J. Math. Soc. Japan
Vol. 44, No. 1, 1992

Motion of hypersurfaces and geometric equations

Dedicated to Professor Noboru Tanaka on his 60th birthday

By Yoshikazu GIGA and Shun'ichi GOTO

(Received Feb. 5, 1990)
(Revised Dec. 28, 1990)

1. Introduction.

We are concerned with the motion of a hypersurface whose speed locally depends on the normal vector field and its derivatives. To be specific let Γ_t denote the hypersurface expressed as the boundary of a bounded open set D_t in $\boldsymbol{R}^n$ $(n \geqq 2)$ at time t. Let $\boldsymbol{n}$ denote the unit exterior normal vector field to $\Gamma_t = \partial D_t$. It is convenient to extend $\boldsymbol{n}$ to a vector field (still denoted by $\boldsymbol{n}$) on a tubular neighborhood of Γ_t such that $\boldsymbol{n}$ is constant in the normal direction of Γ_t. Let $V = V(t, x)$ denote the speed of Γ_t at $x \in \Gamma_t$ in the exterior normal direction. The equation for Γ_t we consider here is of form

$$V = f(t, x, \boldsymbol{n}(x), \nabla \boldsymbol{n}(x)) \quad \text{on } \Gamma_t, \tag{1.1}$$

where f is a given function and ∇ stands for spatial derivatives. Material science provides a lot of examples of (1.1) where Γ_t is an interface bounding two phases of materials (see [**2**, **11**, **12**] and references therein). For example if

$$V = -\operatorname{div} \boldsymbol{n}, \tag{1.2}$$

the hypersurface Γ_t moves by its $(n-1$ times) mean curvature and (1.2) is known as the mean curvature flow equation. We note that this equation arises as a singular limit of some reaction-diffusion equations [**3**, **17**]. It is also important to consider anisotropic properties of materials. A typical model (cf. [**11**, **12**]) is

$$\beta(\boldsymbol{n})V = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(\frac{\partial H}{\partial p_i}(\boldsymbol{n})\Big) + c, \tag{1.3}$$

where H is convex on $\boldsymbol{R}^n$ and positively homogeneous of degree one, β is a positive function on a unit sphere S^{n-1} in $\boldsymbol{R}^n$ and c is a constant. The equation (1.3) includes (1.2) as a particular example with $H(p) = |p|$, $\beta = 1$ and $c = 0$. We remark that in general the right hand side of (1.3) is *not* expressed as a function of principal curvatures $\kappa_1, \cdots, \kappa_{n-1}$ of Γ_t and $\boldsymbol{n}$. In other words

$$V = g(\kappa_1, \cdots, \kappa_{n-1}, \boldsymbol{n}) \tag{1.4}$$

excludes (1.3), although (1.4) itself is interesting.

A fundamental analytic question to (1.1) is to construct a global-in-time unique solution $\{\Gamma_t\}_{t\geq 0}$ for a given initial data Γ_0 (allowing that Γ_t becomes empty in a finite time). There are a couple of approaches depending on description of (hyper) surfaces. A classical approach appeals to a parametrization of Γ_t. For the mean curvature flow equation (1.2) Huisken [**13**] constructed a unique smooth solution Γ_t which shrinks to a point in a finite time provided that Γ_0 is uniformly convex and C^2 and that $n\geqq 3$. A similar result is proved by Gage and Hamilton [8] when $n=2$. Moreover, Grayson [**10**] proved that any embedded curve moved by (1.2) never becomes singular unless it shrinks to a point. However, for $n\geqq 3$ even embedded surface may develop singularities before it shrinks to a point. Even when $n=2$ such singularities may develop if we consider

$$V = -\operatorname{div} \boldsymbol{n}+c \tag{1.5}$$

with some constant instead of (1.2). Angenent [**1**] constructed a weak solution across singularities for a class of parabolic equation (1.1) provided that $n=2$ (see also [**2**]); however, f is assumed 'symmetric' so that the orientation of the curve does not affect its evolution. In particular (1.5) is excluded (unless $c=0$). The uniqueness of his weak solution is not claimed in [**1**]. It also seems difficult to track the evolution of Γ_t across singularities by a parametrization of Γ_t when $n\geqq 3$.

