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Twisting and knot types

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Introduction.

Let K be an unoriented smooth knot in the oriented 3-sphere S^3, and V a solid torus endowed with a preferred framing which contains K in its interior. Let f_n be a twisting homeomorphism, that is an orientation preserving homeomorphism of V satisfying $f_n(m)=m$ and $f_n(l)=l+nm$ in $H_1(\partial V)$, where (m, l) is a meridian-longitude pair of V. (Throughout this paper a longitude means a preferred longitude, and we shall not distinguish notationally between a homeomorphism and an isomorphism on a homology group induced by it and also often identify a homology class with a curve representing it.) These define the new knot $f_n(K)$ in S^3, which is denoted by $K_{V,n}$. We call this operation *twisting*. For two knots K_1 and K_2, we write $K_1 \sim K_2$ if there exists a homeomorphism of S^3 carrying K_1 to K_2. In particular if this homeomorphism preserves the orientation of S^3, then we write $K_1 \cong K_2$. Note that $K_1 \cong K_2$ if and only if K_1 and K_2 are ambient isotopic in S^3. It is easy to see that for two homeomorphisms f_n, g_n satisfying the above condition, we have $f_n(K) \cong g_n(K)$ in S^3. The wrapping number of K in V is defined to be the geometric intersection number of K with a meridian disk in V, and we denote it by $w_V(K)$. $w_V(K)=0$ is the same as saying that K is contained in a 3-ball in V. Clearly $K_{V,n} \cong K$ when $w_V(K) \leqq 1$. The origin of twisting of knots goes back to the example given by Whitehead [**20**]. In his paper, he constructed knots depicted in Figure 1, and proved that T_m and T_n are distinct knots for $m \neq n$

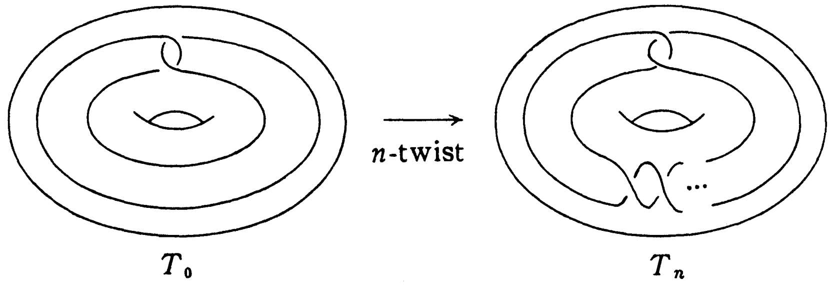

Figure 1

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by using their Alexander polynomials. (Compare the example in Remark 4.3.)

Twisting is a natural move and one of the important notions in knot theory, which leads us to the following fundamental problem.

PROBLEM. *A knot K in V is assumed to have $w_V(K)\geqq 2$. Can "twisting" change knot type of K?*

In Section 2, we treat this problem in the case when V is knotted and answer affirmatively by Theorems 2.1 (in the strong equivalence $\cong$) and 2.8 (in the weak equivalence $\sim$). On the other hand, if V is unknotted, then there exist examples such that $K_{V,n}\cong K$ for some nonzero integer n (Example 3.1). Sections 3 and 4 are devoted to the case when V is unknotted. Our purpose in Section 3 is to prove Theorem 3.2 and give a precise information in the case when the wrapping number of K in V equals the winding number of K in V (i.e. the algebraic intersection number of K with a meridian disk in V) by Theorems 3.10 and 3.11. In Section 4 we shall particularly concern ourselves with the case when the original knot is trivial and answer the above problem by Theorem 4.2 as an implicit corollary of Litherland [**9**] and Gabai [**2**]. This also answers Problem 1.18 (B) in [**7**] by Martin (solved by Bleiler-Scharlemann [**1**]) and its generalization. Recently Mathieu has obtained independently Theorem 4.2 in his thesis [**10**]. Throughout this paper $N(X)$, ∂X and $\operatorname{int} X$ denote the tubular neighborhood of X, the boundary of X and the interior of X respectively.

1. Knots in a solid torus.

In this section we shall study knots in a solid torus V. Any knot K in V is assumed to have $w_V(K)\geqq 2$.

LEMMA 1.1. *$V-\operatorname{int} N(K)$ is a boundary irreducible Haken manifold.*

PROOF. The fact $w_V(K)\neq 0$ implies the required result. ■

We will use the torus decomposition of $V-\operatorname{int} N(K)$ in the sense of Jaco-Shalen [**4**], Johannson [**5**] and Thurston's uniformization theorem [**11**]. They assert that $V-\operatorname{int} N(K)$ is uniquely decomposed by tori into pieces each of which is Seifert fibred or admits a complete hyperbolic structure of finite volume in its interior. In particular, for Seifert pieces, we have the following.

LEMMA 1.2. *Each Seifert piece is either a torus knot space, a cable space or a composing space (see [**4**]).*

PROOF. We may consider that V is a knotted solid torus in S^3. Then the Seifert piece in $V-\operatorname{int} N(K)$ is a Seifert fibred manifold in the knot space

$S^3-\operatorname{int} N(K)$ with incompressible boundary. Hence the result follows from Lemma VI 3.4 [4]. ■

Let f be an orientation preserving homeomorphism of V, f is said to be *faithful* if $f|_{\partial V}$ induces *identity* or *−identity* of $H_1(\partial V)$. The following two lemmas are useful but their proofs are easy and omitted here.

LEMMA 1.3. *Let T be a torus and α_1, α_2 essential simple loops in T which satisfies $\alpha_1 \neq \pm\alpha_2$ in $H_1(T)$. Suppose that f is a homeomorphism of T satisfying $f(\alpha_1)=\alpha_1$, $f(\alpha_2)=\alpha_2$ in $H_1(T)$. Then f is isotopic to the identity.*

LEMMA 1.4. *Let f be an orientation preserving homeomorphism of V. If f^N is faithful for some integer $N>0$, then f is also faithful.*

The next theorem is a fundamental result on knot theory in a solid torus.

THEOREM 1.5. *Let V be a solid torus and K a knot in V satisfying $w_V(K) \geqq 2$. Suppose that f is an orientation preserving homeomorphism of V with $f(K)=K$. Then f must be faithful.*

PROOF. The proof of this theorem will be done by the induction on the closed Haken number of the exterior $V-\operatorname{int} N(K)$, $h(V-\operatorname{int} N(K))$ (see [3]). Let $\{T_1, \cdots, T_s\}$ be a family of tori which defines the torus decomposition of $V-\operatorname{int} N(K)$ in the sense of Jaco-Shalen, Johannson and $P_0, \cdots, P_u$ are decomposing pieces each of which is Seifert fibred or hyperbolic. Moreover a Seifert piece has a form as in Lemma 1.2. For convenience we assume P_0 contains ∂V. Since T_i has a meridian disk in V, there exists a meridian m_i of T_i. In addition if T_i does not separate $\partial N(K)$ and ∂V, then T_i has a longitude l_i which bounds an orientable surface in $V-\operatorname{int} N(K)$. We may assume that $f(N(K))=N(K)$, and by the uniqueness of the torus decomposition, we can assume $\{T_1, \cdots, T_s\}$ is invariant under f. Hence for simplicity we may assume that after an iteration of f, f^N satisfies $f^N(T_i)=T_i$, $f^N(P_j)=P_j$, $f^N(m)=m$, $f^N(m_i)=m_i$. Moreover if T_i does not separate $\partial N(K)$ and ∂V, then we may assume that $f^N(l_i)=l_i$.

