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Growth properties of p-th hyperplane means of Green potentials in a half space

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1. Introduction.

Recently, for a Green potential v on the unit ball of $\boldsymbol{R}^n$, Gardiner [1] studied the limiting behavior of $\mathcal{M}_p(v, r)$, which is the p-th order mean of v over the sphere of radius r centered at the origin. In this paper we are concerned with Green potentials Gf in the half space $D=\{x=(x', x_n)\in\boldsymbol{R}^{n-1}\times\boldsymbol{R}^1; x_n>0\}$, where $n\geqq 2$ and f is a nonnegative measurable function on D satisfying

$$\int_D x_n^\alpha f(x)^q dx < \infty,$$

where $q\geqq 1$ and $\alpha\leqq 2q-1$. For $p>0$ and a nonnegative Borel measurable function u on D, define $M_p(u, r)=\left(\int_{R^{n-1}} u(x', r)^p dx'\right)^{1/p}$; in case $p=\infty$, define $M_\infty(u, r)=\sup\{u(x', r); x'\in\boldsymbol{R}^{n-1}\}$. Our aim in this paper is to prove that

$$\lim_{x_n\downarrow 0} x_n^{(n-2q+\alpha)/q-(n-1)/p} M_p(Gf, x_n)=0,$$

or, more weakly,

$$\liminf_{x_n\downarrow 0} x_n^{(n-2q+\alpha)/q-(n-1)/p} M_p(Gf, x_n)=0$$

for p satisfying a suitable condition; the power of x_n is shown to be best possible. In case $q=1$, our theorems below give versions of Gardiner's results in [1] to the half space.

2. Preliminary lemmas.

Now we give some notation and terminologies needed later. Let $G(x, y)$ denote the Green function in the half space D, that is,

$$G(x, y)=\begin{cases} |x-y|^{2-n}-|\bar{x}-y|^{2-n} & \text{in case } n\geqq 3, \\ \log(|\bar{x}-y|/|x-y|) & \text{in case } n=2, \end{cases}$$

where $\bar{x}=(x', -x_n)$ for $x=(x', x_n)$. We define the Green potential $G\mu$ of a nonnegative (Radon) measure μ on D by setting

$$G\mu(x)=\int_D G(x, y)d\mu(y).$$

If μ has a density $f\in L^1_{loc}(D)$, then we write Gf instead of $G\mu$.

It is easy to see that $G\mu\not\equiv\infty$ if and only if

$$\int_D (1+|y|)^{-n}y_n d\mu(y)<\infty.$$

The symbols $K, K_1, K_2, \cdots$, will be used to denote various constants independent of the variables in question. We use the convention that $1/0=\infty$.

First we show a fundamental tool in our discussions.

LEMMA 1. *If $(n-1)/n<p<(n-1)/(n-2)$, then*

$$\left(\int_{R^{n-1}} G(x, y)^p dx'\right)^{1/p} \leqq Kx_n y_n(x_n+y_n)^{-n+(n-1)/p};$$

if $n>2$ and $(n-1)/(n-2)<p$, then

$$\left(\int_{R^{n-1}} G(x, y)^p dx'\right)^{1/p}$$
$$\leqq Kx_n y_n\{(x_n+y_n)^{-n+(n-1)/p}+|x_n-y_n|^{2-n+(n-1)/p}(x_n+y_n)^{-2}\};$$

if $n>2$ and $p=(n-1)/(n-2)$, then

$$\left(\int_{R^{n-1}} G(x, y)^p dx'\right)^{1/p} \leqq Kx_n y_n(x_n+y_n)^{-2}\{1+[\log((x_n+y_n)/|x_n-y_n|)]^{1/p}\},$$

where K is a positive constant independent of x and y.