To overcome this difficulty one way would be to describe surfaces in a weak sense such as varifolds in geometric measure theory. For (1.2) Brakke [**4**] constructed a global varifold solution for arbitrary initial data. Unfortunately, the uniqueness of such a solution is not known. Another way is to describe a surface Γ_t as a level set of a function u satisfying a second order evolution equation in $\boldsymbol{R}^n$:

$$\partial_t u+F(t, x, \nabla u, \nabla^2 u) = 0\,, \tag{1.6}$$

where $\partial_t=\partial/\partial t$ and $\nabla^2 u$ denotes the Hessian matrix of u in space variables. This idea is introduced by Osher and Sethian [**18**] for a numerical calculation of (1.5) and independently by Chen and the authors [**5**]. In [**5**] we introduced a weak notion for solution Γ_t of (1.1) through viscosity solutions of (1.6). We constructed a *unique* global weak solution $\{\Gamma_t\}_{t\geq 0}$ with arbitrary initial data for a certain class of (1.1) including (1.2), (1.3) and (1.5) (where H is C^2 outside the origin and β is continuous). Almost at the same time Evans and Spruck [7] constructed the same solution but only for (1.2). We note that Tso [19] applies a variant of a level surface approach to (1.4) when $-g$ is the Gauss-Kronecker curvature. He constructed smooth Γ_t shrinking to a point in a finite time provided that Γ_0 is uniformly convex and C^2. The corresponding result to (1.2) is proved by Huisken [13] as is explained in the second paragraph.

Our main goal is to clarify the class of equations of form (1.1) to which the level surface approach in [5] yields a unique global weak solution $\{\Gamma_t\}_{t\geq 0}$ with a given initial data. We first derive (1.6) from (1.1). Suppose that $u>0$ in D_t and $u=0$ on Γ_t. If u is C^2 and $\nabla u\neq 0$ near Γ_t, we see

$$\boldsymbol{n} = -\frac{\nabla u}{|\nabla u|} \quad \text{on } \Gamma_t . \tag{1.7a}$$

Unfortunately, the vector field $\boldsymbol{m}=-\nabla u/|\nabla u|$ near Γ_t is not constant in the normal direction of Γ_t, so our $\nabla \boldsymbol{n}$ in (1.1) may not agree with $\nabla \boldsymbol{m}$ on Γ_t. It turns out that

$$\nabla \boldsymbol{n} = \nabla \boldsymbol{m} - \boldsymbol{n}\otimes(\boldsymbol{n}\cdot\nabla)\boldsymbol{m} \qquad \text{on } \Gamma_t ,$$

where $\boldsymbol{m}$ and $\boldsymbol{n}$ are regarded as row vectors and $\otimes$ denotes a tensor product of vectors in $\boldsymbol{R}^n$. A direct calculation yields

$$\nabla \boldsymbol{n} = -\frac{1}{|\nabla u|}Q_{\bar{p}}(\nabla^2 u), \qquad \bar{p} = \frac{\nabla u}{|\nabla u|} \quad \text{and} \tag{1.7b}$$

$$Q_{\bar{p}}(X) = R_{\bar{p}} X R_{\bar{p}} \qquad \text{with } R_{\bar{p}} = I - \bar{p}\otimes\bar{p} ,$$

where X is an $n\times n$ real symmetric matrix and I denotes the identity matrix. It follows from (1.7a, b) and $V=\partial_t u/|\nabla u|$ that (1.1) is formally equivalent to (1.6) on Γ_t with

$$F(t, x, p, X) = -|p| f\Big(t, x, -\bar{p}, -\frac{1}{|p|}Q_{\bar{p}}(X)\Big), \qquad \bar{p} = \frac{p}{|p|}, \tag{1.8}$$

where p is a nonzero vector in $\boldsymbol{R}^n$. A direct calculation shows that F in (1.8) has the scaling invariance

$$F(t, x, \lambda p, \lambda X + p\otimes y + y\otimes p) = \lambda F(t, x, p, X) \tag{1.9}$$

$$\text{for all } \lambda>0, \quad p\in \boldsymbol{R}^n\setminus\{0\}, \quad y\in\boldsymbol{R}^n, \ X\in \boldsymbol{S}_n ,$$

where $\boldsymbol{S}_n$ denotes the space of all $n\times n$ real symmetric matrices. We say F is *strongly geometric* if F satisfies (1.9). In this paper we shall show that every strongly geometric F is of the form (1.8) with some f and f is (essentially) uniquely determined by F. This shows that the concept "strongly geometric" is very natural to study the equation (1.1) by our level surface approach. Clearly,

F is *geometric* in the sense of [5], i.e.,

$$F(t,\ x,\ \lambda p,\ \lambda X+\sigma p\otimes p)=\lambda F(t,\ x,\ p,\ X) \tag{1.9'}$$

$$\text{for all } \lambda>0,\ \sigma\in\boldsymbol{R},\ p\in\boldsymbol{R}^{n}\setminus\{0\},\ X\in\boldsymbol{S}_{n}$$

if F is strongly geometric. The converse is true provided that F is degenerate elliptic and continuous in X for $p\neq 0$. It will turn out that the results in [5] yield a unique global weak solution $\{\Gamma_t\}_{t\geq 0}$ of (1.1) with an arbitrary initial data Γ_0 provided that f is degenerate elliptic, continuous and grows linearly in $\nabla\boldsymbol{n}$, where f is assumed to be independent of x. We present our theory in [5] for geometric parabolic equations under simpler assumptions of F but slightly stronger then those of [5]. When F is independent of t and x, both assumptions are equivalent. We thank to the referee for valuable comments especially on the form (1.7b) and relation between (1.9) and (1.9').

