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The Gorensteinness of symbolic Rees algebras for space curves

By Shiro GOTO, Koji NISHIDA and Yasuhiro SHIMODA*

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1. Introduction.

Let A be a regular local ring and $\mathfrak{p}$ a prime ideal in A. We put $R_s(\mathfrak{p})=\sum_{n\geq 0}\mathfrak{p}^{(n)}t_n$ (here t denotes an indeterminate over A) and call it the symbolic Rees algebra of $\mathfrak{p}$. Our purpose is to discuss when $R_s(\mathfrak{p})$ is a Gorenstein ring.

The problem when $R_s(\mathfrak{p})$ is Noetherian is, of course, more fundamental and has been studied by many authors from several points of view (cf. [**3**, **4**, **5**, **10**, **11**, **12**, **13**, **14**, **16**, **19**, **20**, **21** and **22**]). The finite generation problem on the A-algebra $R_s(\mathfrak{p})$ was raised by R. C. Cowsik [**3**], showing that $\mathfrak{p}$ is a set-theoretic complete intersection in A if $\dim A/\mathfrak{p}=1$ and if $R_s(\mathfrak{p})$ is a finitely generated A-algebra. If A is not regular, there is no hope in general of $R_s(\mathfrak{p})$ being Noetherian (cf. e.g., [**5**, Sect. 5]), as was firstly noticed by D. Rees [**19**] in his construction of a counterexample to the Zariski problem. Nevertheless even though A is regular the rings $R_s(\mathfrak{p})$ are not necessarily Noetherian. P. Roberts [**20**] gave such examples, passing to Nagata's counterexamples [**18**] to the 14-th problem of Hilbert. And as far as we know, Cowsik's problem seems still open for general prime ideals $\mathfrak{p}=\mathfrak{p}(n_1, n_2, n_3)$ of $A=k[[X, Y, Z]]$ (the formal power series ring over a field k) defining space monomial curves $X=t^{n_1}$, $Y=t^{n_2}$ and $Z=t^{n_3}$ with $GCD(n_1, n_2, n_3)=1$.

We look now at a prime ideal $\mathfrak{p}$ of height 2 in a 3-dimensional regular local ring A with maximal ideal $\mathfrak{m}$. Then in his remarkable paper [**12**] C. Huneke gave the following criterion for $R_s(\mathfrak{p})$ to be a Noetherian ring:

($*$) *If there exist $f\in\mathfrak{p}^{(k)}$ and $g\in\mathfrak{p}^{(l)}$ with positive integers k, l such that $\mathrm{length}_A(A/(f, g, x)A)=kl\cdot\mathrm{length}_A(A/\mathfrak{p}+xA)$ for some $x\in\mathfrak{m}\setminus\mathfrak{p}$, then $R_s(\mathfrak{p})$ is Noetherian. When the field $A/\mathfrak{m}$ is infinite, the converse is also true.*

With this criterion Huneke explored prime ideals $\mathfrak{p}=\mathfrak{p}(n_1, n_2, n_3)$ and guaranteed that $R_s(\mathfrak{p})$ is Noetherian, if $\min(n_1, n_2, n_3)=4$.

In the present paper we would like to succeed Huneke's research, mainly asking for similar practical criteria as his for $R_s(\mathfrak{p})$ to be Gorenstein. It might

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be worthy to be noted that $R_s(\mathfrak{p})$ is a Gorenstein ring once it is Cohen-Macaulay, because it is a quasi-Gorenstein ring (cf. [**22**, Corollary 3.4]). The criterion which we shall prove in this paper is based on Huneke's condition (*) above and can be summarized into the following

THEOREM (1.1). *Let A be a regular local ring of* $\dim A=3$ *and $\mathfrak{p}$ a prime ideal of A with* $\dim A/\mathfrak{p}=1$. *Assume that there exist $f\in\mathfrak{p}^{(k)}$ and $g\in\mathfrak{p}^{(l)}$ satisfying Huneke's condition* (*). *Then the following two conditions are equivalent.*

(1) *$R_s(\mathfrak{p})$ is a Gorenstein ring.*

(2) *$A/(f, g)+\mathfrak{p}^{(n)}$ are Cohen-Macaulay for all $1\leqq n\leqq k+l-2$.*

When this is the case, the A-algebra $R_s(\mathfrak{p})$ is generated by $\{\mathfrak{p}^{(n)}t^n\}_{1\leqq n\leqq k+l-2}$, ft^k and gt^l, and the rings $A/fA+\mathfrak{p}^{(n)}$, $A/gA+\mathfrak{p}^{(n)}$ and $A/(f, g)+\mathfrak{p}^{(n)}$ are Cohen-Macaulay for all $n\geqq 1$.

The condition (2) in Theorem (1.1) is naturally satisfied when $k, l\leqq 2$ and as an immediate consequence of Theorem (1.1) and [**12**, Proof of Corollary (3.6)], we have

COROLLARY (1.2). *Let $\mathfrak{p}$ be a prime ideal of a regular local ring A of* $\dim A=3$. *Then $R_s(\mathfrak{p})$ is a Gorenstein ring, if the multiplicity of $A/\mathfrak{p}$ is equal to* 3.

Let us explain how to organize this paper. Theorem (1.1) will be proved in Section 3. Section 2 is devoted to some preliminary step which we throughout need. By means of Theorem (1.1) we are able to prove that $R_s(\mathfrak{p})$ are Gorenstein for the space monomial curves $\mathfrak{p}=\mathfrak{p}(n_1, n_2, n_3)$ with $n_1=4$. However as the proof is somewhat long, we shall postpone the detail to the subsequent paper [**6**]. Instead we will explore some concrete examples to see how the criterion (1.1) works. In Section 4 we will establish a criterion for $R_s(\mathfrak{p})$ to be a Noetherian ring, which leads us to a certain class of space monomial curves $\mathfrak{p}=\mathfrak{p}(n_1, n_2, n_3)$ with Gorenstein symbolic Rees algebras (cf. (4.1) and (4.7)). Unfortunately $R_s(\mathfrak{p})$ are not necessarily Cohen-Macaulay even for the space monomial curves $\mathfrak{p}$, although they are Noetherian. We will analyze one example in Section 5.

Throughout this paper let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. For each prime ideal $\mathfrak{p}$ of A we define

$$R_s(\mathfrak{p})=\sum_{n\geqq 0}\mathfrak{p}^{(n)}t^n\,,\qquad R_s'(\mathfrak{p})=\sum_{n\in\mathbf{Z}}\mathfrak{p}^{(n)}t^n\ (=R_s(\mathfrak{p})[t^{-1}])$$
$$\text{and}\quad G_s(\mathfrak{p})=R_s'(\mathfrak{p})/t^{-1}R_s'(\mathfrak{p})\,,$$

where t stands for an indeterminate over A and $\mathfrak{p}^{(n)}=A$ for $n\leqq 0$. Similarly we put, for arbitrary ideals I of A,

$$R(I)=\sum_{n\geqq 0}I^n t^n,\quad R'(I)=\sum_{n\in \boldsymbol{Z}}I^n t^n\ (=R(I)[t^{-1}])\quad \text{and}\quad G(I)=R'(I)/t^{-1}R'(I),$$

where $I^n=A$ for $n\leqq 0$.

For each A-module M we denote by $l_A(M)$ the length of M.

2. The depth of the ring $R_s(\mathfrak{p})$.

Let A be a regular local ring of $\dim A=3$ and $\mathfrak{p}$ a prime ideal of A with $\dim A/\mathfrak{p}=1$. In this section we assume that $R_s(\mathfrak{p})$ is a Noetherian ring. Hence $G_s(\mathfrak{p})$ is Noetherian as

$$G_s(\mathfrak{p})=R_s(\mathfrak{p})[u]/uR_s(\mathfrak{p})[u].$$

where $u=t^{-1}$. We consider $R_s(\mathfrak{p})$ and $R'_s(\mathfrak{p})=R_s(\mathfrak{p})[u]$ to be $\boldsymbol{Z}$-graded rings whose graduations $\{[R_s(\mathfrak{p})]_n\}_{n\in \boldsymbol{Z}}$ and $\{[R'_s(\mathfrak{p})]_n\}_{n\in \boldsymbol{Z}}$ are given by $[R_s(\mathfrak{p})]_n=\mathfrak{p}^{(n)}t^n$ for $n\geqq 0$, $[R_s(\mathfrak{p})]_n=(0)$ for $n<0$ and $[R'_s(\mathfrak{p})]_n=\mathfrak{p}^{(n)}t^n$ for all $n\in \boldsymbol{Z}$.

Let $\mathfrak{M}=\mathfrak{m}R_s(\mathfrak{p})+\sum_{n>0}\mathfrak{p}^{(n)}t^n$ be the unique graded maximal ideal of $R_s(\mathfrak{p})$ and we respectively denote by depth $R_s(\mathfrak{p})$ and depth $G_s(\mathfrak{p})$ the depth of the local rings $R_s(\mathfrak{p})_{\mathfrak{M}}$ and $G_s(\mathfrak{p})_{\mathfrak{M}}$. The purpose is to prove the following

THEOREM (2.1). depth $R_s(\mathfrak{p})$=depth $G_s(\mathfrak{p})+1$.

