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# Extremal almost periodic states on $C^*$-algebras

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## 1. Introduction.

Let $(A, G, \alpha)$ be a $C^*$-dynamical system, namely, a triple consisting of a $C^*$-algebra $A$, a locally compact group $G$ and a group homomorphism $\alpha$ from $G$ into the automorphism group of $A$ such that $G\ni t\rightarrow\alpha_t(x)$ is continuous for each $x$ in $A$. Assume that $A$ is unital for a while (it will be irrelevant in Section 2 whether or not a $C^*$-algebra possesses the identity). Then the state space of $A$ is weakly* compact. In decomposition theory of states (cf. [**1**, 4.1-4.4]), we are interested in decomposing a given state as a convex combination of states which are extremal points of some closed convex subset of the state space endowed with the weak* topology. The closed convex subset might be given directly by some physical requirement. In the covariant situation, usually the set of $\alpha$-invariant states is considered as such a closed convex subset. Extremal points in the set of $\alpha$-invariant states are called ergodic states (or $\alpha$-ergodic states), and some of their characterizations are given in [**1**, Theorems 4.3.17 and 4.3.20].

Now assume that $G$ is a locally compact *abelian* group. Recall that a state $\varphi$ of $A$ is called an *almost periodic state* if, for each $x$ in $A$, the function $G\ni t\rightarrow\varphi(\alpha_t(x))$ is the uniform limit of a family of finite linear combinations of characters of $G$. Then we turn our attention to considering the decomposition of a given state into the weak* closure of almost periodic states (cf. [**1**], [**2**]). Here note that every $\alpha$-invariant state is automatically almost periodic. At the first stage in this paper, we shall examine conditions under which an $\alpha$-ergodic state becomes an extremal point in the weak* closure of almost periodic states. When an $\alpha$-ergodic state becomes an extremal point in the weak* closure of almost periodic states, such a state shall be named an *ergodic state of almost periodic type*, together with the explicit definition, in Section 2. We shall consider also the class of states corresponding to centrally ergodic states (see [**1**, §4.3.2] for the definition of a centrally ergodic state), and every state belonging to such a class shall be called a *centrally ergodic state of almost periodic type*, whose explicit definition shall be given later. In the latter half of this

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paper, it is shown that centrally ergodic states of almost periodic type $\varphi$ and $\psi$ are quasi-equivalent if and only if $(\varphi+\psi)/2$ is a centrally ergodic state of almost periodic type.

## 2. Ergodic states of almost periodic type.

Let $(A, G, \alpha)$ be a $C^*$-dynamical system where $G$ is a locally compact abelian group. Let $\varphi$ be an $\alpha$-invariant state of $A$ and $(\pi_\varphi, u^\varphi, H_\varphi, \xi_\varphi)$ be the GNS covariant representation associated with $\varphi$, that is, $\pi_\varphi$ is a representation of $A$ on the Hilbert space $H_\varphi$ with the canonical cyclic vector $\xi_\varphi$ and $u^\varphi$ is a strongly continuous unitary representation of $G$ on $H_\varphi$ defined by

$$u_t^\varphi(\pi_\varphi(x))\xi_\varphi = \pi_\varphi(\alpha_t(x))\xi_\varphi$$

for $x \in A$ and $t \in G$. Note that

$$\pi_\varphi(\alpha_t(x)) = u_t^\varphi(\pi_\varphi(x))u_t^{\varphi*}.$$

Then the spectral decomposition of $u^\varphi$ is given by

$$u_t^\varphi = \int_{\hat{G}} \overline{\langle t, \gamma\rangle} dP_\varphi(\gamma),$$

where $dP_\varphi$ denotes the projection-valued measure on the dual group $\hat{G}$ of $G$. For simplicity, we use the notation

$$p_\varphi(\gamma) = P_\varphi(\{\gamma\}).$$

Then the point spectrum $\sigma(u^\varphi)$ of $u^\varphi$ is defined by

$$\sigma(u^\varphi) = \{\gamma \in \hat{G} \mid p_\varphi(\gamma) \neq 0\},$$

and this definition implies that $\gamma \in \sigma(u^\varphi)$ if and only if there exists a non-zero eigenvector $\eta_\gamma$ in $H_\varphi$ such that

$$u_t^\varphi \eta_\gamma = \overline{\langle t, \gamma\rangle} \eta_\gamma$$

for all $t$ in $G$. Define the projection $p_\varphi$ on $H_\varphi$ by

$$p_\varphi = \sum_{\gamma \in \hat{G}} p_\varphi(\gamma),$$

and we refer to $p_\varphi H_\varphi$ as the subspace of *$u^\varphi$-almost periodic vectors.*

