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On microhyperbolic mixed problems

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Introduction.

Let us consider the following Dirichlet problem on $\Omega=\{t\in\boldsymbol{R};\ t>0\}\times\boldsymbol{R}_x^n$.

$$\begin{cases} P\,u(t,\,x)=f(t,\,x) & \text{in } \Omega, \\ u(+0,\,x)=g(x) & \text{on } \partial\Omega. \end{cases} \tag{0.1}$$

Here $\partial\Omega=\{t=0\}\times\boldsymbol{R}_x^n$, and P is an analytic differential operator of order 2 defined on $\bar{\Omega}=\Omega\cup\partial\Omega$ of the form

$$P=D_t^2+A_1(t,\,x,\,D_x)D_t+A_2(t,\,x,\,D_x) \tag{0.2}$$

with $D_t=\partial/\partial t$, $D_x=(\partial/\partial x_1,\,\cdots,\,\partial/\partial x_n)$. We study the problem (0.1) in the space of hyperfunctions, and thus $f(t,\,x)$ and $g(x)$ are hyperfunctions defined on Ω and $\partial\Omega$ respectively. Moreover we assume that $f(t,\,x)$ is mild on $t=+0$. This means that $f(t,\,x)$ belongs to a class of hyperfunctions for which the boundary values $D_t^j f(+0,\,x)$ $(j=0,\,1,\,2,\,\cdots)$ to $\partial\Omega$ are well-defined (see §1). Under this assumption, it follows from the non-charactericity of $\partial\Omega$ that every solution on Ω to the first equation of (0.1) becomes mild, and in particular that the second equation of (0.1) makes sense.

Let $u(t,\,x)$ be a hyperfunction solution to (0.1) in Ω. Then taking the canonical extensions $\tilde{u}(t,\,x)$ and $\tilde{f}(t,\,x)$ of $u(t,\,x)$ and $f(t,\,x)$ respectively, we get the identities

$$P\tilde{u}(t,\,x)=\tilde{f}(t,\,x)+g(x)\delta'(t)+(D_t u(+0,\,x)+A_1(0,\,x,\,D_x)g(x))\cdot\delta(t) \tag{0.3}$$

and

$$tP\tilde{u}(t,\,x)=t\tilde{f}(t,\,x)-g(x)\delta(t) \quad \text{in } \boldsymbol{R}_t\times\boldsymbol{R}_x^n. \tag{0.4}$$

Here the correspondence $u\to\tilde{u}$ is a well-defined operation on mild hyperfunctions, which is similar to the cut-off operation by the Heaviside function $Y(t)$.

Conversely, it is easy to see that every hyperfunction solution to (0.4) with condition $\operatorname{supp}\tilde{u}\subset\{t\geqq 0\}$ gives a solution to (0.1). Thus we can reduce the Dirichlet problem (0.1) to studying the local or global cohomology groups of the complex of sheaves

$$(0.5)\qquad 0 \longrightarrow \Gamma_{\{t\geqq 0\}}(\mathcal{B}_{t,x}) \overset{tP}{\longrightarrow} \Gamma_{\{t\geqq 0\}}(\mathcal{B}_{t,x}) \longrightarrow 0\,.$$

In other words, we study $\ker(tP)$ and $\operatorname{coker}(tP)$ in (0.5). Here $\Gamma_{\{t\geqq 0\}}(\mathcal{B}_{t,x})$ denotes the sheaf of hyperfunctions in $(t, x)\in \boldsymbol{R}_t\times\boldsymbol{R}_x^n$ supported by $\{t\geqq 0\}$.

On the other hand, mixed problems are also formulated in a similar manner. Take another hypersurface H, for example $\{x_1=0\}$, which is non-characteristic for P and transversal to $\partial\Omega$, and pose an initial condition for the solution u to (0.1):

$$(0.6)\qquad u(t, 0, x')=v_0(t, x'),\quad D_{x_1}u(t, 0, x')=v_1(t, x')\qquad \text{on } H\cap\Omega\,.$$

(Here $x'=(x_2, \cdots, x_n)$.) Then we treat $u_\pm=u(t, x)Y(\pm x_1)$ instead of u as before and get the equations

$$(0.7)\qquad Pu_\pm=h_\pm(t, x)\qquad \text{in } \Omega.$$

Here the functions $\{h_\pm(t, x)\}$ are linear combinations of $f(t, x)Y(\pm x_1)$, $v_0(t, x')\delta^{(j)}(x_1)$ $(j=0, 1)$ and $v_1(t, x')\delta(x_1)$. For example, assume that $f\equiv 0$ and that v_0 and v_1 are mild at $t=+0$. Then $u_\pm$ is also well-defined and also mild at $t=+0$. In this case, however, the authors do not know whether the Dirichlet data $g_\pm:=u_\pm(+0, x)$ can be determined only by the data f, g, v_0 and v_1. We know only that

$$(0.8)\qquad g_+(x)+g_-(x)=g(x)\quad \text{and}\quad \operatorname{supp} g_\pm(x)\subset\{\pm x_1\geqq 0\}.$$

This ambiguity comes from the degeneracy of (0.4) on $\{t=0\}$. By this reason, we can not expect that $\tilde{u}(t, x)$ depends analytically on x_1 at $t=x_1=0$. Thus we do not know whether $\tilde{u}(t, x)Y(\pm x_1)$ are well-defined.

Nevertheless, once we give a decomposition $g_\pm(x)$ of $g(x)$ satisfying (0.8), the solvability of our mixed problem reduces to that of the problems

$$(0.9)_\pm\qquad \begin{cases} Pu_\pm(t, x)=h_\pm(t, x) & \text{in } \Omega\\ u_\pm(+0, x)=g_\pm(x) & \text{on } \partial\Omega\\ \operatorname{supp} u_\pm(t, x)\subset\{\pm x_1\geqq 0\}. \end{cases}$$

For the uniqueness of solution $u(t, x)$ to (0.1), we need additional conditions on solutions. For example, it suffices to assume that $D_t u(+0, x)$ depends analytically on x_1 near $\{x_1=0\}$, or that $u(t, x)$ is C^2 differentiable up to $\{t=+0\}$. In any case, it is important to study the problems $(0.9)_\pm$, and thus we call them "mixed problems" in this paper.

Since $h_\pm$ and $g_\pm$ in $(0.9)_\pm$ are given hyperfunctions on Ω and $\partial\Omega$ satisfying

$$\operatorname{supp} h_\pm(t, x)\subset\{\pm x_1\geqq 0\},\qquad \operatorname{supp} g_\pm(x)\subset\{\pm x_1\geqq 0\},$$

respectively, the problems $(0.9)_\pm$ are equivalent to calculating the local or global cohomology groups of the complexes of sheaves

$$(0.10)_{\pm} \qquad 0 \longrightarrow \Gamma_{\{t\geq 0, \pm x_1\geq 0\}}(\mathcal{B}_{t,x}) \xrightarrow{tP} \Gamma_{\{t\geq 0, \pm x_1\geq 0\}}(\mathcal{B}_{t,x}) \longrightarrow 0\,.$$

In fact if the above sequence $(0.10)_{\pm}$ is exact at $\mathring{q}=(0, 0, \mathring{x}')$, then the problem $(0.9)_{\pm}$ localized near $\mathring{q}$ is uniquely solvable. Thus we obtained a cohomological formulation of mixed problems. Moreover since

$$\begin{aligned}\Gamma_{\{t\geq 0\}}(\mathcal{B}_{t,x}) &= \mathcal{H}^n_{\{\mathrm{Im}\, z=0\}}(\Gamma_{\{t\geq 0\}}(\mathcal{B}_t\mathcal{O}_z)) \\ &= \boldsymbol{R}\Gamma_{\{\mathrm{Im}\, z=0\}}(\Gamma_{\{t\geq 0\}}(\mathcal{B}_t\mathcal{O}_z))[n]\,,\end{aligned}$$

the exactness of $(0.10)_{\pm}$ at $\mathring{q}$ is equivalent to

$$(0.11) \qquad \boldsymbol{R}\Gamma_{\{\pm x_1\geq 0, \mathrm{Im}\, z=0\}}(\mathcal{F})|_{\mathring{q}} = 0\,.$$