This work was done while the second author was a graduate student of the Department of Mathematics of Hokkaido University. This work is partly supported by the Japan Ministry of Education, Science and Culture through grant no. 01740076 and 01540092.

2. Geometric equations.

The equation (1.6) is called *(strongly) geometric* if F is (strongly) geometric. We observe in this section that there is roughly a one-to-one correspondence from a strongly geometric equation to (1.1). Indeed we shall show at least formally that every level surface of a function u moves by (1.1) for some f if and only if (1.6) is strongly geometric. Moreover, f is uniquely determined by F.

For $\bar{p}\in S^{n-1}$ we introduce a linear operator $Q_{\bar{p}}$ from $\boldsymbol{M}_n$ into itself defined by

$$Q_{\bar{p}}(X)=R_{\bar{p}}XR_{\bar{p}},\quad R_{\bar{p}}=I-\bar{p}\otimes\bar{p},\qquad \text{for } X\in\boldsymbol{M}_n, \tag{2.1}$$

where $\boldsymbol{M}_n$ denotes the space of all $n\times n$ real matrices. We note that the right hand side of (2.1) appears in (1.8).

LEMMA 2.1. (i) *The operator $Q_{\bar{p}}$ is a projection, i.e., $Q_{\bar{p}}^2=Q_{\bar{p}}$.*

(ii) *Let $L_{\bar{p}}$ denote a vector subspace of $\boldsymbol{S}_n$ defined by*

$$L_{\bar{p}}=\{\bar{p}\otimes y+y\otimes\bar{p}\,;\ y\in\boldsymbol{R}^n\}.$$

It holds

$$\boldsymbol{S}_n\cap\ker Q_{\bar{p}}=L_{\bar{p}}\,. \tag{2.2}$$

PROOF. (i) This follows directly from (2.1) if we observe

$$(x\otimes\bar{p})(\bar{p}\otimes y)=x\otimes y \tag{2.3}$$

with $x=y=\bar{p}$.

(ii) By (2.3) it is clear that $L_{\bar{p}}$ is contained in the kernel of $Q_{\bar{p}}$. It remains to prove that $Q_{\bar{p}}(X)=O$ for $X\in \boldsymbol{S}_n$ implies $X\in L_{\bar{p}}$. For an orthogonal matrix U it follows from the definition (2.1) that

$$U^{-1}Q_{\bar{p}}(X)U=Q_{\bar{q}}(Y),\quad \bar{q}=\bar{p}U,\quad Y=U^{-1}XU,\qquad X\in \boldsymbol{M}_n.$$

We take U so that $\bar{q}=(1, 0, \cdots, 0)$ and observe that $Q_{\bar{q}}(Y)=O$ implies

$$Y=\begin{pmatrix} y_1 & y_2'\cdots y_n' \\ \vdots & \\ y_n & O \end{pmatrix}\qquad \text{with } y_j,\ y_j'\in \boldsymbol{R}.$$

If Y is symmetric, we see $y_j=y_j'$ for $j\geqq 2$. Since $X\in \boldsymbol{S}_n$ implies $Y\in \boldsymbol{S}_n$, we now conclude that for $X\in \boldsymbol{S}_n$ the condition $Q_{\bar{p}}(X)=O$ implies $Y=\bar{q}\otimes y+y\otimes\bar{q}$ which is the same as $X\in L_{\bar{p}}$. □

We next introduce a (smooth) vector bundle E over S^{n-1} of the form

$$E=\{(\bar{p},\ Q_{\bar{p}}(X));\ \bar{p}\in S^{n-1},\ X\in \boldsymbol{S}_n\}. \tag{2.4}$$

The bundle E is a subbundle of a trivial bundle $S^{n-1}\times \boldsymbol{S}_n$ and its fibre dimension equals $n(n-1)/2$. Let Q be a bundle map

$$Q: S^{n-1}\times \boldsymbol{S}_n \longrightarrow E$$

defined by

$$Q(\bar{p},\ X)=(\bar{p},\ Q_{\bar{p}}(X)).$$

Let L be a bundle over S^{n-1} of form

$$L=\{(\bar{p},\ X);\ \bar{p}\in S^{n-1},\ X\in L_{\bar{p}}\}. \tag{2.5}$$

The bundle L is a subbundle of $S^{n-1}\times \boldsymbol{S}_n$. Since Q is surjective, Lemma 2.1 provides a characterization of E as a quotient bundle.