We remark that in our situation $h(V-\operatorname{int} N(K))\geqq 3$.

First step. $h(V-\operatorname{int} N(K))=3$.

In this case $V-\operatorname{int} N(K)$ consists of only one piece P_0. When P_0 is a cable space, we have $f^N(t)=t$ for a regular fibre t of P_0 because its Seifert fibration is unique (Lemma VI. 18 [3]). Also $f^N(m)=m$ holds for a meridian m of V. We easily see that $t\neq\pm m$, so by Lemma 1.3, f^N is the identity of $H_1(\partial V)$ and f is faithful by Lemma 1.4. If P_0 is a composing space, it is a one fold composing space (i.e. homeomorphic to $S^1\times S^1\times I$) and K is a core of V. This contradicts the fact $w_V(K)\geqq 2$.

Now we consider the case when P_0 is hyperbolic. By Mostow's rigidity theorem, $f^N|_{\mathrm{int}\, P_0}$ is homotopic to a unique isometry φ of int P_0. Since $Isom(\mathrm{int}P_0)$ is a compact Lie group, and which is discrete by Mostow's rigidity theorem, it is a finite group (see [**11**]). Hence φ^n is the identity for some integer $n>0$. It follows that $f^{nN}|_{\mathrm{int}\, P_0}$ is homotopic to the identity. Using Waldhausen's result in [**19**], by an isotopic deformation we may assume $f^{nN}|_{\mathrm{int}\overline{P_0}}$ is the identity, where $\overline{P_0}$ denotes a compact submanifold in P_0 obtained from P_0 by truncating the open collar neighborhood of P_0. Hence f is faithful by Lemma 1.4.

Second step.

When P_0 is a cable space or hyperbolic, the result holds by the same argument in First step. Suppose P_0 is a composing space, that is, $P_0=\Delta\times S^1$ where Δ is a disk with k-holes ($k\geqq 2$). We set $\partial\Delta=a_0\cup\cdots\cup a_k$, $T_i=a_i\times S^1$ ($i=1,\cdots,k$), $T_0=\partial V$ and assume T_1 separates ∂V and $\partial N(K)$. In this case, for a regular fibre $t=\{*\}\times S^1$, $t=m_i$ and T_i has a longitude l_i for $i=2,\cdots,k$. It follows that $f^N|_{T_i}$ is isotopic to the identity for $i=2,\cdots,k$ by Lemma 1.3. Since the fact $\partial N(K)=T_1$ implies $w_V(K)=1$, we have $\partial N(K)\neq T_1$. Let W be a solid torus in V bounded by T_1, then $f^N(W)=W$. In addition, since the boundary of a meridian disk of W coincides with a regular fibre t of P_0, the union of the meridian disk of W and a saturated annulus A in P_0 such that $\partial A\subset\partial V\cup T_1$ and $\partial A\supset t$ becomes a meridian disk of V. Thereby we see $w_W(K)\geqq 2$, and we can check $h(W-\mathrm{int}\,N(K))<h(V-\mathrm{int}\,N(K))$. Hence we can use the induction hypothesis and can conclude that $f^N|_W$ is faithful. In particular f^N satisfies $f^N(m_1)=m_1$, thus $f^N|_{T_1}$ is the identity of $H_1(T_1)$. Then we have:

$$\begin{aligned} f^N(a_0)-a_0 &= f^N(a_0)+a_1+\cdots+a_k \\ &= f^N(a_0)+f^N(a_1)+\cdots+f^N(a_k) \\ &= f^N(a_0+\cdots+a_k)=f^N(0)=0 \end{aligned}$$

In this way we get $f^N(t)=t$, $f^N(a_0)=a_0$. Since $a_0\neq\pm t$, $f^N|_{\partial V}$ is the identity by Lemma 1.3 and this implies f is faithful by Lemma 1.4. This completes the proof. ■

2. Twisting along knotted solid tori.

We now state the main result in this section, which gives the complete answer to Problem in the case when V is knotted.

THEOREM 2.1. *Let K be a knot in S^3 and V a knotted solid torus containing K with $w_V(K)\geqq 2$. For any nonzero integer n, $K_{V,n}\not\cong K$.*

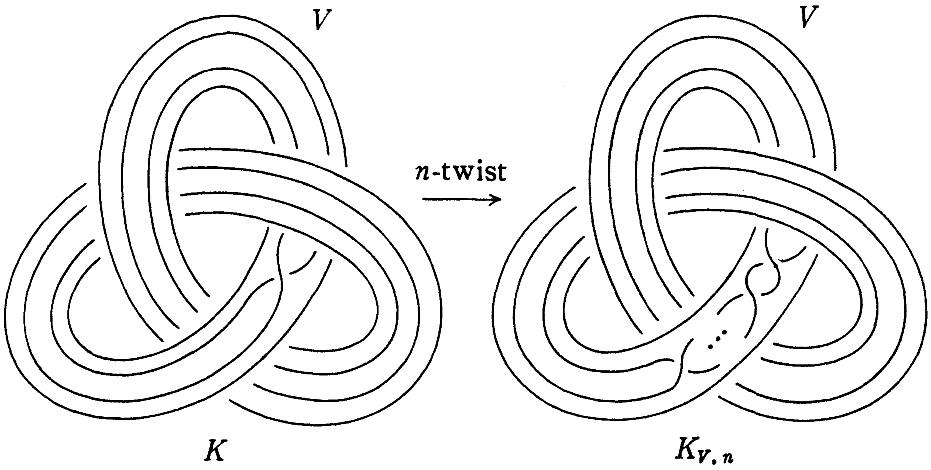


Figure 2

REMARK 2.2. This theorem implies also that if $m \neq n$, then $K_{V,m} \not\cong K_{V,n}$, because $K_{V,m}$ is obtained from $K_{V,n}$ by $(m-n)$-twist along V.

PROOF OF THEOREM 2.1. In our situation, K and $K_{V,n}(=f_n(K))$ are knots in a knotted solid torus $V \subset S^3$ with $w_V(K) \geqq 2$. Supposing $K \cong K_{V,n}$ then there exists an orientation preserving homeomorphism g of S^3 such that $g(K_{V,n})=K$. Then $f=g|_V \circ f_n$ is an orientation preserving homeomorphism from V to $V'=g(V)$ with $f(K)=K$.

