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Newton polygons and formal Gevrey indices in the Cauchy-Goursat-Fuchs type equations

Dedicated to Professor Mutsuhide Matsumura on his 60th birthday

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Introduction.

In a recent paper [14], Yonemura has studied divergent formal power series solutions of the Cauchy problem to non Kowalevskian equation, and characterized its formal Gevrey index by a Newton polygon associated with the operator. It is a version to partial differential equations of results in ordinary differential equations studied by Ramis [12], where an index theory of ordinary differential operators in a category of formal Gevrey functions was studied.

In this paper, we shall extend and give refinements of Yonemura's result to integrodifferential operators, which we call of Cauchy-Goursat-Fuchs type. Precisely, we shall study a unique solvability of integrodifferential equations in a category of convergent power series or formal power series with formal Gevrey index. It should be mentioned that our main interest is in the divergent formal solutions, but the results obtained in the category of holomorphic functions are new.

It is well known that the Cauchy-Kowalevski theorem does not hold for non Kowalevskian equations. Precisely, the formal power series solution of the Cauchy problem to non Kowalevskian equation does not converge in general even if the Cauchy data and the right hand side of the equation converge. This was the main subject of the inverse problem of Cauchy-Kowalevski theorem (c.f. Miyake [8] and Mizohata [10]).

Nevertheless, we shall study the divergent formal power series solutions. To make clear the motivation of the study, we shall give two examples below.

EXAMPLE 0.1. Let us consider the Cauchy problem,

$$D_t u = t^{\sigma} D_x{}^m u, \qquad u(x, 0) = \varphi(x) \equiv \frac{1}{1-x}, \tag{0.1}$$

where $(x, t) \in \boldsymbol{C}^2$, $(D_x, D_t) = (\partial/\partial x, \partial/\partial t)$ and σ $(\geqq 0)$ and m $(\geqq 2)$ are integers. Then this Cauchy problem does not have a holomorphic solution in any neighbourhood of the origin. Indeed, (0.1) has a unique formal solution,

$$u=\sum_{k=0}^{\infty} D_x{}^{km}\varphi(x)\frac{t^{k(\sigma+1)}}{k!(\sigma+1)^k} \underset{\text{put}}{=} \sum_j u_j(x)\frac{t^j}{j!}.$$

Since $u_{k(\sigma+1)}(0)=(km)!\{k(\sigma+1)\}!/k!(\sigma+1)^k$, it holds that

$$u_{k(\sigma+1)}(0)\sim C\Big(\frac{m}{\sigma+1}\Big)^{km} k^{-(m-1)/2(\sigma+1)}\{k(\sigma+1)\}!^{(\sigma+m)/(\sigma+1)},$$

for some positive constant C from Stirling's formula. The exponent $(\sigma+m)/(\sigma+1)$ (>1) indicates the rate of divergence of the formal solution, and we call it the formal Gevrey index of formal solution of the Cauchy problem (0.1).

Such an observation to construct a Cauchy data for which the formal solution diverges was studied by Miyake [8] and Kitagawa and Sadamatsu [7].

Next example is a special case of the characteristic Cauchy problems studied by Hasegawa [3].

EXAMPLE 0.2. We consider the Cauchy problem,

$$(0.2)\qquad \begin{aligned} &\{ta(t)D_t{}^2+tD_xD_t-D_t+D_x{}^2\}u=0,\\ &u(x,0)=\varphi(x). \end{aligned}$$

Here, $a(t)$ and $\varphi(x)$ are holomorphic at the origin. Hasegawa proved the following results:

(i) If $a(0)\neq 0$ and $la(0)\neq 1$ for any $l=1, 2, \cdots$, then (0.2) has a unique holomorphic solution in a neighbourhood of the origin.

(ii) If $a(0)=0$, then there is a Cauchy data such that the formal solution of (0.2) diverges in any neighbourhood of the origin. She actually constructed such Cauchy data $\varphi(x)$ that the formal Gevrey index of formal solution is at least 2.

The case (i) is a typical case of Fuchs type equation, and many and various studies have been done after the work of Hasegawa [3, 4]. However, the case (ii) has been analized no more, because such a problem has no holomorphic solutions.

We shall make clear the meaning of formal power series solutions to integrodifferential equations including the above examples.

We explain the reason of studying the unique solvability of integrodifferential equations instead of studying the unique solvability of the Cauchy-Goursat problems for partial differential equations.

Let $x=(x_1, \cdots, x_p)\in \boldsymbol{C}^p$ and $t=(t_1, \cdots, t_q)\in \boldsymbol{C}^q$ be complex variables, and $D_x=(D_{x_1}, \cdots, D_{x_p})$ and $D_t=(D_{t_1}, \cdots, D_{t_q})$ denote the usual symbols of differentiations. Let

$$L(x, t; D_x, D_t) = \sum_{\alpha, j} a_{\alpha j}(x, t) D_x{}^{\alpha} D_t{}^{j} \qquad ((\alpha, j) \in \boldsymbol{N}^p \times \boldsymbol{N}^q)$$

be a partial differential operator with holomorphic coefficients in a neighbourhood of the origin of $\boldsymbol{C}_{x,t}^{p+q}$, where $\boldsymbol{N}=\{0, 1, 2, \cdots\}$. Then the Cauchy-Goursat problem for the operator L is formulated as follows:

$$\text{(0.3)} \qquad \begin{aligned} &L(x, t; D_x, D_t) u(x, t) = f(x, t), \\ &u(x, t) - w(x, t) = O(x^{\beta} t^{l}) \qquad \text{at } (x, t) = (0, 0). \end{aligned}$$

Here $(\beta, l) \in \boldsymbol{N}^p \times \boldsymbol{N}^q$ and $w(x, t)$ is the Cauchy-Goursat data.

Now we change the unknown function u to U by

$$u(x, t) = D_x^{-\beta} D_t^{-l} U(x, t) + w(x, t).$$

Then the unique solvability of the problem (0.3) is equivalent to the unique solvability of the following integrodifferential equation:

$$\text{(0.4)} \qquad \sum_{\alpha, j} a_{\alpha j}(x, t) D_x^{\alpha-\beta} D_t^{j-l} U(x, t) = F(x, t).$$

Here $D_x^{-\beta} U = D_{x_1}^{-\beta_1} \cdots D_{x_p}^{-\beta_p} U$ $(\beta = (\beta_1, \cdots, \beta_p))$ is defined by

$$\text{(0.5)} \qquad D_{x_j}^{-1} U(x) = \int_0^{x_j} U(x_1, \cdots, \xi_j, \cdots, x_p) d\xi_j.$$

It is the same for $D_t^{-l} U$. Note that $D_x{}^{\alpha} D_x^{-\beta} = D_x^{\alpha-\beta}$ holds for any $\alpha, \beta \in \boldsymbol{N}^p$.

Such a change of problem enables us to make a systematic study of many problems in partial differential equations such as the Cauchy problems, Goursat problems and the Fuchs type equations.

In Section 1, we shall state our results after giving some definitions of function spaces and integrodifferential operators considered in this paper. Sections 2 and 3 are devoted to investigate properties of function spaces of formal power series with formal Gevrey index and integrodifferential operators acting on such spaces. Fundamental ideas developed in these sections are those in Miyake [**9**], and they enable us to determine the existence domain of solutions in a precise form. Our theorems are, then, proved in Sections 4 and 5 in much precise forms.

At the end of this section, the author would like to thank the referee for his careful reading of the manuscript and useful comments.

1. Statement of Results.

1.1. List of notations. Let $x=(x_1, \cdots, x_p) \in \boldsymbol{C}^p$ and $t=(t_1, \cdots, t_q) \in \boldsymbol{C}^q$ be complex variables. For $x \in \boldsymbol{C}^p$, we set

$$|x| = x_1 + \cdots + x_p \quad \text{and} \quad \|x\| = |x_1| + \cdots + |x_p|.$$

We denote by $\boldsymbol{C}\{x\}$ the set of convergent power series in x. For a domain $\Omega\subset\boldsymbol{C}^p$, $\mathcal{O}(\Omega)$ denotes the set of holomorphic functions in Ω, and $\mathcal{O}(\bar{\Omega}):=\mathcal{O}(\Omega)\cap C(\bar{\Omega})$, that is, holomorphic in Ω and continuous up to the boundary of Ω. For $X>0$, $\mathcal{O}(\|x\|<X)$ denotes the set of holomorphic functions in a domain $\Omega=\{x\in\boldsymbol{C}^p;\ \|x\|<X\}$, and $\mathcal{O}(\|x\|\leqq X)=\mathcal{O}(\bar{\Omega})$. Similar notations will be used frequently for functions defined in a domain of $\boldsymbol{C}^{p+q}_{x,t}$.

For a ring A, $A[[x]]$ denotes the set of formal power series in x with coefficients in A.

We denote by $\boldsymbol{N}$ and $\boldsymbol{Z}$ the set of non negative integers and the set of integeres, respectively.

For a multi-index $\alpha=(\alpha_1,\cdots,\alpha_p)\in\boldsymbol{Z}^p$, an integrodifferential $D_x{}^\alpha U(x)$ of $U(x)\in\boldsymbol{C}[[x]]$ is defined as follows:

Set $U(x)=\sum_{\beta\in\boldsymbol{N}^p}U_\beta x^\beta/\beta!$. Then,

$$D_x{}^\alpha U(x):=\sum_\beta{}' U_\beta\frac{x^{\beta-\alpha}}{(\beta-\alpha)!}.$$

Here the summation is taken over $\beta\in\boldsymbol{N}^p$ such that $\beta-\alpha\in\boldsymbol{N}^p$. It is the same for $D_t{}^jU(t)$ with $j=(j_1,\cdots,j_q)\in\boldsymbol{Z}^q$.

For $\alpha=(\alpha_1,\cdots,\alpha_p)\in\boldsymbol{Z}^p$, we set $|\alpha|=\sum_{i=1}^p\alpha_i\in\boldsymbol{Z}$. For two multi-indices $\alpha=(\alpha_1,\cdots,\alpha_p)$ and $\beta=(\beta_1,\cdots,\beta_p)$ in $\boldsymbol{Z}^p$, $\alpha\geqq\beta$ (resp. $\alpha>\beta$) means $\alpha_i\geqq\beta_i$ for any $i=1,2,\cdots,p$ (resp. $\alpha\geqq\beta$ and $|\alpha|>|\beta|$).

1.2. Formal Gevrey class G^s. For $U(x,t)\in\boldsymbol{C}[[x,t]]$, we set

$$U(x,t)=\sum_{\beta,l}U_{\beta l}\frac{x^\beta t^l}{\beta!l!},\qquad U_l(x)=\sum_\beta U_{\beta l}\frac{x^\beta}{\beta!}\in\boldsymbol{C}[[x]],\tag{1.1}$$

where $U_{\beta l}\in\boldsymbol{C}$, $\beta\in\boldsymbol{N}^p$ and $l\in\boldsymbol{N}^q$.