For $\gamma \in \sigma(u^\varphi)$, $p_\varphi(\gamma)$ is the projection from $H_\varphi$ onto the subspace formed by the vectors invariant under the unitary rpresentation $\gamma u^\varphi$ of $G$ defined by $G \ni t \to \langle t, \gamma\rangle u_t^\varphi$. It then follows from the Alaoglu-Birkhoff mean ergodic theorem ([**1**, Proposition 4.3.4] or [3, 7.12.3]) that $p_\varphi(\gamma)$ is strongly approximated by convex combinations of $\gamma u^\varphi$. We therefore see that $p_\varphi(\gamma) \in u_G^{\varphi\prime\prime}$, and hence $p_\varphi \in u_G^{\varphi\prime\prime}$, equivalently $u_G^{\varphi\prime} \subset \{p_\varphi\}'$. We shall often use this fact without comment.

DEFINITION 2.1. An $\alpha$-invariant state $\varphi$ of $A$ is called an *ergodic state of almost periodic type* if

$$\pi_{\varphi}(A)'\cap\{p_{\varphi}\}'=\boldsymbol{C}\cdot 1\,.$$

Note that in the case when $A$ is unital, every ergodic state of almost periodic type is an extremal point in the weak* closure of all almost periodic states. In fact, if $\pi_{\varphi}(A)'\cap\{p_{\varphi}\}'=\boldsymbol{C}\cdot 1$, the maximal orthogonal measure $\mu$ corresponding to $\pi_{\varphi}(A)'\cap\{p_{\varphi}\}'$ is pseudosupported by the extremal points in the weak* closure of all almost periodic states ([**1**, Proposition 4.3.41]). Since the dimension of $\pi_{\varphi}(A)'\cap\{p_{\varphi}\}'$ is one, $\mu$ is the one point measure at $\varphi$ (cf. [**1**, Theorem 4.1.25]), from which it easily follows that $\varphi$ is extremal in the weak* closure of all almost periodic states.

Recall here that an $\alpha$-invariant state $\varphi$ of $A$ is said to be a *G-central state of almost periodic type* if for each $x, y\in A$, $z\in\pi_{\varphi}(A)'$, $\gamma\in\hat{G}$ and $\xi, \eta\in p_{\varphi}H_{\varphi}$, the following is satisfied:

$$\inf|(\pi_{\varphi}([x', y])z\xi\,|\,\eta)|=0\,,$$

where the infimum is taken over all $x'$ in the convex hull of $\{\langle t, \gamma\rangle\alpha_t(x)\,|\,t\in G\}$ (see [**2**, 2.2]). The notion of a $G$-central state of almost periodic type was introduced in [**2**] in order to consider the subcentral decomposition of an $\alpha$-invariant state into almost periodic states. An $\alpha$-invariant state $\varphi$ is said to be $G_{\Gamma}$*-abelian* when $z$ is chosen as 1 in the above definition (see [**1**, Definition 4.3.29]).

THEOREM 2.2. *Let $(A, G, \alpha)$ be a $C^*$-dynamical system where $G$ is a locally compact abelian group. Let $\varphi$ be an $\alpha$-invariant state on $A$. Consider the following conditions:*

(1) *$p_{\varphi}$ has rank one.*

(2) *$\varphi$ is an ergodic state of almost periodic type.*

(3) $\pi_{\varphi}(A)''\cap\pi_{\varphi}(A)'\cap\{p_{\varphi}\}'=\boldsymbol{C}\cdot 1$.

*Then it follows that* (1)$\Rightarrow$(2)$\Rightarrow$(3). *If $\varphi$ is a $G_{\Gamma}$-abelian state, then* (2)$\Rightarrow$(1). *If $\varphi$ is a G-central state of almost periodic type, then* (3)$\Rightarrow$(1).