PROOF. First we give a proof in case $n\geqq 3$. In this case

$$G(x, y)\leqq K_1 x_n y_n|x-y|^{2-n}|\bar{x}-y|^{-2},$$

and hence

$$\left(\int_{R^{n-1}} G(x, y)^p dx'\right)^{1/p}$$
$$\leqq K_1 x_n y_n\left(\int_{R^{n-1}}[(|x'|^2+(x_n-y_n)^2)^{(2-n)/2}(|x'|^2+(x_n+y_n)^2)^{-1}]^p dx'\right)^{1/p}$$
$$\leqq K_2 x_n y_n I(a, b),$$

where $a=|x_n-y_n|$, $b=|x_n+y_n|$ and

$$I(a, b)=\left(\int_0^\infty [(r+a)^{2-n}(r+b)^{-2}]^p r^{n-2}dr\right)^{1/p}.$$

If $(n-1)/n<p<(n-1)/(n-2)$, then, since $a\leqq b$, we have

$$I(a, b)\leqq\left(\int_0^b [(r+a)^{2-n}(r+b)^{-2}]^p r^{n-2}dr\right)^{1/p}$$
$$+\left(\int_b^\infty [(r+a)^{2-n}(r+b)^{-2}]^p r^{n-2}dr\right)^{1/p}$$

$$\leqq b^{-2}\left(\int_0^b r^{(2-n)p+n-2}dr\right)^{1/p}+\left(\int_b^\infty r^{-np+n-2}dr\right)^{1/p}$$
$$\leqq K_3 b^{-n+(n-1)/p}.$$

The remaining cases can be proved similarly.

In case $n=2$, for any $\varepsilon>0$, we can find $K(\varepsilon)>0$ such that

$$G(x, y) \leqq K(\varepsilon)x_2y_2|x-y|^{-\varepsilon}|\bar{x}-y|^{\varepsilon-2}.$$

Thus the same arguments as above are also applicable to obtain the required result.

For $0<\beta<n$, we define an outer capacity by setting

$$C_\beta(E)=\inf \mu(\boldsymbol{R}^n), \qquad E\subset \boldsymbol{R}^n,$$

where the infimum is taken over all nonnegative measures μ on $\boldsymbol{R}^n$ such that $\int|x-y|^{\beta-n}d\mu(y)\geqq 1$ for every $x\in E$.

In case $\beta=n$, for a set $E\subset\boldsymbol{R}^n$, define

$$C_n^{(p)}(E)=\inf \mu(\boldsymbol{R}^n),$$

where the infimum is taken over all nonnegative measures μ on $\boldsymbol{R}^n$ such that

$$\int_{B(x,1)}[\log(2/|x-y|)]^{1/p}d\mu(y)\geqq 1 \qquad \text{for every } x\in E.$$

Here $B(x, a)$ denotes the open ball with radius a and center at x.

For simplicity, let $\boldsymbol{R}_+$ denote the open interval $(0, \infty)$.

LEMMA 2. *Let* $0<\beta<1$ *and* μ *be a nonnegative measure on* $\boldsymbol{R}_+$ *such that* $\mu(\boldsymbol{R}_+)<\infty$. *Then there exists a set* $E\subset\boldsymbol{R}_+$ *such that*

$$\lim_{x\to 0,\, x\in\boldsymbol{R}_+-E} x^\beta\int_{\boldsymbol{R}_+}|x-y|^{-\beta}d\mu(y)=0$$

and

$$\textstyle\sum_j 2^{j\beta}C_{1-\beta}(E_j)<\infty,$$

where $E_j=\{x\in E;\ 2^{-j}\leqq x<2^{-j+1}\}$.

PROOF. For $x>0$, we write $\int|x-y|^{-\beta}d\mu(y)=u_1(x)+u_2(x)$, where

$$u_1(x)=\int_{\{y;|x-y|<x/2\}}|x-y|^{-\beta}d\mu(y)$$

and

$$u_2(x)=\int_{\{y\in\boldsymbol{R}_+;|x-y|\geqq x/2\}}|x-y|^{-\beta}d\mu(y).$$

If $|x-y|\geqq x/2$, then $x^\beta|x-y|^{-\beta}\leqq 2^\beta$, so that we can apply Lebesgue's dominated convergence theorem to obtain

$$\lim_{x\downarrow 0} x^{\beta}u_2(x)=0.$$

For each positive integer j, we define

$$E_j=\{x\,;\ 2^{-j}\leqq x<2^{-j+1},\ 2^{-j\beta}u_1(x)>a_j^{-1}\},$$

where $\{a_j\}$ is a sequence of positive integers so chosen that

$$\lim_{j\to\infty} a_j=\infty \quad \text{and} \quad \sum_j a_j\mu(D_j)<\infty \quad \text{with} \quad D_j=(2^{-j-1},\ 2^{-j+2}).$$

Then it follows from the definition of $C_{1-\beta}$ that

$$C_{1-\beta}(E_j)\leqq a_j 2^{-j\beta}\mu(D_j).$$

If we set $E=\cup_j E_j$, then we see easily that E has the required properties.