LEMMA 2.2. *The vector bundle E is isomorphic to the quotient bundle*

$$S^{n-1}\times \boldsymbol{S}_n/L=\{(\bar{p},\ [X]);\ \bar{p}\in S^{n-1},\ [X]\in \boldsymbol{S}_n/L_{\bar{p}}\}.$$

We now turn to study relation (1.8) of f and F. Since our argument is pointwise in t and x we suppress t, x-dependence of f and F in this section. The expression (1.7b) of $\nabla \boldsymbol{n}$ shows that our f in (1.1) needs to be defined only on E not whole $S^{n-1}\times \boldsymbol{M}_n$. We thus consider the space $\mathcal{F}$ of all real valued functions f defined on E. To each f we correspond a function F on $(\boldsymbol{R}^n\setminus\{0\})\times\boldsymbol{S}_n$ by (1.8), i.e.,

$$F(p, X)=-|p|f\Big(-\bar{p}, -\frac{1}{|p|}Q_{\bar{p}}(X)\Big), \qquad \bar{p}=\frac{p}{|p|}.$$

Let $\mathcal{G}$ denote the set of all strongly geometric real valued function F defined on $(\boldsymbol{R}^n\setminus\{0\})\times\boldsymbol{S}_n$. Lemma 2.2 now shows that the concept "strongly geometric" is very natural.

THEOREM 2.3. *The mapping $f\mapsto F$ is a bijection from $\mathcal{F}$ to $\mathcal{G}$.*

PROOF. Let $\mathcal{G}'$ be the set of all real valued functions F' on $S^{n-1}\times\boldsymbol{S}_n$ satisfying

$$F'(\bar{p}, X+\bar{p}\otimes y+y\otimes\bar{p})=F'(\bar{p}, X) \quad \text{for all } y\in\boldsymbol{R}^n,\ (\bar{p}, X)\in S^{n-1}\times\boldsymbol{S}_n. \tag{2.6}$$

By (1.9) we see the mapping $F'\mapsto F$ defined by

$$F(p, X)=|p|F'\Big(\bar{p}, \frac{X}{|p|}\Big), \qquad \bar{p}=\frac{p}{|p|}$$

gives a bijection from $\mathcal{G}'$ to $\mathcal{G}$. By the definition (2.5) of L and (2.6) one may identify $F'\in\mathcal{G}'$ with a function on $S^{n-1}\times\boldsymbol{S}_n/L$. By Lemma 2.2 the mapping $f\mapsto F'$ defined by

$$F'(\bar{p}, X)=-f(-\bar{p}, -Q_{\bar{p}}(X))$$

gives a bijection from $\mathcal{F}$ to $\mathcal{G}'$ since $Q_{\bar{p}}=Q_{-\bar{p}}$. Since the mapping $f\mapsto F$ is a composition of $f\mapsto F'$ and $F'\mapsto F$, it gives a bijection from $\mathcal{F}$ to $\mathcal{G}$. □

By Theorem 2.3 we see every level surface of a function u moves by (1.1) for some f if and only if (1.6) is strongly geometric at least formally, where F is uniquely determined from f by (1.8).

REMARK 2.4. A function F is strongly geometric if F is geometric, degenerate elliptic and continuous in X for $p\neq 0$. Here we say F is degenerate elliptic if F satisfies

$$F(p, X)\leqq F(p, Y) \qquad \text{for } X\geqq Y,\ X, Y\in\boldsymbol{S}_n,$$

where $\boldsymbol{S}_n$ is equipped with the usual ordering.

Indeed, to show (1.9) it suffices to prove

$$F(p, X+p\otimes y+y\otimes p)=F(p, X), \qquad p\neq 0 \tag{2.7}$$

for y orthogonal to p since (1.9′) is assumed. We may assume $y\neq 0$. We set $\bar{p}=p/|p|$ and $\bar{y}=y/|y|$. An elementary calculation shows that

$$\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \leqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \leqq \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

provided $ab \geqq 1$, $cd \geqq 1$, $a>0$, $c<0$. This estimate yields

$$c\bar{p}\otimes\bar{p}+d\bar{y}\otimes\bar{y} \leqq \bar{p}\otimes\bar{y}+\bar{y}\otimes\bar{p} \leqq a\bar{p}\otimes\bar{p}+b\bar{y}\otimes\bar{y} \quad \text{in } \boldsymbol{S}_n$$

which deduces

$$\begin{aligned} F(p,\ X+\mu b\bar{y}\otimes\bar{y}) &= F(p,\ X+\mu(a\bar{p}\otimes\bar{p}+b\bar{y}\otimes\bar{y})) \\ &\leqq F(p,\ X+p\otimes y+y\otimes p) \\ &\leqq F(p,\ X+\mu(c\bar{p}\otimes\bar{p}+d\bar{y}\otimes\bar{y})) \\ &= F(p,\ X+\mu d\bar{y}\otimes\bar{y}), \qquad \mu=|y|\cdot|p|, \end{aligned} \tag{2.8}$$

since F is degenerate elliptic and geometric. Keeping the relation $ab \geqq 1$, $cd \geqq 1$, $a>0$, $c<0$ we send b, d to zero in (2.8) and obtain (2.7) since F is continuous in X for $p \neq 0$.