Let $s\geqq1$, $X>0$ and $T>0$. Then we define $U(x,t)\in G^s(X,T)$ $(\subset\boldsymbol{C}[[x,t]])$ by

$$\|U\|^{(s)}_{X,T}:=\sup_{\beta,l}|U_{\beta l}|\frac{X^{|\beta|}T^{|l|}}{(|\beta|+s|l|)!}<\infty,\tag{1.2}$$

where $(|\beta|+s|l|)!:=\Gamma(|\beta|+s|l|+1)$. It is obvious that $G^s(X,T)$ becomes a Banach space by the norm $\|\cdot\|^{(s)}_{X,T}$.

It will be shown in Section 2 that

$$G^s(X,T)\subset\mathcal{O}(\|x\|<X)[[t]],\quad G^1(X,T)\subset\mathcal{O}\Big(\frac{\|x\|}{X}+\frac{\|t\|}{T}<1\Big).$$

Moreover, for any Y with $0<Y<X$, there are positive constants R and C satisfying

$$\max_{\|x\|\leqq Y}|U_l(x)|\leqq C\frac{|l|!^s}{R^{|l|}}\qquad\text{for any } l\in\boldsymbol{N}^q.$$

This is the reason why we call $G^s(X,T)$ the formal Gevrey class.

Now we define G^s for $1 \leqq s \leqq \infty$ by

$$(1.3)\qquad \begin{aligned} G^s &= \bigcup_{X,T>0} G^s(X, T), \qquad \text{if } 1 \leqq s < \infty, \\ G^\infty &= \bigcup_{X>0} \mathcal{O}(\|x\|<X)[[t]]. \end{aligned}$$

By the definition, it holds that $G^1 = \boldsymbol{C}\{x, t\}$. We shall study more precisely the formal Gevrey class in Section 2.

1.3. Cauchy-Goursat-Fuchs type operator. Let $t \cdot D_t = (t_1 D_{t_1}, t_2 D_{t_2}, \cdots, t_q D_{t_q})$, and set

$$(t \cdot D_t)^j = (t_1 D_{t_1})^{j_1} (t_2 D_{t_2})^{j_2} \cdots (t_q D_{t_q})^{j_q}$$

for $j = (j_1, \cdots, j_q) \in \boldsymbol{N}^q$.

An integrodifferential operator $L_m(x, t; D_x, D_t)$ written in the following form is said to be of Cauchy-Goursat-Fuchs type with weight m $(\in \boldsymbol{N})$:

$$(1.4)\qquad L_m = P_m(t \cdot D_t) + Q(x, t; D_x, D_t) + R(x, t; D_x, D_t).$$

Here, P_m is an operator of Euler type of order m, that is,

$$P_m(t \cdot D_t) = \begin{cases} \sum_{|j| \leq m} a_j (t \cdot D_t)^j & \text{if } m \geqq 1, \\ 1 & \text{if } m = 0, \end{cases}$$

where $j \in \boldsymbol{N}^q$ and $a_j \in \boldsymbol{C}$.

$$Q = \sum_{\substack{|\sigma| = |j| \\ |\alpha| \leq \min\{m - |j|, 0\}}} b_{\alpha j}^{(\sigma)}(x) t^\sigma D_x{}^\alpha D_t{}^j,$$

where $(\sigma, \alpha, j) \in \boldsymbol{N}^q \times \boldsymbol{Z}^p \times \boldsymbol{Z}^q$ and $b_{\alpha j}^{(\sigma)}(x) \in \boldsymbol{C}[[x]]$.

$$R = \sum_{\alpha, j} \sum_{|\sigma| > |j|} c_{\alpha j}^{(\sigma)}(x, t) t^\sigma D_x{}^\alpha D_t{}^j \qquad (c_{\alpha j}^{(\sigma)}(x, 0) \not\equiv 0 \ \text{ if } c_{\alpha j}^{(\sigma)} \not\equiv 0),$$

where $(\sigma, \alpha, j) \in \boldsymbol{N}^q \times \boldsymbol{Z}^p \times \boldsymbol{Z}^q$ and $c_{\alpha j}^{(\sigma)}(x, t) \in \boldsymbol{C}[[x, t]]$.

Our restriction on Q that $|\alpha| \leqq \min\{m - |j|, 0\}$ is often made in the study of Fuchs type equations (cf. Baouendi and Goulaouic [1]). Therefore, we exclude an operator $L_1 = tD_t - D_x$, but treat an operator $L_0 = I - tD_t D_x^{-1}$, where $(x, t) \in \boldsymbol{C}^2$. Note that these operators are equivalent, since $L_0 = L_1 D_x^{-1}$. The difference of these operators will be made clear by Remark 1.4.

We define the Gevrey index s_0 of the operator L_m by

$$(1.5)\qquad s_0 = \max\left\{1, \max\left\{\frac{|\sigma| + |\alpha| - m}{|\sigma| - |j|}; c_{\alpha j}^{(\sigma)}(x, 0) \not\equiv 0\right\}\right\}.$$

An equivalent definition by using a Newton polygon associated with the operator L_m will be given in §1.5.

In the case $s_0 = 1$, the operator L_m is often said to be Kowalevskian, and in the case $s_0>1$, it is said to be non Kowalevskian.

Let $s_0 \leqq s \leqq \infty$. Then the principal part of L_m in the category G^s is defined as follows:

G^{s_0}-principal part $= \{P_m(t\cdot D_t),\ b_{\alpha j}^{(\sigma)}(x)t^\sigma D_x{}^\alpha D_t{}^j$ with $|\alpha|=\min\{m-|j|, 0\}$, $c_{\alpha j}^{(\sigma)}(x, t)t^\sigma D_x{}^\alpha D_t{}^j$ with (σ, α, j) satisfying (1.5)$\}$.

If $s_0<s\leqq\infty$, then

G^s-principal part $= \{P_m(t\cdot D_t),\ b_{\alpha j}^{(\sigma)}(x)t^\sigma D_x{}^\alpha D_t{}^j$ with $|\alpha|=\min\{m-|j|, 0\}\}$.

1.4. Statement of results. Let L_m be an operator of Cauchy-Goursat-Fuchs type with weight m. Then our main interest in this paper is to study the mapping,

$$L_m : G^\infty/G^s \longrightarrow G^\infty/G^s \qquad (1\leqq s<\infty). \tag{1.6$_s$}$$

Here, we always assume that the coefficients of L_m are in the function space compatible with the mapping.

We first give the following,

THEOREM 1.1 (Regularity). *Let s_0 be the Gevrey index of L_m, and assume there is a positive constant δ such that:*

(i)

$$\min_{|\tau|=1} \Big|\sum_{|j|=m} a_j\, \tau^j\Big| > \delta \qquad (\tau\in\bar{\boldsymbol{R}}_+{}^q,\ \bar{\boldsymbol{R}}_+:=[0, \infty)). \tag{1.7}$$

(ii) *There are $\xi\in\boldsymbol{R}_+{}^p$ and $\tau\in\boldsymbol{R}_+{}^q$ $(\boldsymbol{R}_+=(0, \infty))$ satisfying*

$$\sum_{|\sigma|=|j|=m,\ |\alpha|=0} |b_{\alpha j}^{(\sigma)}(0)|\,\tau^{j-\sigma}\xi^\alpha<\delta\,. \tag{1.8}$$

Then the mapping (1.6)$_s$ *is bijective for every $s_0\leqq s<\infty$.*

The following theorem is an immediate consequence of the proof of the above theorem.

THEOREM 1.2 (Existence). *Let L_m and s_0 be as in Theorem* 1.1. *We assume there is a positive constant δ such that:*

(i)

$$|P_m(l)|> \delta(|l|+1)^m \qquad \textit{for any } l\in\boldsymbol{N}^q\,. \tag{1.9}$$

(ii) *There are $\xi\in\boldsymbol{R}_+{}^p$ and $\tau\in\boldsymbol{R}_+{}^q$ satisfying*

$$\sum_{|\sigma|=|j|\leqq m,\ |\alpha|=0} |b_{\alpha j}^{(\sigma)}(0)|\,\tau^{j-\sigma}\xi^\alpha<\delta\,. \tag{1.10}$$

Then the mappings,

$$L_m : G^s \longrightarrow G^s\ (s_0\leqq s\leqq\infty) \quad \textit{and} \quad L_m : \boldsymbol{C}[[x, t]] \longrightarrow \boldsymbol{C}[[x, t]]$$

are all bijective.

It will be proved that the "existence domain" of solutions depends only on the G^s-principal part of L_m. Such a precise description of results will be given in the proofs of theorems (see Theorems 4.1 and 5.1).

The condition (1.7) or (1.9) is sometimes said to be a condition on the indicial polynomial when $q=1$, and the Poincaré type condition when $q\geqq 2$. The condition (1.8) or (1.10) is a kind of spectral condition assumed in the study of Goursat problems.

REMARK 1.3. In the conditions (1.8) and (1.10) of the above theorems, we may assume $\xi=(1, \cdots, 1)$ and $\tau=(1, \cdots, 1)$, without loss of generality. Indeed, it suffices to transform the variables $(x, t)\mapsto(y, s)$ by $y_j=\xi_j x_j$ $(j=1, \cdots, p)$ and $s_j=\tau_j t_j$ $(j=1, \cdots, q)$. Hence, we replace them as follows:

$$(1.8) \qquad \sum_{|\sigma|=|j|=m,\, |\alpha|=0} |b_{\alpha j}^{(\sigma)}(0)|<\delta .$$

$$(1.10) \qquad \sum_{|\sigma|=|j|\leqq m,\, |\alpha|=0} |b_{\alpha j}^{(\sigma)}(0)|<\delta .$$

REMARK 1.4. The equation, $\{I-tD_tD_x^{-1}\}U(x, t)=F(x, t)\in G^s$ is uniquely solvable in G^s for every $1\leqq s\leqq\infty$ by Theorem 1.2. By changing $u=D_x^{-1}U$, this is equivalent to the unique solvability in G^s of the Cauchy problem,

$$\{D_x-tD_t\}u(x, t)=F(x, t)\in G^s, \qquad u(0, t)=0 .$$

This shows that we consider only an integrodifferential operator L_m reduced from the uniquely solvable Cauchy-Goursat problem.

REMARK 1.5. From the proof of Theorem 1.1, we shall see that the following holds: Let $p=0$, and assume the conditions (1.7) and (1.8). Let $G^s[[t]]:=G^s\cap \boldsymbol{C}[[t]]$. Then the mapping, $L_m(t; D_t): G^s[[t]]\to G^s[[t]]$ has finite dimensional kernel and cokernel for every $s_0\leqq s\leqq\infty$ and it holds that

$$\dim_{\boldsymbol{C}} \operatorname{Ker}(L_m; G^s[[t]])=\dim_{\boldsymbol{C}} \operatorname{Coker}(L_m; G^s[[t]]) .$$

1.5. Newton polygon and the Gevrey index. The following definition of the Newton polygon is an analogue to the one in Yonemura [**14**], where only partial differential operators were studied (see, also, Ramis [**12**]).