From now on, we prove that f sends *longitude* of V to $\pm$*longitude* of V'. Such an orientation preserving homeomorphism between two solid tori is said to be *faithful*.

Our first step is to study the geometric situation of V and V'. The following lemma which is a consequence of Satz 1, § 18 in [**15**] gives us complete information which we need in this step.

LEMMA 2.3. *By an ambient isotopy of S^3 which leaves K fixed, we can deform V' so that either*

(1) $\partial V \cap \partial V' = \emptyset$, *or*

(2) *there exist meridian disks D_1 and D_2 of both V and V' such that the closure of one component of $V' - \bigcup_{i=1}^2 D_i$ is a knotted 3-ball in the closure of some component of $V - \bigcup_{i=1}^2 D_i$ (see Figure 3).*

We continue to use the symbol V' and f to denote the results which are deformed by this ambient isotopy, and we assume that $f(N(K))=N(K)$ for some tubular neighborhood $N(K)$. We remark that the resulting homeomorphism f is faithful if and only if the original homeomorphism f is faithful.

In case (1), we divide further into three cases.

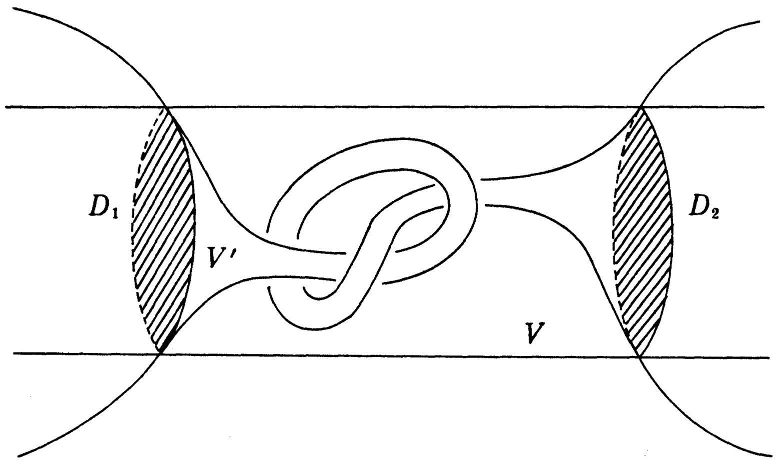


Figure 3

(a) $V' \subset \operatorname{int} V$

(b) $V \subset \operatorname{int} V'$

(c) $V \cup V' = S^3$.

If (b) occurs, by exchanging V and f for V' and f^{-1}, we can regard (b) as (a).

Case (1)-(a). When $T_1 = \partial V$ and $T_2 = \partial V'$ are parallel in S^3, we can modify f so that $V = V'(=f(V))$ by an ambient isotopy which leaves K fixed. In this case, we can conclude that f is faithful by Theorem 1.5.

Suppose T_1 and T_2 are not parallel in S^3. Since $V' = f(V) \subset \operatorname{int} V$, we can consider f^s for any integer $s>0$, and we put $T_s = f^{s-1}(T_1)$. Then we have:

SUBLEMMA 2.4. *$\{T_1, \cdots, T_s\}$ is a collection of disjoint nonparallel incompressible tori in $V - \operatorname{int} N(K)$.*

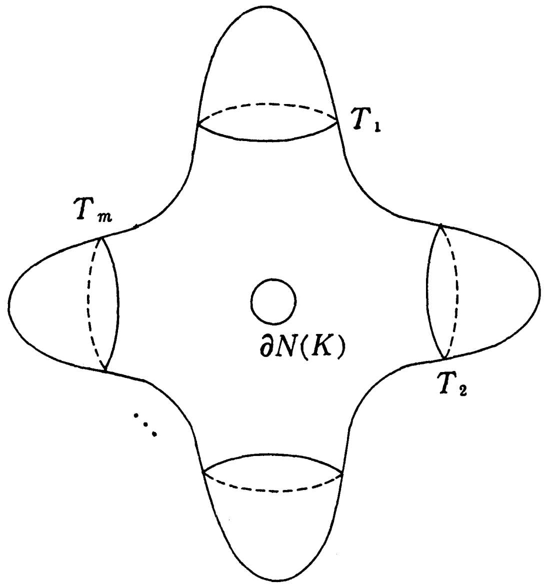
The proof of this is essentially the same as that of Sublemma 1 in [8], and we omit it. Since the above sublemma holds for any integer $s>0$, this contradicts Haken's finiteness theorem (see Theorem III.20 [3]).

Case (1)-(c). In this case $T_2 = f(T_1)(=\partial V')$ is contained in $\operatorname{int} V$ and it does not separate $T_1(=\partial V)$ and $\partial N(K)$. We suppose that for $i=1, \cdots, m-1$, $T_{i+1} = f(T_i)$ can be defined and $T_2, \cdots, T_m$ are contained in $\operatorname{int} V$ and each of which does not separate $T_1(=\partial V)$ and $\partial N(K)$ (see Schematic 1).

Consider $T_{m+1} = f(T_m)$. Let V_{m+1} be the solid torus bounded by T_{m+1} containing $N(K)$ in S^3. We note that $w_{V_{m+1}}(K) \geqq 1$. For two solid tori V and V_{m+1}, as in Lemma 2.3, by an ambient isotopy of S^3 which leaves K fixed, V_{m+1} can be arranged so that either

(1′) $\partial V \cap \partial V_{m+1} = \emptyset$, or

(2′) there exist meridian disks D_1 and D_2 of both V and V_{m+1} such that



Schematic 1

the closure of one component X' of $V_{m+1}-\bigcup_{i=1}^{2}D_i$ is a knotted 3-ball in the closure of some component X of $V-\bigcup_{i=1}^{2}D_i$.

Moreover we can assume that $N(K)$, $T_2, \cdots, T_m$ are also fixed through the above ambient isotopy.

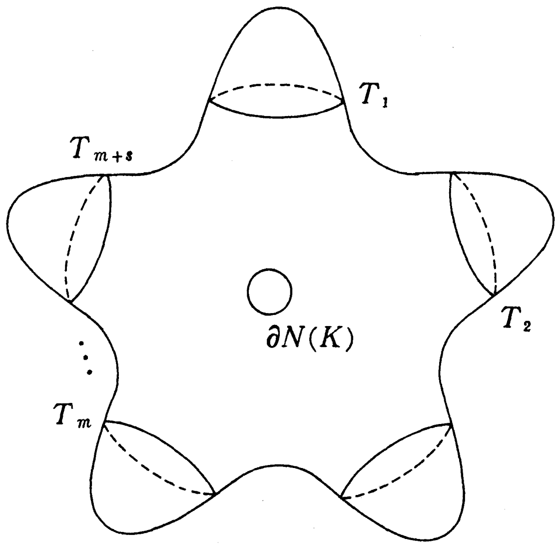
If necessary we modify f by this ambient isotopy and denote the resulting homeomorphism from V to V' by the same symbol f, and we continue to use the symbol T_{m+1} to denote the boundary of the isotoped V_{m+1}. Clearly the resulting homeomorphism f is faithful if and only if the original one is faithful. For this resulting f, we can still assume that $f(N(K))=N(K)$ and $f(T_i)=T_{i+1}$ $(i=1, \cdots, m)$ by the choice of an ambient isotopy.