REMARK 2.5. In Introduction we extend a unit normal vector field $\boldsymbol{n}$ to an open neighborhood of the hypersurface Γ_t so that $\boldsymbol{n}\cdot\nabla\boldsymbol{n}=0$, i.e., $\boldsymbol{n}$ is constant on the normals to the hypersurface. From this choice it follows that $\nabla\boldsymbol{n}$ is given by

$$\nabla\boldsymbol{n} = 0\oplus(-L),$$

where L is the Weingarten map of the hypersurface; the direct sum corresponds to the decomposition of the tangent space $T_x\boldsymbol{R}^n=\boldsymbol{R}^n$ at $x\in\Gamma_t$:

$$\boldsymbol{R}^n = \langle\boldsymbol{n}\rangle\oplus\langle\boldsymbol{n}\rangle^{\perp}, \qquad \langle\boldsymbol{n}\rangle^{\perp}=T_x\Gamma_t,$$

where $\langle\boldsymbol{n}\rangle$ is the normal vector space of Γ_t at x. Thus $\nabla\boldsymbol{n}$ is identified with a self adjoint linear transformation on the tangent space $T_x\Gamma_t$ of the hypersurface. The space of such transformations has dimension $n(n-1)/2$, which agrees with the fibre dimension of E defined in (2.4). When we consider $f(\boldsymbol{n}, \nabla\boldsymbol{n})$ for the function $f: E\to\boldsymbol{R}$, one implicitly assume that $\nabla\boldsymbol{n}$ has $n(n-1)/2$ independent components. We remark that an eigenvalue of L is called a principal curvature and that the trace of L equals $n-1$ times the mean curvature of the hypersurface.

3. Existence and uniqueness of weak solutions.

We shall clarify the class of hypersurface evolution equations (1.1) to which our theory of geometric parabolic equations developed in [**5**] yields a unique global weak solution for a given initial data. We shall also simplify the assumptions of [**5**]. We first define a weak solution $\{(\Gamma_t, D_t)\}_{t\geqq 0}$ of (1.1) through a viscosity solution of (1.6) similarly to [**5**]. As in [**5**] we discuss the case when Γ_t is compact.

DEFINITION 3.1. Let D_0 be a bounded open set and $\Gamma_0(\subset \boldsymbol{R}^n \setminus D_0)$ be a compact set containing ∂D_0. Let $\{(\Gamma_t, D_t)\}_{t\geqq 0}$ be a family of compact sets and bounded open sets in $\boldsymbol{R}^n$. Suppose that for some $\alpha<0$ there is a viscosity solution $u\in C_\alpha([0, T]\times \boldsymbol{R}^n)$ for (1.6) with (1.8) in $(0, \infty)\times \boldsymbol{R}^n$ such that zero level surface of $u(t, \cdot)$ at time $t\geqq 0$ equals Γ_t and that the set D_t where $u>0$ is bounded open. If $(\Gamma_t, D_t)|_{t=0}=(\Gamma_0, D_0)$, we say $\{(\Gamma_t, D_t)\}_{t\geqq 0}$ is a *weak solution* of (1.1) with initial data (Γ_0, D_0). Here $T>0$ is arbitrary and $v\in C_\alpha(A)$ means $v-\alpha$ is continuous and has compact support in A.

Instead of giving a definition of a viscosity solution we just remark that a viscosity solution is a kind of weak solutions satisfying the comparision principle for nonlinear degenerate elliptic equations. A fundamental theory is established by Jensen [**16**] and Ishii [**14**] (see also [**15**] and [**6**]). Since our F in (1.8) is not continuous at $p=0$ even if f is continuous, we were forced to extend their theory. We here reproduce results on geometric parabolic equations in [**5**]. We consider (1.6) in $(0, \infty)\times \boldsymbol{R}^n$ with F independent of x. The function F is assumed to satisfy the following conditions.

(F0) $F: J=(0, \infty)\times(\boldsymbol{R}^n\setminus\{0\})\times \boldsymbol{S}_n\to \boldsymbol{R}$ is geometric, i.e., F satisfies (1.9′).

(F1) $F: J\to \boldsymbol{R}$ is continuous.

(F2) F is degenerate elliptic i.e.,

$$F(t, p, X)\leqq F(t, p, Y) \qquad \text{if } X\geqq Y\,.$$

(F3) $-\infty<F_*(t, 0, O)=F^*(t, 0, O)<\infty$.

(F4) Let T be a positive number. It holds

$$(-)\quad F_*(t, p, -I)\leqq c_-(|p|)$$
$$(+)\quad F^*(t, p, I)\geqq -c_+(|p|)$$

for all $0<t<T$ with some $c_\pm(\sigma)\in C^1[0, \infty)$ and $c_0>0$ (depending only on T) such that $c_\pm(\sigma)\geqq c_0$ for all $\sigma\geqq 0$.