Here $\mathcal{B}_t\mathcal{O}_z$ is the sheaf of hyperfunctions in $(t, z=x+\sqrt{-1}y)\in \boldsymbol{R}_t\times \boldsymbol{C}^n_x$ depending holomorphically on z, and $\mathcal{F}$ is the complex of sheaves

$$(0.12) \qquad \mathcal{F}: 0 \longrightarrow \Gamma_{\{t\geq 0\}}(\mathcal{B}_t\mathcal{O}_z) \xrightarrow{tP} \Gamma_{\{t\geq 0\}}(\mathcal{B}_t\mathcal{O}_z) \longrightarrow 0\,.$$

On the other hand, the condition (0–11) is almost equivalent to

$$(0.13) \qquad SS(\boldsymbol{R}\Gamma_{\{y=0\}}(\mathcal{F})) \not\ni (0, 0, \mathring{x}'\,;\, \pm dx_1)\,.$$

(The condition (0.13) is stronger than (0.11).) As for the definition of micro-supports $SS(\cdot)$ of sheaves, refer to § 2.1 and Kashiwara-Schapira [**9**]. Further, if we have an estimate for $SS(\mathcal{F})$, we can derive an estimate for $SS(\boldsymbol{R}\Gamma_{\{y=0\}}(\mathcal{F}))$ by using the formula due to Kashiwara-Schapira (see (2.6) in § 2.1). Hence the problem finally reduces to estimating the micro-support of $\mathcal{F}$.

We remark here that the complex $\mathcal{F}$ is quasi-isomorphic to just a sheaf

$$(0.14) \qquad \mathcal{G} = (\mathcal{O}^P_{w,z}\cap w\cdot \mathcal{O}_{w,z})|_{\tilde{M}_+}\,,$$

where $\mathcal{O}_{w,z}$ is the sheaf of holomorphic functions of $(w, z)\in \boldsymbol{C}_w\times \boldsymbol{C}^n_z$, $\mathcal{O}^P_{w,z}$ is the solution sheaf of the equation $P(w, z, D_w, D_z)u(w, z)=0$ with value in $\mathcal{O}_{w,z}$, and

$$\tilde{M}_+ = \{(w, z)\in \boldsymbol{C}_w\times \boldsymbol{C}^n_z\,;\, \mathrm{Im}\, w=0,\ \mathrm{Re}\, w\geqq 0\}.$$

For the morphism tP in (0.12) is surjective, and we can identify $\ker(tP)$ with $\mathcal{G}$ as

$$(0.15) \qquad \mathcal{G}\ni u(w, z) \longmapsto u(t, z)\cdot Y(t)\in \ker(tP)\,.$$

Therefore $\ker(tP)$ and $\mathrm{coker}(tP)$ in $(0.10)_{\pm}$ are respectively isomorphic to

$$\mathcal{H}^n_{\{y=0, \pm x_1\geq 0\}}(\mathcal{G}) \quad \text{and} \quad \mathcal{H}^{n+1}_{\{y=0, \pm x_1\geq 0\}}(\mathcal{G})\,.$$

But it is easier to treat the complex $\mathcal{F}$ than $\mathcal{G}$ because some global cohomology groups of $\mathcal{O}_{w,z}|_{\tilde{M}_+}$ don't vanish.

Our result on the micro-support of $\mathcal{F}$ is given by

THEOREM 0.1. *Take a point* $p=(\mathring{t}, \mathring{z}; \mathring{\tau}dt+\mathrm{Re}(\mathring{\zeta}dz))\in T^*(\boldsymbol{R}_t\times \boldsymbol{C}_z^n)$ *with* $\mathring{\zeta}\neq 0$ *and* $\mathring{t}\geqq 0$. *Then* $p\notin SS(\mathcal{F})$ *if*

(i) $\mathring{t}>0$, *and* $\sigma(P)(\mathring{t}, \mathring{z}, \theta+\mathring{\tau}, \mathring{\zeta})\neq 0$ *for any* $\theta\in\sqrt{-1}\boldsymbol{R}$,

or

(ii) $\mathring{t}=0$, *and the equation* $\sigma(P)(\mathring{t}, \mathring{z}, \theta+\mathring{\tau}, \mathring{\zeta})=0$ *in* θ *has one root in* $\{\mathrm{Re}\,\theta>0\}$ *and the other one in* $\{\mathrm{Re}\,\theta<0\}$.

REMARK. This theorem will be generalized in Theorem 1.1 to m-th order operators with general boundary conditions. In that case, we will utilize some system of equations with $(m+1)$ generators instead of $\mathcal{M}=\mathcal{D}/\mathcal{D}tP$, and we will need additional conditions of Shapiro-Lopatinski's type.

REMARK. As seen in (0.14), Theorem 0.1 is almost equivalent to the following problem of Martinez's type. Let $f(w, z)$ and $g(z)$ be holomorphic respectively on some neighborhoods of

$$\Omega_\delta=\{(w, z)\in\boldsymbol{C}\times\boldsymbol{C}^n;\ \mathrm{Im}\,w=0,\ \mathrm{Re}\,w\geqq 0,\ |w|+|z-\mathring{z}|<\delta,\ \varphi(\mathrm{Re}\,w, \mathrm{Re}\,z, \mathrm{Im}\,z)<0\}$$

and

$$\omega_\delta=\{z\in\boldsymbol{C}^n;\ |z-\mathring{z}|<\delta,\ \varphi(0, \mathrm{Re}\,z, \mathrm{Im}\,z)<0\},$$

for a small $\delta>0$. Here $\varphi(t, x, y)$ is a real valued C^ω function defined in a neighborhood of $(0, \mathring{x}, \mathring{y})$ such that

$$\varphi=0 \quad\text{and}\quad (\varphi_t, \varphi_x, \varphi_y)=(\mathring{\tau}, \mathrm{Re}\,\mathring{\zeta}, -\mathrm{Im}\,\mathring{\zeta})$$

at $(0, \mathring{x}, \mathring{y})$. In this situation, our result also claims that *for any* $f(w, z)$, $g(z)$ *as above, there exists a holomorphic function* $u(w, z)$ *defined in a neighborhood of* $\Omega_{\delta'}$ *with some positive* δ' $(\delta'\leqq\delta)$ *satisfying*

$$(0.16)\qquad \begin{cases} P(w, z, D_w, D_z)u(w, z)=f(w, z) & on\ \Omega_{\delta'} \\ u(0, z)=g(z) & on\ \omega_{\delta'}\,. \end{cases}$$

Moreover the difference of two solutions u, u' *of* (0.16) *extends holomorphically to* $(0, \mathring{z})$.

In [**15**], A. Martinez proves the last statement given above concerning the uniqueness of solutions by using the method of J. Sjöstrand. (He also treated more general problems under general boundary conditions.)

We prove Theorem 0.1 (or Theorem 1.1) by reducing the problem of type (0.16) to an elliptic boundary value problem on real analytic manifold $\{(t, x, y);\ \varphi(t, x, y)=0\}$. More precisely we treat a degenerate elliptic problem, for example

$$(0.17)\qquad \begin{cases} tP(t, x, D_t, D_x)u(t, x) = f(t, x), \\ \qquad\qquad \operatorname{supp} u(t, x) \subset \{t \geqq 0\}. \end{cases}$$

Here P is an operator of the form (0.2), $f(t, x)$ is a given hyperfunction with $\operatorname{supp} f \subset \{t \geqq 0\}$, and P is elliptic on the singular spectrum of f. Then the problem is to find a condition on $\sigma(P)$ for the problem (0.17) to have a unique solution $u(t, x)$ in some sense locally at a point $(t, x) = (\mathring{t}, \mathring{x})$. Easily to see, it suffices to assume that for any $(0, \mathring{x}; i\mathring{\tau}, i\mathring{\eta}) \in SS(f)$, the equation $\sigma(P)(0, \mathring{x}, \theta, i\mathring{\eta}) = 0$ in θ has one root in $\{\operatorname{Re}\theta > 0\}$ and the other root in $\{\operatorname{Re}\theta < 0\}$. This is the reason why the condition (ii) in Theorem 0.1 appears.