First we treat the case (1)-(c)-(1′), that is, $T_{m+1}\cap T_1=\emptyset$.

CLAIM. *If T_{m+1} is contained in* int V, *then we can assume that it separates T_1 and $\partial N(K)$.*

PROOF OF CLAIM. If T_{m+1} separates T_1 and $\partial N(K)$, then we have nothing to do. Suppose that T_{m+1} is contained in int V and it does not separate T_1 and $\partial N(K)$ either. Then $T_2(=\partial V')$, $\cdots$, T_m, T_{m+1} are contained in int V and each of which does not separate T_1 and $\partial N(K)$. We remark that taking m as $m+1$, our situation can be reduced essentially to the former situation.

After the inductive procedure above, we have $T_2, \cdots, T_{m+s}=f(T_{m+s-1})$, which are contained in int V and each of which does not separate T_1 and $\partial N(K)$, and $T_{m+s+1}=f(T_{m+s})$ can not be deformed so that "T_{m+s+1} is contained in int V and it does not separate T_1 and $\partial N(K)$" by such an isotopy as above for some integer $s>0$. The existence of such an integer s can be shown as follows. If such an integer s does not exist, then we get a collection of disjoint nonparallel incompressible tori $\{T_1, \cdots, T_{m+s}\}$ in $S^3-\operatorname{int} N(K)$ for any $s>0$ (see Schematic 2). This contradicts Haken's finiteness theorem.



Schematic 2

Taking m as $m+s$, the required result follows. ■

It follows from this claim that if (1)-(c)-(1') occurs T_{m+1} is contained in $\operatorname{int} V$ and separates T_1 and $\partial N(K)$, or T_{m+1} is contained in S^3-V. Suppose that T_{m+1} and T_1 are parallel in S^3, then we may assume $T_{m+1}=T_1$ by an isotopy (which leaves $N(K), T_2, \cdots, T_m$ fixed). This is the just situation in the following lemma.

LEMMA 2.5. *Let V_i be a knotted solid torus such that $V_1 \cup V_2=S^3$ and $\partial V_1 \cap \partial V_2=\emptyset$. Let K be a knot in V_i with $w_{V_i}(K)\geqq 2$ and f an orientation preserving homeomorphism from V_1 to V_2 satisfying $f^m(\partial V_1)=\partial V_1$ for some integer $m>0$ and $f(K)=K$. Then f is faithful.*

PROOF. We induct on the closed Haken number of the exterior $V_1-\operatorname{int} N(K)$, $h(V_1-\operatorname{int} N(K))$. Let $T_1=\partial V_1$, $T_2=\partial V_2$, $T_i=f^{i-1}(T_1)$ and V_i the solid torus in S^3 bounded by T_i containing $N(K)$. For the meridian-logitude pair (m_i, l_i) of V_i $(i=1, \cdots, m)$, $f(m_1)=\varepsilon_1 m_2$, $f(m_2)=\varepsilon_2 m_3, \cdots, f(m_m)=\varepsilon_m m_1$ and $f(l_2)=\varepsilon_2 l_3, \cdots,$ $f(l_m)=\varepsilon_m l_1$ $(\varepsilon_i=\pm 1)$ holds. Then $f(l_1)$ is presented by $\varepsilon_1(l_2+\alpha m_2)$ for some integer α. It suffices to show that α equals zero. We remark that $f|_{V_1\cap\cdots\cap V_m}$ is a homeomorphism of $V_1\cap \cdots \cap V_m$ with $f(K)=K$ by the assumption of Lemma 2.5, and we may assume that $f(N(K))=N(K)$ for some tubular neighborhood

$N(K)(\subset V_1\cap\cdots\cap V_m)$. For $V_1\cap\cdots\cap V_m-\operatorname{int}N(K)$, we have the following sublemma which is essentially the same as Lemma 1.1 and the proof is omitted.

SUBLEMMA 2.6. *$V_1\cap\cdots\cap V_m-\operatorname{int}N(K)$ is a boundary irreducible Haken manifold.*

Now consider the torus decomposition of $V_1\cap\cdots\cap V_m-\operatorname{int}N(K)$ in the sense of Jaco-Shalen and Johannson. Let $\{J_0,\cdots,J_s\}$ be a family of tori which defines a torus decomposition with decomposing pieces $P_0,\cdots,P_u$. Here we assume P_0 contains ∂V_1. Each Seifert piece is either a torus knot space, a cable space or a composing space (see Lemma 1.2). For some (even) integer $N>0$, we can assume f^{mN} satisfies $f^{mN}(J_0)=J_0,\cdots,f^{mN}(J_s)=J_s$, $f^{mN}(T_i)=T_i$, $f^{mN}(m_i)=m_i$, $f^{mN}(l_i)=l_i+N\alpha m_i(i=1,\cdots,m)$ if necessary.

In our situation $h(V_1-\operatorname{int}N(K))\geqq 4$.

First step. $h(V_1-\operatorname{int}N(K))=4$.

In this case $V_1\cap\cdots\cap V_m-\operatorname{int}N(K)$ consists of only one piece P_0, which is a composing space or hyperbolic, because the number of components of ∂P_0 is greater than or equal three. When P_0 is a composing space then we have $w_V(K)=1$, and this is a contradiction. If P_0 is hyperbolic, then by Mostow's rigidity theorem, the argument in Theorem 1.5 implies that $f^{mpN}|_{T_1}$ is isotopic to the identity for some integer $p>0$. This implies $\alpha=0$ and $f(l_1)=\varepsilon_1 l_2$, hence f is faithful.

Second step.