Let $I_j=[2^{-j},\ 2^{-j+1})$. Then we have

$$\int_{I_j}|x-y|^{-\beta}dx\leqq 2\int_0^{2^{-j}/2}|x|^{-\beta}dx=2\,(1-\beta)^{-1}(2^{-j-1})^{1-\beta}\equiv A_\beta 2^{j(\beta-1)}.$$

If $\int|x-y|^{-\beta}d\mu(y)\geqq 1$ on I_j, then

$$\int_{I_j}dx\leqq\int_{I_j}\left(\int|x-y|^{-\beta}d\mu(y)\right)dx=\int\left(\int_{I_j}|x-y|^{-\beta}dx\right)d\mu(y)$$
$$\leqq A_\beta 2^{j(\beta-1)}\mu(\boldsymbol{R}_+),$$

which implies that $2^{\beta j}C_{1-\beta}(I_j)\geqq A_\beta^{-1}>0$. Thus $I_j-E_j\neq\infty$ for large j, and hence Lemma 2 gives the following result.

COROLLARY. *If μ and β are as in Lemma 2, then*

$$\liminf_{x\downarrow 0} x^{\beta}\int_{\boldsymbol{R}_+}|x-y|^{-\beta}d\mu(y)=0.$$

Similarly, we can prove the following results, which deal with the case $\beta=0$.

LEMMA 3. *Let $p>0$ and let μ be a nonnegative measure on $\boldsymbol{R}_+$ such that $\mu(\boldsymbol{R}_+)<\infty$. Then there exists a set $E\subset\boldsymbol{R}_+$ such that*

$$\lim_{x\to 0,\, x\in\boldsymbol{R}_+-E}\int_{\boldsymbol{R}_+}[\log(|x+y|/|x-y|)]^{1/p}d\mu(y)=0$$

and

$$\sum_j C_1^{(p)}(E_j)<\infty.$$

COROLLARY. *If p and μ are as in Lemma 3, then*

$$\liminf_{x\downarrow 0}\int_{\boldsymbol{R}_+}[\log(|x+y|/|x-y|)]^{1/p}d\mu(y)=0.$$

3. p-th order hyperplane mean.

For a number $p\geqq 1$, let $p'=p/(p-1)$. We begin with giving versions of Theorems 1 and 2 in [**1**] in the case of half space. We give proofs of our results for the sake of completeness.

THEOREM 1. *Let μ be a nonnegative measure on D such that $\int_D y_n^\alpha d\mu(y)<\infty$. If $1\leqq p<(n-1)/(n-2)$ and $-(n-1)/p'<\alpha\leqq 1$, then*

$$\lim_{r\to 0} r^{(n-1)/p'+(\alpha-1)} M_p(G\mu,\ r)=0.$$

PROOF. First, by Minkowski's inequality, we have

$$M_p(G\mu,\ x_n)\leqq \int_D\left(\int_{R^{n-1}}[G(x,\ y)]^p dx'\right)^{1/p} d\mu(y)$$

and hence Lemma 1 yields

$$M_p(G\mu,\ x_n)\leqq K_1\int_D [x_n y_n^{1-\alpha}(x_n+y_n)^{-n+(n-1)/p}] y_n^\alpha d\mu(y).$$

Since $x_n^{(n-1)/p'+(\alpha-1)}[x_n y_n^{1-\alpha}(x_n+y_n)^{-n+(n-1)/p}]$ is not larger than 1 and it tends to zero as $x_n\to 0$ for fixed y_n, Lebesgue's dominated convergence theorem implies that $x_n^{(n-1)/p'+(\alpha-1)}M_p(G\mu,\ x_n)$ tends to zero as $x_n\to 0$.