As a direct corollary of Theorem 0.1 and the formula (2.6) in § 2.1 due to Kashiwara-Schapira [**9**], we have

THEOREM 0.2. *Take points* $q_\pm = (0, 0, \mathring{x}'; \pm dx_1) \in T^*(\boldsymbol{R}_t \times \boldsymbol{R}^n_x)$. *Then*

$$q_\pm \notin SS(\boldsymbol{R}\Gamma_{\{y=0\}}(\mathcal{F}))$$

if

(i) *$\sigma(P)$ is hyperbolic with respect to x_1; that is, for some $\delta > 0$,*

$$\sigma(P)(t, x, \sqrt{-1}\tau, \sqrt{-1}\eta_1 \pm 1, \sqrt{-1}\eta') \neq 0$$

$$\text{on } \{0 \leqq t \leqq \delta,\ |x_1| + |x' - \mathring{x}'| < \delta,\ \tau \in \boldsymbol{R},\ \eta = (\eta_1, \eta') \in \boldsymbol{R}^n\},$$

and

(ii) *the equation $\sigma(P)(0, 0, \mathring{x}', \theta, \pm 1, 0, \cdots, 0) = 0$ in θ has one positive real root and the other negative real root.*

As we explained before, this implies the unique prolongation property across $\{x_1 = 0\}$ for solutions of the Dirichlet problem (0.1), and also the unique solvability of the mixed problem $(0.9)_\pm$. Moreover this theorem can be microlocalized with respect to x by using the microlocalization functor along $\{y = 0\}$ instead of $\boldsymbol{R}\Gamma_{\{y=0\}}(\cdot)$. Hence we obtain a unique prolongation theorem for microlocal solutions to (0.1) and also the solvability of the microlocal mixed problem.

The plan of this paper is as follows: In § 1 we state our main theorems, and at the same time we define the sheaf $\hat{C}_{\boldsymbol{R}_+ \times N}$ to formulate microlocally the well-posedness of mixed problems. This sheaf is a variation of the sheaf of mild microfunctions. In § 2, we prepare and recall some notions and tools which are indispensable for the proof of main theorems. In particular, we review the theory of microlocal elliptic boundary value problems in § 2.4. There we give related propositions similar to (0.17), which will become crucial later. We will reduce our problem of vanishings of $\mathcal{B}\mathcal{O}$-solutions complex to those on a real analytic hypersurface in $\boldsymbol{R}_t \times \boldsymbol{C}^n_z$ in the course of proof of the main theorems. To this end, we characterize in § 2.5 the boundary values of sections of $\mathcal{B}_z\mathcal{O}_z$ to a real analytic hypersurface of the form $L = \{\operatorname{Im} z_1 = \varphi(t, \operatorname{Re} z, \operatorname{Im} z')\}$ by conditions on hyperfunctions on L. Then we need to pose conditions not only on their singular spectrums but also on their second singular spectrums along an involutive submanifold. This difficulty comes from the degeneracy of the partial Cauchy-Riemann system $\bar{\partial}_z$ on $\{\pm\sqrt{-1}dt\}$. In §3, we give the proof of the main theorems.

1. Statement of the main theorems.

Let $M=(-T, T)\times N$ be an open subset of $\boldsymbol{R}_t\times\boldsymbol{R}_x^n$ with a complexification X in $\boldsymbol{C}_w\times\boldsymbol{C}_z^n$ for a small $T>0$ and an open subset N in $\boldsymbol{R}_x^n$. We set

$$M_+ = \{(t, x)\in M; t\geqq 0\}.$$

We often identify N with the boundary $\{t=0\}\times N$. Let P and B_j $(1\leqq j\leqq m_+)$ be differential operators with real analytic coefficients on M of order m and m_j respectively:

$$P(t, x, D_t, D_x) = D_t^m+\sum_{k=0}^{m-1}A_k(t, x, D_x)\cdot D_t^k$$

and

$$B_j(x, D_t, D_x) = \sum_{k=0}^{m_j}B_{jk}(x, D_x)\cdot D_t^k \qquad (1\leqq j\leqq m_+).$$

Here $(D_t, D_x)=(\partial/\partial t, \partial/\partial x)$, $0\leqq m_+\leqq m$, $0\leqq m_j<m$ $(j=1, \cdots, m_+)$ and A_k's [resp. B_{jk}'s] are differential operators of order $\leqq m-k$ [resp. m_j-k]. We extend the definition of B_{jk} as 0 for $m_j<k\leqq m-1$.

We consider the boundary value problem for $u\in\mathscr{B}((0, T)\times N)$:

$$(1.1)\qquad \begin{cases} Pu = f(t, x) \\ B_ju|_{t\to+0} = g_j(x) \qquad (1\leqq j\leqq m_+) \end{cases}$$

where $f\in\mathscr{B}((0, T)\times N)$ and $g_j\in\mathscr{B}(N)$ $(1\leqq j\leqq m_+)$, and we assume that f is mild on $\{t=+0\}$. We formulate this problem as illustrated in Introduction. To this purpose, we put $f=0$, $g_j=0$ $(1\leqq j\leqq m_+)$. Then by the theory of non-characteristic boundary value problems due to Komatsu-Kawai [**14**] and Schapira [**19**], we find that the canonical extension $\tilde{u}(t, x)$ of any hyperfunction solution $u\in\mathscr{B}((0, T)\times N)$ to (1.1) satisfies the equation

$$(1.2)\qquad P\tilde{u}(t, x) = \sum_{k=0}^{m-1}Q_k(x, D_t, D_x)\cdot(D_t^ku(+0, x)\cdot\delta(t)),$$

where

$$(1.3)\qquad Q_k = D_t^{m-k-1}+\sum_{0\leq j+l\leq m-k-2}(-)^l\binom{j+l}{l}\cdot(\partial/\partial t)^lA_{j+l+k+1}(0, x, D_x)D_t^j.$$

Taking this identity into account, we define a coherent $\mathscr{D}_X$ module $\mathfrak{M}$ with generators $U, U_0, \cdots, U_{m-1}$ by the relations

$$(1.4)\qquad \mathfrak{M}:\begin{cases} PU = Q_0U_0+\cdots+Q_{m-1}U_{m-1}, \\ \sum_{k=0}^{m_j} B_{jk}(x, D_x)U_k = 0 \quad (1\leqq j\leqq m_+), \\ tU_k = 0 \quad (0\leqq k\leqq m-1). \end{cases}$$

Precisely U corresponds to the canonical extension $\tilde{u}(t, x)$ of u to M and

$$(1.5)\qquad U_k = D_t^k u(+0, x)\cdot\delta(t) \quad (0\leqq k\leqq m-1).$$

In the case of Dirichlet problem *i.e.* $B_j(x, D_t, D_x)=D_t^{j-1}$ $(1\leqq j\leqq m_+)$, $\mathfrak{M}$ reduces to $\mathcal{D}_X/\mathcal{D}_X t^{m-m_+}P$. Thus the microlocal analysis for (1.1) has been reduced to that of $\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{M_+}(\mathcal{B}_M))$. For that purpose, we estimate the microsupport of the solution sheaves to $\mathfrak{M}$ with value in hyperfunctions with holomorphic parameters.