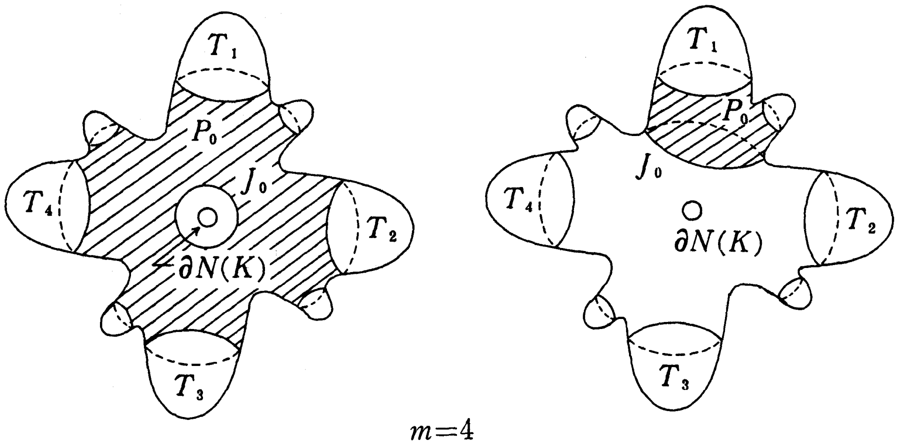
When P_0 is hyperbolic, the required result follows from the argument above. Suppose that P_0 is a cable space and $\partial P_0=T_1\cup J_0$. First we note that a meridian m_1 is not a regular fibre t of P_0. Since a Seifert fibration of P_0 is unique, $f^{mN}(t)=t$. Also we have $f^{mN}(m_1)=m_1$, thus by Lemma 1.3, f^{mN} is the identity of $H_1(T_1)$. Hence we get $\alpha=0$ and f is faithful.

Now we suppose that P_0 is a composing space. In this case a meridian m_1 of V_1 coincides with a regular fibre t of P_0. If ∂P_0 contains $\partial N(K)$, then the union of a meridian disk of $N(K)$ (whose boundary coincides with a regular fibre t) and a saturated annulus A in P_0 such that $\partial A\subset T_1\cup\partial N(K)$ and $\partial A\supset t$ becomes a meridian disk of V_1. This implies $w_{V_1}(K)=1$ and we have a contradiction. Hence ∂P_0 does not contain $\partial N(K)$. Let J_0 denote the component of ∂P_0 separating P_0 and $\partial N(K)$. We divide into two cases (see Schematic 3).

(I) J_0 separates T_i $(i=1,\cdots,m)$ and $\partial N(K)$, or

(II) otherwise.

In case (I), it turns out that ∂P_0 contains $T_1,\cdots,T_m$. Hence $f(P_0)=P_0$ and $f(J_0)=J_0$ because J_0 is a unique component of ∂P_0 separating P_0 and $\partial N(K)$. Let W be the solid torus in V_1 bounded by J_0. Then $f(W)=W$ and we have the following.



$m=4$

Schematic 3

SUBLEMMA 2.7. *Let C_W be a core of W, then we have $w_{V_1}(C_W)=1$ (see Figure 4). In addition $w_W(K)\geqq 2$ holds.*

PROOF OF SUBLEMMA 2.7. A meridian of W coincides with a regular fibre of P_0. Take a saturated annulus A in P_0 with $\partial A\subset T_1\cup J_0$. The union of a meridian disk in W and the annulus A becomes a meridian disk of V_1. Thus we get $w_{V_1}(C_W)=1$. If $w_W(K)\leqq 1$, then we have $w_{V_1}(K)\leqq 1$, and this is a contradiction. Hence $w_W(K)\geqq 2$. ■

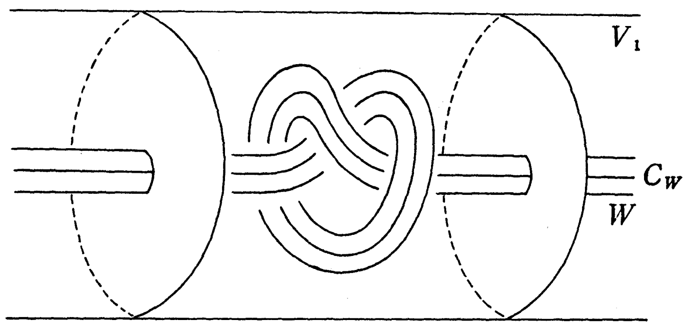


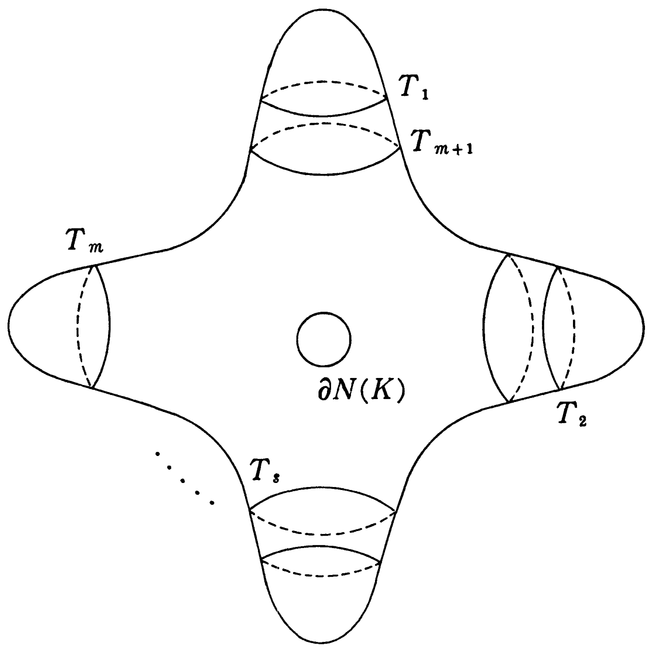
Figure 4

Since f is an orientation preserving homeomorphism of W such that $f(K)=K$ and $w_W(K)\geqq 2$, $f|_W$ is faithful by Theorem 1.5. Namely for a longitude l_W of W, $f(l_W)=\varepsilon l_W$ ($\varepsilon=\pm 1$) holds. In $V_1-\operatorname{int} W$, we have $l_1=\varepsilon' l_W$ ($\varepsilon'=\pm 1$) by Sublemma 2.7, also we have $l_2=\varepsilon'' l_W$ ($\varepsilon''=\pm 1$) in $V_2-\operatorname{int} W=f(V_1-\operatorname{int} W)$. It follows that in $V_2-\operatorname{int} W=f(V_1)-\operatorname{int} f(W)$, $f(l_1)=\varepsilon' f(l_W)=\varepsilon\varepsilon' l_W=\varepsilon\varepsilon'\varepsilon'' l_2$. Hence f is faithful.

In case (II), let W be a solid torus in V_1 bounded by J_0. Then as in Sublemma 2.7, C_W (a core of W) satisfies $w_{V_1}(C_W)=1$ and $w_W(K)\geqq 2$. Also we can check that W and $f(W)$ satisfy the condition of V_1 and V_2 in Lemma 2.5. In addition $h(W-\text{int}\,N(K))<h(V_1-\text{int}\,N(K))$ holds. Thus by the induction hypothesis, $f|_W$ is faithful. Applying the same argument in case (I), we conclude that f is faithful and the proof of Lemma 2.5 is completed. ■

Regarding $V=V_1$ and $V'=V_2$, we can conclude that f is faithful.

Assume now T_{m+1} separates T_1 and $\partial N(K)$ and is not parallel to T_1, then for any integer $s>0$, $\{T_1, \cdots, T_s=f^{s-1}(T_1)\}$ is a collection of disjoint nonparallel incompressible tori in $S^3-\text{int}\,N(K)$ (see Schematic 4). This contradicts Haken's finiteness theorem.