Let

$$P(x, t; D_x, D_t)=\sum_{\alpha, j} a_{\alpha j}(x, t)D_x{}^{\alpha}D_t{}^{j}$$

be an integrodifferential operator with coefficients $a_{\alpha j}\in\boldsymbol{C}[[x, t]]$. We set

$$a_{\alpha j}(x, t)=\sum_{\sigma\in\boldsymbol{N}^q} a_{\alpha j}^{(\sigma)}(x)t^{\sigma}, \qquad a_{\alpha j}^{(\sigma)}(x)\in\boldsymbol{C}[[x]] .$$

For a point $(u, v)\in\boldsymbol{R}^2$, we put

$$Q(u, v) = \{(r, s) \in \boldsymbol{R}^2 ; r \leqq u, s \geqq v\}.$$

Then the Newton polygon N(P) of P is defined by

$$\mathrm{N}(P) = \mathrm{ch}\{Q(|\alpha|+|j|, |\sigma|-|j|) ; a_{\alpha j}^{(\sigma)} \neq 0\},$$

where ch$\{\cdot\}$ denotes the convex hull of the elements in $\{\cdot\}$.

Now, return to the operator L_m of Cauchy-Goursat-Fuchs type with weight m.

Let $0=k_0<k_1<\cdots<k_i=+\infty$ be the slopes of sides of N(L_m). Then the Gevrey index s_0 of L_m defined by (1.5) is equal to

$$s_0 = 1 + \frac{1}{k_1}. \tag{1.11}$$

Here, we define $s_0=1$ if $k_1=+\infty$. We remark that G^{s_0}-principal part is nothing but $P_m(t \cdot D_t)$, $b_{\alpha j}^{(\sigma)}(x) t^\sigma D_x{}^\alpha D_t{}^j$ with $|\alpha| = \min\{m-|j|, 0\}$ and the collection of operators in $R(x, t ; D_x, D_t)$ lying on the side with slope k_1 of N(L_m).

1.6. Some comments. Yonemura [**14**] studied the solvability in G^s of the Cauchy problem to non Kowalevskian equations. His case is reduced to L_0 with $q=1$, $j<0$ and $\alpha \in \boldsymbol{N}^p$. On the other hand, Ōuchi [**11**] studied the Cauchy problem for non Kowalevskian equations, and he characterized asymptotic meaning of the formal power series solutions.

The Fuchs type equations or characteristic Cauchy problems studied by Baouendi and Goulaouic [**1**], Tahara [**13**] and others are reduced to L_m of Kowalevski type with $q=1$, $j<0$ and $\alpha \in \boldsymbol{N}^p$, and their main interest was Theorem 1.2. We note that Hasegawa [**3, 4**] studied the Goursat problems not only the Cauchy problems to Fuchs type equations, which is somewhat similar to our formulation.

Recently, Igari [**5, 6**] studied the characteristic Cauchy problem which is somewhat different from [**1, 3, 4, 13**]. Among others, he proved a unique solvability of the characteristic Cauchy problem, which corresponds to Theorem 1.2 for L_1 of Kowalevski type.

At the end, we cite Bengel and Gérard [**2**] and Yoshino [**15**], where the convergence of formal power series solutions of Fuchs type equations of multiple variables was studied under the Poincaré or Siegel condition.

2. Banach space $G^s(X, T ; k ; m)$.

We use the notations defined in §1.2.

Let $s \geqq 1$, X, $T>0$ and $k, m \in \boldsymbol{N}$. Then we define $U(x, t) \in G^s(X, T ; k ; m)$ $(\subset \boldsymbol{C}[[x, t]])$ by

$$\|U\|_{X,T;k;m}^{(s)} := \sup_{\beta, l} |U_{\beta l}| \frac{|l|!^m X^{|\beta|} T^{|l|}}{\{|\beta|+(s+m)|l|+k\}!} < \infty, \tag{2.1}$$

where $\{|\beta|+(s+m)|l|+k\}!=\Gamma(|\beta|+(s+m)|l|+k+1)$.

It is obvious that $G^s(X, T; k; m)$ becomes a Banach space by the norm $\|\cdot\|_{X,T;k;m}^{(s)}$ and

$$G^s(X, T) = G^s(X, T; 0; 0) \subset G^s(X, T; k; m) \qquad \text{for any } k, m \in \boldsymbol{N}.$$

Conversely, there is a positive constant ε independent of X and T such that $G^s(X, T; k; m) \subset G^s(\varepsilon X, \varepsilon T)$. Indeed, it is sufficient to see that by Stirling's formula there are positive constants C and R such that

$$\frac{\{|\beta|+(s+m)|l|+k\}!}{|l|!^m(|\beta|+s|l|)!} \leqq CR^{|\beta|+|l|} \qquad \text{for any } \beta \text{ and } l.$$

Therefore, it holds that

$$G^s = \bigcup_{X,T>0} G^s(X, T; k; m) \qquad \text{for any fixed } k \text{ and } m. \tag{2.2}$$

By the definition, the following properties are obvious:

(A) If $X'<X$ and $T'<T$, then

$$G^s(X, T; k; m) \hookrightarrow G^s(X', T'; 0; m) \qquad \text{for any } k \in \boldsymbol{N}. \tag{2.3}$$

(B) If $X' \leqq X$, $T' \leqq T$ and $k' \geqq k$, then

$$\begin{gathered} G^s(X, T; k; m) \hookrightarrow G^s(X', T'; k'; m). \\ \|\cdot\|_{X,T;k;m}^{(s)} \geqq \|\cdot\|_{X',T';k';m}^{(s)} \end{gathered} \tag{2.4}$$

Let $U(x, t) \in G^s(X, T; k; m)$. Then we easily obtain

$$U_l(x) = \sum_\beta U_{\beta l}\frac{x^\beta}{\beta!} \ll \frac{\|U\|}{T^{|l|}|l|!^m}\frac{\{(s+m)|l|+k\}!}{(1-|x|/X)^{(s+m)|l|+k+1}},$$

where $\|\cdot\|$ stands for the norm and $U(x) \ll \Xi(x)$ means $U(x)$ is majorized by $\Xi(x)$. Hence, we have

$$G^s(X, T; k; m) \subset \mathcal{O}(\|x\|<X)[[t]]. \tag{2.5}$$

Moreover, for any Y with $0<Y<X$, there are positive constants C and R such that

$$\max_{\|x\| \leqq Y} |U_l(x)| \leqq C\frac{|l|!^s}{R^{|l|}} \qquad \text{for any } l \in \boldsymbol{N}^q. \tag{2.6}$$

Conversely, we can prove the following,

PROPOSITION 2.1. *Let $U_l(x) \in \mathcal{O}(\|x\| \leqq Y)$ $(l \in \boldsymbol{N}^q)$ satisfy the inequality (2.6). Then $U(x, t) := \sum_l U_l(x)t^l/l! \in G^s(Y, R; k_0; 0)$, where $k_0 = \min\{k \in \boldsymbol{N}; k \geqq (p+s-1)/2\}$. Hence, $U(x, t) \in G^s(X, T)$ for any $X<Y$ and $T<R$.*

PROOF. Set $U_l(x)=\sum_\beta U_{\beta l}x^\beta/\beta!$. Then by Cauchy's integral formula on the polycircle $\prod_{j=1}^p\{|x_j|=\xi_j Y\}$ $(\xi_j>0, \sum_j \xi_j=1)$, we obtain

$$|U_{\beta l}| \leqq \frac{C}{Y^{|\beta|}R^{|l|}} \frac{|l|!^s \beta!}{\xi^\beta},$$

where $\xi=(\xi_1, \cdots, \xi_p)\in \boldsymbol{R}_+{}^p$ and $\xi^\beta=\xi_1^{\beta_1}\cdots\xi_p^{\beta_p}$. Since ξ^β takes its maximum in $\{\xi\in\boldsymbol{R}_+{}^p;\ |\xi|=1\}$ at $\xi=(\beta_1/|\beta|, \cdots, \beta_p/|\beta|)$, by Stirling's formula we have

$$\begin{aligned}|U_{\beta l}| &\leqq C\frac{(2\pi)^{(p+s-1)/2}}{Y^{|\beta|}R^{|l|}}\frac{|\beta|^{p/2}|l|^{s/2}}{(|\beta|+s|l|)^{1/2}}(|\beta|+s|l|)! \\ &\leqq C\frac{(2\pi)^{(p+s-1)/2}}{Y^{|\beta|}R^{|l|}}(|\beta|+s|l|+k_0)!,\end{aligned}$$

by the definition of k_0. □

Let

(2.7) $$G^1(X) := G^1(X, T)\cap \boldsymbol{C}[[x]].$$

Then $G^1(X)$ becomes a Banach space by the induced norm $\|\cdot\|_X^{(1)}$ from $G^1(X, T)$, and $G^1(X)\subset\mathcal{O}(\|x\|<X)$ by (2.5). The following lemma is a special case of Proposition 2.1, but it is useful to write down.

LEMMA 2.2. *Let $\kappa>1$. Then $\mathcal{O}(\|x\|\leqq\kappa X)\subset G^1(X)$ and for $a(x)\in\mathcal{O}(\|x\|\leqq\kappa X)$ it holds that*

(2.8) $$\|a\|_X^{(1)}\leqq c(p, \kappa)\max_{\|x\|\leqq\kappa X}|a(x)|,$$

where $c(p, \kappa)$ is a positive constant depending only on p and κ, and non increasing in κ.

PROOF. By the same way as Proposition 2.1, we have

$$\|a\|_X^{(1)} \leqq \sup_\beta\left\{\frac{(2\pi|\beta|)^{(p-1)/2}}{\kappa^{|\beta|}}\right\}\max_{\|x\|\leqq\kappa X}|a(x)|. \qquad \square$$

To make clear the reason of introducing of the parameters k and m, we prove the following,

PROPOSITION 2.3.

(i) $G^1(X, T; k; m)\subset\mathcal{O}\left(\frac{\|x\|}{X}+(m+1)\left(\frac{\|t\|}{T}\right)^{1/(m+1)}<1\right)$.

(ii) *For any κ with $\kappa>1$, we have*

$$\mathcal{O}\left(\frac{\|x\|}{X}+(m+1)\left(\frac{\|t\|}{T}\right)^{1/(m+1)}\leqq\kappa\right)\subset G^1(X, T; 0; m).$$

(iii) $\mathcal{O}\left(\frac{\|x\|}{X}+(m+1)\left(\frac{\|t\|}{T}\right)^{1/(m+1)}\leqq 1\right)\subset G^1\left(X, T; \left[\frac{p+q+m}{2}\right]; m\right)$.