To be more precise, we set

$$(1.6)\qquad \tilde{M} = \boldsymbol{R}_t\times \boldsymbol{C}_z^n\cap X, \qquad \tilde{M}_+ = \{(t, z)\in\tilde{M}\,;\, t\geqq 0\}$$

and

$$(1.7)\qquad \mathcal{BO} = \boldsymbol{R}\Gamma_{\tilde{M}}(\mathcal{O}_X)[1],$$

where the complex in (1.7) is concentrated in degree 0. (We follow Hartshorne [**2**] and Kashiwara-Schapira [**9**] for the notions of derived categories, derived functors and micro-supports of sheaves.)

We estimate the micro-support of $\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{\tilde{M}_+}(\mathcal{BO}))$ by the following Theorem 1.1 where $\mathfrak{M}$ is defined on X as a $\mathcal{D}_X$ module.

We take a coordinate system of $T^*\tilde{M}$ as $(t, z\,;\,\tau dt+\mathrm{Re}(\zeta\cdot dz))$ with $\tau\in\boldsymbol{R}$ and $z=x+\sqrt{-1}y$, $\zeta=\xi+\sqrt{-1}\eta\in\boldsymbol{C}^n$.

THEOREM 1.1. *Take a point* $\rho_0=(\mathring{t}, \mathring{z}\,;\,\mathring{\tau}dt+\mathrm{Re}(\mathring{\zeta}\cdot dz))\in T^*\tilde{M}\setminus\tilde{M}$ *with* $\mathring{\zeta}\neq 0$. *Assume the condition* (A1) *in case* $\mathring{t}>0$, *and the conditions* (A1) *and* (A2) *in case* $\mathring{t}=0$;

(A1) $H(\theta)=\sigma_m(P)(\mathring{t}, \mathring{z}, \mathring{\tau}+\theta, \mathring{\zeta})=0$ *has* $m-m_+$ *roots with respect to* θ *in* $\{\mathrm{Re}\,\theta>0\}$ *and* m_+ *roots in* $\{\mathrm{Re}\,\theta<0\}$.

(A2) *Under the assumption* (A1), *we decompose*

$$H(\theta) = H_+(\theta)\cdot H_-(\theta)$$

so that $H_+(\theta)\neq 0$ *on* $\{\mathrm{Re}\,\theta\leqq 0\}$ *and* $H_-(\theta)\neq 0$ *on* $\{\mathrm{Re}\,\theta\geqq 0\}$. *Define the Lapatinskii polynomial*

$$(1.8)\qquad R_j(\mathring{z}, \mathring{\tau}, \mathring{\zeta})(\theta) = \sum_{k=0}^{m_+-1}\beta_{jk}(\mathring{z}, \mathring{\tau}, \mathring{\zeta})\cdot(\theta+\mathring{\tau})^k$$

$$\equiv \sigma_{m_j}(B_j)(\mathring{z}, \theta+\mathring{\tau}, \mathring{\zeta}) \qquad \text{modulo } H_-(\theta)$$

for $j=1, \cdots, m_+$, *and assume*

$$\det(\beta_{j,k-1}(\mathring{z}, \mathring{\tau}, \mathring{\zeta}))_{1\leq j,k\leq m_+} \neq 0.$$

Then $\rho_0 \notin SS(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O})))$.

REMARK. i) When $\mathring{t}>0$, the condition (A1) can be replaced by a weaker condition that $H(\theta)=\sigma_m(P)(\mathring{t}, \mathring{z}, \mathring{\tau}+\theta, \mathring{\zeta})=0$ has no pure imaginary roots with respect to θ (see Kashiwara-Kawai [**6**]).

ii) For 0-th cohomology group, the above theorem corresponds to the Zerner type theorem for holomorphic boundary value problems due to A. Martinez [**15**]. (Refer also to Introduction.)

iii) Inspired by Kashiwara-Kawai [**6**], we took one parameter t as a real variable to simplify the problem.

By Theorem 1.1, we can give theorems concerning hyperfunction solutions to the hyperbolic mixed problems. Theorem 1.2 asserts the unique prolongation of solutions to boundary value problems, and Theorem 1.3 the unique solvability of mixed problems in our sense.

THEOREM 1.2. *We assume the following conditions* (H1) *and* (H2). *We set* $\mathring{x}^*=(1, 0, \cdots, 0)\in\boldsymbol{R}^n$:

(H1) $\sigma_m(P)$ *is hyperbolic with respect to* x_1, *that is*

$$\sigma_m(P)(t, x, \theta, \sqrt{-1}\eta+\varepsilon\mathring{x}^*) \neq 0$$

on $\{0<\varepsilon<\delta, 0\leqq t<\delta, |x-\mathring{x}|<\delta, |\eta|<\delta, \theta\in\sqrt{-1}\boldsymbol{R}\}$ *for some positive* δ.

(H2) *The equation* $H(\theta)=\sigma_m(P)(0, \mathring{x}, \theta, \mathring{x}^*)=0$ *with respect to* θ *has* $m-m_+$ *positive real roots and* m_+ *negative real roots. Moreover we assume*

$$\det(\beta_{j,k-1}(x+\sqrt{-1}\varepsilon y, \varepsilon\tau, \varepsilon\xi+\sqrt{-1}\eta))_{1\leq j,k\leq m_+} \neq 0$$

on $\{0<\varepsilon<\delta, |x-\mathring{x}|+|\eta|<\delta, |\xi-\mathring{x}^*|+|y|+|\tau|<\delta\}$ *for* (β_{jk}) *defined in* (1.8). *Let* u *be a hyperfunction solution to*

$$(1.9)\qquad \begin{cases} P(t, x, D_t, D_x)u(t, x)=0 \\ B_j(x, D_t, D_x)u(+0, x)=0 \qquad (1\leqq j\leqq m_+) \end{cases}$$

defined on $\{(t, x)\in M; 0<t<\delta, |x-\mathring{x}|<\delta, x_1<\mathring{x}_1\}$. *Then* u *extends uniquely to* $\{0<t<\delta', |x-\mathring{x}|<\delta'\}$ *as a hyperfunction solution to* (1.9) *for a small* $\delta'>0$.

Before stating the further results precisely, we recall the sheaf $\mathring{\mathcal{B}}_{N|M_+}$ [resp. $\mathring{C}_{N|M_+}$] of mild hyperfunctions on N [resp. microfunctions on $\sqrt{-1}\mathring{T}^*N$]. ($\mathring{\mathcal{B}}_{N|M_+}$ [resp. $\mathring{C}_{N|M_+}$] was once denoted as $\hat{\mathcal{B}}_{N|M_+}$ [resp $\hat{C}_{N|M_+}$].) Roughly $\mathring{\mathcal{B}}_{N|M_+}$ is the sheaf of hyperfunctions over $\{0<t<\delta, |x-\mathring{x}|<\delta\}$ with trace at $\{t=0\}$, and $\mathring{C}_{N|M_+}$ is its microlocalization. See [**11**], [**12**], [**24**], §2.4 and §2.6 of this paper: that is, there are sheaf morphisms

$$
(1.10)\qquad
\begin{array}{ll}
\text{Trace}: & \begin{array}{l}\mathring{\mathscr{B}}_{N|M_+}\ni f(t,x)\longmapsto f(+0,x)\in\mathscr{B}_N\\ \mathring{\mathcal{C}}_{N|M_+}\ni f(t,x)\longmapsto f(+0,x)\in\mathcal{C}_N\end{array}\\[2ex]
\text{ext}: & \begin{array}{l}\mathring{\mathscr{B}}_{N|M_+}\ni f(t,x)\longmapsto f(t,x)\cdot Y(t)\in\Gamma_{M_+}(\mathscr{B}_M)|_N\\ \mathring{\mathcal{C}}_{N|M_+}\ni f(t,x)\longmapsto f(t,x)\cdot Y(t)\in\iota_*(\mathcal{C}_M|_{\sqrt{-1}(T^*M\times_M N\setminus T^*_N M)})\end{array}
\end{array}
$$

where $\iota\colon\sqrt{-1}(T^*M\underset{M}{\times}N\setminus T^*_N M)\to\sqrt{-1}\mathring{T}^*N$ is the projection.