Here I denotes the identity matrix and $F_*: \bar{J}\to \boldsymbol{R}\cup\{\pm\infty\}$ is the lower semicontinuous relaxation of $F: J\to \boldsymbol{R}$, i.e.,

$$F_*(z)=\lim_{\varepsilon\downarrow 0}\inf_{\substack{|w-z|<\varepsilon\\ w\in J}} F(w), \qquad z=(t, p, X)\in \bar{J}\,.$$

The function F^* is defined by $F^*=-(-F)_*$.

We note that (F0)-(F2) imply that F is strongly geometric by Remark 2.4.

PROPOSITION 3.2 ([**5**, Theorems 6.8 and 7.1]). *Assume that* (F0)-(F4).

(i) *Let* $\alpha<0$. *For* $a\in C_\alpha(\boldsymbol{R}^n)$ *there is a unique global viscosity solution* u_a *of* (1.6) *such that* $u_a(0, x)=a(x)$ *and that* u_a *is in* $C_\alpha([0, T]\times\boldsymbol{R}^n)$ *for every* $T>0$.

(ii) *Let* Γ_t *denote the zero level surface of* $u_a(t, \cdot)$ *and* D_t *denote the set where* $u_a(t, \cdot)>0$. *The family* $\{(\Gamma_t, D_t)\}_{t\geq 0}$ *is uniquely determined by* (Γ_0, D_0) *and independent of* α *and* a.

By Theorem 2.3 (F0) follows from the condition that F is expressed as in (1.8) with $f:(0, \infty)\times E\to\boldsymbol{R}$ where E is the bundle defined by (2.4). Proposition 3.2 yields a unique global solution of (1.1) (cf. [**5**, Theorem 7.3]).

PROPOSITION 3.3. *Assume that* F *defined in* (1.8) *satisfies* (F1)-(F4). *Suppose that* D_0 *is a bounded open set and* Γ_0 $(\subset\boldsymbol{R}^n\setminus D_0)$ *is a compact set containing* ∂D_0. *Then there is a unique global weak solution* $\{(\Gamma_t, D_t)\}_{t\geq 0}$ *of* (1.1) *with initial data* (Γ_0, D_0).

REMARK 3.4. Proposition 3.2 is based on the comparison principle for viscosity solutions in a bounded domain. It turns out that the proof in [**5**] of the comparison principle can be simplified if we appeal to a maximum principle of Crandall and Ishii [**6**]. We give a simplified proof in our paper with Ishii and Sato [**9**] as well as extensions to the case when F depends on x and the domain is unbounded.

We seek simple conditions on f so that Proposition 3.3 is applicable to (1.1). For this purpose we first study conditions (F1)-(F4). It is convenient to introduce

$$M(s)=\sup_{\substack{|p|\leq 1\\ p\neq 0}} F(s, p, -I), \qquad m(s)=\inf_{\substack{|p|\leq 1\\ p\neq 0}} F(s, p, I). \tag{3.1}$$

LEMMA 3.5. *Assume that* F *satisfies* (F0) *and* (F2).

(i) *For* $t\geqq 0$ *it holds*

$$F^*(t, 0, O)=\lim_{\varepsilon\downarrow 0}\Big(\varepsilon\sup_{\substack{|t-s|<\varepsilon\\ s>0}} M(s)\Big), \qquad F_*(t, 0, O)=\lim_{\varepsilon\downarrow 0}\Big(\varepsilon\inf_{\substack{|t-s|<\varepsilon\\ s>0}} m(s)\Big).$$

(ii) *If* $M^*(t)<\infty$ *(resp.* $m_*(t)>-\infty$*), then* $F^*(t, 0, O)=0$ $(F_*(t, 0, O)=0)$.

(iii) *If* F *is independent of* t, *the following three conditions are equivalent.*

(a) $F^*(0, O)<\infty$ *(resp.* $F_*(0, O)>-\infty$*)*

(b) $M<\infty$ $(m>-\infty)$

(c) $F^*(0, O)=0$ $(F_*(0, O)=0)$.

PROOF. (i) If $|X|$ denotes the operator norm of $X\in\boldsymbol{S}_n$, the estimate $|X|\leqq\varepsilon$ implies

$$-\varepsilon I \leqq X \leqq \varepsilon I .$$

Since F is degenerate elliptic by (F2), we see

$$\sup_{|X|\leqq\varepsilon} F(s,\ p,\ X) \leqq F(s,\ p,\ -\varepsilon I), \qquad (s,\ p,\ X)\in J .$$

The converse inequality is trivial since $|-\varepsilon I|=\varepsilon$. We thus observe that

$$\sup_{\substack{|p|\leqq\varepsilon\\ p\neq 0}} \sup_{|X|\leqq\varepsilon} F(s,\ p,\ X) = \sup_{\substack{|p|\leqq\varepsilon\\ p\neq 0}} F(s,\ p,\ -\varepsilon I) = \varepsilon \sup_{\substack{|p|\leqq\varepsilon\\ p\neq 0}} F(s,\ p/\varepsilon,\ -I) = \varepsilon M(s)$$

since F is geometric by (F0). This yields the first identity of (i). The second identity is parallelly proved.