PROOF. (i) Let $\varphi(z)=1/(1-z)$ and $\varphi^{(k)}(z)=(d/dz)^k\varphi(z)$, where $z\in\boldsymbol{C}$. Then for $U(x, t)\in G^1(X, T; k; m)$, it holds that

$$U_l(x)\ll \frac{\|U\|}{T^{|l|}|l|!^m}\varphi^{((1+m)|l|+k)}\Big(\frac{|x|}{X}\Big).$$

Hence we have

$$U(x, t)\ll \|U\|\sum_{r=0}^{\infty}\varphi^{((1+m)r+k)}\Big(\frac{|x|}{X}\Big)\frac{|t|^r}{T^r r!^{m+1}}$$

$$\ll C\|U\|\sum_{r=0}^{\infty}\varphi^{((1+m)r+k)}\Big(\frac{|x|}{X}\Big)(m+1)^{(m+1)r}\frac{(|t|/T)^r}{\{(1+m)r\}!},$$

for some positive constant C. Since $\sum_{r=0}^{\infty}\varphi^{(r+k)}(|x|/X)z^r/r!=\varphi^{(k)}((|x|/X)+z)$ is holomorphic in $|z|<1-\|x\|/X$, we obtain the assertion.

(ii) Let $U(x, t)\in\mathcal{O}(\|x\|/X+(m+1)(\|t\|/T)^{1/(1+m)}\leqq\kappa)$ and set

$$\|U\|_\infty=\max\Big\{|U(x, t)|\,;\,\frac{\|x\|}{X}+(m+1)\Big(\frac{\|t\|}{T}\Big)^{1/(m+1)}\leqq\kappa\Big\}.$$

Let $0<s<\kappa X$, $\xi=(\xi_1, \cdots, \xi_p)\in\boldsymbol{R}_+^p$ with $|\xi|=1$ and $\tau=(\tau_1, \cdots, \tau_q)\in\boldsymbol{R}_+^q$ with $|\tau|=1$. Then by Cauchy's integral formula on the polycircle

$$\prod_{j=1}^{p}\{|x_j|=\xi_j s\}\times\prod_{j=1}^{q}\Big\{|t_j|=\tau_j\Big(\kappa-\frac{s}{X}\Big)^{m+1}(m+1)^{-(m+1)}T\Big\},$$

we obtain

$$|U_{\beta l}|\leqq\|U\|_\infty\frac{\beta!\,l!\,(m+1)^{(m+1)|l|}}{\xi^\beta\tau^l s^{|\beta|}(\kappa-s/X)^{(m+1)|l|}T^{|l|}}.$$

Since ξ^β, τ^l and $s^{|\beta|}(\kappa-s/X)^{(m+1)|l|}$ take their maximums at $\xi=(\beta_1/|\beta|, \cdots, \beta_p/|\beta|)$, $\tau=(l_1/|l|, \cdots, l_q/|l|)$ and $s=\kappa X|\beta|/(|\beta|+(m+1)|l|)$ respectively, we have

$$|U_{\beta l}|\leqq\frac{\|U\|_\infty}{X^{|\beta|}T^{|l|}}\frac{1}{\kappa^{|\beta|+(m+1)|l|}}\frac{\beta!\,l!}{\beta^\beta l^l}\frac{\{|\beta|+(m+1)|l|\}^{|\beta|+(m+1)|l|}}{|l|^{m|l|}}$$

Hence, by Stirling's formula we obtain the following inequality.

$$\|U\|^{(1)}_{X,T;0;m}\leqq\|U\|_\infty\times\max_{\beta,l}\frac{\{2\pi(|\beta|+|l|)\}^{(p+q+m-1)/2}}{\kappa^{|\beta|+(m+1)|l|}}.$$

This proves (ii).

(iii) is an immediate consequence of the above proof. □

Note that the above proposition shows

$$\mathcal{O}\Big(\frac{\|x\|}{X}+(m+1)\Big(\frac{\|t\|}{T}\Big)^{1/(m+1)}<1\Big)=\bigcap_{\substack{0<X'<X\\0<T'<T}}G^1(X', T'; 0; m).$$

This shows the parameter m plays the role to determine the shape of domain of holomorphy of solutions. The parameter k does not play any role to determine the shape of domain of holomorphy, but will play an important role to make clear that the existence domain of solutions depends only on the "principal part" of the operator.

The idea of introducing of the parameter $k\in\boldsymbol{N}$ was found in Wagschal [**16**] and Miyake [**9**] in the study of the Goursat problems, and the idea of introducing of the parameter $m\ (=1)$ was found in Igari [**6**] in the study of the characteristic Cauchy problems.

We summarize some notations used in the proof below.

For $\alpha,\ \beta\in\boldsymbol{N}^{r}$ with $\beta\leqq\alpha$, $\binom{\alpha}{\beta}:=\dfrac{\alpha!}{\beta!(\alpha-\beta)!}$.

For $\lambda=(\lambda_{1},\ \cdots,\ \lambda_{r})\in\boldsymbol{R}^{r}$ and $\sigma=(\sigma_{1},\ \cdots,\ \sigma_{r})\in\boldsymbol{N}^{r}$,

$$[\lambda]_{\sigma}:=\prod_{i=1}^{r}\lambda_{i}(\lambda_{i}-1)\cdots(\lambda_{i}-\sigma_{i}+1),$$

where $[\lambda]_{\sigma}:=1$ if $\sigma=0$. For $\lambda\geqq 0$, $\lambda!:=\Gamma(\lambda+1)$. The following relation will be used frequently.

$$[\lambda+\sigma]_{\sigma}\lambda!=(\lambda+\sigma)! \qquad \text{for } \lambda\geqq 0 \text{ and } \sigma\in\boldsymbol{N}.$$

LEMMA 2.4. *Let* $\rho>1$ *and* $U(x,t)\in G^{s}(X, T; k; m)$.

(i) *If* $a(x,t)\in G^{s}(\rho X, \rho T; 0; m)$, *then* $aU\in G^{s}(X, T; k; m)$ *and*

$$\|aU\|_{X,T;k;m}^{(s)}\leqq\left(\frac{\rho}{\rho-1}\right)^{2}\|a\|_{\rho X,\rho T;0;m}^{(s)}\|U\|_{X,T;k;m}^{(s)}. \tag{2.9}$$

(ii) *If* $a(x)\in\mathcal{O}(\|x\|<X)$, *then* $aU\in G^{s}(Y, T; k; m)$ *for any* Y *with* $Y<X$, *and we have*

$$\|aU\|_{Y,T;k;m}^{(s)}\leqq\{|a(0)|+\varepsilon(Y)\}\|U\|_{Y,T;k;m}^{(s)}, \tag{2.10}$$

where $\lim_{Y\downarrow 0}\varepsilon(Y)=0$.

REMARK. If $s=1$ and $m=0$ in the lemma, then

$$\|aU\|_{X,T;k;0}^{(1)}\leqq\frac{\rho}{\rho-1}\|a\|_{\rho X,\rho T;0;0}^{(1)}\|U\|_{X,T;k;0}^{(1)} \tag{2.9$'$}$$

holds. It is the same in the case $a\equiv a(x)\in G^{1}(\rho X)$.

PROOF. (i) We put $a(x,t)=\sum a_{\beta l}x^{\beta}t^{l}/\beta!l!$, $U(x,t)=\sum U_{\beta l}x^{\beta}t^{l}/\beta!l!$ and $aU=\sum V_{\beta l}x^{\beta}t^{l}/\beta!l!$. Then

$$V_{\beta l}=\sum_{0\leqq\gamma\leqq\beta}\sum_{0\leqq n\leqq l}a_{\gamma n}U_{\beta-\gamma,l-n}\binom{\beta}{\gamma}\binom{l}{n}.$$

Here $0=(0,\cdots,0)$ in $\boldsymbol{N}^{p}$ or $\boldsymbol{N}^{q}$. By the definition of the norm, we have

$$|V_{\beta l}|X^{|\beta|}T^{|l|}|l|!^m \leqq \|a\|\|U\| \times \sum_{0\leqq\gamma\leqq\beta}\sum_{0\leqq n\leqq l}\frac{\{|\gamma|+(s+m)|n|\}!\{|\beta|-|\gamma|+(s+m)(|l|-|n|)+k\}!}{\rho^{|\gamma|+|n|}} \times\binom{\beta}{\gamma}\binom{l}{n}\binom{|l|}{|n|}^m .$$

Here we omit the indices in the norms. Considering the inequality,

$$\sum_{|\gamma|=u}\sum_{|n|=v}\binom{\beta}{\gamma}\binom{l}{n}\binom{|l|}{v}^m=\binom{|\beta|}{u}\binom{|l|}{v}^{m+1}$$
$$\leqq\binom{|\beta|+(m+1)|l|}{u+(m+1)v}=\frac{[|\beta|+(m+1)|l|]_{u+(m+1)v}}{\{u+(m+1)v\}!},$$

we obtain

$$|V_{\beta l}|X^{|\beta|}T^{|l|}|l|!^m \leqq \|a\|\|U\| \times\sum_{u=0}^{|\beta|}\sum_{v=0}^{|l|}\frac{\{u+(s+m)v\}!\,\{|\beta|-u+(s+m)(|l|-v)+k\}!}{\rho^{u+v}} \times\frac{[|\beta|+(m+1)|l|]_{u+(m+1)v}}{\{u+(m+1)v\}!}.$$

Now our purpose is to prove the following inequality.

$$\{u+(s+m)v\}!\,\{|\beta|-u+(s+m)(|l|-v)+k\}! \times\frac{[|\beta|+(m+1)|l|]_{u+(m+1)v}}{\{u+(m+1)v\}!}\leqq\{|\beta|+(s+m)|l|+k\}!\,. \tag{2.11}$$

Since the case $s=1$ or $v=0$ is trivial, we consider the case $s>1$ and $v>0$. The following inequality is easily obtained.

$$\{|\beta|-u+(s+m)(|l|-v)+k\}!\,[|\beta|+(m+1)|l|]_{u+(m+1)v}$$
$$=\{|\beta|+(s+m)|l|-(s-1)v+k\}!\,\frac{[|\beta|+(m+1)|l|]_{u+(m+1)v}}{[|\beta|+(s+m)|l|-(s-1)v+k]_{u+(m+1)v}}$$
$$\leqq\{|\beta|+(s+m)|l|-(s-1)v+k\}!\,.$$

Next, by the formula $\Gamma(x)\Gamma(y)=\Gamma(x+y)B(x, y)$,

$$\frac{\{u+(m+s)v\}!}{\{u+(m+1)v\}!}=\frac{\Gamma((s-1)v)}{B((s-1)v,\ u+(m+1)v+1)}$$

and

$$\{|\beta|+(s+m)|l|-(s-1)v+k\}!\,\Gamma((s-1)v)$$
$$=\{|\beta|+(s+m)|l|+k\}!\,B((s-1)v,\ |\beta|+(s+m)|l|-(s-1)v+k+1).$$

Since $u+(m+1)v+1\leqq|\beta|+(s+m)|l|-(s-1)v+k+1$, it holds that

$$B((s-1)v,\ u+(m+1)v+1)$$
$$\geqq B((s-1)v,\ |\beta|+(s+m)|l|-(s-1)v+k+1).$$

Combining these inequalities, we obtain (2.11). Hence,

$$|V_{\beta l}|X^{|\beta|}T^{|l|}|l|!^m \leqq \|a\|\|U\|\{|\beta|+(s+m)|l|+k\}!\sum_{u=0}^{|\beta|}\sum_{v=0}^{|l|}\frac{1}{\rho^{u+v}}$$

$$\leqq \|a\|\|U\|\{|\beta|+(s+m)|l|+k\}!\left(\frac{\rho}{\rho-1}\right)^2.$$

This proves (2.9).