Schematic 4

When T^{m+1} is contained in S^3-V and not parallel to T_1, then considering preimages $f^{-j}(T_1)$ of T_1, we get arbitrarily many disjoint nonparallel incompressible tori in $S^3-\text{int}\,N(K)$. Thus we have again a contradiction by Haken's finiteness theorem.

Now consider the case when (1)-(c)-(2′) holds. Let $Y=V-\text{int}\,X$, $W'=Y\cup X'$ and $W=W'-\text{int}\,N(\partial W')$. Then we have $w_W(K)\geqq 2$ and $h(W-\text{int}\,N(K))<h(V-\text{int}\,N(K))$. Applying the inductive argument, we can conclude that $f|_W: W\to f(W)$ is faithful. Hence $f(l_W)=\varepsilon l_{f(W)}$ ($\varepsilon=\pm 1$) holds, where l_W and $l_{f(W)}$ are longitudes of W and $f(W)$ respectively. In $V-\text{int}\,W$, we have $l=\varepsilon' l_W$ ($\varepsilon'=\pm 1$) for a longitude l of V. Also we have $l'=\varepsilon'' l_{f(W)}$ ($\varepsilon''=\pm 1$) in $V'-\text{int}\,f(W)=f(V-\text{int}\,W)$ for a longitude l' of V'. It follows that in $f(V_1)-\text{int} f(W)$, $f(l)=\varepsilon' f(l_W)=\varepsilon\varepsilon' l_{f(W)}=\varepsilon\varepsilon'\varepsilon'' l'$. Hence f is faithful.

In case (2), by the argument in (1)-(c)-(2′), we can conclude that f is faithful.

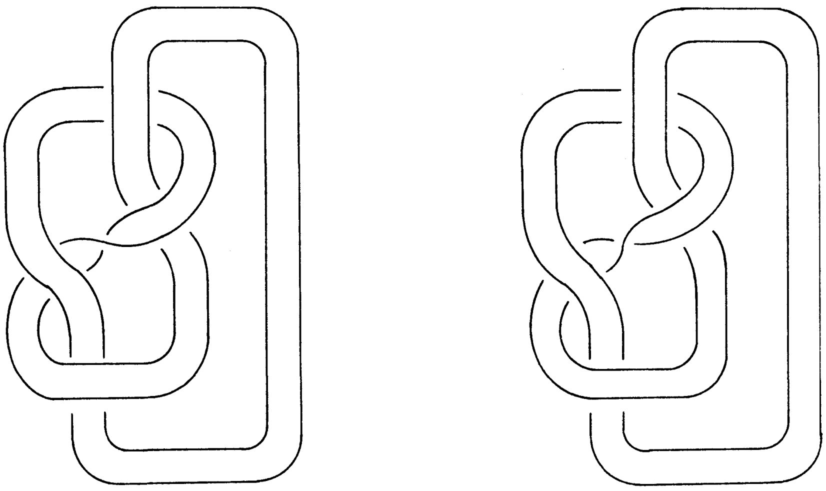
In this way we see that $f=g|_V \circ f_n$ is faithful. From this we can conclude that f_n is also faithful and $n=0$. This completes the proof of Theorem 2.1. ■

As for weak equivalence $\sim$, we have:

THEOREM 2.8. *Let K be any knot in S^3 and V a knotted solid torus with $w_V(K)\geqq 2$. Then there is at most one nonzero integer n such that $K_{V,n}\sim K$.*

PROOF. Suppose that we have $K_{V,m}\sim K$ and $K_{V,n}\sim K$ $(m, n\neq 0)$. Since m, $n\neq 0$, Theorem 2.1 implies $K_{V,m}\not\cong K$ and $K_{V,n}\not\cong K$. From this, we get $K_{V,m}\cong K_{V,n}$, then by Remark 2.2 $m=n$ holds. ■

EXAMPLE 2.9. Let K be a (2, 1)-cable of the figure eight knot k, and V a knotted solid torus which is a tubular neighborhood of k containing K. Then by Theorem 2.1, $K_{V,-1}\not\cong K$, but we have $K_{V,-1}\sim K$.



K $\qquad$ $K_{V,-1}$

Figure 5

3. Twisting along unknotted solid tori.

When a solid torus V is unknotted, Theorem 2.1 is not true. To start with, we give the following example.

EXAMPLE 3.1. In Figure 6, $K_{V,1}\cong K$ does hold.

This example also demonstrates that for any knot K there exists an unknotted solid torus V with $w_V(K)=2$ such that $K_{V,1}\cong K$.

In this section first we prove:

THEOREM 3.2. *Let K be a knot in S^3 and V an unknotted solid torus con-*

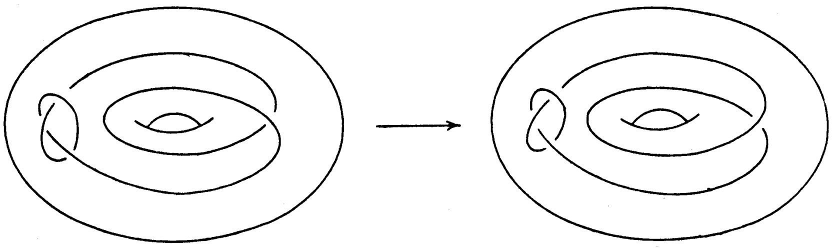


Figure 6

taining K with $w_V(K)\geqq 2$. Then there are at most finitely many integers n_i such that $K_{V,n_i}\sim K$.

This theorem is follows from Propositions 3.4, 3.10 and 3.13. In our setting, $S^3-\operatorname{int} V$ is also an unknotted solid torus V^*. We denote the core of V^* by J. When we perform $(-1/n)$-Dehn surgery on J, the result is again S^3 because J is trivial and K can be considered as a knot in this resultant S^3. We denote this knot by K_n^*. The next lemma is an interpretation of the twisting and we omit the proof.

LEMMA 3.3. *For two knots $K_{V,n}$ and K_n^*, we have $K_{V,n}\cong K_n^*$.*

From this, the exterior of $K_{V,n}$ and $V_J\cup_{m_J=l m^{-n}}(V-\operatorname{int} N(K))$ are homeomorphic, where m_J is a meridian of V_J. Recall that $V-\operatorname{int} N(K)$ is a boundary irreducible Haken manifold by Lemma 1.1. We consider the torus decomposition of $V-\operatorname{int} N(K)$ in the sense of Jaco-Shalen, Johannson. Then by Lemma 1.2, each Seifert piece is either a torus knot space, a cable space or a composing space. As in the previous section, P_0 denotes the piece in $V-\operatorname{int} N(K)$ containing ∂V. First consider the case when P_0 is a cable space. In this case the following holds.