(ii) This follows immediately from (i) since it always holds $M_*(t)>-\infty$ and $m^*(t)<\infty$.

(iii) By (i) the condition (b) follows from (a). By (ii) the condition (b) implies (c). Clearly (c) implies (a) and the proof is now complete. □

We consider a slightly stronger condition than (F1) on the continuity of F in t.

(F1′) $F: [0, \infty)\times(\boldsymbol{R}^n\setminus\{0\})\times\boldsymbol{S}_n\to\boldsymbol{R}$ is continuous.

LEMMA 3.6. *Assume that F satisfies* (F1′). *Let M and m be as in* (3.1). *The condition* (F4−) *(resp.* (F4+)) *is equivalent to*

(3.2−) $\qquad M^*(t) < \infty \qquad$ *for* $t\geqq 0$.

((3.2+) $\qquad m_*(t) > -\infty \qquad$ *for* $t>0$.)

PROOF. We only prove that (F4−) is equivalent to (3.2−) since the other equivalence is parallelly proved. The condition (F4−) implies

$$M(t) \leqq \sup_{|p|\leqq 1} c_-(|p|) \qquad \text{for } 0\leqq t\leqq T$$

which yields (3.2−). Since $M^*(t)$ is upper semicontinuous, (3.2−) implies that

$$\sup_{0\leqq t\leqq T} M(t) = c_T < \infty .$$

This yields (F4−) since $F(t,\ p,\ -I)$ is bounded on

$$[0,\ T]\times\{p\in\boldsymbol{R}^n;\ 1\leqq|p|\leqq R\}$$

for every $R>1$ by (F1′). □

LEMMA 3.7. *Assume that F satisfies* (F0), (F1′) *and* (F2).

(i) *The conditions* (3.2±) *imply* (F3)-(F4).

(ii) *If F is independent of t, then*

(3.3) $$M<\infty \quad \text{and} \quad m>-\infty$$

is equivalent to (F3)-(F4). *Here* M *and* m *are defined by* (3.1).

PROOF. This follows from a combination of Lemmas 3.5 and 3.6. □

We now rewrite our conditions in terms of f when F is of the form (1.8). The condition (F1′) is clearly equivalent to

(f1′) $f:[0, \infty)\times E\to \boldsymbol{R}$ is continuous, where E is the bundle defined by (2.4).

The condition (F2) is clearly equivalent to

(f2) $f(t, -\bar{p}, -Q_{\bar{p}}(X))\geqq f(t, -\bar{p}, -Q_{\bar{p}}(Y))$ for $X\geqq Y$, $\bar{p}\in S^{n-1}$ and $t\geqq 0$.

This condition means that f is degenerate elliptic. By (1.8) and (3.1) we observe that

(3.4)
$$M(s)=-\inf_{0<\rho<1}\rho\inf_{|\bar{p}|=1}f\left(s, -\bar{p}, \frac{I-\bar{p}\otimes\bar{p}}{\rho}\right)$$
$$m(s)=-\sup_{0<\rho<1}\rho\sup_{|\bar{p}|=1}f\left(s, -\bar{p}, \frac{-I+\bar{p}\otimes\bar{p}}{\rho}\right).$$

It is easy to see that (3.3) is equivalent to

(3.5)
$$\liminf_{\rho\downarrow 0}\rho\inf_{|\bar{p}|=1}f\left(-\bar{p}, \frac{I-\bar{p}\otimes\bar{p}}{\rho}\right)>-\infty$$
$$\limsup_{\rho\downarrow 0}\rho\sup_{|\bar{p}|=1}f\left(-\bar{p}, \frac{-I+\bar{p}\otimes\bar{p}}{\rho}\right)<\infty.$$

This condition (and also (3.3)) is fulfilled if $f=f(\bar{p}, Z)$ is positively homogeneous of degree one in Z, where $(\bar{p}, Z)\in E$, i.e.

(3.6) $$f(\bar{p}, \lambda Z)=\lambda f(\bar{p}, Z) \qquad \text{for all } \lambda>0.$$

By Lemma 3.7 Proposition 3.3 deduces the unique existence of global weak solutions under conditions easier to check.