When $a\equiv a(x)$ the inequality (2.9)′ holds, because of the absence of v in the above summation. When $s=1$ and $m=0$, we obtain (2.9)′ by putting $|\gamma|+|n|=u$ instead of putting $|\gamma|=u$ and $|n|=v$ in the above proof.

(ii) Let $a(x)\in\mathcal{O}(\|x\|<X)$ and $U(x, t)\in G^s(X, T; k; m)$. Then $aU\in G^s(Y; T; k; m)$ for any Y with $Y<X$. Let ρ and κ be fixed constants with $\rho, \kappa>1$ and $\kappa\rho Y<X$. Then by (i),

$$\|aU\|_{Y,T;k;m}^{(s)} \leqq \left\{|a(0)|+\frac{\rho}{\rho-1}\|a(x)-a(0)\|_{\rho Y}^{(1)}\right\}\|U\|_{Y,T;k;m}^{(s)}.$$

On the other hand, by Lemma 2.2 we have

$$\|a(x)-a(0)\|_{\rho Y}^{(1)} \leqq c(p, \kappa)\max_{\|x\|\leqq\kappa\rho Y}|a(x)-a(0)|\downarrow 0 \quad \text{as } Y\downarrow 0. \qquad \square$$

The next lemma is easy, but it is useful in the proof of the solvability in G^∞ of integrodifferential equations.

LEMMA 2.5. *Let $\alpha\in\mathbf{Z}^p$ satisfy $|\alpha|\leqq 0$. Then the mapping*

$$D_x^\alpha: G^s(X, T; k; m)\longrightarrow G^s(X, T; k; m)$$

is bounded and the operator norm is estimated by

$$(2.12) \qquad \|D_x^\alpha\|_{X,T;k;m}^{(s)} \leqq \left(\frac{X}{k}\right)^{-|\alpha|}.$$

PROOF. Let $D_x^\alpha U=\sum V_{\beta l}x^\beta t^l/\beta!l!$. Then we have $V_{\beta l}=U_{\beta+\alpha,l}$. This implies the result immediately. $\square$

3. Operator of the form $t^\sigma D_x^\alpha D_t^j P_m^{-1}$.

Let an operator $P_m(t\cdot D_t)=\sum_{|j|\leqq m}a_j(t\cdot D_t)^j$ ($j\in\mathbf{N}^q$, $a_j\in\mathbf{C}$) satisfy the condition (1.7). Then there is $N_0\in\mathbf{N}$ such that

$$(3.1) \qquad |P_m(l)|>\delta|l|^m \quad \text{for any } l\in\mathbf{N}^q \text{ with } |l|\geqq N_0.$$

For $N\in\mathbf{N}$, we set

$$(3.2) \qquad \mathbf{C}[[x, t]][N]:=\{U\in\mathbf{C}[[x, t]];\ U_{\beta l}=0 \text{ if } |l|<N\}.$$

Then the mapping,

$$P_m(t\cdot D_t): \boldsymbol{C}[[x, t]][N] \longrightarrow \boldsymbol{C}[[x, t]][N]$$

is bijective for any $N \geqq N_0$, and its inverse P_m^{-1} is given by

$$P_m^{-1}U = \sum_{\beta, l} P_m(l)^{-1} U_{\beta l} \frac{x^\beta t^l}{\beta! l!} \qquad (|l| \geqq N).$$

In what follows, we always assume $N \geqq N_0$. We set

$$G^s(X, T; k; m)[N] = G^s(X, T; k; m) \cap \boldsymbol{C}[[x, t]][N]. \tag{3.3}$$

Then $G^s(X, T; k; m)[N]$ is a Banach space by the norm $\|\cdot\|^{(s)}_{X, T; k; m; N}$ induced from $G^s(X, T; k; m)$.

Let us study an operator

$$t^\sigma D_x{}^\alpha D_t{}^j P_m^{-1} \qquad ((\sigma, \alpha, j) \in \boldsymbol{N}^q \times \boldsymbol{Z}^p \times \boldsymbol{Z}^q, \quad |\sigma| \geqq |j|) \tag{3.4}$$

acting on $G^s(X, T; k; m)[N]$. For $U \in \boldsymbol{C}[[x, t]][N]$, we set

$$t^\sigma D_x{}^\alpha D_t{}^j P_m^{-1} U = \sum_{\beta, l} V_{\beta l} \frac{x^\beta t^l}{\beta! l!}.$$

Then by an easy calculation we obtain

$$V_{\beta l} = \frac{[l]_\sigma}{P_m(l+j-\sigma)} U_{\beta+\alpha, l+j-\sigma}, \tag{3.5}$$

where $\beta+\alpha \in \boldsymbol{N}^p$, $l+j-\sigma \in \boldsymbol{N}^q$ $(|l|+|j|-|\sigma| \geqq N)$ and $l \geqq \sigma$.

Hence for $U \in G^s(X, T; k; m)[N]$, it holds that

$$\begin{aligned} |V_{\beta l}| \frac{X^{|\beta|} T^{|l|} |l|!^m}{\{|\beta|+(s+m)|l|+k\}!} &\leqq \|U\| \frac{T^{|\sigma|-|j|}}{X^{|\alpha|}} \frac{[l]_\sigma}{|P_m(l+j-\sigma)|} \\ &\times \frac{|l|!^m}{\{|l|+|j|-|\sigma|\}!^m} \frac{\{|\beta|+|\alpha|+(s+m)(|l|+|j|-|\sigma|)+k\}!}{\{|\beta|+(s+m)|l|+k\}!}. \end{aligned} \tag{3.6}$$

From this inequality, we obtain the estimate below for the operator norm of the mapping

$$t^\sigma D_x{}^\alpha D_t{}^j P_m^{-1}: G^s(X, T; k; m)[N] \longrightarrow G^s(X, T; k; m)[N].$$

(A) *The case* $|\sigma| = |j| \leqq m$ *and* $|\alpha| \leqq 0$.

$$\|t^\sigma D_x{}^\alpha D_t{}^j P_m^{-1}\|^{(s)}_{X, T; k; m; N} < \delta^{-1} N^{|\sigma|-m} \left(\frac{X}{k}\right)^{-|\alpha|}. \tag{3.7}$$

(B) *The case* $|\sigma| = |j| > m$ *and* $|\alpha| + |j| \leqq m$.
Let $\rho = m - |\alpha| - |\sigma|$ $(\geqq 0)$. *Then we have*

$$\|t^\sigma D_x{}^\alpha D_t{}^j P_m^{-1}\| < \delta^{-1} X^{-|\alpha|} k^{-\rho}. \tag{3.8}$$

Here and in what follows we omit the indices in the operator norm for the simplicity.

Indeed, it suffices to notice the following inequality.

$$|l|^{|\sigma|-m}\{|\beta|+|\alpha|+(s+m)|l|+k\}!$$
$$<\{|\beta|+(s+m)|l|+|\alpha|+|\sigma|-m+k\}!.$$

(C) *The case* $|\sigma|>|j|$ *and* $s\geqq(|\sigma|+|\alpha|-m)/(|\sigma|-|j|)$.
Let $\rho=s(|\sigma|-|j|)-(|\sigma|+|\alpha|-m)$ $(\geqq 0)$. *Then we have*

$$\|t^{\sigma}D_x^{\alpha}D_t^{j}P_m^{-1}\|<CT^{|\sigma|-|j|}X^{-|\alpha|}k^{-\rho}, \tag{3.9}$$

for some positive constant C independent of X and T.

PROOF. The following inequality holds for some positive constant C_1.

$$\frac{[l]_{\sigma}}{|P_m(l+j-\sigma)|}\frac{|l|!^{m}}{\{|l|+|j|-|\sigma|\}!^{m}}\leqq C_1|l|^{(|\sigma|-|j|)m+|\sigma|-m}.$$

Since $|\sigma|>|j|$, it holds that $(|\sigma|-|j|)m+|\sigma|-m\geqq 0$. Put

$$\{|\beta|+(s+m)|l|+k\}!=[|\beta|+(s+m)|l|+k]_{(|\sigma|-|j|)m+|\sigma|-m}$$
$$\times\{|\beta|+(s+m)|l|+k-(|\sigma|-|j|)m-|\sigma|+m\}!.$$

The following inequality is obvious.

$$\frac{|l|^{(|\sigma|-|j|)m+|\sigma|-m}}{[|\beta|+(s+m)|l|+k]_{(|\sigma|-|j|)m+|\sigma|-m}}\leqq C_2,$$

where C_2 is a positive constant. On the other hand, since

$$\rho=\{(|\sigma|-|j|)(s+m)-|\alpha|\}-\{(|\sigma|-|j|)m+|\sigma|-m\},$$

we have

$$\frac{\{|\beta|+|\alpha|+(s+m)(|l|+|j|-|\sigma|)+k\}!}{\{|\beta|+(s+m)|l|+k-(|\sigma|-|j|)m-|\sigma|+m\}!}\leqq C_3k^{-\rho},$$

for some positive constant C_3. □

4. Proof of Theorem 1.1.

For $X, T>0$ and $1\leqq s\leqq\infty$, we define

$$\mathfrak{G}^{s}(X,T;m)=\bigcap_{0<X'<X}\bigcap_{0<T'<T}G^{s}(X',T';0;m)\qquad\text{if } 1\leqq s<\infty,$$
$$\mathfrak{G}^{\infty}(X,T;m)=\mathfrak{G}^{\infty}(X)=\mathcal{O}(\|x\|<X)[[t]]. \tag{4.1}$$

As mentioned after the proof of Proposition 2.3,

$$\mathfrak{G}^{1}(X,T;m)=\mathcal{O}\Big(\frac{\|x\|}{X}+(m+1)\Big(\frac{\|t\|}{T}\Big)^{1/(m+1)}<1\Big).$$

We shall prove the following theorem, which is a precise form of Theorem 1.1.