THEOREM 2. *Let μ be as in Theorem 1. If $n\geqq 3$, $(n-1)/(n-2)\leqq p<(n-1)/(n-3)$ and $-(n-1)/p'<\alpha\leqq 1$, then*

$$\liminf_{r\to 0} r^{(n-1)/p'+(\alpha-1)} M_p(G\mu,\ r)=0.$$

PROOF. We give a proof only in case $p>(n-1)/(n-2)$. By Minkowski's inequality and Lemma 1, we obtain

$$M_p(G\mu,\ x_n)\leqq \int_D\left(\int_{R^{n-1}}[G(x,\ y)]^p dx'\right)^{1/p} d\mu(y)$$

and

$$\left(\int_{R^{n-1}} G(x,\ y)^p dx'\right)^{1/p}\leqq K_1 x_n y_n[(x_n+y_n)^{-n+(n-1)/p} + (x_n+y_n)^{-2}|x_n-y_n|^{2-n+(n-1)/p}].$$

Hence we establish

$$x_n^{(n-1)/p'+(\alpha-1)}M_p(G\mu,\ x_n)\leqq K_1[I_1(x_n)+I_2(x_n)],$$

where

$$I_1(x_n)=\int_D [x_n^{(n-1)/p'+\alpha} y_n^{1-\alpha}|x_n+y_n|^{-n+(n-1)/p}] y_n^\alpha d\mu(y)$$

and

$$I_2(x_n)=\int_D [x_n^{(n-1)/p'+\alpha} y_n^{1-\alpha}(x_n+y_n)^{-2}|x_n-y_n|^{2-n+(n-1)/p}] y_n^\alpha d\mu(y).$$

In view of the proof of Theorem 1, we see that $I_1(x_n)$ tends to zero as $x_n \to 0$.

On the other hand, if $0 < y_n \leqq 2x_n$, then

$$x_n^{(n-1)/p'+\alpha} y_n^{1-\alpha} |x_n - y_n|^{2-n+(n-1)/p} (x_n + y_n)^{-2} \leqq K_3 x_n^{\beta} |x_n - y_n|^{-\beta},$$

where $\beta = n-2-(n-1)/p$; if $y_n > 2x_n$, then

$$x_n^{(n-1)/p'+\alpha} y_n^{1-\alpha} |x_n - y_n|^{2-n+(n-1)/p} (x_n + y_n)^{-2} \leqq K_4 (x_n/y_n)^{(n-1)/p'+\alpha} \leqq K_5 .$$

Here note that $0 < \beta < 1$. Consequently, by the Corollary to Lemma 2 and Lebesgue's dominated convergence theorem, we find that $\liminf_{x_n \to 0} I_2(x) = 0$ (cf. Proof of Lemma 2).

LEMMA 4. *If $0 < p < n/(n-2)$, $\alpha + p > -1$ and $1 - n + (\alpha + n)/p < 0$, then*

$$\left(\int_D G(x, y)^p y_n^{\alpha} dy\right)^{1/p} \leqq K x_n^{2-n+(\alpha+n)/p}$$

with a positive constant K independent of x.

PROOF. Consider the sets

$$D(x) = \{y \in D;\ y_n > 2x_n\} \quad \text{and} \quad E(x) = \{y \in D;\ y_n \leqq 2x_n\}.$$

If $y \in D(x)$, then $G(x, y) \leqq K_1 x_n (y_n - x_n) |x - y|^{-n}$, so that we have by polar coordinates with the origin at x

$$\begin{aligned}\left(\int_{D(x)} G(x, y)^p y_n^{\alpha} dy\right)^{1/p} &\leqq K_1 x_n \left(\int_{D(x)} [(y_n - x_n)|x-y|^{-n}]^p (y_n - x_n)^{\alpha} dy\right)^{1/p} \\ &\leqq K_2 x_n \left(\int_{x_n}^{\infty} r^{(1-n)p+\alpha} r^{n-1} dr\right)^{1/p} \\ &\leqq K_3 x_n^{2-n+(\alpha+n)/p} .\end{aligned}$$

On the other hand, if $y \in E(x) - B(x, x_n/2)$, then, letting $z = (x', 0)$, we see that $G(x, y) \leqq K_4 x_n y_n (|z-y| + x_n)^{-n}$, so that

$$\begin{aligned}\left(\int_{E(x)} G(x, y)^p y_n^{\alpha} dy\right)^{1/p} &\leqq K_5 x_n^{\alpha/p} \left(\int_{B(x, x_n/2)} |x-y|^{p(2-n)} dy\right)^{1/p} \\ &\quad + K_5 x_n \left(\int_{E(x)} [y_n (|z-y| + x_n)^{-n}]^p y_n^{\alpha} dy\right)^{1/p} \\ &\leqq K_6 x_n^{2-n+(\alpha+n)/p} + K_6 x_n \left(\int_0^{\infty} (r + x_n)^{-np} r^{p+\alpha+n-1} dr\right)^{1/p} \\ &\leqq K_7 x_n^{2-n+(\alpha+n)/p} .\end{aligned}$$

Thus Lemma 4 is proved.