REMARK (2.2). As $\dim R_s(\mathfrak{p})=4$ and $\dim G_s(\mathfrak{p})=3$, by Theorem (2.1) we see that $R_s(\mathfrak{p})$ is a Gorenstein ring if $G_s(\mathfrak{p})$ is Cohen-Macaulay (cf. [**22**, Corollary 3.4]). Since $G_s(\mathfrak{p})=\bigoplus_{n\geqq 0}\mathfrak{p}^{(n)}/\mathfrak{p}^{(n+1)}$ is an integral domain, $u=t^{-1}$ is a prime element of $R'_s(\mathfrak{p})$ with $R'_s(\mathfrak{p})[u^{-1}]=A[t, t^{-1}]$ factorial. Hence $R'_s(\mathfrak{p})$ is a factorial ring by Nagata's theorem so that $R'_s(\mathfrak{p})$ is a Gorenstein ring, if it is Cohen-Macaulay (cf. [**17**]). Thus $G_s(\mathfrak{p})$ is a Gorenstein ring, when it is Cohen-Macaulay.

We divide the proof of Theorem (2.1) into a few parts. First choose an integer $k>0$ so that $[\mathfrak{p}^{(k)}]^n=\mathfrak{p}^{(kn)}$ for all $n\geqq 0$ (this choice is possible, as $R_s(\mathfrak{p})$ is Noetherian, cf. [**1**, Ch. 3, Sect. 1.3]).

PROPOSITION (2.3). *$R(\mathfrak{p}^{(k)})$ is a Cohen-Macaulay ring.*

PROOF. We many assume $A/\mathfrak{m}$ to be infinite. Then depth $A/[\mathfrak{p}^{(k)}]^n=1$ for any $n\geqq 1$ and so by Burch's theorem [**2**] there exist $f, g\in \mathfrak{p}^{(k)}$ such that $[\mathfrak{p}^{(k)}]^{r+1}=(f, g)[\mathfrak{p}^{(k)}]^r$ for some $r\geqq 0$. Choose $x\in \mathfrak{m}\setminus\mathfrak{p}$ and put $B=A/xA$, $J=(\mathfrak{p}^{(k)}+xA)/xA$. Let * denote the reduction mod xA. Then letting $e=l_A(A/\mathfrak{p}+xA)$, we have

$$l_B(B/J^{n+1})=ek^2\binom{n+2}{2}-e\binom{k}{2}(n+1)$$

for all $n\geqq 0$ (cf. e.g., [**12**, Proof of Theorem 3.1]), whence $J^2=(f^*, g^*)J$ by [**12**, Theorem 2.1]. Therefore $G(J)$ is a Cohen-Macaulay ring by [**23**, Proposition 3.1] so that $R(J)$ is Cohen-Macaulay too (cf. [7, Remark (3.10)]). Because $xA\cap\mathfrak{p}^{(n)}=x\mathfrak{p}^{(n)}$ for all $n\geqq 0$, we get an isomorphism

$$R(\mathfrak{p}^{(k)})/xR(\mathfrak{p}^{(k)}) \cong R(J)$$

of B-algebras which guarantees that $R(\mathfrak{p}^{(k)})$ is a Cohen-Macaulay ring.

Let $\mathrm{H}^i_{\mathfrak{M}}(*)$ denote the i-th local cohomology functor of $R_s(\mathfrak{p})$ relative to $\mathfrak{M}$.

COROLLARY (2.4). $[\mathrm{H}^i_{\mathfrak{M}}(R_s(\mathfrak{p}))]_{kn}=(0)$ *for all* $n\in \boldsymbol{Z}$ *and* $i\leqq 3$.

PROOF. Let $S=\sum_{n\geqq 0}\mathfrak{p}^{(kn)}t^{kn}$ be the Veronesean subring of $R_s(\mathfrak{p})$ with order k and let $\mathfrak{N}$ denote the unique graded maximal ideal of S. Then we get isomorphisms

$$[\mathrm{H}^i_{\mathfrak{N}}(S)]_n \cong [\mathrm{H}^i_{\mathfrak{M}}(R_s(\mathfrak{p}))]_{kn}$$

of A-modules for all n, $i\in\boldsymbol{Z}$ (cf. [8, (3.1.1)]). Because S is, by (2.3), a Cohen-Macaulay ring of $\dim S=4$, we have the required vanishing.

Let $I=\sum_{n>0}\mathfrak{p}^{(n)}t^n$ $(=[R_s(\mathfrak{p})]_+)$ and we have two exact sequences

$$(\#)\quad 0\longrightarrow I\longrightarrow R_s(\mathfrak{p})\longrightarrow A\longrightarrow 0$$
$$(\#\#)\quad 0\longrightarrow I(1)\longrightarrow R_s(\mathfrak{p})\longrightarrow G_s(\mathfrak{p})\longrightarrow 0$$

of graded $R_s(\mathfrak{p})$-modules, where $I(1)$ stands for the graded module I shifted in degree 1, i.e., $[I(1)]_n=I_{n+1}$ $(n\in\boldsymbol{Z})$.

Apply local cohomology functors $\mathrm{H}^i_{\mathfrak{M}}(*)$ to the exact sequence $(\#)$ to get isomorphisms $\mathrm{H}^i_{\mathfrak{M}}(I)\cong\mathrm{H}^i_{\mathfrak{M}}(R_s(\mathfrak{p}))$ for $i\leqq 2$ as well as the embedding $\mathrm{H}^3_{\mathfrak{M}}(I)\subseteq\mathrm{H}^3_{\mathfrak{M}}(R_s(\mathfrak{p}))$. Then because $\mathrm{H}^i_{\mathfrak{M}}(R_s(\mathfrak{p}))=(0)$ for $i\leqq 1$ (recall that $R_s(\mathfrak{p})$ is normal, cf. [22, Lemma 2.5]), we have by the exact sequence $(\#\#)$ that $\mathrm{H}^1_{\mathfrak{M}}(G_s(\mathfrak{p}))\subseteq[\mathrm{H}^2_{\mathfrak{M}}(I)](1)=[\mathrm{H}^2_{\mathfrak{M}}(R_s(\mathfrak{p}))](1)$, whence depth $R_s(\mathfrak{p})=2$ if depth $G_s(\mathfrak{p})=1$.

LEMMA (2.5). *Suppose that* depth $G_s(\mathfrak{p})\geqq 2$. *Then* $\mathrm{H}^2_{\mathfrak{M}}(R_s(\mathfrak{p}))=(0)$.

PROOF. We have by the exact sequence $(\#\#)$ the embedding $[\mathrm{H}^2_{\mathfrak{M}}(I)](1)\subseteq\mathrm{H}^2_{\mathfrak{M}}(R_s(\mathfrak{p}))$ and so $[\mathrm{H}^2_{\mathfrak{M}}(R_s(\mathfrak{p}))](1)\subseteq\mathrm{H}^2_{\mathfrak{M}}(R_s(\mathfrak{p}))$, as $\mathrm{H}^2_{\mathfrak{M}}(I)\cong\mathrm{H}^2_{\mathfrak{M}}(R_s(\mathfrak{p}))$. Hence for each $n\in\boldsymbol{Z}$ we get an embedding

$$[\mathrm{H}^2_{\mathfrak{M}}(R_s(\mathfrak{p}))]_{n+1}\subseteq[\mathrm{H}^2_{\mathfrak{M}}(R_s(\mathfrak{p}))]_n$$

of A-modules so that the vanishing $\mathrm{H}^2_{\mathfrak{M}}(R_s(\mathfrak{p}))=(0)$ follows, because $[\mathrm{H}^2_{\mathfrak{M}}(R_s(\mathfrak{p}))]_{kn}=(0)$ for all $n\in\boldsymbol{Z}$ (cf. (2.4)).

PROOF OF THEOREM (2.1). If depth $G_s(\mathfrak{p})=2$, then $\mathrm{H}^i_{\mathfrak{M}}(R_s(\mathfrak{p}))=(0)$ $(i\leqq 2)$ by (2.5) and we get depth $R_s(\mathfrak{p})=3$ because of the embedding $\mathrm{H}^2_{\mathfrak{M}}(G_s(\mathfrak{p}))\subseteq[\mathrm{H}^3_{\mathfrak{M}}(R_s(\mathfrak{p}))](1)$ that follows from the exact sequences $(\#)$ and $(\#\#)$.