THEOREM 4.1. *Let L_m and s_0 be as in Theorem* 1.1 *and the conditions in Theorem* 1.1 *are satisfied. We assume the coefficients of L_m belong to $G^s(X_0, T_0; 0; m)$ $(s_0 \leqq s \leqq \infty)$.*

(i) *Let us consider the case $s=s_0$. Then there are positive constants X_1 and T_1 such that the mapping,*

$$L_m : \mathfrak{G}^{\infty}(X_1)/\mathfrak{G}^{s_0}(X_1, T_1; m) \longrightarrow \mathfrak{G}^{\infty}(X_1)/\mathfrak{G}^{s_0}(X_1, T_1; m) \tag{4.2$_{s_0}$}$$

is bijective. Here, X_1 and T_1 depend only on the G^{s_0}-principal part of L_m.

(ii) *Let us consider the case $s_0 < s < \infty$. Then there is a positive constant X_1 such that the mapping,*

$$L_m : \mathfrak{G}^{\infty}(X_1)/\mathfrak{G}^{s}(X_1, T_0; m) \longrightarrow \mathfrak{G}^{\infty}(X_1)/\mathfrak{G}^{s}(X_1, T_0; m) \tag{4.2$_s$}$$

is bijective. Here, X_1 depends only on the G^s-principal part.

For $N \in \boldsymbol{N}$, we set

$$\mathfrak{G}^s(X, T; m)[N] = \mathfrak{G}^s(X, T; m) \cap \boldsymbol{C}[[x, t]][N] \qquad (1 \leqq s \leqq \infty), \tag{4.3}$$

Let the operator L_m satisfy the conditions in Theorem 4.1, and $N_0 \in \boldsymbol{N}$ satisfy the condition (3.1). Then $P_m P_m^{-1}$ is the identity operator in the space $\mathfrak{G}^s(X, T; m)[N]$ for any $1 \leqq s \leqq \infty$ and $N \geqq N_0$. Hence, Theorem 4.1 is proved by showing the following proposition.

PROPOSITION 4.2. *Let L_m and s_0 be as in Theorem* 4.1 *and the conditions in Theorem* 4.1 *are satisfied.*

(i) *There are positive constants X_1, T_1 and $N_1 \in \boldsymbol{N}$ such that the mapping,*

$$L_m P_m^{-1} : \mathfrak{G}^{s_0}(X_1, T_1; m)[N_1] \longrightarrow \mathfrak{G}^{s_0}(X_1, T_1; m)[N_1] \tag{4.4$_{s_0}$}$$

is bijective.

(ii) *Let $s_0 < s \leqq \infty$. Then there are a positive constant X_1 and $N_1 \in \boldsymbol{N}$ such that the mapping,*

$$L_m P_m^{-1} : \mathfrak{G}^{s}(X_1, T_0; m)[N_1] \longrightarrow \mathfrak{G}^{s}(X_1, T_0; m)[N_1] \tag{4.4$_s$}$$

is bijective.

We first consider the case $s_0 \leqq s < \infty$. We set

$$L_m P_m^{-1} = I - A, \qquad A = \sum_{i=1}^{6} A_i. \tag{4.5}$$

Here, A_i $(i=1, 2, \cdots, 6)$ are defined as follows.

$$A_1 = \sum_{|\sigma|=|j|\leqq m,\, |\alpha|=0} b_{\alpha j}^{(\sigma)}(x) t^{\sigma} D_x{}^{\alpha} D_t{}^{j} P_m^{-1}.$$

$$A_2 = \sum_{|\sigma|=|j|>m,\, |\alpha|+|j|=m} b_{\alpha j}^{(\sigma)}(x) t^{\sigma} D_x{}^{\alpha} D_t{}^{j} P_m^{-1}.$$

$$A_3 = \sum_{|\sigma|>|j|}' c_{\alpha j}^{(\sigma)}(x, t) t^\sigma D_x{}^\alpha D_t{}^j P_m^{-1},$$

where the summation is taken over $(\sigma, \alpha, j) \in \boldsymbol{N}^q \times \boldsymbol{Z}^p \times \boldsymbol{Z}^q$ such that

$$s(|\sigma|-|j|) = |\sigma|+|\alpha|-m. \tag{4.6}$$

Hence, $A_3=0$ when $s>s_0$. Note that A_i $(i=1, 2, 3)$ consist of the G^s-principal part defined in § 1.3.

$$A_4 = \sum_{|\sigma|=|j|\leqq m, |\alpha|<0} b_{\alpha j}^{(\sigma)}(x) t^\sigma D_x{}^\alpha D_t{}^j P_m^{-1}.$$

$$A_5 = \sum_{|\sigma|=|j|>m, |\alpha|+|j|<m} b_{\alpha j}^{(\sigma)}(x) t^\sigma D_x{}^\alpha D_t{}^j P_m^{-1}.$$

$$A_6 = \sum_{|\sigma|>|j|}'' c_{\alpha j}^{(\sigma)}(x, t) t^\sigma D_x{}^\alpha D_t{}^j P_m^{-1},$$

where the summation is taken over $(\sigma, \alpha, j) \in \boldsymbol{N}^q \times \boldsymbol{Z}^p \times \boldsymbol{Z}^q$ such that

$$s(|\sigma|-|j|) > |\sigma|+|\alpha|-m. \tag{4.7}$$

We define a non negative constant r by

$$r = \max\left\{0, \max\left\{\frac{|\alpha|}{|\sigma|-|j|}; (\sigma, \alpha, j) \text{ satisfies (4.6)}\right\}\right\}. \tag{4.8}$$

Then we have the following,

LEMMA 4.3. *Let L_m and s_0 be as in Theorem 4.1 and the conditions in Theorem 4.1 are satisfied.*

(i) *There are positive constants R_1 and ε_1, and $N_1 \in \boldsymbol{N}$ such that the mapping,*

$$A: G^{s_0}(R, \varepsilon R^r; k; m)[N_1] \longrightarrow G^{s_0}(R, \varepsilon R^r; k; m)[N_1] \tag{4.9$_{s_0}$}$$

becomes a contraction mapping for any $0<R\leqq R_1$ and $0<\varepsilon\leqq\varepsilon_1$, if we choose sufficiently large $k\in\boldsymbol{N}$ which depends on R, ε and N_1.

(ii) *Let $s_0<s<\infty$. Then there are a positive constant X_1 and $N_1\in\boldsymbol{N}$ such that the mapping,*

$$A: G^s(X, T; k; m)[N_1] \longrightarrow G^s(X, T; k; m)[N_1] \tag{4.9$_s$}$$

becomes a contraction mapping for any $0<X\leqq X_1$ and $0<T<T_0$, if we choose sufficienlly large $k\in\boldsymbol{N}$ which depends on X, T and N_1.

PROOF. By using results obtained in the previous sections, we can estimate each operator norm $\|A_i\|_{X,T;k;m;N}^{(s)}$ $(i=1, \cdots, 6)$ as follows. Here, N is taken so that $N\geqq N_0$.

(A) By (3.7) and the results in Section 2, we obtain the following estimate for any X with $0<X<X_0$.

$$\|A_1\|^{(s)}_{X,T;k;m;N} < \delta^{-1} \sum_{\substack{|\alpha|=0 \\ |\sigma|=|j|=m}} \{|b^{(\sigma)}_{\alpha j}(0)| + \varepsilon(X)\}$$

$$+\delta^{-1}\frac{X_0}{X_0-X} \sum_{\substack{|\alpha|=0 \\ |\sigma|=|j|<m}} \|b^{(\sigma)}_{\alpha j}\|^{(1)}_{X_0} N^{|\sigma|-m},$$

where $\lim_{X\downarrow 0}\varepsilon(X)=0$. In view of the assumption (1.8) and $|\sigma|<m$ in the summation we can choose $0<\varepsilon_2<1$, $X_2>0$ and $N_1\in \boldsymbol{N}$ so that

$$\|A_1\|^{(s)}_{X,T;k;m;N_1} < 1-\varepsilon_2 \tag{4.10}$$

holds for any $0<X\leqq X_2$, $T>0$ and $k\in \boldsymbol{N}$.

(B) By (3.8) and the results in Section 2, we have for any $0<X\leqq X_2$, $T>0$ and $k\in \boldsymbol{N}$,

$$\|A_2\|^{(s)}_{X,T;k;m;N_1} \leqq \delta^{-1}\frac{X_0}{X_0-X} \sum_{\substack{|\sigma|=|j|>m \\ |\alpha|+|j|=m}} \|b^{(\sigma)}_{\alpha j}\|^{(1)}_{X_0} X^{-|\alpha|}.$$

Since $|\alpha|<0$ in this case, for any ε_3 with $0<\varepsilon_3<\varepsilon_2$ there is $X_1>0$ ($X_1\leqq X_2$) such that

$$\|A_2\|^{(s)}_{X,T;k;m;N_1} < \varepsilon_3, \tag{4.11}$$

holds for any $0<X\leqq X_1$, $T>0$ and $k\in \boldsymbol{N}$.

(C) Let $0<X\leqq X_1$ ($<X_0$), $0<T<T_0$ and put $\rho=\min\{X_0/X, T_0/T\}$. Then by (3.9) and Lemma 2.4, we have the following inequality for any $k\in \boldsymbol{N}$.

$$\|A_3\|^{(s)}_{X,T;k;m;N_1} \leqq C\left(\frac{\rho}{\rho-1}\right)^2 \sum_{|\sigma|>|j|}{}' \|c^{(\sigma)}_{\alpha j}\|^{(s)}_{\rho X,\rho T;0;m} T^{|\sigma|-|j|} X^{-|\alpha|}$$

$$\leqq C\left(\frac{\rho}{\rho-1}\right)^2 \sum_{|\sigma|>|j|}{}' \|c^{(\sigma)}_{\alpha j}\|^{(s)}_{X_0,T_0;0;m} T^{|\sigma|-|j|} X^{-|\alpha|},$$

where C is a positive constant.

By the definition of the constant r,

$$\|A_3\|^{(s)}_{X,T;k;m;N_1} \leqq \sum_{|\sigma|>|j|}{}' C_{\sigma\alpha j} X^{r(|\sigma|-|j|)-|\alpha|}\left(\frac{T}{X^r}\right)^{|\sigma|-|j|}.$$

Since $r(|\sigma|-|j|)-|\alpha|\geqq 0$, there are positive constants R_1 ($\leqq X_1$) and ε_1 such that

$$\|A_1+A_2+A_3\|^{(s)}_{R,\varepsilon R^r;k;m;N_1} < 1, \tag{4.12}$$

holds for any $0<R\leqq R_1$, $0<\varepsilon\leqq \varepsilon_1$ and $k\in \boldsymbol{N}$.