Now we give our main theorems.

THEOREM 3. *Let* $1 \leqq q \leqq p$, $\alpha < 2q-1$ *and*

$$(n-2q)/q(n-1) < 1/p < (n-q+\alpha)/q(n-1).$$

If f *is a nonnegative measurable function on* D *such that* $\int_D y_n^\alpha f(y)^q dy < \infty$, *then*

$$\lim_{x_n \to 0} x_n^{(n-2q+\alpha)/q-(n-1)/p} M_p(Gf, x_n) = 0.$$

PROOF. The case $q=1$ was proved by Theorem 1, so we assume that $q>1$. Let (δ, β) be taken so that

$$q(n-1)\delta+q-n < \beta < q(n-1)\delta+\alpha-q(n-1)/p,$$

$$\alpha-q\delta < \beta < -q\delta+2q-1$$

and

$$(n-2q)/q(n-2) < \delta < (n-1)/p(n-2).$$

Then note that

$$(1-\delta)q' < [1-(n-2q)/q(n-2)]q' = n/(n-2),$$

$$(1-\delta)q'-\beta q'/q > -1$$

and

$$(1-n)(1-\delta)q'-\beta q'/q+n < 0.$$

Hence, by Hölder's inequality and Lemma 4, we have

$$\begin{aligned} Gf(x) &\leqq \left(\int_D G(x, y)^{(1-\delta)q'} y_n^{-\beta q'/q} dy\right)^{1/q'} \left(\int_D G(x, y)^{\delta q} y_n^\beta f(y)^q dy\right)^{1/q} \\ &\leqq K_1 x_n^{(2-n)(1-\delta)+n/q'-\beta/q} \left(\int_D G(x, y)^{\delta q} y_n^\beta f(y)^q dy\right)^{1/q}. \end{aligned}$$

Using Minkowski's inequality, we obtain

$$\begin{aligned} M_p(Gf, x_n)^q \leqq K_2 &[x_n^{(2-n)(1-\delta)+n/q'-\beta/q}]^q \\ &\times \int_D \left(\int_{R^{n-1}} G(x, y)^{\delta p} dx'\right)^{q/p} y_n^\beta f(y)^q dy. \end{aligned}$$

Since $(n-1)/n < \delta p < (n-1)/(n-2)$, by Lemma 1, we find

$$\left(\int_{R^{n-1}} G(x, y)^{\delta p} dx'\right)^{q/p} \leqq K_3 [x_n y_n (x_n+y_n)^{-n+(n-1)/\delta p}]^{\delta q},$$

so that

$$\begin{aligned} M_p(Gf, x_n)^q \leqq K_4 \int_D \{ &[x_n^{(2-n)(1-\delta)+n/q'-\beta/q}]^q \\ &\times [x_n y_n (x_n+y_n)^{-n+(n-1)/\delta p}]^{\delta q} y_n^{\beta-\alpha}\} y_n^\alpha f(y)^q dy. \end{aligned}$$

Therefore,

$$x_n^{(n-2q+\alpha)/q-(n-1)/p} M_p(Gf,\ x_n) \leqq K_4\Big(\int_D \{[x_n^{(n-1)(\delta-1/p)+(\alpha-\beta)/q}] y_n^{\delta-(\alpha-\beta)/q} \times [(x_n+y_n)^{-\delta n+(n-1)/p}]\}^q y_n^\alpha f(y)^q dy\Big)^{1/q}.$$

Noting that $(n-1)(\delta-1/p)+(\alpha-\beta)/q>0$, we can show that the left hand side tends to zero as $x_n \to 0$, as in the proof of Theorem 1.