PROOF. (1)$\Rightarrow$(2). Since $p_{\varphi}=p_{\varphi}(0)$ and $p_{\varphi}(0)\xi_{\varphi}=\xi_{\varphi}$, it follows from cyclicity of $\xi_{\varphi}$ for $\pi_{\varphi}(A)''$ that $\pi_{\varphi}(A)\cup\{p_{\varphi}\}$ is irreducible.

(2)$\Rightarrow$(3). This is obvious.

We first assume that $\varphi$ is $G_{\Gamma}$-abelian. We now show that the implication (2)$\Rightarrow$(1). Since $\pi_{\varphi}(A)'\cap u_G^{\varphi}{}'\subset\pi_{\varphi}(A)'\cap\{p_{\varphi}\}'=\boldsymbol{C}\cdot 1$, $\varphi$ is $\alpha$-ergodic. Thus it follows from [**1**, Theorem 4.3.31] that for $\gamma\in\sigma(u^{\varphi})$, there exists a unitary element $v_{\gamma}\in\pi_{\varphi}(A)'$ such that

$$u_t^{\varphi}v_{\gamma}u_t^{\varphi *}=\overline{\langle t, \gamma\rangle}v_{\gamma}$$

for all $t \in G$ and that

$$\{v_\gamma \in \pi_\varphi(A)' \mid \gamma \in \sigma(u^\varphi)\}'' = \pi_\varphi(A)' \cap \{p_\varphi\}' = \boldsymbol{C} \cdot 1 .$$

Hence we have

$$v_\gamma = u_t^\varphi v_\gamma u_t^{\varphi *} = \overline{\langle t, \gamma \rangle} v_\gamma$$

for all $t$ in $G$, which means that $\gamma = 0$. We thus conclude that $p_\varphi = p_\varphi(0)$. Since every $G_\Gamma$-abelian state is automatically $G$-abelian (see [**1**, Definition 4.3.6] for the definition of a $G$-abelian state), it follows from [**1**, Theorem 4.3.17] that ergodicity of $\varphi$ implies that $p_\varphi(0)$ has rank one.

Next we assume that $\varphi$ is a $G$-central state of almost periodic type and show the implication (3)⇒(1). Since every $G$-central state of almost periodic type is $G_\Gamma$-abelian, we have only to prove the implication (3)⇒(2). Since $\pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap u_G^{\varphi\prime} \subset \pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}' = \boldsymbol{C} \cdot 1$, $\varphi$ is centrally ergodic. Since $\varphi$ is automatically a $G$-central state (see [**1**, Definition 4.3.6] for the definition of a $G$-central state), central ergodicity of $\varphi$ implies that $\varphi$ is $\alpha$-ergodic (see [**1**, Theorem 4.3.14 (3)]). It therefore follows from [**2**, Theorem 2.4] that

$$\pi_\varphi(A)' \cap \{p_\varphi\}' = \pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}' = \boldsymbol{C} \cdot 1 .$$

Thus we complete the proof. Q.E.D.

Let $m$ be an invariant mean on $G$. For an $\alpha$-invariant state $\varphi$ on $A$ and each $\gamma \in \hat{G}$, we define a linear map $Q_\gamma^\varphi$ from $\pi_\varphi(A)''$ onto the closed subspace

$$\{x \in \pi_\varphi(A)'' \mid \bar{\alpha}_t(x) \equiv u_t^\varphi x u_t^{\varphi *} = \overline{\langle t, \gamma \rangle} x \text{ for all } t \in G\}$$

by

$$\langle Q_\gamma^\varphi(x), \phi \rangle = m(\langle (\gamma\bar{\alpha})(x), \phi \rangle)$$

for $x \in \pi_\varphi(A)''$ and $\phi \in \pi_\varphi(A)''_*$, where $\gamma\bar{\alpha}$ is defined by

$$(\gamma\bar{\alpha})_t(x) = \langle t, \gamma \rangle \bar{\alpha}_t(x)$$

for all $x \in \pi_\varphi(A)''$. Here it is significant to note that $Q_\gamma^\varphi$ maps the center of $\pi_\varphi(A)''$ into itself. This fact immediately follows from the definition of $Q_\gamma^\varphi$ and will be used in the proof of Lemma 2.4.