Note that $A_3=0$ when $s>s_0$.

(D) By (3.7), (3.8), (3.9) and the results in Section 2, for any fixed X and T, there is a positive constant η such that

$$\|A_i\|^{(s)}_{X,T;k;m;N_1} = O(k^{-\eta}) \qquad (i=4, 5, 6). \tag{4.13}$$

Hence, we can choose sufficiently large $k\in \boldsymbol{N}$ which depends on R, ε

and N_1 $(0<R\leqq R_1,\ 0<\varepsilon\leqq\varepsilon_1)$ so that A becomes a contraction mapping in $G^{s_0}(R, \varepsilon R^r; k; m)[N_1]$.

It is the same in the case $s_0<s<\infty$. □

Next, we consider the case $s=\infty$. We write $L_m P_m^{-1}$ as follows.

$$L_m P_m^{-1} = I-B, \qquad B=\sum_{i=1}^{5} B_i. \tag{4.14}$$

Here B_i $(i=1, 2, \cdots, 5)$ are defined as follows.

$$B_1 = \sum_{|\sigma|=|j|=m,\, |\alpha|=0} b_{\alpha j}^{(\sigma)}(x) t^\sigma D_x{}^\alpha D_t{}^j P_m^{-1}.$$

$$B_2 = \sum_{|\sigma|=|j|<m,\, |\alpha|=0} b_{\alpha j}^{(\sigma)}(x) t^\sigma D_x{}^\alpha D_t{}^j P_m^{-1}.$$

$$B_3 = \sum_{|\sigma|=|j|\leqq m,\, |\alpha|<0} b_{\alpha j}^{(\sigma)}(x) t^\sigma D_x{}^\alpha D_t{}^j P_m^{-1}.$$

$$B_4 = \sum_{|\sigma|=|j|>m,\, |\alpha|+|j|\leqq m} b_{\alpha j}^{(\sigma)}(x) t^\sigma D_x{}^\alpha D_t{}^j P_m^{-1}.$$

$$B_5 = R(x, t; D_x, D_t) P_m^{-1}.$$

Let $U(x, t)=\sum_l U_l(x) t^l/l! \in \mathcal{O}(\|x\|<X)[[t]][N_0]$ and put

$$L_m P_m^{-1} U = \sum_{|l|\geqq N_0} F_l(x) \frac{t^l}{l!}.$$

Then,

$$F_l(x) = U_l(x) - \sum_{i=1}^{4} \sum_{|\sigma|=|j|}{}^{(l,i)} b_{\alpha j}^{(\sigma)}(x) \frac{[l]_\sigma}{P_m(l+j-\sigma)} D_x{}^\alpha U_{l+j-\sigma}(x) \tag{4.15}$$
$$-\mathcal{R}^{(l)}(x, D_x{}^\alpha U_n(x);\ |n|<|l|,\ \text{finite number of } \alpha),$$

where the summation $\sum^{(l,i)}$ is taken over $(\sigma, \alpha, j)\in \boldsymbol{N}^q\times\boldsymbol{Z}^p\times\boldsymbol{Z}^q$ with $|\sigma|=|l|$ and $l+j-\sigma\in\boldsymbol{N}^q$ corresponding to the expression of B_i $(i=1, \cdots, 4)$.

Let $\mathcal{U}^{(N)}(x)$ be a column vector defined by

$$\mathcal{U}^{(N)}(x) = {}^t(U_l(x);\ |l|=N), \tag{4.16}$$

of length $(q+N-1)!/(q-1)!N!$ $(=\sharp\{l\in\boldsymbol{N}^q;\ |l|=N\})$.

Then the relation (4.15) implies a sequence of systems of integrodifferential equations for $\mathcal{U}^{(N)}(x)$ of the form,

$$\{I-\mathcal{B}^{(N)}(x, D_x{}^\alpha)\}\mathcal{U}^{(N)}(x) = \mathcal{R}^{(N)}(x, D_x{}^\alpha U_l(x);\ |l|<N) - \mathcal{F}^{(N)}(x), \tag{4.17}$$

for $N\geqq N_0$. Here $\mathcal{F}^{(N)}(x)$ is a vector defined from $F_l(x)$ $(|l|=N)$. Let

$$\mathcal{G}^{(N)}(X; k) := \{\mathcal{U}^{(N)}(x);\ U_l(x)\in G^1(X; k),\ |l|=N\},$$

where $G^1(X; k) := G^s(X, T; k; m)\cap \boldsymbol{C}[[x]]$. Then $\mathcal{G}^{(N)}(X; k)$ is a Banach space with norm

(4.18) $$\|\mathcal{U}^{(N)}\|_{X;k}^{(1)}=\max_{|l|=N}\|U_l\|_{X;k}^{(1)}.$$

Under these preparations, we can prove the following,

LEMMA 4.4. *Let the coefficients of* L_m *belong to* $G^1(X_0)[[t]]$. *Then there are a positive constant* X_1, $N_1\in\boldsymbol{N}$ *and* $k(X_1, N)\in\boldsymbol{N}$ $(N\geqq N_1)$ *such that the mapping,*

(4.19) $$\mathcal{B}^{(N)}: \mathcal{G}^{(N)}(X; k(X_1, N)) \longrightarrow \mathcal{G}^{(N)}(X; k(X_1, N))$$

becomes a contraction mapping for any $X\leqq X_1$ *and* $N\geqq N_1$. *Here,* X_1 *and* N_1 *depend only on* B_1 *and* B_2.

PROOF. Note that by the conditions (1.7) and (1.8),

(4.20) $$\sum_{\substack{|\sigma|=|j|=m\\|\alpha|=0}}^{(l,1)}|b_{\alpha j}^{(\sigma)}(0)|\frac{[l]_\sigma}{|P_m(l+j-\sigma)|}<1,$$

holds for any $l\in\boldsymbol{N}^q$ with $|l|\geqq N_0$.

(A) Let $|l|=N\geqq N_0$ and $0<X<X_0$. Then by Lemma 2.5, we have

$$\begin{aligned}&\left\|\sum_{\substack{|\sigma|=|j|=m\\|\alpha|=0}}^{(l,1)}b_{\alpha j}^{(\sigma)}(x)\frac{[l]_\sigma}{P_m(l+j-\sigma)}D_x{}^\alpha U_{l+j-\sigma}(x)\right\|_{X;k}^{(1)}\\&\leqq\sum_{\substack{|\sigma|=|j|=m\\|\alpha|=0}}^{(l,1)}\frac{[l]_\sigma}{|P_m(l+j-\sigma)|}\{|b_{\alpha j}^{(\sigma)}(0)|+\varepsilon(X)\}\|D_x{}^\alpha U_{l+j-\sigma}\|_{X;k}^{(1)}\\&\leqq\sum_{\substack{|\sigma|=|j|=m\\|\alpha|=0}}^{(l,1)}\frac{[l]_\sigma}{|P_m(l+j-\sigma)|}\{|b_{\alpha j}^{(\sigma)}(0)|+\varepsilon(X)\}\|\mathcal{U}^{(N)}\|_{X;k}^{(1)}.\end{aligned}$$

In view of (4.20) and $\lim_{X\downarrow 0}\varepsilon(X)=0$, there are positive constants X_1 $(<X_0)$ and ε_1 $(0<\varepsilon_1<1)$ such that

(4.21) $$\begin{aligned}&\left\|\sum_{\substack{|\sigma|=|j|=m\\|\alpha|=0}}^{(l,1)}b_{\alpha j}^{(\sigma)}(x)\frac{[l]_\sigma}{P_m(l+j-\sigma)}D_x{}^\alpha U_{l+j-\sigma}(x)\right\|_{X;k}^{(1)}\\&\qquad\leqq(1-\varepsilon_1)\|\mathcal{U}^{(N)}\|_{X;k}^{(1)},\end{aligned}$$

holds for any $X\leqq X_1$, $k\in\boldsymbol{N}$ and l with $|l|=N$.

(B) For any $X\leqq X_1$, $|l|=N$ and $k\in\boldsymbol{N}$, we have

$$\begin{aligned}&\left\|\sum_{\substack{|\sigma|=|j|<m\\|\alpha|=0}}^{(l,2)}b_{\alpha j}^{(\sigma)}(x)\frac{[l]_\sigma}{P_m(l+j-\sigma)}D_x{}^\alpha U_{l+j-\sigma}(x)\right\|_{X;k}^{(1)}\\&\leqq\delta^{-1}\frac{X_0}{X_0-X}\|\mathcal{U}^{(N)}\|_{X;k}^{(1)}\sum_{\substack{|\sigma|=|j|<m\\|\alpha|=0}}^{(l,2)}\|b_{\alpha j}^{(\sigma)}\|_{X_0}^{(1)}N^{|\sigma|-m}.\end{aligned}$$

Since $|\sigma|<m$ in this case, for any ε_2 with $0<\varepsilon_2<\varepsilon_1$, we can choose $N_1\in\boldsymbol{N}$ $(N_1\geqq N_0)$ so that

$$(4.22)\qquad \Big\| \sum_{\substack{|\sigma|=|j|<m\\ |\alpha|=0}}^{(l,2)} b_{\alpha j}^{(\sigma)}(x)\frac{[l]_\sigma}{P_m(l+j-\sigma)} D_x{}^\alpha U_{l+j-\sigma}(x)\Big\|_{X;k}^{(1)} < \varepsilon_2 \|\mathcal{U}^{(N)}\|_{X;k}^{(1)}$$

holds for any $X \leqq X_1$, l with $|l|=N\geqq N_1$ and $k\in \boldsymbol{N}$.

(C) For any $X\leqq X_1$ and l with $|l|=N\geqq N_1$, we have

$$\Big\| \sum_{\substack{|\sigma|=|j|\leqq m\\ |\alpha|<0}}^{(l,3)} b_{\alpha j}^{(\sigma)}(x)\frac{[l]_\sigma}{P_m(l+j-\sigma)} D_x{}^\alpha U_{l+j-\sigma}(x)\Big\|_{X;k}^{(1)}$$
$$\leqq \delta^{-1}\frac{X_0}{X_0-X}\|\mathcal{U}^{(N)}\|_{X;k}^{(1)} \sum_{\substack{|\sigma|=|j|\leqq m\\ |\alpha|<0}}^{(l,3)} \|b_{\alpha j}^{(\sigma)}\|_{X_0}^{(1)} N^{|\sigma|-m}\Big(\frac{k}{X}\Big)^{|\alpha|}.$$

Since $|\alpha|<0$ and $|\sigma|\leqq m$ in this case, for any ε_3 with $0<\varepsilon_3<\varepsilon_1-\varepsilon_3$, there is $k_0\in\boldsymbol{N}$ such that

$$(4.23)\qquad \Big\| \sum_{\substack{|\sigma|=|j|\leqq m\\ |\alpha|<0}}^{(l,3)} b_{\alpha j}^{(\sigma)}(x)\frac{[l]_\sigma}{P_m(l+j-\sigma)} D_x{}^\alpha U_{l+j-\sigma}(x)\Big\|_{X;k}^{(1)} < \varepsilon_3 \|\mathcal{U}^{(N)}\|_{X;k}^{(1)}$$

holds for any $X\leqq X_1$, $k\geqq k_0$ and $N\geqq N_1$.