THEOREM 1.3. *We assume the same conditions as in Theorem* 1.2. *Let* $f(t,x)$ *and* $g_j(x)$ $(1\leqq j\leqq m_+)$ *be hyperfunctions respectively defined on* $\{0<t<\delta,\ |x-\mathring{x}|<\delta\}$ *and* $\{|x-\mathring{x}|<\delta\}$ *such that* $f(t,x)$ *is mild on* $\{t=+0\}$ *and that*

$$\operatorname{supp}(f)\subset\{x_1\geqq\mathring{x}_1\}\quad\textit{and}\quad\operatorname{supp}(g_j)\subset\{x_1\geqq\mathring{x}_1\}\quad(1\leqq j\leqq m_+).$$

Then there exists a unique hyperfunction solution $u(t,x)$ *on* $\{0<t<\delta',\ |x-\mathring{x}|<\delta'\}$ *to the problem*

$$
(1.11)\qquad
\begin{cases}
P(t,x,D_t,D_x)u(t,x)=f(t,x)\\
B_j(x,D_t,D_x)u(+0,x)=g_j(x)\qquad(1\leqq j\leqq m_+)\\
\operatorname{supp}(u)\subset\{x_1\geqq\mathring{x}_1\},
\end{cases}
$$

for some small $\delta'>0$.

REMARK. The assumptions in Theorem 1.3 is well-known since Sakamoto [**16**] in case P and B_j's are with constant coefficients. Kajitani-Wakabayashi [**3**] obtained a similar result in Gevrey classes.

We microlocalize the results given in Theorem 1.2 and Theorem 1.3. To describe the theorems elegantly, we introduce the sheaf $\mathring{\mathcal{C}}_{R_+\times N}$ on $\mathring{T}^*_M\tilde{M}$ (see § 2.6). Hereafter we make an identification

$$T^*_M\tilde{M}\simeq(-T,T)\times\sqrt{-1}T^*N,$$

through which we regard $\sqrt{-1}T^*N$ as $\{t=0\}\times\sqrt{-1}T^*N$. Moreover the triple

$$M\longrightarrow\tilde{M}\longrightarrow X$$

induces a projection

$$
\begin{array}{ccc}
\hat{\iota}\colon \mathring{T}^*_M X\setminus T^*_{\tilde{M}}X & \longrightarrow & \mathring{T}^*_M\tilde{M}\\
\Cup & & \Cup\\
(t,x;\sqrt{-1}(\tau,\eta)) & \longmapsto & (t,x;\sqrt{-1}\eta).
\end{array}
$$

Then the sheaf $\mathring{\mathcal{C}}_{R_+\times N}$ of $\mathscr{D}_X$-modules associated to the product structure $(-T,T)\times N$ of M can be given as

$$
(1.12)\qquad
\mathring{\mathcal{C}}_{R_+\times N}=
\begin{cases}
0 & \text{on } \mathring{T}^*_M\tilde{M}\cap\{t<0\}\\
\mathring{\mathcal{C}}_{N|M_+} & \text{on } \mathring{T}^*_M\tilde{M}\cap\{t=0\}\\
\hat{\iota}_!(\mathcal{C}_M|_{\mathring{T}^*_M X\setminus T^*_{\tilde{M}}X}) & \text{on } \mathring{T}^*_M\tilde{M}\cap\{t>0\}.
\end{cases}
$$

For we have a natural $\mathscr{D}_X$-homomorphism:

$$\hat{C}_{N|M_+} \longrightarrow k_* \hat{\imath}_! (C_M|_{\mathring{T}^*_M X \setminus T^*_{\tilde{M}} X})|_{\{t=0\}}$$

with the embedding $k: \mathring{T}^*_{\tilde{M}} \tilde{M} \cap \{t>0\} \to \mathring{T}^*_{\tilde{M}} \tilde{M}$ (see (2.10) and recall that $C_{N|X} \subset \Gamma_{N \times_M T^*_M X}(C_M)$ on $\mathring{T}^*_N X \cap \mathring{T}^*_{\tilde{M}} X$). The sheaf $\hat{C}_{R_+ \times N}$ can be constructed in another way (see § 2.6). Roughly the sheaf $\hat{C}_{R_+ \times N}$ gives the cotangential decomposition with respect to the variable x of hyperfunctions which are mild at $t=+0$ and depend analytically on t in $\{t>0\}$. Indeed, take an open subset U in M and a hyperfunction $f(t, x)$ defined on $U \cap \{t>0\}$. Then if $f(t, x)$ is mild on $U \cap \{t=0\}$ from $t=+0$ and satisfies the condition

$$SS(f) \cap \{(t, x; \pm\sqrt{-1}dt); t>0, (t, x) \in U\} = \emptyset,$$

$f(t, x)$ gives a section $[f(t, x)]$ of $\hat{C}_{R_+ \times N}$ on $U \times_M \mathring{T}^*_{\tilde{M}} \tilde{M}$. In this case, for a point $(\mathring{t}, \mathring{x}) \in U$ with $\mathring{t} \geqq 0$,

$$[f(t, x)] = 0 \qquad \text{on } \mathring{T}^*_{\tilde{M}} \tilde{M}|_{(\mathring{t}, \mathring{x})}$$

if and only if $f(t, x)$ is analytic at $(\mathring{t}, \mathring{x})$ (or extends analytically to a full neighborhood of $(0, \mathring{x})$ in case $\mathring{t}=0$). Further if we take the canonical extension $\tilde{f}(t, x)$ of $f(t, x)$, we can write

(1.13) $\qquad$ (the support of $[f(t, x)]$ in $\hat{C}_{R_+ \times N}$) $= \hat{\imath}(SS(\tilde{f}(t, x)) \setminus T^*_{\tilde{M}} X)$.

In particular, for any hyperfunction solution $u(t, x)$ of (1.9), we have

(1.14) $\qquad$ (the support of $[u(t, x)]$ in $\hat{C}_{R_+ \times N}$)

$$= \bigcup_{j=0}^{m-1} \{(0, x; \sqrt{-1}\eta); (x; \sqrt{-1}\eta) \in SS(D_t^j u(+0, x))\}$$

$$\cup \hat{\imath}(SS(u(t, x)) \cap \{t>0\} \setminus T^*_{\tilde{M}} X).$$

THEOREM 1.4. *We take a real valued* C^1 *function* $\psi(t, x, \eta)$ *defined in a neighborhood of* $(t=0, x=\mathring{x}, \eta=\mathring{\eta}) \in \boldsymbol{R} \times \mathring{T}^* N$ *such that* $\psi(0, \mathring{x}, \mathring{\eta})=0$, $d\psi(0, \mathring{x}, \mathring{\eta}) \wedge dt \neq 0$ *and that* ψ *is homogeneous of order* 0 *with respect to* η. *We assume the following conditions* (S1) *and* (S2) *with* $\mathring{t}^* dt + \mathring{x}^* dx + \mathring{\eta}^* d\eta = d\psi(0, \mathring{x}, \mathring{\eta})$.

(S1) $\qquad$ *There exists a small number* $\delta>0$ *such that*

$$\sigma_m(P)(t, x+\sqrt{-1}\varepsilon\mathring{\eta}^*, \theta+\varepsilon\mathring{t}^*, \sqrt{-1}\eta+\varepsilon\mathring{x}^*) \neq 0$$

on $\{0<\varepsilon<\delta, 0 \leqq t<\delta, |x-\mathring{x}|<\delta, |\eta-\mathring{\eta}|<\delta, \theta \in \sqrt{-1}\boldsymbol{R}\}$,

(S2) *the equation for* θ

$$\sigma_m(P)(0, \mathring{x}+\sqrt{-1}\varepsilon\mathring{\eta}^*, \theta+\varepsilon\mathring{t}^*, \sqrt{-1}\mathring{\eta}+\varepsilon\mathring{x}^*) = 0$$

has $m-m_+$ *roots with* $\operatorname{Re}\theta>0$ *and* m_+ *roots with* $\operatorname{Re}\theta<0$ *for any* $0<\varepsilon<\delta$. *Moreover*

$$\det(\beta_{j,k-1}(x+\sqrt{-1}\varepsilon\eta^*,\ \varepsilon t^*,\ \sqrt{-1}\eta+\varepsilon x^*))_{1\le j,k\le m_+}\neq 0$$

on $\{0<\varepsilon<\delta,\ |x-\mathring{x}|+|\eta-\mathring{\eta}|<\delta,\ |\eta^*-\mathring{\eta}^*|+|x^*-\mathring{x}^*|+|t^*-\mathring{t}^*|<\delta\}$ *for* (β_{jk}) *defined in* (1.8).