THEOREM 4. *Let* $n \geqq 3$, $1 \leqq q \leqq p$, $\alpha<2q-1$,

$$(n-3)(n-2q)/q(n-1)(n-2) < 1/p < (n-q+\alpha)/q(n-1)$$

and

$$1/p \leqq (n-2q)/q(n-1).$$

If f is a nonnegative measurable function on D such that $\int_D y_n^\alpha f(y)^q dy < \infty$, then

$$\liminf_{x_n \to 0} x_n^{(n-2q+\alpha)/q-(n-1)/p} M_p(Gf,\ x_n) = 0.$$

PROOF. Since the case $q=1$ was proved by Theorem 2, we may assume that $q>1$. Let $(\delta,\ \beta)$ be chosen so that

$$q(n-1)\delta+q-n<\beta<q(n-1)\delta+\alpha-q(n-1)/p,$$
$$\alpha-q\delta<\beta<-q\delta+2q-1$$

and

$$(n-2q)/q(n-2)<\delta<\min\{(n-1)/p(n-3),\ 1/q(n-2)+(n-1)/p(n-2)\}.$$

By the proof of Theorem 3, we have

$$M_p(Gf,\ x_n)^q \leqq K_1[x_n^{(2-n)(1-\delta)+n/q'-\beta/q}]^q \times \int_D \Big(\int_{R^{n-1}} G(x,\ y)^{\delta p} dx'\Big)^{q/p} y_n^\beta f(y)^q dy$$

and, since $(n-1)/(n-2) \leqq p(n-2q)/q(n-2)<\delta p<(n-1)/(n-3)$,

$$\Big(\int_{R^{n-1}} G(x,\ y)^{\delta p} dx'\Big)^{q/p} \leqq K_2[x_n y_n(|x_n-y_n|^{2-n+(n-1)/\delta p}(x_n+y_n)^{-2} + (x_n+y_n)^{-n+(n-1)/\delta p})]^{\delta q}.$$

Consequently,

$$\begin{aligned} & x_n^{(n-2q+\alpha)/q-(n-1)/p} M_p(Gf,\ x_n) \\ & \leqq K_4\Big(\int_D \{[x_n^{(n-1)(\delta-1/p)+(\alpha-\beta)/q}] y_n^{\delta-(\alpha-\beta)/q} [(x_n+y_n)^{-\delta n+(n-1)/p}]\}^q y_n^\alpha f(y)^q dy \\ & \quad + \int_D \{[x_n^{q(n-1)(\delta-1/p)+\alpha-\beta}] y_n^{\delta q-(\alpha-\beta)} (x_n+y_n)^{-2\delta q} \\ & \quad \times |x_n-y_n|^{\delta q(2-n)+q(n-1)/p}\} y_n^\alpha f(y)^q dy\Big)^{1/q} \\ & = K_4[I_1(x_n)+I_2(x_n)]^{1/q}. \end{aligned}$$

In the proof of Theorem 3, we proved that $I_1(x_n)$ tends to zero as $x_n \to 0$. Letting $\gamma = \delta q(n-2) - q(n-1)/p$, we note that $0 < \gamma < 1$. If $y_n \leqq 2x_n$, then

$$[x_n^{q(n-1)(\delta-1/p)+\alpha-\beta}]y_n^{\delta q-(\alpha-\beta)}(x_n+y_n)^{-2\delta q}|x_n-y_n|^{-\gamma} \leqq K_5[x_n^{\gamma}|x_n-y_n|^{-\gamma}]$$

and if $y_n \geqq 2x_n$, then

$$[x_n^{q(n-1)(\delta-1/p)+\alpha-\beta}]y_n^{\delta q-(\alpha-\beta)}(x_n+y_n)^{-2\delta q}|x_n-y_n|^{-\gamma}$$
$$\leqq K_6(x_n/y_n)^{q(n-1)(\delta-1/p)+\alpha} \leqq K_7.$$

Hence, by the Corollary to Lemma 2 and Lebesgue's dominated convergence theorem, we see that $\liminf_{x_n \downarrow 0} I_2(x_n) = 0$. Thus Theorem 4 is established.