Assume depth $G_s(\mathfrak{p})=3$. We must show that $\mathrm{H}^3_{\mathfrak{M}}(R_s(\mathfrak{p}))=(0)$. Let $a(G_s(\mathfrak{p}))=$

$\max\{n\in \boldsymbol{Z}\,|\,[\mathrm{H}^3_{\mathfrak{M}}(G_s(\mathfrak{p}))]_n\neq(0)\}$. Then we have $a(G_s(\mathfrak{p}))=-2$ by [8, (3.1.6)] and [**15**, Proposition (1.10)], because $A_{\mathfrak{p}}\otimes_A G_s(\mathfrak{p})=G(\mathfrak{p}A_{\mathfrak{p}})$ is a polynomial ring with two variables over the field $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Therefore from the exact sequence ($^{\#\#}$) it follows that $[\mathrm{H}^3_{\mathfrak{M}}(I)]_{n+1}\cong[\mathrm{H}^3_{\mathfrak{M}}(R_s(\mathfrak{p}))]_n$ for all $n\geqq-1$, while we have by the exact sequence ($^{\#}$) the embedding $[\mathrm{H}^3_{\mathfrak{M}}(I)]_n\subseteqq[\mathrm{H}^3_{\mathfrak{M}}(R_s(\mathfrak{p}))]_n$ for each $n\in\boldsymbol{Z}$. Hence

$$[\mathrm{H}^3_{\mathfrak{M}}(R_s(\mathfrak{p}))]_n \subseteqq [\mathrm{H}^3_{\mathfrak{M}}(R_s(\mathfrak{p}))]_{n+1}$$

for $n\geqq-1$ which guarantees $[\mathrm{H}^3_{\mathfrak{M}}(R_s(\mathfrak{p}))]_n=(0)$ for $n\geqq-1$, because $[\mathrm{H}^3_{\mathfrak{M}}(R_s(\mathfrak{p}))]_n=(0)$ for all $n\gg0$.

Let $n\leqq-2$ be an integer. Then we have by the exact sequence ($^{\#\#}$) that $[\mathrm{H}^3_{\mathfrak{M}}(I)]_n\subseteqq[\mathrm{H}^3_{\mathfrak{M}}(R_s(\mathfrak{p}))]_{n-1}$. On the other hand we see by the exact sequence ($^{\#}$) $[\mathrm{H}^3_{\mathfrak{M}}(I)]_n\cong[\mathrm{H}^3_{\mathfrak{M}}(R_s(\mathfrak{p}))]_n$ as $\mathrm{H}^3_{\mathfrak{M}}(A)=[\mathrm{H}^3_{\mathfrak{M}}(A)]_0$. Hence

$$[\mathrm{H}^3_{\mathfrak{M}}(R_s(\mathfrak{p}))]_n \subseteqq [\mathrm{H}^3_{\mathfrak{M}}(R_s(\mathfrak{p}))]_{n-1}\,,$$

which yields by (2.4) that $[\mathrm{H}^3_{\mathfrak{M}}(R_s(\mathfrak{p}))]_n=(0)$ for all $n\leqq-2$. Thus $\mathrm{H}^3_{\mathfrak{M}}(R_s(\mathfrak{p}))=(0)$ and depth $R_s(\mathfrak{p})=4$, which completes the proof of Theorem (2.1).

3. Proof of Theorem (1.1).

Let A be a regular local ring of $\dim A=3$ and $\mathfrak{p}$ a prime ideal of A with $\dim A/\mathfrak{p}=1$. The purpose of this section is to prove Theorem (1.1). Let $f\in\mathfrak{p}^{(k)}$ and $g\in\mathfrak{p}^{(l)}$ ($k, l>0$) such that $l_A(A/(f, g, x))=kl\cdot l_A(A/\mathfrak{p}+xA)$ for some $x\in\mathfrak{m}\setminus\mathfrak{p}$.

We begin with the following

PROPOSITION (3.1) (cf. [**12**]). (1) *(f, g) is a $\mathfrak{p}$-primary ideal of A.*

(2) $[G_s(\mathfrak{p})]_+=\mathrm{Rad}\,((ft^k, gt^l)G_s(\mathfrak{p}))$.

(3) *ft^k, gt^l forms a $G(\mathfrak{p}A_{\mathfrak{p}})$-regular sequence.*

PROOF. Passing to the elements f^l, g^k of $\mathfrak{p}^{(kl)}$, we may assume that $k=l$. We put $B=A/xA$ and $J=(\mathfrak{p}^{(k)}+xA)/xA$ and let $*$ denote the reduction mod xA. Then by [**12**, Proof of Theorem 3.1] we get $[\mathfrak{p}^{(k)}]^n=\mathfrak{p}^{(kn)}$ and the equality

$$l_B(B/J^{n+1}) = ek^2\binom{n+2}{2}-e\binom{k}{2}(n+1)$$

for all $n\geqq0$ (here $e=l_A(A/\mathfrak{p}+xA)$). Therefore by [**12**, Theorem 2.1] we have $J^2=(f^*, g^*)J$, that is $[\mathfrak{p}^{(k)}]^2\subseteqq(f, g)\mathfrak{p}^{(k)}+xA$, whence we get $[\mathfrak{p}^{(k)}]^2=(f, g)\mathfrak{p}^{(k)}+x[\mathfrak{p}^{(k)}]^2$ because $[\mathfrak{p}^{(k)}]^2=\mathfrak{p}^{(2k)}$ and $x\notin\mathfrak{p}$. Thus $[\mathfrak{p}^{(k)}]^2=(f, g)\mathfrak{p}^{(k)}$ by Nakayama's lemma so that $\mathfrak{p}=\mathrm{Rad}\,((f, g))$, whence we have the assertion (1). The assertions (2) and (3) follow from the equality $[\mathfrak{p}^{(k)}]^2=(f, g)\mathfrak{p}^{(k)}$, too.

Let $\mathfrak{M}$ be the unique graded maximal ideal of $G_s(\mathfrak{p})$, i.e., $\mathfrak{M}=\mathfrak{m}G_s(\mathfrak{p})+[G_s(\mathfrak{p})]_+$. Then by (3.1) we get $\mathfrak{M}=\mathrm{Rad}\,((ft^k, gt^l, x)G_s(\mathfrak{p}))$ and so ft^k, gt^l, x

mod $uR_s'(\mathfrak{p})$ forms a homogeneous system of parameters for the graded ring $G_s(\mathfrak{p})$. Therefore we have

COROLLARY (3.2). *$G_s(\mathfrak{p})$ is a Cohen-Macaulay ring if and only if ft^k, gt^l, x forms a $G_s(\mathfrak{p})$-regular sequence.*

COROLLARY (3.3). *$fA\cap\mathfrak{p}^{(n)}=f\mathfrak{p}^{(n-k)}$ for all $n\in\boldsymbol{Z}$.*

PROOF. Since ft^k, gt^l is a regular sequence for $G(\mathfrak{p}A_\mathfrak{p})$, we have $fA_\mathfrak{p}\cap\mathfrak{p}^nA_\mathfrak{p}=f\mathfrak{p}^{n-k}A_\mathfrak{p}$ (cf., e.g., [**23**]) so that $fA\cap\mathfrak{p}^{(n)}\subseteqq fA\cap f\mathfrak{p}^{n-k}A_\mathfrak{p}=f\mathfrak{p}^{(n-k)}$. Hence the assertion.

PROPOSITION (3.4). *$\mathfrak{p}^{(k+l-1)}\subseteqq(f, g)$ but $\mathfrak{p}^{(k+l-2)}\not\subset(f, g)$.*

PROOF. We put $a=\max\{n\in\boldsymbol{Z}\mid[G(\mathfrak{p}A_\mathfrak{p})/(ft^k, gt^l)G(\mathfrak{p}A_\mathfrak{p})]_n\neq(0)\}$. Then as $G(\mathfrak{p}A_\mathfrak{p})$ is a polynomial ring, we have by (3.1) (3) and [8, (3.1.6)] that $a=k+l-2$, i.e., $\mathfrak{p}^{k+l-1}A_\mathfrak{p}\subseteqq(f, g)A_\mathfrak{p}$ and $\mathfrak{p}^{k+l-2}A_\mathfrak{p}\not\subset(f, g)A_\mathfrak{p}$, which yields the assertion because (f, g) is $\mathfrak{p}$-primary by (3.1) (1).

LEMMA (3.5). *$(f, g)\cap\mathfrak{p}^{(n)}=f\mathfrak{p}^{(n-k)}+g\mathfrak{p}^{(n-l)}$ for each $n\leqq k+l$.*

PROOF. Let $\varphi\in(f, g)\cap\mathfrak{p}^{(n)}$ and write $\varphi=fa+gb$ with $a, b\in A$. Then since $(f, g)\cap\mathfrak{p}^{(n)}\subseteqq(f, g)A_\mathfrak{p}\cap\mathfrak{p}^nA_\mathfrak{p}=f\mathfrak{p}^{n-k}A_\mathfrak{p}+g\mathfrak{p}^{n-l}A_\mathfrak{p}$ by (3.1) (3), we have $\varphi=f\alpha+g\beta$ for some $\alpha\in\mathfrak{p}^{n-k}A_\mathfrak{p}$ and $\beta\in\mathfrak{p}^{n-l}A_\mathfrak{p}$. Choose $s\in A\setminus\mathfrak{p}$ so that $s\alpha\in\mathfrak{p}^{(n-k)}$ and $s\beta\in\mathfrak{p}^{(n-l)}$. Then $s\varphi=fs\alpha+gs\beta=fsa+gsb$, whence $sb-s\beta\in fA$ because f, g is an A-regular sequence (cf. (3.1) (1)). Therefore we have $sb\in fA+\mathfrak{p}^{(n-l)}$ so that $b\in\mathfrak{p}^{(n-l)}$, because $f\in\mathfrak{p}^{(n-l)}$ (recall that $n-l\leqq k$) and $s\notin\mathfrak{p}$. Similarly we get $a\in\mathfrak{p}^{(n-k)}$, which proves $\varphi\in f\mathfrak{p}^{(n-k)}+g\mathfrak{p}^{(n-l)}$ as required.

