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Complete metrics of negative Ricci curvature

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Gao and Yau have constructed metrics of negative Ricci curvature on every compact 3-manifold ([1], [2], [3]). They however used techniques peculiar to 3-manifolds and it is hard to see how their method is applicable to general higher dimensional manifolds. In this paper we use simple triangulation argument to construct metrics of negative Ricci curvature on the complement of a point, which will be a partial evidence for affirmative answer to the question whether every manifold with dimension $\geqq 3$ can admit a metric with negative Ricci curvature (Problem 24 of [4]).

THEOREM. *For any connected closed manifold M of dimension $\geqq 2$ and a point p of M, $M \setminus \{p\}$ admits a complete metric of negative Ricci curvature.*

Note that the conclusion is false if Ricci curvature is replaced by sectional curvature. For example, take $M=\boldsymbol{R}P^n$, $n \geqq 3$.

§ 1. Preliminaries

LEMMA. *Let g and $\bar{g}$ be metrics on an n-manifold which are conformaly related as $\bar{g}=e^{-2u}g$ for some smooth function u. Then,*

$$\operatorname{Ric}(\bar{g}) \leqq (n-2)\nabla^2 u+(\Delta u)g+\operatorname{Ric}(g), \tag{1}$$

where Hessian etc. in the right side are taken with respect to g. Assume further that $n \geqq 2$, $u=u(t)$ for some other function t and that $\ddot{u}=(d/dt)^2 u \leqq 0$. Then,

$$\operatorname{Ric}(\bar{g}) \leqq \ddot{u}\,|dt|^2 g+\dot{u}((n-2)\nabla^2 t+(\Delta t)g)+\operatorname{Ric}(g). \tag{2}$$

PROOF. Both inequalities follow immediately from the formula; $\operatorname{Ric}(\bar{g})=(n-2)\nabla^2 u+(\Delta u)g+(n-2)(du\otimes du-|du|^2 g)+\operatorname{Ric}(g)$.

PROPOSITION 1. *Let D be a d-dimensional disk in $\boldsymbol{R}^n$ and g a metric of $\boldsymbol{R}^n$. Suppose $n>d$, $n \geqq 2$ and $\operatorname{Ric}(g)<0$ in a neighborhood of ∂D. Then, there exists another metric $\bar{g}$ such that $\bar{g}=g$ near ∂D and $\operatorname{Ric}(\bar{g})<0$ in a neighborhood of D.*

PROOF. Put $D(r)=\{(x, 0)\in \boldsymbol{R}^d \times \boldsymbol{R}^{n-d}\,;\,|x|<r\} \subset \boldsymbol{R}^n$, where $|x|=(\sum_{j=1}^d (x_j)^2)^{1/2}$. We may assume $D=D(3)$ and $\operatorname{Ric}(g)<0$ on $D \setminus D(1)$.

Define $v\in C^{\infty}(\boldsymbol{R}^{n})=C^{\infty}(\boldsymbol{R}^{d}\times\boldsymbol{R}^{n-d})$ as $v(x, y)=|y|^{2}=\sum_{j=1}^{n-d}(y_{j})^{2}$. Since $dv=0$ on D, $\nabla^{2}v|_{D}$ is independent of the choice of the metric. The Hessian of v with respect to the Euclidean metric is $\geqq$ and $\neq 0$. Therefore, $\nabla^{2}v\geqq 0$ on D, and $\Delta v\geqq a>0$ on D for some constant a.

Let $w\in C^{\infty}(\boldsymbol{R}^{n})$ be a nonnegative function such that $w(x, y)=0$ if $|x|\geqq 2$, and $w(x, y)=b$ if $|x|\leqq 1$, where b is a constant such that $\mathrm{Ric}(g)<abg$ on D.