Let u *be any* $\hat{C}_{R_+\times N}$ *solution to the boundary value problem*

$$(1.15)\qquad \begin{cases} P(t,x,D_t,D_x)u(t,x)=0 \\ B_j(x,D_t,D_x)u(+0,x)=0 \qquad (1\leqq j\leqq m_+) \end{cases}$$

on

$$\{(t,x;\sqrt{-1}\eta);\ \psi(t,x,\eta)<0,\ |t|+|x-\mathring{x}|+|\eta-\mathring{\eta}|<\delta\}.$$

Then u *extends uniquely to* $(0,\mathring{x};\sqrt{-1}\mathring{\eta})$ *as a* $\hat{C}_{N|M_+}$ *solution to* (1.15).

REMARK. i) The Dirichlet problem, i.e. $B_j=D_t^{j-1}$ ($j=1,\cdots,m_+$), always satisfies the latter part of (S2) concerning Lopatinskii polynomials.
ii) The uniqueness part in the above theorem is equivalent to Sjöstrand's result in [28]. Precisely, if a solution u of the equation (1.15) defined in a neighborhood W of $(0,\mathring{x};\sqrt{-1}\mathring{\eta})$ vanishes in $\{\psi(t,x,\eta)<0\}\cap W$, then u vanishes at $(0,\mathring{x};\sqrt{-1}\mathring{\eta})$ as a germ of $\hat{C}_{R_+\times N}$ (see (1.14)). Moreover the boundary value problem

$$\begin{cases} P(t,x,D_t,D_x)u=\{D_t^2-(\sqrt{-1}t+x_1)D_{x_2}^2\}u=0 \\ u(+0,x)=0 \end{cases}$$

with $\psi(t,x,\eta)=\eta_1$ and $\rho_1=(\mathring{t}=0,\ \mathring{x}=0;\ \mathring{\eta}=(0,1);\ d\eta_1)$ satisfies the assumption of Theorem 1.4. However it does not belong to the class of Sjöstrand [28]: recall that our assumptions are concerned with only $t\geqq 0$ part.

Finally we give a theorem concerning the solvability of microhyperbolic mixed problems.

THEOREM 1.5. *We take a real valued* C^1 *function* $\psi(t,x,\eta)$ *as in Theorem* 1.4, *and assume the same conditions as in Theorem* 1.4. *Let* $f(t,x)$ [*resp.* $(g_j)x))_{1\le j\le m_+}$] *be a section of* $\hat{C}_{R_+\times N}$ [*resp.* $C_N^{m_+}$] *defined in a neighborhood of* $(0,\mathring{x};\sqrt{-1}\mathring{\eta})$ [*resp.* $(\mathring{x};\sqrt{-1}\mathring{\eta})$] *satisfying*

$$\operatorname{supp}(f)\subset\{\psi(t,x,\eta)\geqq 0\},\quad \operatorname{supp}(g_j)\subset\{\psi(0,x,\eta)\geqq 0\}\quad (j=1,\cdots,m_+).$$

Then there exists a unique section $u(t,x)$ *of* $\hat{C}_{R_+\times N}$ *defined in a neighborhood of* $(0,\mathring{x};\sqrt{-1}\mathring{\eta})$ *satisfying*

$$(1.16)\qquad \begin{cases} P(t,x,D_t,D_x)u(t,x)=f(t,x), \\ B_j(x,D_t,D_x)u(+0,x)=g_j(x) \qquad (1\leqq j\leqq m_+), \\ \operatorname{supp}(u)\subset\{\psi(t,x,\eta)\geqq 0\}. \end{cases}$$

2. Preliminaries.

2.1. Microlocal study of sheaves. We fix some notation concerning microlocal study of sheaves due to Kashiwara-Schapira [**9**], [**10**]. Let X be a C^1 manifold of dimension n. Then $D(X)$ denotes the derived category of complexes of sheaves of modules on X, and $D^+(X)$ denotes the subcategory of $D(X)$ consisting of complexes with cohomologies bounded from below. See Hartshorne [**2**] for the notions of derived categories and derived functors.

We recall the notion of microsupport due to Kashiwara-Schapira [**9**]. Let $F \in Ob(D^+(X))$, and $\mathring{p}=(\mathring{x}, \mathring{\xi}\cdot dx) \in T^*X$. Then we give

DEFINITION 2.1. $\mathring{p} \notin SS(F)$ if one of the following equivalent conditions (μS1) or (μS2) is satisfied.

(μS1) If there exists an open neighborhood U of $\mathring{p}$ such that for any $a \in M$ and any real C^1 function f defined in a neighborhood of a with $(a; df(a)) \in U$, we have

$$\boldsymbol{R}\Gamma_{\{f(x) \geqq f(a)\}}(F)|_a = 0 .$$

(μS2) If we take a local coordinate system around $\mathring{x}$, there exist a neighborhood V of $\mathring{x}$, an $\varepsilon>0$ and a proper closed convex cone G in $\boldsymbol{R}^n$ with $0 \in G$ and $G \setminus \{0\} \subset \{\gamma; \langle \gamma, \mathring{\xi} \rangle < 0\}$, such that for

$$H=\{x; \langle x-\mathring{x}, \mathring{\xi} \rangle \geqq -\varepsilon\} \quad \text{and} \quad L=\{x; \langle x-\mathring{x}, \mathring{\xi} \rangle = -\varepsilon\},$$

we have

$$\boldsymbol{R}\Gamma(H \cap (a+G), F) \xrightarrow{\sim} \boldsymbol{R}\Gamma(L \cap (a+G), F)$$

for all $a \in V$.

We give some remarks: we may assume in (μS1) that f is real analytic in a coordinate system to show $\mathring{p} \notin SS(F)$. This fact is shown in [**9**]. Moreover it can be supposed in (μS1) that $\{f<0\}$ is strictly convex in a neighborhood of a. Precisely, we fix a local coordinate system x in a neighborhood V' of $\mathring{x}$ and give

PROPOSITION 2.2. *Assume that there exists an open subset U of $\mathring{p}$ such that for any $a \in V'$ and any real C^ω and strictly convex function f of x defined in a neighborhood of a with $(a; df(a)) \in U$, we have*

$$RF_{\{f(x) \geqq f(a)\}}(F)|_a = 0 .$$

Then $\mathring{p} \notin SS(F)$ follows.

PROOF. We may assume $\mathring{\xi}=(1, 0, \cdots, 0)$ and $\mathring{x}=0$. Set $x=(x_1, x')$ with $x'=(x_2, \cdots, x_n)$. Then it suffices to show (μS2) under the assumption of this proposition. Putting

$$G=\{x\in \boldsymbol{R}^n\,;\, x_1\leqq -\delta|x'|\},$$

we take small numbers δ, $\varepsilon>0$ and a neighborhood V of 0 in $\boldsymbol{R}^n$ such that

$$(\{x_1\geqq -\varepsilon\}\cap(V+G))\times(-G^\circ)\subset U\,. \tag{2.1}$$

Here G° denotes the polar set of G. We set H and L as in (μS2).