COROLLARY (3.6). *$A/fA+\mathfrak{p}^{(k+l-1)}$ is a Cohen-Macaulay ring.*

PROOF. Consider the exact sequence

$$0\longrightarrow fA+\mathfrak{p}^{(k+l-1)}\longrightarrow(f, g)\longrightarrow(f, g)/fA+\mathfrak{p}^{(k+l-1)}\longrightarrow 0$$

of A-modules and notice that $fA+\mathfrak{p}^{(k+l-1)}=fA+g\mathfrak{p}^{(k-1)}$ by (3.4) and (3.5). Then as

$$\begin{aligned}(f, g)/fA+\mathfrak{p}^{(k+l-1)}&=(f, g)/fA+g\mathfrak{p}^{(k-1)}\cong gA/fgA+g\mathfrak{p}^{(k-1)}\\&\cong A/fA+\mathfrak{p}^{(k-1)}=A/\mathfrak{p}^{(k-1)},\end{aligned}$$

we have $\operatorname{depth}_A(f, g)/fA+\mathfrak{p}^{(k+l-1)}\geqq1$ so that $\operatorname{depth}_A fA+\mathfrak{p}^{(k+l-1)}=2$ by the above exact sequence, because $\operatorname{depth}_A(f, g)=2$. Thus $\operatorname{depth}_A A/fA+\mathfrak{p}^{(k+l-1)}=1$ and $A/fA+\mathfrak{p}^{(k+l-1)}$ is Cohen-Macaulay.

Our proof of Theorem (1.1) is based on the next

PROPOSITION (3.7). *Suppose that the ring $A/fA+\mathfrak{p}^{(n)}$ is Cohen-Macaulay for each $1\leqq n\leqq k+l-2$. Then we have the following assertions.*

(1) *The ring* $A/fA+\mathfrak{p}^{(n)}$ *is Cohen-Macaulay for any* $n\geqq 1$.
(2) $(f, g)\cap\mathfrak{p}^{(n)}=f\mathfrak{p}^{(n-k)}+g\mathfrak{p}^{(n-l)}$ *for all* $n\in\boldsymbol{Z}$.
(3) ft^{k}, gt^{l} *forms a* $G_{s}(\mathfrak{p})$*-regular sequence.*
(4) $R_{s}(\mathfrak{p})=A[\{\mathfrak{p}^{(n)}t^{n}\}_{1\leqq n\leqq k+l-2}, ft^{k}, gt^{l}]$.

PROOF. As the assertions (3) and (4) immediately follow from the assertion (2) (cf. (3.3) and (3.4)), it is enough to see the assertions (1) and (2). We will prove them by induction on n. By (3.5) and (3.6) we may assume that $n\geqq k+l$ and that the assertions (1) and (2) are true for all smaller n. Hence $A/fA+\mathfrak{p}^{(n-l)}$ is a Cohen-Macaulay ring. Let us consider the exact sequence

$$(\#)\quad 0\longrightarrow f\mathfrak{p}^{(n-k)}\longrightarrow f\mathfrak{p}^{(n-k)}+g\mathfrak{p}^{(n-l)}\longrightarrow f\mathfrak{p}^{(n-k)}+g\mathfrak{p}^{(n-l)}/f\mathfrak{p}^{(n-k)}\longrightarrow 0$$

of A-modules and recall the isomorphisms

$$\begin{aligned} f\mathfrak{p}^{(n-k)}+g\mathfrak{p}^{(n-l)}/f\mathfrak{p}^{(n-k)} &\cong g\mathfrak{p}^{(n-l)}/f\mathfrak{p}^{(n-k)}\cap g\mathfrak{p}^{(n-l)}=g\mathfrak{p}^{(n-l)}/fg\mathfrak{p}^{(n-k-l)}\\ &\cong \mathfrak{p}^{(n-l)}/f\mathfrak{p}^{(n-k-l)}=\mathfrak{p}^{(n-l)}/fA\cap\mathfrak{p}^{(n-l)}. \qquad \text{(by (3.3))}\end{aligned}$$

Then we see by the exact sequence

$$0\longrightarrow \mathfrak{p}^{(n-l)}/fA\cap\mathfrak{p}^{(n-l)}\longrightarrow A/fA\longrightarrow A/fA+\mathfrak{p}^{(n-l)}\longrightarrow 0$$

that $\mathrm{depth}_{A}f\mathfrak{p}^{(n-k)}+g\mathfrak{p}^{(n-l)}/f\mathfrak{p}^{(n-k)}=2$ and so we get by the sequence (#) $\mathrm{depth}_{A}[f\mathfrak{p}^{(n-k)}+g\mathfrak{p}^{(n-l)}]=2$, because $\mathrm{depth}_{A}f\mathfrak{p}^{(n-k)}=\mathrm{depth}_{A}\mathfrak{p}^{(n-k)}=2$. Hence the ideal $J=f\mathfrak{p}^{(n-k)}+g\mathfrak{p}^{(n-l)}$ is $\mathfrak{p}$-primary so that $J\supseteqq(f, g)\cap\mathfrak{p}^{(n)}$ as $JA_{\mathfrak{p}}=(f, g)A_{\mathfrak{p}}\cap\mathfrak{p}^{n}A_{\mathfrak{p}}$ by (3.1) (3). Thus $J=(f, g)\cap\mathfrak{p}^{(n)}$, which proves the assertion (2). Notice that $\mathfrak{p}^{(n)}=f\mathfrak{p}^{(n-k)}+g\mathfrak{p}^{(n-l)}$ as $(f, g)\supseteqq\mathfrak{p}^{(n)}$ by (3.4).

To see the assertion (1) consider the exact sequence

$$0\longrightarrow fA+\mathfrak{p}^{(n)}\longrightarrow (f, g)\longrightarrow (f, g)/fA+\mathfrak{p}^{(n)}\longrightarrow 0.$$

Then because $\mathfrak{p}^{(n)}=f\mathfrak{p}^{(n-k)}+g\mathfrak{p}^{(n-l)}$ by (2), similarly as in the proof of (3.6) we have

$$(f, g)/fA+\mathfrak{p}^{(n)}\cong A/fA+\mathfrak{p}^{(n-l)}$$

so that $\mathrm{depth}_{A}fA+\mathfrak{p}^{(n)}=2$, which yields $A/fA+\mathfrak{p}^{(n)}$ is a Cohen-Macaulay ring. This completes the proof of Proposition (3.7).

COROLLARY (3.8). *Suppose that* $k=1$ *or* $k=2\leqq l$. *Then* depth $R_{s}(\mathfrak{p})\geqq 3$ *and*

$$R_{s}(\mathfrak{p})=A[\{\mathfrak{p}^{(n)}t^{n}\}_{1\leqq n\leqq l}].$$

PROOF. Because $gA+\mathfrak{p}^{(n)}=\mathfrak{p}^{(n)}$ for $1\leqq n\leqq k+l-2$, the second assertion follows from (3.7) (4). See (2.1) and (3.7) (3) for the first assertion.

We are now ready to prove Theorem (1.1).

PROOF OF THEOREM (1.1).

First assume that $R_s(\mathfrak{p})$ is a Gorenstein ring. Then $G_s(\mathfrak{p})$ is a Cohen-Macaulay ring so that ft^k, gt^l, x forms a $G_s(\mathfrak{p})$-regular sequence (cf. (2.1) and (3.2)). Hence $(f, g)\cap\mathfrak{p}^{(n)}=f\mathfrak{p}^{(n-k)}+g\mathfrak{p}^{(n-l)}$ for all $n\in\boldsymbol{Z}$. Recall the canonical isomorphisms

$$[G_s(\mathfrak{p})/(ft^k, gt^l)G_s(\mathfrak{p})]_n \cong \mathfrak{p}^{(n)}/(f\mathfrak{p}^{(n-k)}+g\mathfrak{p}^{(n-l)}+\mathfrak{p}^{(n+1)})$$
$$= \mathfrak{p}^{(n)}/[(f, g)+\mathfrak{p}^{(n+1)}]\cap\mathfrak{p}^{(n)}$$

of A-modules and we find x is a non-zerodivisor on $\mathfrak{p}^{(n)}/[(f, g)+\mathfrak{p}^{(n+1)}]\cap\mathfrak{p}^{(n)}$, whence

$$\operatorname{depth}_A\mathfrak{p}^{(n)}/[(f, g)+\mathfrak{p}^{(n+1)}]\cap\mathfrak{p}^{(n)}\geqq 1$$

for each $n\geqq 0$. Therefore considering the exact sequence

$$0 \to \mathfrak{p}^{(n)}/[(f, g)+\mathfrak{p}^{(n+1)}]\cap\mathfrak{p}^{(n)} \to A/(f, g)+\mathfrak{p}^{(n+1)} \to A/(f, g)+\mathfrak{p}^{(n)} \to 0\,,$$

we see by induction on n that the local rings $A/(f, g)+\mathfrak{p}^{(n)}$ are Cohen-Macaulay for all $n\geqq 1$.