Now put $u(x, y)=-v(x, y)w(x, y)$ and $\bar{g}=e^{-2u}g$. Note that $v=0$ and $dv=0$ on D, and we have from (1) of Lemma

$$\mathrm{Ric}(\bar{g})\leqq-(n-2)w\nabla^{2}v-(w\Delta v)g+\mathrm{Ric}(g)\leqq-awg+\mathrm{Ric}(g)\qquad\text{on } D.$$

Therefore, $\mathrm{Ric}(\bar{g})<0$ on $D\setminus D(1)$ since $\mathrm{Ric}(g)<0$ on $D\setminus D(1)$. Also $\mathrm{Ric}(\bar{g})<0$ on $D(1)$ since $w=b$ on $D(1)$. Thus $\bar{g}$ has negative Ricci curvature in a neightborhood of D. Clearly $\bar{g}=g$ in a neighborhood of $D\setminus D(2)$.

REMARK. If the proposition is true in the case when $n=d\geqq 3$, our theorem implies the existence of negatively Ricci curved metrics on any compact manifold of dimension $\geqq 3$.

PROPOSITION 2. *Let N be a compact manifold with boundary ∂N and g a metric of N with* $\mathrm{Ric}(g)<0$. *Then there is a $u\in C^{\infty}(N\setminus\partial N)$ such that $e^{-2u}g$ is a complete metric of $N\setminus\partial N$ with negative Ricci curvature.*

PROOF. Let $\partial N\times[0, 1)\subset N$ be a collar neighborhood of ∂N, and $t:\partial N\times[0, 1)\to[0, 1)$ be the projection. Let $a>0$, $b>0$ and $0<\varepsilon<1/2$ be constants such that $a<|dt|^{2}$, $(n-2)\nabla^{2}t+(\Delta t)g<bg$ on $\partial N\times[1/2, 1]\subset N$ and $\varepsilon<a/b$.

Define $u\in C^{\infty}(N\setminus\partial N)$ as

$$u(x)=\begin{cases}-e^{2/\varepsilon}\displaystyle\int_{t(x)}^{\varepsilon}\frac{1}{s}e^{1/(s-\varepsilon)}ds & \text{if } x\in\partial N\times(0, \varepsilon)\subset N\\ 0 & \text{otherwise.}\end{cases}$$

It is easy to see that u is smooth and $e^{-u(x)}>\varepsilon/2t(x)$ if $t(x)<\varepsilon/2$, which implies that $e^{-2u}g$ is a complete metric of $N\setminus\partial N$. On the other hand, $\dot{u}=du/dt>0$ and $\ddot{u}<0$ on $\partial N\times(0, \varepsilon)$. Hence, it follows from (2) of Lemma that

$$\mathrm{Ric}(e^{-2u}g)<(a\ddot{u}+b\dot{u})g<\frac{1}{t}e^{2/\varepsilon}e^{1/(t-\varepsilon)}\Big(-\frac{a}{t}+b\Big)g<0\qquad\text{on } \partial N\times(0, \varepsilon).$$

Therefore $\mathrm{Ric}(e^{-2u}g)<0$ on $N\setminus\partial N$ because $\mathrm{Ric}(e^{-2u}g)=\mathrm{Ric}(g)<0$ on $N\setminus\partial N\times[0, \varepsilon)$.

§2. Proof of Theorem

We fix a triangulation of M and denote its d-skeleton by M_{d}; $M_{0}\subset M_{1}\subset\cdots\subset M_{n}=M$, $n\geqq 2$. Obviously we have a metric g_{0} of M which has negative curvature in a neighborhood of M_{0}. Suppose that $d<n$ and g_{d-1} is a metric

of M with $\mathrm{Ric}(g_{d-1})<0$ on M_{d-1}. We apply Proposition 1 to each d-dimensional simplex to get a metric g_d of M with $\mathrm{Ric}(g_d)<0$ on M_d. Thus, by induction, we get a metric $g=g_{n-1}$ whose Ricci curvature is negative in a neighborhood U of M_{n-1}. We may assume $U=M\setminus$disjoint open n-balls. Removing further a neighborhood of curves each of which connects one of the removed balls with every other ball, we have $N\subset U$ such that $N=M\setminus$an open n-ball. Applying Proposition 2 to $(N, g|_N)$, we have a complete metric of $N\setminus\partial N$ with negative Ricci curvature. Clearly, $N\setminus\partial N$ is diffeomorphic to $M\setminus$a point, and the proof is completed.

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