq i\leqq A)$, and

$\alpha_{a_{i-1}}$ and α_{a_i} are boundary components of a single pantalon $(1\leqq i\leqq A)$.

For a subset $J=\{j_1^0, \cdots, j_r^0\}\subset\{1, \cdots, 3g-3\}$, we define the obstacle number $\nu(J, P)$ of a $\mathcal{P}$-path $P=(a_0, \cdots, a_A)$ with respect to the subset J as follows:

$$\nu(J, P):=\max\{0\}\cup\{i_2-i_1+1\,;\ a_i\in J\ (i_1\leqq\forall i\leqq i_2)\}$$

We prove the following inductively on ν.

$$\left.\begin{array}{l} J=\{j_1^0, \cdots, j_r^0\}\subset\{1, \cdots, 3g-3\}\\ P=(a_0, \cdots, a_A)\colon \mathcal{P}\text{-path}\\ a_0,\ a_A\notin J\\ \nu(J, P)\leqq\nu \end{array}\right. \tag{4.13}$$
$$\Rightarrow \Theta_{a_0,i_2,\cdots,i_p,j_1^0,\cdots,j_r^0}=\Theta_{a_A,i_2,\cdots,i_p,j_1^0,\cdots,j_r^0}$$

From (4.13), Lemma 15 follows immediately.

The case $\nu=0$. Take an arbitrary $J=\{j_1^0, \cdots, j_r^0\}$. It suffices to show (4.13) for a $\mathcal{P}$-path $P=(a_0, a_1)$ satisfying $\nu(J, P)=0$. Let β be a simple closed curve which intersects α_{a_1} and does not intersect $\alpha_j (j\neq a_1)$ as in Lemma 12. Evaluating (4.12) for the curve β,

$$0=\frac{\partial^2 f(l_\beta)}{\partial k_{a_0}\partial s_{a_1}}\sum_{j_1,\cdots,j_r\neq a_0,a_1}\{(\Theta_{i_0,i_2,\cdots,i_p,j_1,\cdots,j_r}-\Theta_{i_1,i_2,\cdots,i_p,j_1,\cdots,j_r})\\ dk_{a_0}ds_{a_1}ds_{j_1}\cdots ds_{j_r}\}$$

Since α_{a_0} and α_{a_1} are boundary components of a pantalon, we have

$$\frac{\partial^2 f(l_\beta)}{\partial k_{a_0}\partial s_{a_1}}\not\equiv 0.$$

Applying the interior product $\iota(\partial/\partial k_{a_0})\iota(\partial/\partial s_{a_1})\iota(\partial/\partial s_{j_1^0})\cdots\iota(\partial/\partial s_{j_r^0})$, we obtain

$$\Theta_{a_1,i_2,\cdots,i_p,j_1^0,\cdots,j_r^0}=\Theta_{a_0,i_2,\cdots,i_p,j_1^0,\cdots,j_r^0},$$

which proves (4.13) for $\nu=0$.

The case $\nu\geqq 1$. Take an arbitrary $J=\{j_1^0, \cdots, j_r^0\}$. It suffices to show (4.13) for a $\mathcal{P}$-path $P=(a_0, \cdots, a_{\nu+1})$ satisfying $a_0, a_{\nu+1}\notin J$ and $a_i\in J(1\leqq i\leqq\nu)$. Renumbering the set J, we may assume

$$a_i=j_i^0 \qquad (1\leqq i\leqq\nu).$$

Let β be a simple closed curve which intersect $\alpha_{a_i}(1\leqq i\leqq\nu+1)$ "along" $\mathcal{P}$-path P and does not intersect $\alpha_j(j\neq a_i, 1\leqq i\leqq\nu)$ (see Figure 5).

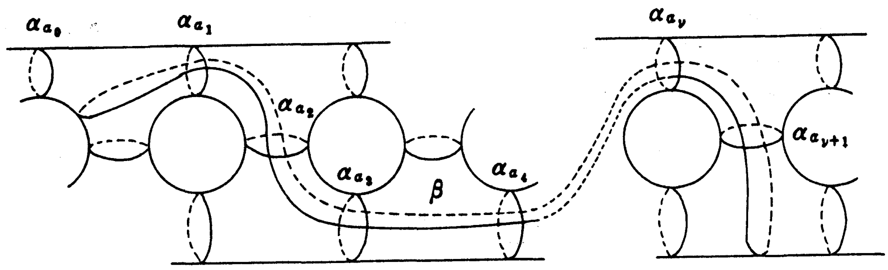


Figure 5.

Evaluating (4.12) for the curve β,

$$0=\sum_{j_1,\cdots,j_r\neq 0}\sum_{i=1}^{\nu+1}(\Theta_{a_i,i_2,\cdots,i_p,j_1,\cdots,j_r}\\ -\Theta_{a_0,i_2,\cdots,i_p,j_1,\cdots,j_r})\frac{\partial^2 f(l_\beta)}{\partial k_{a_0}\partial s_{a_i}}dk_{a_0}ds_{a_i}ds_{j_1}\cdots ds_{j_r}.$$

Applying the interior product $\iota(\partial/\partial k_{a_0})\iota(\partial/\partial s_{a_{\nu+1}})\iota(\partial/\partial s_{j_1^0})\cdots\iota(\partial/\partial s_{j_r^0})$, we obtain

$$(4.14)\qquad 0=\sum_{i=1}^{\nu}\sum_{j_1,\cdots,j_r\neq 0}(\Theta_{a_i,i_2,\cdots,i_p,j_1^0,\cdots,j_{i-1}^0,a_{\nu+1},j_{i+1}^0,\cdots,j_r^0} -\Theta_{a_0,i_2,\cdots,i_p,j_1^0,\cdots,j_{i-1}^0,a_{\nu+1},j_{i+1}^0,\cdots,j_r^0})\frac{\partial^2 f(l_\beta)}{\partial k_{a_0}\partial s_{a_i}} +(\Theta_{a_{\nu+1},i_2,\cdots,i_p,j_1^0,\cdots,j_r^0}-\Theta_{a_0,i_2,\cdots,i_p,j_1^0,\cdots,j_r^0})\frac{\partial^2 f(l_\beta)}{\partial k_{a_0}\partial s_{a_{\nu+1}}}.$$

Now we have

$$\nu(\{j_1^0,\cdots,j_{i-1}^0,a_{\nu+1},j_{i+1}^0,\cdots,j_r^0\},(a_0,\cdots,a_i))=i-1.$$

Hence, by the inductive assumption, the i-th term of (4.14) vanishes for $1\leqq i\leqq\nu$. From the definition of the curve β, we have $\partial^2 f(l_\beta)/\partial k_{i_0}\partial s_{a_{\nu+1}}\neq 0$. Therefore

$$\Theta_{a_{\nu+1},i_2,\cdots,i_p,j_1^0,\cdots,j_r^0}=\Theta_{a_0,i_2,\cdots,i_p,j_1^0,\cdots,j_r^0},$$

which completes the inductive proof of (4.13).

(4.13) implies that the function $\Theta_{i_1,\cdots,i_p,j_1,\cdots,j_r}$ does not depend upon the indices $i_1,\cdots,i_p$. This proves Lemma 15.

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