THEOREM 2.3. *Let $(A, G, \alpha)$ be a $C^*$-dynamical system where $G$ is a locally compact abelian group. Let $\varphi$ be an $\alpha$-invariant state on $A$. Consider the following conditions:*

(1) $\pi_\varphi(A)'' \cap \{p_\varphi\}' = \boldsymbol{C} \cdot 1$.

(2) *$p_\varphi$ has rank one.*

(3) *$\varphi$ is an ergodic state of almost periodic type.*

*Then it follows that* (1)⇒(2)⇒(3). *Moreover,* (1) *implies that $\xi_\varphi$ is separating for $\pi_\varphi(A)''$. Conversely, if $\xi_\varphi$ is separating for $\pi_\varphi(A)''$, then conditions* (1)-(3)

*are equivalent.*

PROOF. Since $\pi_\varphi(A)''\cap u_G^{\varphi\prime}\subset\pi_\varphi(A)''\cap\{p_\varphi\}'$, it follows from [**1**, Theorem 4.3.20] that condition (1) implies that $\xi_\varphi$ is separating for $\pi_\varphi(A)''$ and that $p_\varphi(0)$ has rank one.

(1)⇒(2). We have only to show that $p_\varphi=p_\varphi(0)$. Note that $Q_\gamma^\varphi(\pi_\varphi(A)'')\subset\pi_\varphi(A)''\cap\{p_\varphi\}'=\boldsymbol{C}\cdot 1$ (see [**2**, Lemma 3.1]). This means that $Q_\gamma^\varphi(\pi_\varphi(A)'')=\{0\}$ for any $\gamma\neq 0$. Since $\xi_\varphi$ is separating for $\pi_\varphi(A)''$, the point spectrum of the automorphism group $\bar{\alpha}$ of $\pi_\varphi(A)''$ coincides with $\sigma(u^\varphi)$ (see [**1**, Theorem 4.3.27]). Hence we see that $\sigma(u^\varphi)=\{0\}$. This means that $p_\varphi=p_\varphi(0)$.

(2)⇒(3). This follows from Theorem 2.2.

We assume that $\xi_\varphi$ is separating for $\pi_\varphi(A)''$ and show the implication (3)⇒(1). Let $S$ be the closed antilinear operator on $H_\varphi$ defined by

$$Sx\xi_\varphi = x^*\xi_\varphi$$

for $x\in\pi_\varphi(A)''$. We then have

$$Su_t^\varphi x\xi_\varphi = Su_t^\varphi xu_t^{\varphi *}\xi_\varphi = u_t^\varphi x^*u_t^{\varphi *}\xi_\varphi = u_t^\varphi Sx\xi_\varphi\,.$$

Since $\pi_\varphi(A)''\xi_\varphi$ is a core for $S$, we obtain that $Su_t^\varphi=u_t^\varphi S$. Hence the uniqueness of the polar decomposition of $S$ shows that $Ju_t^\varphi=u_t^\varphi J$, which means that $Jp_\varphi(\gamma)=p_\varphi(\gamma)J$ for all $\gamma\in\hat{G}$, i.e., $Jp_\varphi=p_\varphi J$, where $J$ denotes the modular conjugation associated with $\xi_\varphi$ (cf. [**1**, §2.5.2] or [**3**, 8.13]). Since $J\pi_\varphi(A)''J=\pi_\varphi(A)'$, we have

$$\pi_\varphi(A)''\cap\{p_\varphi\}' = J\{\pi_\varphi(A)'\cap\{p_\varphi\}'\}J\,.$$

Since $\pi_\varphi(A)'\cap\{p_\varphi\}'=\boldsymbol{C}\cdot 1$, we obtain condition (1). Q.E.D.

LEMMA 2.4. *Let $(A, G, \alpha)$ be a $C^*$-dynamical system where $G$ is a locally compact abelian group. Let $\varphi$ and $\omega$ be $\alpha$-invariant states on $A$. Assume that $\pi_\varphi$ and $\pi_\omega$ are quasi-equivalent. Then $\pi_\varphi(A)''\cap\pi_\varphi(A)'\cap\{p_\varphi\}'$ is isomorphic to $\pi_\omega(A)''\cap\pi_\omega(A)'\cap\{p_\omega\}'$.*