Conversely assume that $A/(f, g)+\mathfrak{p}^{(n)}$ is a Cohen-Maculay ring for any $1\leqq n\leqq k+l-2$. We need the following

CLAIM. The rings $A/fA+\mathfrak{p}^{(n)}$ and $A/gA+\mathfrak{p}^{(n)}$ are Cohen-Macaulay for all $1\leqq n\leqq k+l-2$.

PROOF OF THE CLAIM. We may only discuss $A/fA+\mathfrak{p}^{(n)}$. It suffices to check that $\operatorname{depth}_A A/fA+\mathfrak{p}^{(n)}\geqq 1$. Consider the exact sequence

$$0 \longrightarrow fA+\mathfrak{p}^{(n)} \longrightarrow (f, g)+\mathfrak{p}^{(n)} \longrightarrow (f, g)+\mathfrak{p}^{(n)}/fA+\mathfrak{p}^{(n)} \longrightarrow 0\,.$$

Then because

$$(f, g)+\mathfrak{p}^{(n)}/fA+\mathfrak{p}^{(n)} \cong (f, g)/fA+(f, g)\cap\mathfrak{p}^{(n)} = (f, g)/fA+g\mathfrak{p}^{(n-l)} \quad \text{(by (3.5))}$$
$$\cong gA/fgA+g\mathfrak{p}^{(n-l)} \cong A/fA+\mathfrak{p}^{(n-l)} = A/\mathfrak{p}^{(n-l)}\,,$$

we have $\operatorname{depth}_A(f, g)+\mathfrak{p}^{(n)}/fA+\mathfrak{p}^{(n)}\geqq 1$ and so $\operatorname{depth}_A fA+\mathfrak{p}^{(n)}=2$ as $\operatorname{depth}_A(f, g)+\mathfrak{p}^{(n)}=2$ by our assumption; hence $A/fA+\mathfrak{p}^{(n)}$ is Cohen-Macaulay.

By this claim and Proposition (3.7) (3) we get ft^k, gt^l is a $G_s(\mathfrak{p})$-regular sequence. Because

$$[G_s(\mathfrak{p})/(ft^k, gt^l)G_s(\mathfrak{p})]_n \cong \mathfrak{p}^{(n)}/f\mathfrak{p}^{(n-k)}+g\mathfrak{p}^{(n-l)}+\mathfrak{p}^{(n+1)}$$

for all $n\geqq 0$ and because $(f, g)\cap\mathfrak{p}^{(n)}=f\mathfrak{p}^{(n-k)}+g\mathfrak{p}^{(n-l)}$ for all $n\in\boldsymbol{Z}$ by (3.7) (2), we have

$$G_s(\mathfrak{p})/(ft^k, gt^l)G_s(\mathfrak{p}) \cong \bigoplus_{n=0}^{k+l-2} \mathfrak{p}^{(n)}/((f, g)+\mathfrak{p}^{(n+1)})\cap\mathfrak{p}^{(n)}$$

as A-modules (cf. (3.4)). Consequently x is $G_s(\mathfrak{p})/(ft^k, gt^l)G_s(\mathfrak{p})$-regular, as the A-module $\mathfrak{p}^{(n)}/((f, g)+\mathfrak{p}^{(n+1)})\cap\mathfrak{p}^{(n)}$ is a submodule of $A/(f, g)+\mathfrak{p}^{(n+1)}$ and as x is, by our assumption, $A/(f, g)+\mathfrak{p}^{(n+1)}$-regular for any $0\leqq n\leqq k+l-2$ (cf. (3.6) too). Thus ft^k, gt^l, x is a $G_s(\mathfrak{p})$-regular sequence, whence by (2.1) and (3.2) $R_s(\mathfrak{p})$ is a Gorenstein ring. As the last assertions now follow from (3.7), this completes the proof of Theorem (1.1).

The rings $A/(f, g)+\mathfrak{p}^{(n)}$ is obviously Cohen-Macaulay for any $1\leqq n\leqq k+l-2$, if $k, l\leqq 2$. Hence we immediately have

COROLLARY (3.9). *$R_s(\mathfrak{p})$ is a Gorenstein ring, if $k, l\leqq 2$.*

PROOF OF COROLLARY (1.2).

We may assume $A/\mathfrak{m}$ to be infinite so that there exist $f, g\in\mathfrak{p}^{(2)}$ with $l_A(A/(f, g, x))=12$ for some $x\in\mathfrak{m}\setminus\mathfrak{p}$ (cf. [**12**, Proof of Corollary 3.6]) and the assertion follows from (3.9).

4. The Gorensteinness of $R_s(\mathfrak{p})$ for certain space monomial curves $\mathfrak{p}$.

Let A be a regular local ring of $\dim A=3$ and $\mathfrak{p}$ a prime ideal of A with $\dim A/\mathfrak{p}=1$. In this section we assume that $\mathfrak{p}$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} X^{\alpha} & Y^{\beta'} & Z^{\gamma'} \\ Y^{\beta} & Z^{\gamma} & X^{\alpha'} \end{bmatrix}$$

where X, Y, Z is a regular system of parameters for A and $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are positive integers. The purpose is to prove the following

THEOREM (4.1). *Suppose that $\alpha'\geqq\alpha$, $\beta'=m\beta$ and $\gamma'\geqq m\gamma$ with $m\in\boldsymbol{Z}$. Then $R_s(\mathfrak{p})$ is a Gorenstein ring.*

Let $a=Z^{\gamma+\gamma'}-X^{\alpha'}Y^{\beta'}$, $b=X^{\alpha+\alpha'}-Y^{\beta}Z^{\gamma'}$, $c=Y^{\beta+\beta'}-X^{\alpha}Z^{\gamma}$ and assume that $\alpha'\geqq\alpha$, $\beta'=m\beta$ and $\gamma'\geqq m\gamma$ with $m\in\boldsymbol{Z}$. We begin with the following

PROPOSITION (4.2). *There exist $d_n\in\mathfrak{p}^{(n)}$ $(1\leqq n\leqq m+1)$ with $d_1=-a$ such that*

$$X^{\alpha}d_n = Y^{(m-n+1)\beta}b^n+(-1)^{n+1}Z^{\gamma'-(n-1)\gamma}a^{n-1}c\,,$$
$$Y^{\beta}d_n = d_{n-1}b+(-1)^{n}X^{\alpha'-\alpha}Z^{\gamma'-(n-1)\gamma}a^{n-2}c^2\,,$$
$$Z^{\gamma}d_n = -d_{n-1}a-X^{\alpha'-\alpha}Y^{(m-n+1)\beta}b^{n-1}c$$

for any $2\leqq n\leqq m+1$.

PROOF. First notice that $X^{\alpha}a+Y^{m\beta}b+Z^{\gamma'}c=0$ and $Y^{\beta}a+Z^{\gamma}b+X^{\alpha'}c=0$. We assume $1\leqq n\leqq m$ and that the choice of $\{d_i\}_{1\leqq i\leqq m+1}$ is already done within n. Then because $X^{\alpha}d_n=Y^{(m-n+1)\beta}b^n+(-1)^{n+1}Z^{\gamma'-(n-1)\gamma}a^{n-1}c$ (this is true for

$n=1$ too, as $d_1=-a$ and as $X^{\alpha}a+Y^{m\beta}b+Z^{\gamma'}c=0$), we have

$$X^{\alpha}d_n a = Y^{\beta}aY^{(m-n)\beta}b^n+(-1)^{n+1}Z^{\gamma'-(n-1)\gamma}a^n c$$
$$= (-Z^{\gamma}b-X^{\alpha'}c)Y^{(m-n)\beta}b^n+(-1)^{n+1}Z^{\gamma'-(n-1)\gamma}a^n c$$

so that $X^{\alpha}(-d_n a-X^{\alpha'-\alpha}Y^{(m-n)\beta}b^n c)=Z^{\gamma}(Y^{(m-n)\beta}b^{n+1}+(-1)^{n+2}Z^{\gamma'-n\gamma}a^n c)$. Hence $X^{\alpha}d_{n+1}=Y^{(m-n)\beta}b^{n+1}+(-1)^{n+2}Z^{\gamma'-n\gamma}a^n c$ and $Z^{\gamma}d_{n+1}=-d_n a-X^{\alpha'-\alpha}Y^{(m-n)\beta}b^n c$ for some $d_{n+1}\in\mathfrak{p}^{(n+1)}$. Because

$$Y^{\beta}ad_{n+1} = (-Z^{\gamma}b-X^{\alpha'}c)d_{n+1}$$
$$= Z^{\gamma}d_{n+1}(-b)+X^{\alpha}d_{n+1}(-X^{\alpha'-\alpha}c)$$
$$= (d_n b+(-1)^{n+1}X^{\alpha'-\alpha}Z^{\gamma'-n\gamma}a^{n-1}c^2)a\,,$$

we get $Y^{\beta}d_{n+1}=d_n b+(-1)^{n+1}X^{\alpha'-\alpha}Z^{\gamma'-n\gamma}a^{n-1}c^2$, too.

LEMMA (4.3). $d_n\equiv(-1)^n Z^{\gamma+n\gamma'} \bmod (c, X)$ *for* $1\leqq n\leqq m+1$.