PROPOSITION 3.4. *Suppose that P_0 is a cable space. Then there is at most one nonzero integer n such that $K_{V,n}\sim K$.*

PROOF. The Seifert fibration of P_0 is unique and we assume a regular fibre is presented by $l^p m^q$ ($p\geqq 2$). Then J is a fibre of index $|q|$ in $E(K)=S^3-\operatorname{int} N(K)$ and J_n^* the dual of J (*i.e.* J in $E(K_{V,n})$) is a fibre of index $|pn+q|$ in $E(K_{V,n})$. Now assume $K_{V,n}\sim K$, then $E(K_{V,n})$ is homeomorphic to $E(K)$. We divide into two cases whether $|q|$ equals 1 or not.

Case (1) $|q|\neq 1$.

First supposing $|pn+q|\neq 1$, then $V_J\cup_{m_J=l m^{-n}}P_0$ and $V_J\cup_{m_J=l}P_0$ are boundary irreducible, irreducible Seifert fibred manifolds. In addition their Seifert fibration is unique. Thus we get $|pn+q|=|q|$. This implies $p=2$, $n=-q$.

Next we suppose $|pn+q|=1$. Set $T_1=\partial P_0-\partial V$, V_1 a solid torus in V bounded by T_1, which contains K. Then V_1 is knotted. On the other hand $f_n(V_1)$ is unknotted. Thus $K_{V,n}$ is a preimage knot of K, so $K_{V,n}\not\sim K$ by Theorem 1 in [**17**].

Case (2) $|q|=1$.

If $|pn+q|=1$, then we get $p=2$, $q=\varepsilon$ and $n=-\varepsilon$ ($\varepsilon=\pm 1$). Now assume $|pn+q|\neq 1$. Then we see that K is a preimage knot of $K_{V,n}$, and again Theorem 1 in [**17**] implies that $K_{V,n}\not\sim K$. This completes the proof. ■

REMARK 3.5. Except for the case when a regular fibre is presented by l^2m^q, $K_{V,n}\sim K$ implies $n=0$. In the exceptional case there is an example as in Figure 6. In general, l^2m^q presents a $(2, q)$-torus knot $T_{2,q}$, and $(-q)$-twist of $T_{2,q}$ is $T_{2,-q}$ and $T_{2,q}\sim T_{2,-q}$ holds.

When P_0 is a composing space, we have:

PROPOSITION 3.6. *Suppose that P_0 is a composing space. Then there is at most one nonzero integer n such that $K_{V,n}\sim K$. In particular $K_{V,n}\cong K$ implies $n=0$.*

PROOF. Let T be a component of ∂P_0 separating ∂V and $\partial N(K)$, and V_T a solid torus bounded by T.

SUBLEMMA 3.7. *A twisting along V can be reduced to that along V_T.*

PROOF OF SUBLEMMA. A meridian of V_T coincides with a regular fibre of P_0. Then a union of a meridian disk of V_T and a saturated annulus joining T and ∂V is a meridian disk of V. Thus the required result holds. ■

In addition it is easy to see that V_T is knotted and $w_{V_T}(K)\geqq 2$. Then we can apply Theorems 2.1 and 2.8 and the result follows. ■

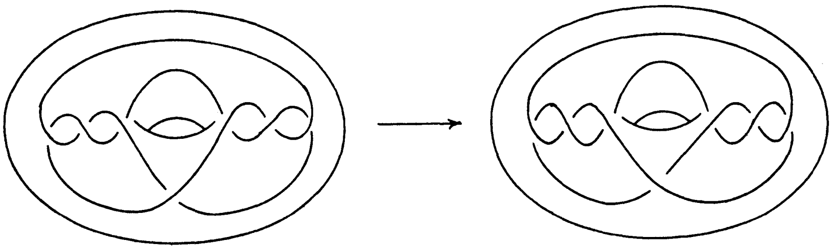
Let assume that P_0 admits a complete hyperbolic structure of finite volume in its interior. In this case we shall use Thurston's hyperbolic Dehn surgery [**18**].

LEMMA 3.8 (Thurston [**18**]). *Let P_0 be hyperbolic. There exists $N_{V,K}$ such that $V_J\cup_{m_J=lm^{-n}}P_0$ is also hyperbolic for $|n|\geqq N_{V,K}$. Moreover for any $\varepsilon>0$, there exists $N_{V,K}(\varepsilon)$ such that J_n^* is a closed geodesic of length $<\varepsilon$ in $V_J\cup_{m_J=lm^{-n}}P_0$ for $|n|\geqq N_{V,K}(\varepsilon)$.*

PROPOSITION 3.9. *Suppose that P_0 is hyperbolic. Then there are at most finitely many integers n_i such that $K_{V,n_i}\sim K$.*

PROOF. Assume that for some $N_{V,K}(\varepsilon)$, J_n^* is the unique shortest closed geodesic in $V_J\cup_{m_J=lm^{-n}}P_0$ provided that $|n|\geqq N_{V,K}(\varepsilon)$. Also we may assume that for some n_0 with $|n_0| \geqq N_{V,K}(\varepsilon)$, $K_{V,n_0} \sim K$. Otherwise, $K_{V,n} \sim K$ implies $|n| < N_{V,K}(\varepsilon)$ and the required result holds. Let ε_0 be the length of $J^*_{n_0}$ in $V_J \cup_{m_J = lm - n_0} P_0$. If $K_{V,n} \sim K$ $(\sim K_{V,n_0})$ for some n with $|n| \geqq N_{V,K}(\varepsilon_0/2)$, then $E(K_{V,n})$, $E(K)$ and $E(K_{V,n_0})$ are homeomorphic. By the uniqueness of the torus decomposition and Mostow's rigidity theorem, the core of V_J in $V_J \cup_{m_J = lm - n_0} P_0$ and that of V_J in $V_J \cup_{m_J = lm - n} P_0$ have the same length. But this is a contradiction. Hence $|n| < N_{V,K}(\varepsilon_0/2)$, and the proof is completed. ∎

Figure 7 ([**6**]) shows an example in this case.



K $\qquad$ $K_{V,1}$

Figure 7

Now we consider the case when the wrapping number of K in V coincides with the winding number of K in V. In this case the signature of knots comes in useful. In [**16**] Shibuya has shown the following theorem.

THEOREM 3.10 ([**16**]). *Let K be a knot in S^3 and V a solid torus containing K such that $w_V(K)$ equals the winding number of K in V. Then $K_{V,m} \cong K_{V,n}$ implies $m=n$ if $w_V(K) \geqq 3$, and $|m-n| \leqq 1$ if $w_V(K)=2$.*

In the case when $w_V(K)=2$, more precisely we get the following.