(D) For any $X\leqq X_1$, $k\geqq k_0$ and l with $|l|=N\geqq N_1$, we have,

$$\Big\| \sum_{\substack{|\sigma|=|j|>m\\ |\alpha|+|j|\leqq m}}^{(l,4)} b_{\alpha j}^{(\sigma)}(x)\frac{[l]_\sigma}{P_m(l+j-\sigma)} D_x{}^\alpha U_{l+j-\sigma}(x)\Big\|_{X;k}^{(1)}$$
$$\leqq \delta^{-1}\frac{X_0}{X_0-X}\|\mathcal{U}^{(N)}\|_{X;k}^{(1)} \sum_{\substack{|\sigma|=|j|>m\\ |\alpha|+|j|\leqq m}}^{(l,4)} \|b_{\alpha j}^{(\sigma)}\|_{X_0}^{(1)} N^{|\sigma|-m}\Big(\frac{k}{X}\Big)^{|\alpha|}.$$

Since $|\alpha|<0$ in this case, for any fixed N ($\geqq N_1$) we can take $k(X_1, N)\in\boldsymbol{N}$ so that

$$(4.24)\qquad \|\mathcal{B}^{(N)}(x\,;D_x)\|_{X;k(X_1,N)}^{(1)} < 1$$

holds for and $X\leqq X_1$. □

PROOF OF PROPOSITION 4.2. (i) Let the coefficients of L_m belong to $G^{s_0}(X_0, T_0\,;0\,;m)$, and take positive constants R_1, ε_1 and $N_1\in\boldsymbol{N}$ as in Lemma 4.3, (i). Then by taking $X_1=R_1$ and $T_1=\varepsilon_1 R_1{}^r$, the mapping $(4.4)_{s_0}$ is bijective. To prove this, we consider the case $r>0$. In this case, it holds that

$$\mathfrak{G}^{s_0}(X_1, T_1\,;m)[N_1] = \bigcap_{0<R<R_1} G^{s_0}(R, \varepsilon_1 R^r\,;0\,;m)[N_1]\,,$$

because $G^s(X, T\,;0\,;m)\subset G^s(X', T'\,;0\,;m)$ for any $X'\leqq X$ and $T'\leqq T$. Let us consider the equation,

$$L_m P_m^{-1} U(x, t) = F(x, t)\in \mathfrak{G}^{s_0}(X_1, T_1\,;m)[N_1]\,.$$

Note $F(x, t)\in G^{s_0}(R, \varepsilon_1 R^r\,;0\,;m)[N_1]\subset G^{s_0}(R, \varepsilon_1 R^r\,;k\,;m)[N_1]$ for any $R<R_1$ and $k\in\boldsymbol{N}$. Then by Lemma 4.3, (i), the above equation has a unique solution U in $G^{s_0}(R, \varepsilon_1 R^r\,;k\,;m)[N_1]$, if we choose sufficiently large $k\in\boldsymbol{N}$. Hence it belongs to $G^{s_0}(S, \varepsilon_1 S^r\,;0\,;m)[N_1]$ for any $0<S<R$. This proves that the uni-

que solution U exists in $\mathfrak{G}^{s_0}(X_1, T_1; m)[N_1]$.

It is the same in the case $r=0$, so we omit the proof.

(ii) Since the case $s_0<s<\infty$ is the same as above, we consider the case $s=\infty$. Let the coefficients of L_m belong to $G^1(X_0)[[t]]$ and take X_1 and N_1 as in Lemma 4.4. Since $\mathfrak{G}^{\infty}(X_1, T_0; m)=\mathcal{O}(\|x\|<X_1)[[t]]$, we have to prove the bijectivity of the mapping,

$$L_m P_m^{-1}: \mathcal{O}(\|x\|<X_1)[[t]][N_1] \longrightarrow \mathcal{O}(\|x\|<X_1)[[t]][N_1].$$

Note that $\mathcal{O}(\|x\|<X_1)=\bigcap_{X_2<X_1} G^1(X_2)$ and $G^1(X_2)\subset G^1(X_2; k)\subset G^1(X_3)$ for any $k\in \boldsymbol{N}$ and $X_2>X_3>0$.

Let us consider the equation,

$$L_m P_m^{-1} U(x, t) = F(x, t) \in \mathcal{O}(\|x\|<X_1)[[t]][N_1].$$

Since $F(x, t)\in G^1(X_2)[[t]][N_1]$ for any $X_2<X_1$, the coefficients $\{U_l(x); |l|=N\}$ $(N\geqq N_1)$ of formal expansion of $U(x, t)$ are uniquely determined in $G^1(X_2; k(X_1; N))$ by Lemma 4.4. Hence $U(x, t)\in G^1(X_3)[[t]][N_1]$ for any X_3 with $X_3<X_2$. This proves our assertion. □

5. Proof of Theorem 1.2.

We shall prove the following theorem, which is a precise form of Theorem 1.2.

THEOREM 5.1. *Let L_m and s_0 be as in Theorem 1.2 and the conditions in Theorem 1.2 are satisfied. We assume the coefficients of L_m belong to $G^s(X_0, T_0; 0; m)$ $(s_0\leqq s\leqq\infty)$.*

(i) *Let us consider the case $s=s_0$. Then there are positive constants X_1 and T_1 such that the mapping,*

$$(5.1)_{s_0} \qquad L_m: \mathfrak{G}^{s_0}(X_1, T_1; m) \longrightarrow \mathfrak{G}^{s_0}(X_1, T_1; m)$$

is bijective.

(ii) *Let us consider the case $s_0<s\leqq\infty$. Then there is a positive constant X_1 such that the mapping,*

$$(5.1)_s \qquad L_m: \mathfrak{G}^{s}(X_1, T_0; m) \longrightarrow \mathfrak{G}^{s}(X_1, T_0; m)$$

is bijective.

(iii) *Let the coefficients of L_m belong to $\boldsymbol{C}[[x, t]]$. Then the mapping,*

$$(5.2) \qquad L_m: \boldsymbol{C}[[x, t]] \longrightarrow \boldsymbol{C}[[x, t]]$$

is bijective.

The proofs of (i) and (ii) are obvious, because Proposition 4.2 holds by putting $N_1=0$. Indeed, it is sufficient to notice the following inequality obtained from the conditions (1.9) and (1.10).

$$(5.3) \qquad \sum_{|\sigma|=|j|\leq m,\, |\alpha|=0} |b_{\alpha j}^{(\sigma)}(0)| \frac{[l]_\sigma}{|P_m(l+j-\sigma)|} < 1 ,$$

for any $l \in \boldsymbol{N}^q$.

PROOF OF THEOREM 5.1. (iii) By the condition (1.9), the mapping, $P_m : \boldsymbol{C}[[x, t]] \to \boldsymbol{C}[[x, t]]$ is bijective. Hence, it is sufficient to prove the unique solvability of the equation,

$$(5.4) \qquad L_m P_m^{-1} U(x, t) = F(x, t) \in \boldsymbol{C}[[x, t]] ,$$

in $\boldsymbol{C}[[x, t]]$. For that purpose, we set

$$(5.5) \qquad L_m P_m^{-1} = I - C , \qquad C = \sum_{i=1}^{4} C_i .$$

Here C_i $(1 \leqq i \leqq 4)$ are defined as follows.

$$C_1 = \sum_{|\sigma|=|j|\leq m,\, |\alpha|=0} b_{\alpha j}^{(\sigma)}(x) t^\sigma D_x{}^\alpha D_t{}^j P_m^{-1} .$$

$$C_2 = \sum_{|\sigma|=|j|\leq m,\, |\alpha|<0} b_{\alpha j}^{(\sigma)}(x) t^\sigma D_x{}^\alpha D_t{}^j P_m^{-1} .$$

$$C_3 = \sum_{|\sigma|=|j|>m,\, |\alpha|+|j|\leq m} b_{\alpha j}^{(\sigma)}(x) t^\sigma D_x{}^\alpha D_t{}^j P_m^{-1} .$$

$$C_4 = R(x, t ; D_x, D_t) P_m^{-1} .$$

We set $U(x, t) = \sum U_{\beta l} x^\beta t^l / \beta ! l !$ and $F(x, t) = \sum F_{\beta l} x^\beta t^l / \beta ! l !$. Then by the expression (5.5) of $L_m P_m^{-1}$, we have the following relations.

$$(5.6) \qquad \begin{aligned} F_{\beta l} = U_{\beta l} - & \sum_{\substack{|\sigma|=|j|\leq m \\ |\alpha|=0}}^{(\beta, l)} b_{\alpha j}^{(\sigma)}(0) \frac{[l]_\sigma}{P_m(l+j-\sigma)} U_{\beta+\alpha, l+j-\sigma} \\ & - \mathcal{R}^{(\beta, l)}(U_{\gamma n} ;\ |\gamma| < |\beta| ,\ |n| = |l|) \\ & - \mathcal{S}^{(\beta, l)}(U_{\gamma n} ;\ |n| < |l|) , \end{aligned}$$

where the summations are taken over (σ, α, j) such that $\beta + \alpha \in \boldsymbol{N}^p$ and $l + j - \sigma \in \boldsymbol{N}^q$ $(l \geqq \sigma)$.

By this relation and (5.3), we can see that the coefficients of the formal power series of $U(x, t)$.

$${}^t(U_{\beta l} ;\ |\beta| = M ,\ |l| = N) \in \boldsymbol{C}^{d(M, N)} ,$$

$(d(M, N) = \# \{(\beta, l) \in \boldsymbol{N}^{p+q} ;\ |\beta| = M ,\ |l| = N\})$ are uniquely determined by the principle of contraction map in $\boldsymbol{C}^{d(M, N)}$. In fact, we use double induction on M and N as follows. For a fixed $N \in \boldsymbol{N}$, we solve $\{U_{\beta l} ;\ |\beta| = M ,\ |l| = N\}$ by induction on $M \in \boldsymbol{N}$, and proceed next N. We omit the detail, since it will be done easily. □

REMARK 5.2. Under the assumptions in Theorem 1.1, we can prove the existence of $N_1 \in \boldsymbol{N}$ for which the mapping,

$$L_m : \boldsymbol{C}[[x, t]][N_1] \longrightarrow \boldsymbol{C}[[x, t]][N_1]$$

is bijective. The proof is the same as above by using the expression (4.14) of $L_m P_m^{-1}$.

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