PROOF. As $d_1=X^{\alpha'}Y^{m\beta}-Z^{\gamma+\gamma'}$, we get the assertion for $n=1$. Let $1\leqq n\leqq m$ and assume that $d_n\equiv(-1)^n Z^{\gamma+n\gamma'} \bmod (c, X)$. Then by (4.2) we see

$$Z^{\gamma}d_{n+1} \equiv -d_n a \equiv (-1)^{n+1}Z^{2\gamma+(n+1)\gamma'} \bmod (c, X)\,.$$

Hence we get $d_{n+1}\equiv(-1)^{n+1}Z^{\gamma+(n+1)\gamma'}$, because c, X, Z^{γ} is an A-regular sequence.

COROLLARY (4.4). *$R_s(\mathfrak{p})$ is a Noetherian ring.*

PROOF. Since $l_A(A/(c, d_{m+1}, X))=l_A(A/(X, Y^{(m+1)\beta}, Z^{\gamma+(m+1)\gamma'}))=(m+1)\beta\{\gamma+(m+1)\gamma'\}$ and $l_A(A/\mathfrak{p}+(X))=\beta\{\gamma+(m+1)\gamma'\}$, we have $l_A(A/(c, d_{m+1}, X))=1\cdot(m+1)\cdot l_A(A/\mathfrak{p}+XA)$ so that $R_s(\mathfrak{p})$ is Noetherian by Huneke's criterion.

PROPOSITION (4.5). *$R_s(\mathfrak{p})$ is a Gorenstein ring and $\mathfrak{p}^{(n)}=\mathfrak{p}^n+(c^{n-i}d_i \mid 2\leqq i\leqq n)$ for $1\leqq n\leqq m+1$. Hence we have*

$$R_s(\mathfrak{p}) = A[\mathfrak{p}t,\ \{d_n t^n\}_{2\leqq n\leqq m+1}]\,.$$

PROOF. Because the multiplicity $e_{XA}(A/cA+\mathfrak{p}^{(n)})$ of the ring $A/cA+\mathfrak{p}^{(n)}$ relative to the parameter X is equal to $n\beta\{\gamma+(m+1)\gamma'\}$, we have the inequalities

$$l_A(A/(c, X, d_n)+\mathfrak{p}^n) \geqq l_A(A/(c, X)+\mathfrak{p}^{(n)})$$
$$\geqq e_{XA}(A/cA+\mathfrak{p}^{(n)}) = n\beta\{\gamma+(m+1)\gamma'\}.$$

On the other hand, as $(c, X, d_n)+\mathfrak{p}^n=(X, Y^{(m+1)\beta}, Y^{n\beta}Z^{n\gamma'}, Z^{\gamma+n\gamma'})$ (cf. (4.3)), we have $l_A(A/(c, X, d_n)+\mathfrak{p}^n)=n\beta\{\gamma+(m+1)\gamma'\}$. Therefore we get that $(c, X)+\mathfrak{p}^{(n)}=(c, X, d_n)+\mathfrak{p}^n$ and that the ring $A/cA+\mathfrak{p}^{(n)}$ is Cohen-Macaulay for each $1\leqq n\leqq m+1$. Hence $R_s(\mathfrak{p})$ is a Gorenstein ring by (1.1) and so we have $\mathfrak{p}^{(n)}=d_nA+\mathfrak{p}^n+c\mathfrak{p}^{(n-1)}+X\mathfrak{p}^{(n)}$, because $(c, X)\cap\mathfrak{p}^{(n)}=c\mathfrak{p}^{(n-1)}+X\mathfrak{p}^{(n)}$ (recall that ct,

$d_{m+1}t^{m+1}$, X is a $G_s(\mathfrak{p})$-regular sequence by (3.2)). Thus we get by Nakayama's lemma $\mathfrak{p}^{(n)}=d_nA+\mathfrak{p}^n+c\mathfrak{p}^{(n-1)}$ for $1\leqq n\leqq m+1$, which implies $\mathfrak{p}^{(n)}=\mathfrak{p}^n+(c^{n-i}d_i \mid 2\leqq i\leqq n)$ as required. This completes the proof of (4.5) as well as that of Theorem (4.1).

Let $A=k[[X, Y, Z]]$ and $S=k[[t]]$ be formal power series ring over a field k and let n_1, n_2, n_3 be positive integers with $GCD(n_1, n_2, n_3)=1$. We denote by $\mathfrak{p}=\mathfrak{p}(n_1, n_2, n_3)$ the kernel of the k-algebra map $f: A\to S$ with $f(X)=t^{n_1}$, $f(Y)=t^{n_2}$ and $f(Z)=t^{n_3}$. Thus $\mathfrak{p}$ is the defining ideal of the space monomial curve $X=t^{n_1}$, $Y=t^{n_2}$ and $Z=t^{n_3}$ and as is well known (cf. [**9**]), $\mathfrak{p}$ is either a complete intersection in A or generated by the maximal minors of a matrix of the form

$$\begin{bmatrix} X^{\alpha} & Y^{\beta'} & Z^{\gamma'} \\ Y^{\beta} & Z^{\gamma} & X^{\alpha'} \end{bmatrix}$$

where α, β, γ, α', β' and γ' are positive integers.

COROLLARY (4.6). *Let $m\geqq 1$ be an integer. Then $R_s(\mathfrak{p})$ is a Gorenstein ring, if $\mathfrak{p}=\mathfrak{p}(m, m+1, m+3)$.*

PROOF. We write $m=3n+q$ with $0\leqq q<3$. If $q=0$, then $\mathfrak{p}=(X^{n+1}-Z^n, Y^3-X^2Z)$ which is a complete intersection in A so that $R_s(\mathfrak{p})=R(\mathfrak{p})$. Hence $R_s(\mathfrak{p})\cong A[T_1, T_2]/fA[T_1, T_2]$ as A-algebras, where $A[T_1, T_2]$ is a polynomial ring and $0\neq f\in A[T_1, T_2]$. Thus $R_s(\mathfrak{p})$ is Gorenstein.

(1) ($q=1$) We may assume $n\geqq 1$ and so $\mathfrak{p}$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} X^2 & Y^2 & Z^n \\ Y & Z & X^n \end{bmatrix}.$$

Hence our assertion follows from (4.1), if $n\geqq 2$. When $n=1$, notice that $\mathfrak{p}$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} Z & X^2 & Y^2 \\ X & Y & Z \end{bmatrix}.$$

(2) ($q=2$) We may assume $n\geqq 1$ so that $\mathfrak{p}$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} Z & Y^2 & X^{n+1} \\ Y & X^2 & Z^n \end{bmatrix},$$

whence the assertion follows from (4.1), if $n\geqq 3$. If $n=1$, then $\mathfrak{p}=\mathfrak{p}(5, 6, 8)$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} X^2 & Z & Y^2 \\ Z & Y & X^2 \end{bmatrix},$$

and so again by (4.1) we get that $R_s(\mathfrak{p})$ is Gorenstein.

Now assume $n=2$, i.e., $\mathfrak{p}=\mathfrak{p}(8, 9, 11)$. Then $\mathfrak{p}$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} X^2 & Y & Z^2 \\ Y^2 & Z & X^3 \end{bmatrix}.$$

Let $a=Z^3-X^3Y$, $b=X^5-Y^2Z^2$ and $c=Y^3-X^2Z$. Then as $X^2a+Yb+Z^2c=Y^2a+Zb+X^3c=0$, we get

$$Yb^2 = (-X^2a-Z^2c)b = -X^2ab-Z^2bc = -X^2ab+cZ(Y^2a+X^3c)$$

whence $Y(b^2-acYZ)=X^2(-ab+c^2XZ)$ so that $X^2d_2=b^2-acYZ$ and $Yd_2=-ab+c^2XZ$ for some $d_2\in\mathfrak{p}^{(2)}$. Because

$$(d_2Z)b = d_2(-Y^2a-X^3c) = (-aY)\cdot Yd_2+(-cX)\cdot X^2d_2$$
$$= a^2Yb-b^2cX = a^2(-X^2a-Z^2c)-b^2cX\,,$$

we have $Z(bd_2+a^2cZ)=X(-a^3X-b^2c)$ whence $Xd_3=bd_2+a^2cZ$ and $Zd_3=-a^3X-b^2c$ with $d_3\in\mathfrak{p}^{(3)}$. Finally because

$$(Yd_3)b = d_3(-X^2a-Z^2c) = (-aX)\cdot Xd_3+(-cZ)\cdot Zd_3 = -abd_2X+b^2c^2Z\,,$$

we get $Yd_3=-ad_2X+bc^2Z=-ad_2X-c^2(Y^2a+X^3c)$ whence $Y(d_3+ac^2Y)=X(-ad_2-c^3X^2)$. Thus $Yd_3'=ad_2+c^3X^2$ for some $d_3'\in\mathfrak{p}^{(3)}$. As $Yd_2\equiv Y^2Z^5$ mod XA, we get $d_2\equiv YZ^5$ mod XA. Hence $Yd_3'\equiv YZ^8$ mod XA and so $d_3'\equiv Z^8$ mod XA. Consequently we see

$$l_A(A/(c, d_3', X)) = 1\cdot 3\cdot l_A(A/\mathfrak{p}+XA) = 24$$

and $R_s(\mathfrak{p})$ is Noetherian. Because

$$l_A(A/(c, d_2, X)+\mathfrak{p}^2) = \mathrm{e}_{XA}(A/cA+\mathfrak{p}^{(2)}) = 16\,,$$

the ring $A/cA+\mathfrak{p}^{(2)}$ is Cohen-Macaulay so that $R_s(\mathfrak{p})$ is Gorenstein by (1.1).