4. The case $p = \infty$.

If μ is a nonnegative measure on D such that $G\mu \not\equiv \infty$, then there exists a set $F \subset D$ such that F is *thin at* ∂D and

$$\lim_{x_n \to 0,\, x \in D-F} x_n^{n-1}(1+|x|)^{-n} G\mu(x) = 0;$$

see Mizuta [**2**]. To define the thinness, we use the capacity $C_G(E) = \inf \mu(D)$, where the infimum is taken over all nonnegative measures μ on D such that $G\mu \not\equiv \infty$ and $G\mu(x) \geqq 1$ for every $x \in E$.

In the present situation we have

THEOREM 5. *Let* $1-n < \alpha \leqq 1$. *If* μ *is as in Theorem* 1, *then there exists a set* $E \subset D$ *such that*

$$\lim_{x_n \to 0,\, x \in D-E} x_n^{n-2+\alpha} G\mu(x) = 0$$

and

$$\sum_j 2^{j(n-2)} C_G(E_j) < \infty,$$

where $E_j = \{x = (x', x_n) \in E;\ 2^{-j} \leqq x_n < 2^{-j+1}\}$.

In Theorem 5, if we let E^* be the projection of E to the half line $l_+ \equiv \{0\} \times \boldsymbol{R}_+$, then we derive the following result.

COROLLARY 1. *If* $n=2$, $-1 < \alpha \leqq 1$ *and* μ *is as in Theorem* 1, *then there exists a set* $E^* \subset l_+$ *such that*

$$\lim_{x_2 \to 0,\, (0, x_2) \in l_+ - E^*} x_2^{\alpha} M_\infty(G\mu, x_2) = 0$$

and

$$\sum_j C_G(E_j^*) < \infty.$$

This Corollary gives the following result, which implies the result of Stoll [**4**].

COROLLARY 2. *If $n=2$, $-1<\alpha\leqq 1$ and μ is as in Theorem 1, then*

$$\liminf_{x_2\to 0} x_2^{\alpha} M_\infty(G\mu, x_2)=0.$$

In case μ has a density f such that $\int_D y_n^\alpha f(y)^q dy<\infty$, we know the following fact (see Mizuta [3]). We also refer the reader to [3] for the definition of capacity $C_{2,q}$.

THEOREM 6. *Let $1<q\leqq n/2$ and $q-n<\alpha<2q-1$. If f is a nonnegative measurable function on D such that $\int_D y_n^\alpha f(y)^q dy<\infty$, then there exists a set $E\subset D$ with the following properties:*

(i) $\lim_{x_n\to 0,\, x\in D-E} x_n^{(n-2q+\alpha)/q} Gf(x)=0$;

(ii) $\sum_j 2^{j(n-2q)} C_{2,q}(E_j\cap G_1;\, D_j\cap G_2)<\infty$ *for any bounded open sets G_1 and G_2 such that $\bar{G}_1\subset G_2$,*

where $D_j=\{x\in D;\, 2^{-j-1}<x_n<2^{-j+2}\}$.

REMARK. If $2q>n$, then, in Theorem 6, E can be taken as the empty set.

Letting E^* be the projection of E and noting the contractive properties of $C_{2,q}$, we can establish the following result.

COROLLARY. *Let $2q>n-1$ and $q-n<\alpha<2q-1$. If f is as in Theorem 6, then there exists a set $E^*\subset l_+$ with the following properties:*

(i) $\lim_{x_n\to 0,\, (0,x_n)\in l_+-E^*} x_n^{(n-2q+\alpha)/q} M_\infty(Gf, x_n)=0$.

(ii) $\sum_j 2^{j(n-2q)} C_{2,q}(E_j^*\cap G_1;\, D_j\cap G_2)<\infty$ *for any bounded open sets G_1 and G_2 such that $\bar{G}_1\subset G_2$;*

in case $2q>n$, E^* can be taken as the empty set.

This Corollary also implies that

$$\liminf_{x_n\downarrow 0} x_n^{(n-2q+\alpha)/q} M_\infty(Gf, x_n)=0.$$

5. Best possibility.

REMARK 1. Our theorems are best possible as to the power of x_n.