We define a family of C^ω functions for $t>0$:

$$\varphi_t(x')=\frac{1+t|x'|^2}{1+\{t(1+(t-1)|x'|^2)\}^{1/2}} \qquad \text{on } W_t$$

with $W_t=\boldsymbol{R}^{n-1}_{x'}$ ($t\geqq 1$) and $W_t=\{x'\in \boldsymbol{R}^{n-1}_{x'}\,;\, |x'|^2<1/(1-t)\}$ ($0<t<1$). First we note that $\varphi_t(x')$ satisfies the following properties:

(i) $$\varphi_t(x')=1-\int_{|x'|^2}^1 \frac{1}{2}\cdot\Big(\frac{t}{1-(1-t)s}\Big)^{1/2} ds \qquad \text{on } W_t$$

and $$\frac{\partial\varphi_t}{\partial t}=-\int_{|x'|^2}^1 \frac{1-s}{4\sqrt{t}\cdot(1-(1-t)s)^{3/2}}ds\leqq 0 \qquad \text{on } W_t\,.$$

(ii) $$\sum_{j,k=2}^n \frac{\partial^2\varphi_t}{\partial x_j\partial x_k}\xi_j\xi_k\geqq\Big(\frac{t}{1-(1-t)|x'|^2}\Big)^{1/2}\cdot\frac{|\xi'|^2}{1+|1-t|\,|x'|^2}$$

for $(x,\xi')\in W_t\times \boldsymbol{R}^{n-1}$.

Set a family of open subsets of X increasing in t:

$$\Omega_t(a)=\Big\{x\,;\, a_1-x_1>(\varepsilon+a_1)\cdot\varphi_t\Big(\frac{\delta(x'-a')}{\varepsilon+a_1}\Big),\ \frac{\delta}{\varepsilon+a_1}(x'-a')\in W_t\Big\}.$$

We show that the natural morphism

$$\boldsymbol{R}\Gamma(\Omega_t(a),F|_H)\xrightarrow{\sim}\boldsymbol{R}\Gamma(\Omega_s(a),F|_H) \tag{2.2}$$

is isomorphic in case $t\geqq s>0$. Put $j_t:\Omega_t(a)\to X$. Then applying the argument similar to $(1)_\omega\to(3)$ of Theorem 3.1.1 of [**9**], it is sufficient to show

$$\boldsymbol{R}\Gamma_{X\setminus\Omega_{t'}(a)}(\boldsymbol{R}j_{t*}j_t^{-1}(F|_H))_y=0 \tag{2.3}$$

for any $y\in(Z_s(a)\setminus\Omega_{t'}(a))\cap H$ and any $t>t'\geqq s>0$. Here we put

$$Z_s(a)=\bigcap_{s'>s}\overline{(\Omega_{s'}(a)\setminus\Omega_s(a))}=\partial\Omega_s(a)\,.$$

Moreover it suffices to show (2.3) in the case $t'=s$, $y\in\partial\Omega_s(a)\cap H$. If $y\in\partial\Omega_s(a)\cap\operatorname{Int}H$, this comes from (2.1) and the assumption of this proposition. For (2.3) is equivalent to $\boldsymbol{R}\Gamma_{X\setminus\Omega_s(a)}(F)|_y=0$. Moreover let $y\in\partial\Omega_s(a)\cap L$. Then remark that $\boldsymbol{R}\Gamma_{X\setminus\Omega_s(a)}(\boldsymbol{R}j_{t*}j_t^{-1}(F|_H))_y$ is a direct summand of $\boldsymbol{R}\Gamma_{X\setminus\Omega_s(a)}(\boldsymbol{R}j_{t*}j_t^{-1}F)_y$. This implies (2.3) if we keep in mind (2.1) and the distinguished triangle:

$$\boldsymbol{R}\Gamma_{X\setminus\Omega_t(a)}(F)|_y\longrightarrow\boldsymbol{R}\Gamma_{X\setminus\Omega_s(a)}(F)|_y\longrightarrow\boldsymbol{R}\Gamma_{X\setminus\Omega_s(a)}(\boldsymbol{R}j_{t*}j_t^{-1}F)|_y\xrightarrow{+1}\,.$$

Hence we get the isomorphism (2.2). Finally taking the projective limit of the isomorphism (2.2) as $t\to\infty$, we deduce the isomorphism

$$\boldsymbol{R}\Gamma((a+\mathrm{Int}\, G), F|_H) \xrightarrow{\sim} \boldsymbol{R}\Gamma(\Omega_t(a), F|_H) \qquad \text{for any } t>0\,.$$

Finally the same argument as in Theorem 3.1.1 of [**9**] leads us to (μS2).

q. e. d.

We denote the canonical 1-form of T^*X as ω_X. Then $d\omega_x$ induces an isomorphism

$$(2.4) \qquad \begin{aligned}(-H)^{-1}: TT^*X &\xrightarrow{\sim} T^*T^*X \\ v &\longmapsto -d\omega_X(\cdot \wedge v)\,.\end{aligned}$$

When we take a local coordinate system of X as $x\in\boldsymbol{R}^n$ and that of T^*X as $(x, \xi\cdot dx)$ with $\xi\in\boldsymbol{R}^n$, then the isomorphism is written by coordinates as

$$(2.5) \qquad (x, \xi\,;\, \tilde{x}\cdot\partial/\partial x+\tilde{\xi}\cdot\partial/\partial\xi) \longmapsto (x, \xi\,;\, \tilde{\xi}\cdot dx-\tilde{x}\cdot d\xi)\,.$$

In case X is a complex manifold, T^*X is endowed with a structure of real homogeneous symplectic manifolds by a 1-form $\omega_{X_R}=\mathrm{Re}\,\omega_X$. Here ω_X is the canonical 1-form of T^*X. Thus we identify TT^*X with T^*T^*X by $(-H^R)^{-1}$. (H^R is the Hamiltonian isomorphism induced by ω_{X_R}.) If we take a coordinate system of T^*X as $(z, \zeta\cdot dz)$ with $z=x+\sqrt{-1}y$, $\zeta=\xi+\sqrt{-1}\eta$, this identification is given explicitly by

$$\begin{aligned}&(x, y, \xi, \eta\,;\, \tilde{x}\partial/\partial x+\tilde{y}\partial/\partial y+\tilde{\xi}\partial/\partial\xi+\tilde{\eta}\partial/\partial\eta)\\ &\qquad\longmapsto (x, y, \xi, \eta\,;\, \tilde{\xi}dx-\tilde{\eta}dy-\tilde{x}d\xi+\tilde{y}d\eta)\,.\end{aligned}$$

Let M be a closed submanifold of X. Then for $F\in Ob(D^+(X))$, we have an estimate of the microsupport of $\boldsymbol{R}\Gamma_M(F)$ as

$$(2.6) \qquad SS(\boldsymbol{R}\Gamma_M(F)) \subset C_{T^*_M X}(SS(F))\cap T^*M\,.$$

Here $C_{T^*_M X}(\cdot)$ denotes the normal cone along $T^*_M X$, which is identified with a conic closed subset in $T^*T^*_M X$ through

$$(-H)^{-1}: T_{T^*_M X}T^*X \longrightarrow T^*T^*_M X\,,$$

and T^*M is identified with a subset of $T^*T^*_M X$ through the embedding $T^*M\to T^*T^*_M X$ induced from $T^*_M X\to M$. See Chapter 1 of Kashiwara-Schapira [**9**] and § 2.2 below for normal cones.