REMARK (4.7). Similarly as (4.6) one can prove that $R_s(\mathfrak{p})$ is a Gorenstein ring for $\mathfrak{p}=\mathfrak{p}(m, m+1, m+2)$ $(m\geqq 1)$.

We close this section with the following

EXAMPLE (4.8). $R_s(\mathfrak{p})$ is a Gorenstein ring for $\mathfrak{p}=\mathfrak{p}(4, 7, 13)$.

PROOF. The ideal $\mathfrak{p}$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} X^2 & Y^2 & Z \\ Y & Z & X^3 \end{bmatrix}.$$

Let $a=Z^2-X^3Y^2$, $b=X^5-YZ$ and $c=Y^3-X^2Z$. Then $X^2a+Y^2b+Zc=Ya+Zb+X^3c=0$. As $(-X^2a-Zc)a=Y^2ab=(-Zb-X^3c)Yb$, we have $X^2(a^2-bcXY)=Z(b^2Y-ac)$ so that $X^2d_2=b^2Y-ac$ and $Zd_2=-bcXY+a^2$ with $d_2\in\mathfrak{p}^{(2)}$. Notice that $Yd_2=c^2X-ab$, because

$$aY\cdot d_2=(-b)\cdot Zd_2+(-cX)\cdot X^2d_2=a(c^2X-ab).$$

Then $(b^2Y-X^2d_2)b=abc=(c^2X-Yd_2)c$ whence $X(bd_2X+c^3)=Y(b^3+cd_2)$ so that $Xd_3=b^3+cd_2$ and $Yd_3=bd_2X+c^3$ with $d_3\in\mathfrak{p}^{(3)}$. Because $d_2\equiv Z^3 \bmod XA$ and $d_3\equiv Y^8 \bmod XA$, we get

$$l_A(A/(X, d_2, d_3))=2\cdot 3\cdot l_A(A/\mathfrak{p}+XA)=24.$$

Hence $R_s(\mathfrak{p})$ is Noetherian. As $l_A(A/(X, d_2)+\mathfrak{p}^2)=\mathrm{e}_{XA}(A/\mathfrak{p}^{(2)})=12$ and as $l_A(A/(X, d_3)+d_2\mathfrak{p}+\mathfrak{p}^3)=\mathrm{e}_{XA}(A/\mathfrak{p}^{(3)})=24$, we see $\mathfrak{p}^{(2)}=d_2A+\mathfrak{p}^2$ and $\mathfrak{p}^{(3)}=d_2\mathfrak{p}+d_3A+\mathfrak{p}^3$. Therefore $d_2A+\mathfrak{p}^{(3)}=(d_2, d_3)+(bc^2, b^2c)$ and

$$l_A(A/(X, d_2)+\mathfrak{p}^{(3)})=\mathrm{e}_{XA}(A/d_2A+\mathfrak{p}^{(3)})=20$$

so that $A/d_2A+\mathfrak{p}^{(3)}$ is Cohen-Macaulay and $R_s(\mathfrak{p})$ is by (1.1) a Gorenstein ring.

5. Example of non-Cohen-Macaulay symbolic Rees algebras.

Unfortunately symbolic Rees algebras are not necessarily Cohen-Macaulay even for the space monomial curves. We will explore the following

EXAMPLE (5.1). Let $\mathfrak{p}=\mathfrak{p}(7, 9, 10)$ and suppose that ch $k=2$. Then depth $R_s(\mathfrak{p})=3$ and $R_s(\mathfrak{p})$ is not a Cohen-Macaulay ring.

PROOF. The ideal $\mathfrak{p}$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} X & Y & Z \\ Y^2 & Z^2 & X^3 \end{bmatrix}.$$

Let $a=Z^3-X^3Y$, $b=X^4-Y^2Z$ and $c=Y^3-XZ^2$. Then $Xa+Yb+Zc=0$ and $Y^2a+Z^2b+X^3c=0$. Because

$$Zb(-Xa-Yb)=Z^2bc=c(-Y^2a-X^3c),$$

we have $X(c^2X^2-abZ)=Y(b^2Z-acY)$ and so $Xd_2=b^2Z-acY$ and $Yd_2=c^2X^2-abZ$ with $d_2\in\mathfrak{p}^{(2)}$. Hence

$$c(Xd_2+acY)=c\cdot b^2Z=b^2(-Xa-Yb)$$

which implies $X(cd_2+ab^2)=Y(-ac^2-b^3)$ so that $Yd_3=cd_2+ab^2$ with $d_3\in\mathfrak{p}^{(3)}$. Notice that $Yd_2\equiv -abZ \bmod X^2A$ and we get $d_2\equiv YZ^5 \bmod X^2A$. Hence

$$Yd_3\equiv(Y^3-XZ^2)YZ^5+Z^3(-Y^2Z)^2\equiv 2Y^4Z^5-XYZ^7\equiv -XYZ^7 \bmod X^2A,$$

because ch $k=2$ by our assumption. Thus $d_3=Xe_3$ for some $e_3\in\mathfrak{p}^{(3)}$. Obviously $e_3\equiv -Z^7 \bmod XA$ and so we have

$$l_A(A/(c,\ e_3,\ X)) = 1\cdot 3\cdot l_A(A/\mathfrak{p}+XA) = 21\,,$$

whence $R_s(\mathfrak{p})$ is Noetherian.

To see that $R_s(\mathfrak{p})$ is not Cohen-Macaulay, it suffices to check that $A/cA+\mathfrak{p}^{(2)}$ is not a Cohen-Macaulay ring (cf. (1.1)). We put $J=d_2A+\mathfrak{p}^2$. Then because $XA+J=XA+(Y,Z)^6$, we have $l_A(A/XA+J)=21$, while

$$l_A(A/XA+\mathfrak{p}^{(2)}) = e_{XA}(A/\mathfrak{p}^{(2)}) = 7\cdot l_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{p}^2A_\mathfrak{p}) = 21\,.$$

Thus $XA+J=XA+\mathfrak{p}^{(2)}$, whence $\mathfrak{p}^{(2)}=d_2A+\mathfrak{p}^2$ by Nakayama's lemma. Consequently $(c, X)+\mathfrak{p}^{(2)}=(Z^6, YZ^5, Y^2Z^4, Y^3)+XA$ so that $l_A(A/(c, X)+\mathfrak{p}^{(2)})=15$, while

$$e_{XA}(A/(cA+\mathfrak{p}^{(2)}) = 7l_{A_\mathfrak{p}}(A_\mathfrak{p}/cA_\mathfrak{p}+\mathfrak{p}^2A_\mathfrak{p}) = 14\,.$$

Hence $A/cA+\mathfrak{p}^{(2)}$ is not Cohen-Macaulay. As depth $G_s(\mathfrak{p})\geqq 2$ by (3.7)(3), we have depth $R_s(\mathfrak{p})=3$ by (2.1).

REMARK (5.2). This example shows that the assumption $k,\ l\leqq 2$ in Corollary (3.9) is the best possible. In her master thesis M. Morimoto provided numerous examples of prime ideals $\mathfrak{p}=\mathfrak{p}(n_1, n_2, n_3)$ whose symbolic Rees algebras $R_s(\mathfrak{p})$ are Noetherian but non-Cohen-Macaulay. The example (5.1) was chosen among her construction.

Appendix by Mayumi Morimoto.

The following example was posed by Moh and shown to be Noetherian by Huneke [**12**, Example 3.7]. The ideal $\mathfrak{p}$ is minimally generated by four elements so that it is not a monomial curve.

PROPOSITION. *Let* $A=\boldsymbol{C}[[X, Y, Z]]$ *and* $S=\boldsymbol{C}[[t]]$ *be formal power series rings and let* $\mathfrak{p}$ *denote the kernel of the* $\boldsymbol{C}$*-algebra map* $f: A\to S$ *with* $f(X)=t^6$, $f(Y)=t^7+t^{10}$ *and* $f(Z)=t^8$. *Then* $R_s(\mathfrak{p})$ *is a Gorenstein ring.*

PROOF. Let

$$a = 2XZ^3-3X^2YZ-2X^4+Y^3-XYZ\,,$$
$$b = X^3Z-2YZ^2+XY^2-X^2Z\,,$$
$$c = X^2Z^2-2X^3Y+Y^2Z-XZ^2 \qquad \text{and}$$
$$d = X^4-Z^3\,.$$

Then $\mathfrak{p}=(a, b, c, d)$ (cf. [**12**, Example 3.7]). Let

$$e = 4X^8-4X^7-4X^5YZ-4X^4YZ-5X^4Z^3+4X^3Y^3$$
$$+6X^3Z^3-X^2Z^3-6X^2Y^2Z^2+2XY^2Z^2+8XYZ^4-Y^4Z\,,$$

$$f = 10X^5Y^2 - 2X^4Y^2 + 2X^5Z - 4X^6Z + 2X^7Z - 2XY^3Z$$
$$-10X^2Y^3Z + X^2YZ^2 + 14X^3YZ^2 - 15X^4YZ^2 - 2Y^2Z^3$$
$$+10XY^2Z^3 + 2XZ^4 - 4X^2Z^4 + 2X^3Z^4 + Y^5 \quad \text{and}$$
$$g = -8X^6Y + XY^4 - 2X^2Y^2Z + 6X^3Y^2Z + X^3Z^2 - 6X^4Z^2$$
$$+5X^5Z^2 - 4Y^3Z^2 + 4XYZ^3 + 4X^2YZ^3 + 4Z^5 - 4XZ^5 .$$

Then $e, f, g \in \mathfrak{p}^{(2)}$, because

$$Xe = 2ad - bc + 4d^2X ,$$
$$Xf = 2c^2 + ab + 2bdX \quad \text{and}$$
$$Xg = b^2 + 4cd .$$

As $bf^2 + 2ae^2 + geb \equiv 0 \bmod (X)$, we get $Xh = bf^2 + 2ae^2 + geb$ for some $h \in \mathfrak{p}^{(5)}$.