THEOREM 3.11. *Let K be a knot in S^3 and V a solid torus containing K such that $w_V(K)$ equals the winding number of K in V. Then there is at most one integer n such that $K_{V,n} \cong K_{V,n+1}$ (see Example 3.1).*

PROOF. Assume that there are integers m, n $(m>n)$ such that $K_{V,m} \cong K_{V,m+1}$ and $K_{V,n} \cong K_{V,n+1}$. Note that $K_{V,m}$ can be obtained by 2-times fusion of $K_{V,n}$ and a torus link L of type $(2, 2(m-n))$, see Fig. 2 in [**16**]. Let $\sigma(k)$ denote the signature of k. As $\sigma(L)=2(m-n)-1$ by Lemma in [**16**], we obtain that $\sigma(K_{V,m})-\sigma(K_{V,n})=2(m-n)-2$ or $2(m-n)$ by Lemma 7.1 in [**12**]. By the same way as above, $\sigma(K_{V,m+1})-\sigma(K_{V,n})=2(m-n)$ or $2(m-n)+2$. Since $K_{V,m} \cong K_{V,m+1}$, $\sigma(K_{V,m})=\sigma(K_{V,m+1})$. Hence we obtain that $\sigma(K_{V,m})-\sigma(K_{V,n})=2(m-n)$. By the same reason as above, we obtain that $\sigma(K_{V,m})-\sigma(K_{V,n+1})=2(m-n)-2$ or $2(m-n-2)$. As $K_{V,n} \cong K_{V,n+1}$, $\sigma(K_{V,n})=\sigma(K_{V,n+1})$. It follows that $\sigma(K_{V,m})-$

$\sigma(K_{V,n})=2(m-n)-2$ or $2(m-n-2)$, which is a contradiction. Thus we get $m=n$. ∎

REMARK 3.12. In theorems 3.10 and 3.11, the condition $K_{V,m}\cong K_{V,n}$ can be replaced by the condition that $K_{V,m}$ and $K_{V,n}$ are cobordant, and these answers the conjecture proposed by Nakanishi.

4. Twisting of unknots.

In this section we shall concern ourselves with the case when an original knot is trivial as in Whitehead's example. To begin with, we remark the following basic fact (see [**14**]).

PROPOSITION 4.1. *Let V be a solid torus in S^3 containing the unknot O with $w_V(O)\geqq 1$. Then V is unknotted.*

Hence, if consider twisting of unknots, the solid torus which defines twisting is necessarily unknotted. Our purpose is to prove the following theorem.

THEOREM 4.2. *Let V be an unknotted solid torus containing the unknot O with $w_V(O)\geqq 2$. Except for the cases as in Figure 8, $O_{V,n}$ is nontrivial for any nonzero integer n. Moreover in the exceptional case, $O_{V_1,n}$ (resp. $O_{V_2,n}$) is trivial only when $n=0$, 1 (resp. $n=0$, -1).*

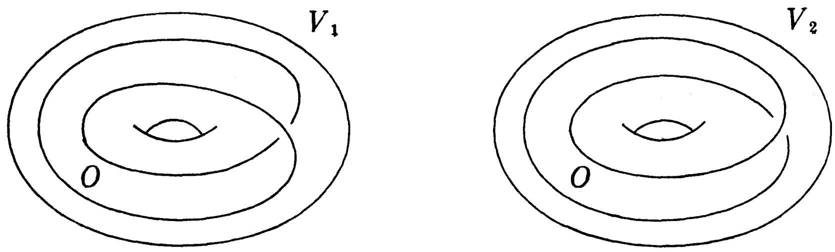


Figure 8

PROOF. Let J be the core of the complementary solid torus (*i.e.* $S^3-\operatorname{int} V$), and O_n^* the knot which is the image of O after performing $(-1/n)$-Dehn surgery on J. Then by Lemma 3.3 $O_{V,n}\cong O_n^*$. Suppose that the winding number of O in V is greater than two. Then Litherland's result [**9**] asserts $O_{V,n}\cong O$ implies $n=0$. Assume the winding number of O in V is not greater than two. Since O is trivial, $V'=S^3-\operatorname{int} N(O)$ is also a solid torus, and J is embedded in V'. Then $w_{V'}(J)\neq 0$ because $w_V(O)\neq 0$. Suppose that $O_{V,n}$ becomes a trivial knot for some integer $n\neq 0$. This implies that the result of $(-1/n)$-Dehn surgery of V' on J is also a solid torus for n. Due to Gabai's theorem [**2**], J is either a 0 or 1-bridge braid in V'. Then the wrapping number of J in V' coincides with the winding number of J in V', which equals that of O in V because these are equal to the absolute value of the linking number of O and J. Hence $w_{V'}(J) \leqq 2$, and using the fact that J is a 0 or 1-bridge braid, we see that J and V' are situated as in Figure 9.

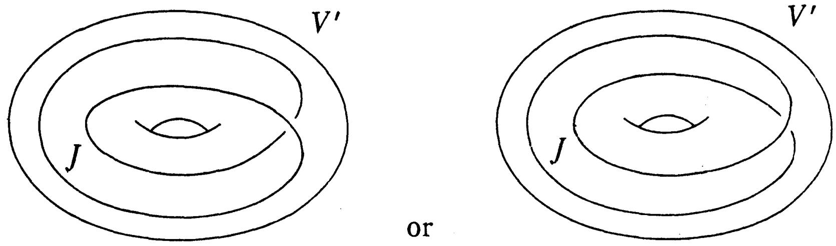


Figure 9

From this we see that O and V as in Figure 8, and in this exceptional case $O_{V_1,n}$ (*resp.* $O_{V_2,n}$) is a torus knot of type $(2, -1+2n)$ (*resp.* $(2, 1+2n)$). Thus we can conclude the desired result. ■

REMARK 4.3. In Whitehead's example, Alexander polynomials are complete invariants for $\{T_n\}$. But in general there is an example (Figure 10) such that Alexander polynomials are not changed under twisting [**13**]. As for Jones polynomials, recently Yokota has studied the behavior of Jones polynomials under twisting [**21**].

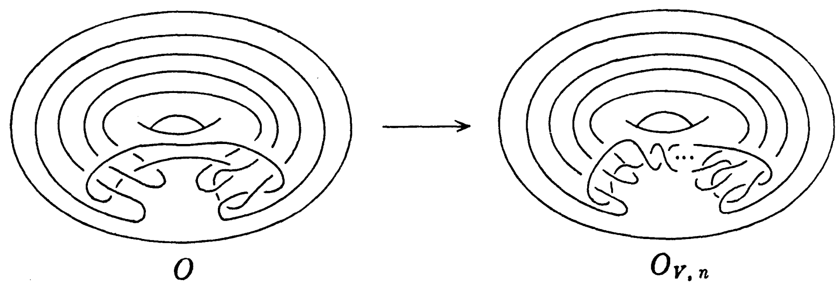


Figure 10

ACKNOWLEDGEMENT. The second named auther wishes to thank Professor M. Sakuma for informing him of Litherland's paper [**9**] and helpful comments. He is also grateful to Professor M. Kato for his warm encouragement.

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