For this, consider the function $f(y)=y_n^a/|y|^b$ for $y\in D\cap B(0, 1)$ and $f(y)=0$ elsewhere. Here $a=-(\alpha+1)/q+\delta$, $b=(n-1)/q$ and $\delta>0$ is chosen sufficiently small. Then, $\int_D y_n^\alpha f(y)^q dy<\infty$. If $x\in D\cap B(0, 1/2)$, then

$$Gf(x)\geqq K_1\int_{B(x,x_n/2)} |x-y|^{2-n} f(y)dy$$
$$\geqq K_2 x_n^a |x|^{-b}\int_{B(x,x_n/2)} |x-y|^{2-n}dy\geqq K_3 x_n^{a+2}|x|^{-b}.$$

Hence, for a number $p>0$ and a point x satisfying $0<x_n<1/4$, we obtain

$$M_p(Gf,\ x_n) \geqq K_3 x_n^{\alpha+2}\left(\int_{\{|x'|<1/4\}} (|x'|^2+x_n^2)^{-bp/2}dx'\right)^{1/p}$$

$$\geqq K_4 x_n^{\alpha+2-b+(n-1)/p} = K_4 x_n^{-(n-2q+\alpha)/q+(n-1)/p+\delta};$$

similarly,

$$M_\infty(Gf,\ x_n) \geqq K_4 x_n^{-(n-2q+\alpha)/q+\delta}.$$

These facts imply the best possibility of our theorems as to the power of x_n.

REMARK 2. If $1/p \leqq (n-2q)/q(n-1)$, then there exists a nonnegative measurable function f on D such that $\int_D y_n^\alpha f(y)^q dy<\infty$ and

$$\limsup_{r\downarrow 0} r^{(n-2q+\alpha)/q-(n-1)/p} M_p(Gf,\ r) = \infty.$$

For this, let

$$e_r = (0, \cdots, 0, r) \in D, \qquad 0 < a_r < 1/4, \qquad \Delta(r) = B(e_r,\ a_r r),$$

and consider the functions

$$f_r(y) = \begin{cases} |y-e_r|^{-n/q}[\log(r/4|y-e_r|)]^{-\beta} & \text{if } y\in\Delta(r) \\ 0 & \text{elsewhere} \end{cases}$$

for $\beta>1/q$. Then

$$\int y_n^\alpha f_r(y)^q dy \leqq K_1 r^\alpha \int_0^{a_r r} [\log(r/4t)]^{-\beta q} t^{-1} dt = K_2 r^\alpha [\log(1/4a_r)]^{-\beta q+1}.$$

If $x\in\Delta(r)$ and $x_n=r$, then

$$Gf_r(x) \geqq K_3 \int_{\Delta(r)} |x-y|^{2-n} f_r(y) dy$$

$$\geqq K_4 |x-e_r|^{2-n} \int_0^{|x-e_r|} t^{-n/q}[(\log(r/4t)]^{-\beta} dt$$

$$\geqq K_5 |x-e_r|^{2-n/q}[\log(r/4|x-e_r|)]^{-\beta}.$$

Hence, in case $(2-n/q)p+n-1<0$, we obtain

$$M_p(Gf_r,\ r) \geqq K_5 \left(\int_{\{x'\in R^{n-1};\, |x'|<a_r r\}} |x'|^{(2-n/q)p}[\log(r/4|x'|)]^{-\beta p} dx'\right)^{1/p} = \infty;$$

and in case $(2-n/q)p+n-1=0$,

$$M_p(Gf_r,\ r) \geqq K_6 [\log(1/4a_r)]^{-\beta+1/p}.$$

If $1/p<(n-2q)/q(n-1)$, then let $r=2^{-j}$, $a_r=1/8$ and $f=\sum f_{2^{-j}}$. If $1/p=(n-2q)/q(n-1)$, then let $b_j=2^{-j\alpha/q}[\log(1/4a_{2^{-j}})]^{-\beta+1/p}$ and note

$$\int_D y_n^\alpha f_{2^{-j}}(y)^q dy \leqq K_2 2^{-j\alpha}[b_j 2^{j\alpha/q}]^{(-\beta q+1)/(-\beta+1/p)}$$

$$\leqq K_2 2^{-j\alpha(1/q-1/p)} b_j^{(\beta q-1)(\beta-1/p)}.$$

Now choose $a_{2^{-j}}$ so that $b_j=j$, and let $f=\sum_j f_{2^{-j}}$. Then f satisfies the required conditions.

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