CLAIM. $\mathfrak{p}^{(2)} = (e, f, g) + \mathfrak{p}^2$, $\mathfrak{p}^{(3)} = \mathfrak{p}\mathfrak{p}^{(2)}$, $\mathfrak{p}^{(4)} = (e, f, g)^2 + \mathfrak{p}\mathfrak{p}^{(3)}$ and $\mathfrak{p}^{(5)} = (h) + \mathfrak{p}\mathfrak{p}^{(4)}$.

PROOF OF THE CLAIM. Notice that

$$e \equiv -Y^4Z ,$$
$$f \equiv Y^5 - 2Y^2Z^3 ,$$
$$g \equiv 4(Z^5 - Y^3Z^2) \quad \text{and}$$
$$h \equiv Y^{12} - 68Y^8Z^5 + 136Y^5Z^8 - 64Y^2Z^{11} \bmod (X).$$

Then as $(X) + \mathfrak{p} = (X) + (Y, Z)^3$, we see $(X) + (e, f, g) + \mathfrak{p}^2 = (X) + (Y, Z)^6 + (Y^4Z, Z^5 - Y^3Z^2, Y^5 - 2Y^2Z^3)$ so that $l_A(A/(X) + (e, f, g) + \mathfrak{p}^2) \leqq 18$. Therefore by the inequality

$$l_A(A/(X) + (e, f, g) + \mathfrak{p}^2) \geqq l_A(A/(X) + \mathfrak{p}^{(2)}) = \mathrm{e}_{XA}(A/\mathfrak{p}^{(2)}) = 18$$

we get $(X) + \mathfrak{p}^{(2)} = (X) + (e, f, g) + \mathfrak{p}^2$ whence $\mathfrak{p}^{(2)} = (e, f, g) + \mathfrak{p}^2$ by Nakayama's lemma. Consequently $(X) + \mathfrak{p}\mathfrak{p}^{(2)} = (X) + (Y, Z)^8$ so that $l_A(A/(X) + \mathfrak{p}\mathfrak{p}^{(2)}) = 36 = \mathrm{e}_{XA}(A/\mathfrak{p}^{(3)})$ whence $\mathfrak{p}^{(3)} = \mathfrak{p}\mathfrak{p}^{(2)}$. As $(X) + (e, f, g)^2 + \mathfrak{p}\mathfrak{p}^{(3)} = (X) + (Y, Z)^{11} + (Y^8Z^2, Y^{10}, Y^9Z - 2Y^6Z^4, Y^4Z^6 - Y^7Z^3, 3Y^5Z^5 - 2Y^2Z^8, Z^{10} - 2Y^3Z^7 + Y^6Z^4)$, we have $l_A(A/(X) + (e, f, g)^2 + \mathfrak{p}\mathfrak{p}^{(3)}) \leqq 60 = \mathrm{e}_{XA}(A/\mathfrak{p}^{(4)})$. Hence $\mathfrak{p}^{(4)} = (e, f, g)^2 + \mathfrak{p}\mathfrak{p}^{(3)}$ so that $\mathfrak{p}^{(5)} = (h) + \mathfrak{p}\mathfrak{p}^{(4)}$, because $(X) + (h) + \mathfrak{p}\mathfrak{p}^{(4)} = (X) + (Y, Z)^{13} + (Y^{12})$.

As $l_A(A/(g, h, X)) = 2 \cdot 5 \cdot l_A(A/(X) + \mathfrak{p}) = 60$, we have $R_s(\mathfrak{p})$ to be Noetherian (cf. [**12**, 3.7]). To see $R_s(\mathfrak{p})$ is Gorenstein, by (1.1) it suffices to show that $A/(g) + \mathfrak{p}^{(n)}$ is Cohen-Macaulay for $n=3$, 4 and 5, or equivalently $l_A(A/(X, g) + \mathfrak{p}^{(n)}) \leqq \mathrm{e}_{XA}(A/(g) + \mathfrak{p}^{(n)})$. These inequalities are directly checked, because

$$\mathrm{e}_{XA}(A/(g) + \mathfrak{p}^{(n)}) = 30 \quad (n=3),$$
$$= 42 \quad (n=4),$$
$$= 54 \quad (n=5)$$

and because by the claim we have already known $\mathfrak{p}^{(n)}$ explicitly. Thus $R_s(\mathfrak{p})$ is a Gorenstein ring.

COROLLARY. $R_s(\mathfrak{p})=A[at, bt, ct, dt, et^2, ft^2, gt^2, ht^5]$.

References

[1] N. Bourbaki, Commutative Algebra, Addison-Wesley Publishing Company, 1972.
[2] L. Burch, Codimension and analytic spread, Proc. Camb. Phil. Soc., **72** (1972), 369-373.
[3] R.C. Cowsik, Symbolic powers and the number of defining equations, Algebra and Its Applications, Lecture Notes in Pure and Appl. Math., **91** (1985), 13-14.
[4] S. Eliahou, Symbolic powers of monomial curves, J. Algebra, **117** (1988), 437-456.
[5] S. Goto, M. Herrmann, K. Nishida and O. Villamayor, On the structure of Noetherian symbolic Rees algebras, Manuscripta Math., **67** (1990), 197-225.
[6] S. Goto, K. Nishida and Y. Shimoda, The Gorensteinness of the symbolic blow-ups for certain space monomial curves, in preparation.
[7] S. Goto and Y. Shimoda, On the Rees algebras of Cohen-Macaulay local rings, Lecture Notes in Pure and Appl. Math., **68** (1982), 201-231.
[8] S. Goto and K. Watanabe, On graded rings I, J. Math. Soc. Japan, **30** (1978), 179-213.
[9] J. Herzog, Generators and relations of abelian semigroups and semigroup rings, Manuscripta Math., **3** (1970), 175-193.
[10] J. Herzog and B. Ulrich, Self-linked curve singularities, Nagoya Math. J., **120** (1990), 129-153.
[11] C. Huneke, On the finite generation of symbolic blow-ups, Math. Z., **179** (1982), 465-472.
[12] C. Huneke, Hilbert functions and symbolic powers, Michigan Math. J., **34** (1987), 293-318.
[13] S. Huckaba, Symbolic powers of prime ideals with applications to hypersurface rings, Nagoya Math. J., **113** (1989), 161-172.
[14] S. Huckaba, Analytic spread modulo an element and symbolic Rees algebras, J. Algebra, **128** (1990), 306-320.
[15] S. Ikeda, On the Gorensteinness of Rees algebras over local rings, Nagoya Math. J., **102** (1986), 135-154.
[16] D. Katz and L.J. Ratliff, Jr., On the symbolic Rees ring of a primary ideal, Comm. in Algebra, **14** (1986), 959-970.
[17] M.P. Murthy, A note on factorial rings, Arch. Math., **15** (1964), 418-420.
[18] M. Nagata, On the fourteenth problem of Hilbert, Proc. Internat. Congress Math., 1958, Cambridge Univ. Press, 1960.
[19] D. Rees, On a problem of Zariski, Illinois J. Math., **2** (1958), 145-149.
[20] P. Roberts, A prime ideal in a polynomial ring whose symbolic blow-up is not Noetherian, Proc. Amer. Math. Soc., **94** (1985), 589-592.
[21] P. Schenzel, Examples of Noetherian symbolic blow-up rings, Rev. Roumaine Math. Pures Appl., **33** (1988), 4, 375-383.
[22] A. Simis and N.V. Trung, The divisor class group of ordinary and symbolic blow-ups, Math. Z., **198** (1988), 479-491.
[23] P. Valabrega and G. Valla, Form rings and regular sequences, Nagoya Math. J., **72** (1978), 93-101.

S. GOTO
Department of Mathematics
School of Science and Technology
Meiji University
Higashimita, Tama-ku
Kawasaki-shi
Japan

K. NISHIDA
Department of Mathematics
Faculty of Science
Chiba University
Yayoi-cho, Chiba-shi
Japan

Y. SHIMODA
College of Liberal Arts and Sciences
Kitasato University
Kitasato, Sagamihara-shi
Japan