THEOREM 3.8. *Assume that* f *is independent of* x *and satisfies* (f1′) *and* (f2). *Assume that* f *satisfies* (3.2±) *with* (3.4) *or that* f *is independent of* t *and satisfies* (3.5). *Let* D_0 *be a bounded open set in* $\boldsymbol{R}^n$ *and let* Γ_0 ($\subset\boldsymbol{R}^n\setminus D_0$) *be a compact set containing* ∂D_0. *Then there is a unique global weak solution* $\{(\Gamma_t, D_t)\}_{t\geqq 0}$ *of* (1.1) *with initial data* (Γ_0, D_0).

REMARK 3.9. The examples (1.2), (1.3) and (1.5) fulfill all the assumptions of Theorem 3.8; here we assume that $H\in C^2(\boldsymbol{R}^n\setminus\{0\})$ is convex and positively homogeneous of degree one and that β is continuous. Indeed, it is easy to check (f1′) and (f2) directly. In these examples f is independent of t and satisfies (3.6). Since (3.6) implies (3.5), our f satisfies all assumptions of Theorem 3.8.

REMARK 3.10. For the mean curvature flow equation (1.2) Evans and Spruck [7] proved that the family $\{\Gamma_t\}_{t\geq 0}$ of the weak solution $\{(\Gamma_t, D_t)\}_{t\geq 0}$ is determined only by Γ_0 and is independent of D_0. In other words there is no need to distinguish interior and exterior bounded by Γ_t. This property holds for more general equation

$$V = f(t, \boldsymbol{n}, \nabla \boldsymbol{n})$$

with f in Theorem 3.8 provided that

$$f(t, -\bar{p}, -Z) = -f(t, \bar{p}, Z), \qquad (\bar{p}, Z) \in E .$$

Instead of giving a proof we remark that this fact is easily proved by combining arguments in [7, 9].

References

[1] S. Angenent, Parabolic equations for curves on surfaces II, intersections, blow up and generalized solutions, Ann. of Math., **133** (1991), 171-215.

[2] S. Angenent and M. Gurtin, Multiphase thermomechanics with interfacial structure. 2. Evolution of an isothermal interface, Arch. Rational Mech. Anal., **108** (1989), 323-391.

[3] L. Bronsard and R. Kohn, Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics, J. Diff. Eq., **90** (1991), 211-237.

[4] K.A. Brakke, The motion of a surface by its mean curvature, Princeton University Press, 1978.

[5] Y.-G. Chen, Y. Giga and S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, J. Diff. Geom., **33** (1991), 749-786. (Announcement: Proc. Japan Acad. Ser. A, **65** (1989), 207-210.)

[6] M.G. Crandall and H. Ishii, The maximum principle for semicontinuous functions, Differential and Integral Equations, **3** (1990), 1001-1014.

[7] L.C. Evans and J. Spruck, Motion of level sets by mean curvature I, J. Diff. Geom., **33** (1991), 635-681.

[8] M. Gage and R. Hamilton, The heat equation shrinking of convex phane curves, J. Diff. Geom., **23** (1986), 69-96.

[9] Y. Giga, S. Goto, H. Ishii and M.-H. Sato, Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains, Indiana Univ. Math. J., **40** (1991). 443-470.

[10] M. Grayson, The heat equation shrinks embedded plane curves to round points, J. Diff. Geom., **26** (1987), 285-314.

[11] M. Gurtin, Towards a nonequilibrium thermodynamics of two-phase materials, Arch. Rational Mech. Anal., **100** (1988), 275-312.

[12] M. Gurtin, Multiphase thermomechanics with interfacial structure. 1. Heat conduction and the capillary balance law, Arch. Rational Mech. Anal., **104** (1988), 195-221.

[13] G. Huisken, Flow by mean curvature of convex surfaces into spheres, J. Diff Geom., **20** (1984), 237-266.

[14] H. Ishii, On uniqueness and existence of viscosity solutions of fully nonlinear

second order elliptic PDE's Comm. Pure Appl. Math., **42** (1989), 15-45.

[15] H. Ishii and P. L. Lions, Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, J. Differential Equations, **83** (1990), 26-78.

[16] R. Jensen, The maximum principle for viscosity solutions of fully nonlinear second-order partial differential equations, Arch. Rational Mech. Anal., **101** (1988), 1-27.

[17] P. de Mottoni and M. Schatzman, Geometrical evolution of developped interfaces, preprint, 1989. (Announcement: Evolution géometric d'interfaces, C. R. Acad. Sci. Paris, **309** (1989), 453-458.)

[18] S. Osher and J. A. Sethian, Front propagating with curvature dependent speed: Algorithms based on Hamilton-Jacobi formulations, J. Comput. Phys., **79** (1988), 12-49.

[19] K. Tso, Deforming a hypersurface by its Gauss-Kronecker curvature, Comm. Pure Appl. Math., **38** (1985), 867-882.

Yoshikazu GIGA
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060
Japan

Shin'ichi GOTO
Department of Applied Science
Faculty of Engineering
Kyushu University 36
Fukuoka 812
Japan