PROOF. Let $\tau$ be an isomorphism from $\pi_\varphi(A)''$ onto $\pi_\omega(A)''$ such that $\tau(\pi_\varphi(x))=\pi_\omega(x)$ for all $x\in A$. Since

$$\tau(u_t^\varphi\pi_\varphi(x)u_t^{\varphi *}) = \tau(\pi_\varphi(\alpha_t(x))) = \pi_\omega(\alpha_t(x)) = u_t^\omega\pi_\omega(x)u_t^{\omega *} = u_t^\omega\tau(\pi_\varphi(x))u_t^{\omega *}$$

for all $x\in A$ and since $\tau$ is $\sigma$-weakly continuous, we see that

$$\tau(u_t^\varphi xu_t^{\varphi *}) = u_t^\omega\tau(x)u_t^{\omega *}$$

for all $x\in\pi_\varphi(A)''$, i.e., $\tau$ is $G$-covariant. Take any element $x$ from $\pi_\varphi(A)''\cap\pi_\varphi(A)'\cap\{p_\varphi\}'$. We then have

$$(p_\varphi(\gamma)x\xi_\varphi\,|\,\eta) = m((\gamma u^\varphi x\xi_\varphi\,|\,\eta)) = m((\gamma u^\varphi xu^{\varphi *}\xi_\varphi\,|\,\eta)) = (Q_\gamma^\varphi(x)\xi_\varphi\,|\,\eta)$$

for all $\eta \in H_\varphi$ and some invariant mean $m$ on $G$. We therefore have

$$p_\varphi(\gamma)x\xi_\varphi = Q_\gamma^\varphi(x)\xi_\varphi .$$

We then assert that

$$x = \sum_\gamma Q_\gamma^\varphi(x) .$$

For $y \in \pi_\varphi(A)''$, in fact, we have

$$\begin{aligned} xy\xi_\varphi &= yx\xi_\varphi = yxp_\varphi\xi_\varphi = yp_\varphi x\xi_\varphi \\ &= y\sum_\gamma p_\varphi(\gamma)x\xi_\varphi = y\sum_\gamma Q_\gamma^\varphi(x)\xi_\varphi = \sum_\gamma Q_\gamma^\varphi(x)y\xi_\varphi . \end{aligned}$$

Since

$$u_t^\omega \tau(Q_\gamma^\varphi(x))u_t^{\omega *} = \tau(u_t^\varphi Q_\gamma^\varphi(x)u_t^{\varphi *}) = \tau(\overline{\langle t, \gamma\rangle} Q_\gamma^\varphi(x)) = \overline{\langle t, \gamma\rangle}\tau(Q_\gamma^\varphi(x)) ,$$

we see that $\tau(Q_\gamma^\varphi(x)) \in Q_\gamma^\varphi(\pi_\omega(A)'') \subset \{p_\omega\}'$ and thus $\tau(x) = \tau(\sum_\gamma Q_\gamma^\varphi(x)) \in \{p_\omega\}'$. Since it is clear that $\tau(x) \in \pi_\omega(A)'' \cap \pi_\omega(A)'$, we conclude that $\tau(x) \in \pi_\omega(A)'' \cap \pi_\omega(A)' \cap \{p_\omega\}'$, from which it follows that $\tau(\pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}') \subset \pi_\omega(A)'' \cap \pi_\omega(A)' \cap \{p_\omega\}'$. Since the above discussions are valid also for $\tau^{-1}$, we obtain the reverse inclusion. Q. E. D.

DEFINITION 2.5. An $\alpha$-invariant state $\varphi$ of $A$ is called a *centrally ergodic state of almost periodic type* if

$$\pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}' = \boldsymbol{C}\cdot 1 .$$

Note that every centrally ergodic state of almost periodic type is always a centrally ergodic state. It is well known that centrally ergodic states $\varphi$ and $\psi$ are quasi-equivalent if and only if $(\varphi+\psi)/2$ is a centrally ergodic state [**1**, Theorem 4.3.19].

THEOREM 2.6. *Let $(A, G, \alpha)$ be a $C^*$-dynamical system where $G$ is a locally compact abelian group. Let $\varphi$ and $\psi$ be centrally ergodic states of almost periodic type. Then $\varphi$ and $\psi$ are quasi-equivalent if and only if $(\varphi+\psi)/2$ is a centrally ergodic state of almost periodic type.*

PROOF. If $(\varphi+\psi)/2$ is a centrally ergodic state of almost periodic type, it is a centrally ergodic state. Hence $\varphi$ and $\psi$ are quasi-equivalent. Thus we have only to show the necessary condition.

Assume that $\varphi$ and $\psi$ are quasi-equivalent. Let $\rho$ be an isomorphism from $\pi_\varphi(A)''$ onto $\pi_\psi(A)''$ such that $\rho(\pi_\varphi(x)) = \pi_\psi(x)$ for all $x \in A$. Then $\rho$ is $G$-covariant from the proof of Lemma 2.4. Define

$$H = H_\varphi \oplus H_\psi, \qquad \pi = \pi_\varphi \oplus \pi_\psi, \qquad u = u^\varphi \oplus u^\psi .$$

Then we denote by $E$ the projection from $H$ onto the closed subspace

$[\pi(A)(\xi_{\varphi}\oplus\xi_{\psi})]$.

Put

$$\omega=(\varphi+\psi)/2.$$

Then the GNS representation $(\pi_{\omega}, u^{\omega}, H_{\omega})$ associated with $\omega$ is identified with the subrepresentation of $(\pi, u, H)$ determined by the projection $E$ in $\pi(A)'$. Any element in $\pi(A)''$ is a $\sigma$-weak limit of elements of $\pi_{\varphi}(A)\oplus\pi_{\psi}(A)$. Since $\rho$ is $\sigma$-weakly continuous, the map from $\pi_{\varphi}(A)''$ onto $\pi(A)''$ defined by

$$\pi_{\varphi}(A)''\ni x\longrightarrow x\oplus\rho(x)\in\pi(A)''$$

is an isomorphism. Define an homomorphism $\tau$ from $\pi_{\varphi}(A)''$ onto $\pi_{\omega}(A)''$ $(=\pi(A)''E)$ by

$$\pi_{\varphi}(A)''\ni x\longrightarrow(x\oplus\rho(x))E\in\pi_{\omega}(A)''.$$

We now assert that $\tau$ is an isomorphism. Since $E\in u_{G}'$ and $\rho$ is $G$-covariant, we have

$$\begin{aligned}\tau(u_t^{\varphi}xu_t^{\varphi*})&=((u_t^{\varphi}xu_t^{\varphi*})\oplus\rho(u_t^{\varphi}xu_t^{\varphi*}))E=(u_t^{\varphi}xu_t^{\varphi*}\oplus u_t^{\psi}\rho(x)u_t^{\psi*})E\\&=(u_t(x\oplus\rho(x))u_t{}^*)E=u_t((x\oplus\rho(x))E)u_t{}^*\\&=u_t\tau(x)u_t{}^*.\end{aligned}$$

Thus $\tau$ is $G$-covariant. Hence the central projection in $\pi_{\varphi}(A)''$ corresponding to the kernel of $\tau$ is $G$-invariant, i.e., the projection belongs to $\pi_{\varphi}(A)''\cap\pi_{\varphi}(A)'\cap u_G^{\varphi}{}'$. Since $\pi_{\varphi}(A)''\cap\pi_{\varphi}(A)'\cap u_G^{\varphi}{}'\subset\pi_{\varphi}(A)''\cap\pi_{\varphi}(A)'\cap\{p_{\varphi}\}'=\boldsymbol{C}\cdot 1$, such a projection is exactly zero. Thus $\tau$ is injective.

Since

$$\tau(\pi_{\varphi}(x))=(\pi_{\varphi}(x)\oplus\rho(\pi_{\varphi}(x)))E=(\pi_{\varphi}(x)\oplus\pi_{\psi}(x))E=\pi(x)E=\pi_{\omega}(x)$$

for all $x\in A$, $\pi_{\varphi}$ and $\pi_{\omega}$ are quasi-equivalent. It therefore follows from Lemma 2.4 that

$$\boldsymbol{C}\cdot 1=\pi_{\varphi}(A)''\cap\pi_{\varphi}(A)'\cap\{p_{\varphi}\}'\cong\pi_{\omega}(A)''\cap\pi_{\omega}(A)'\cap\{p_{\omega}\}'.$$

Therefore $\omega$ is a centrally ergodic state of almost periodic type. Q.E.D.

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