Moreover $\mu_M(F)$ denotes Sato's microlocalization of F along M, which is an object of $D^+(T^*_M X)$. See Chapter 2 of Kashiwara-Schapira [**9**] for its definition. We quote an important formula for the microsupport of $\mu_M(F)$. Explicitly, we have

$$SS(\mu_M(F)) \subset C_{T_M^*X}(SS(F)). \tag{2.7}$$

In case M is an open subset of $\boldsymbol{R}_x^n$ with a complex neighborhood X in $\boldsymbol{C}_z^n$, take coordinates of T^*X [resp. T_M^*X] as $(z=x+\sqrt{-1}y\,;\zeta=\xi+\sqrt{-1}\eta)$ [resp. $(x,\sqrt{-1}\eta)$]. Then $(\mathring{x},\sqrt{-1}\mathring{\eta}\,;\mathring{x}^*dx+\mathring{\eta}^*d\eta)\notin C_{T_M^*X}(SS(F))$ if and only if there exists a positive δ such that

$$\{(x+\sqrt{-1}\varepsilon y\,;\varepsilon\xi+\sqrt{-1}\eta)\in T^*X\,;\ |x-\mathring{x}|+|\eta-\mathring{\eta}|<\delta,$$
$$|y-\mathring{\eta}^*|+|\xi-\mathring{x}^*|<\delta\}\cap SS(F)=\emptyset$$

for any ε with $0<\varepsilon<\delta$.

2.2. Normal cones. Let X be a C^1 manifold and M be a closed submanifold of X. We take a local coordinate system around $q_0\in M$ of X as $(t,x)\in \boldsymbol{R}_t^d\times\boldsymbol{R}_x^{n-d}$ so that $M=\{t=0\}$ and $q_0=(0,x_0)$. We take coordinates of T_MX as $(x,\tilde{t}\cdot\partial/\partial t)$.

For a subset Z in X, $v_0=(x_0,\tilde{t}_0)\notin C_M(Z)$ if and only if there exists a positive number δ such that

$$\{(\varepsilon\tilde{t},x)\in X\,;\ |x-x_0|<\delta,\ |\tilde{t}-\tilde{t}_0|<\delta,\ 0<\varepsilon<\delta\}\cap Z=\emptyset.$$

See Kashiwara-Schapira [8], [9] for details about normal cones.

2.3. Second microlocal analysis. Let M be a real analytic manifold with a complexification X, and Σ a regular involutive submanifold in $\mathring{T}_M^*X$. Take a complexification Λ of Σ in T^*X and set $\tilde{\Sigma}$ as the union of all bicharacteristic leaves of Λ passing through Σ which is called a partial complexification of Σ. On $\tilde{\Sigma}$ there exists the sheaf $\mathcal{C}_{\tilde{\Sigma}}$ of microfunctions along $\tilde{\Sigma}$, which is isomorphic to a sheaf of microfunctions with holomorphic parameters through a quantized contact transformation and defined up to isomorphisms.

M. Kashiwara constructed the sheaf $\mathcal{C}_{\tilde{\Sigma}}^2$ of 2-microfunctions along Σ on $T_{\Sigma}^*\tilde{\Sigma}$, which satisfies the exact sequences:

$$0\longrightarrow \mathcal{C}_{\tilde{\Sigma}}|_{\Sigma}\longrightarrow \mathcal{B}_{\Sigma}^2\longrightarrow \mathring{\pi}_{\Sigma*}(\mathcal{C}_{\tilde{\Sigma}}^2|_{\mathring{T}_{\Sigma}^*\tilde{\Sigma}})\longrightarrow 0$$

and

$$0\longrightarrow \mathcal{C}_M|_{\Sigma}\longrightarrow \mathcal{B}_{\Sigma}^2.$$

Here we set $\mathcal{B}_{\Sigma}^2=\mathcal{C}_{\tilde{\Sigma}}^2|_{\Sigma}$ and $\mathring{\pi}_{\Sigma}:\mathring{T}_{\Sigma}^*\tilde{\Sigma}\to\Sigma$. Moreover we have a canonical spectrum morphism

$$Sp_{\Sigma}^2:\mathring{\pi}_{\Sigma}^{-1}\mathcal{B}_{\Sigma}^2\longrightarrow \mathcal{C}_{\tilde{\Sigma}}^2|_{\mathring{T}_{\Sigma}^*\tilde{\Sigma}}.$$

Then we define the second singular spectrum of u along Σ for $u\in\mathcal{B}_{\Sigma}^2$ (or

$u \in \mathcal{C}_{M}|_{\Sigma})$ by

$$SS^2_{\Sigma}(u) = \mathrm{supp}\,(Sp^2_{\Sigma}(u))\,.$$

Especially when M is an open subset of $\boldsymbol{R}^{n-d}_t \times \boldsymbol{R}^d_x$ $(\subset \boldsymbol{C}^{n-d}_w \times \boldsymbol{C}^d_z)$ and Σ is written as

$$\Sigma = \{(t,\, x\,;\, \sqrt{-1}(\tau dt + \xi dx)) \in T^*_M X\,;\, \xi = 0\}\,,$$

$\mathcal{C}_{\tilde{\Sigma}}$ is nothing but the sheaf of microfunctions with holomorphic parameter z. Moreover $\mathcal{C}^2_{\Sigma}$ is given by

$$\mathcal{C}^2_{\Sigma} = \mu_{\Sigma}(\mathcal{C}_{\tilde{\Sigma}})[d]\,.$$

See Kashiwara-Laurent [**7**] for details about 2-microfunctions, and also Bony-Schapira [**1**] and Kashiwara-Schapira [**8**] for the relation between second singular spectrums and micro-characteristic varieties.

2.4. Elliptic boundary value problems. We recall here the microlocal theory of elliptic boundary value problems due to Schapira [**20**] and Kataoka [**11**], [**12**].

Let $M = (-T,\, T) \times N$ be an open subset of $\boldsymbol{R}_t \times \boldsymbol{R}^n_x$ with a complexification X in $\boldsymbol{C}_w \times \boldsymbol{C}^n_z$ for some $T > 0$ and an open subset N in $\boldsymbol{R}^n_x$. We set

$$M_+ = \{(t,\, x) \in M\,;\, t \geqq 0\}$$

and identify N with the boundary $\{t=0\} \times N$.

Following Kashiwara-Kawai [**4, 5**] and Kataoka [**13**], we introduce sheaves $\mathcal{C}_{N|X}$ and $\mathcal{C}_{M_+|X}$:

$$\mathcal{C}_{N|X} = \mathcal{H}^{n+1}_{S^*_N X}(\pi^{-1}_{N|X}\mathcal{O}_X)\,, \qquad \mathcal{C}_{M_+|X} = \mathcal{H}^{n+1}_{S^*_{M_+}X}(\pi^{-1}_{M_+|X}\mathcal{O}_X) \tag{2.8}$$

which are considered as sheaves on $\mathring{T}^*_N X$ and $\mathring{T}^*_{M_+} X$ respectively. Here

$$(2.9)\qquad \begin{aligned} T^*_N X &= \{(x\,;\, \theta dw + \sqrt{-1}\eta dx) \in N \times \boldsymbol{C} \times \sqrt{-1}\boldsymbol{R}^n\}\,, \\ T^*_{M_+} X &= \{(t,\, x\,;\, \theta dw + \sqrt{-1}\eta dx) \in \boldsymbol{R} \times N \times \boldsymbol{C} \times \sqrt{-1}\boldsymbol{R}^n\,; \\ &\qquad\qquad t \geqq 0,\ \mathrm{Re}\,\theta \geqq 0,\ t\cdot(\mathrm{Re}\,\theta) = 0\}\,, \end{aligned}$$

which are conic closed subsets of T^*X depending only on N and M_+. Further $S^*_N X$ and $S^*_{M_+} X$ are the corresponding sphere bundles. We note that they are also written as

$$\begin{aligned} \mathcal{C}_{N|X} &= \mu_N(\mathcal{O}_X)[n+1]\,, \\ \mathcal{C}_{M_+|X} &= \mu\,\mathrm{hom}\,(\boldsymbol{Z}_{M_+},\, \mathcal{O}_X)[n+1]\,. \end{aligned}$$

by using the microlocalization functor μ_N (see § 2.1). Refer to Kashiwara-Schapira [**9**] for the definition of the bifunctor $\mu\,\mathrm{hom}\,(\cdot,\, \cdot)$.

Fundamental properties of these sheaves are as follows (see [**13**] for more details).

PROPOSITION 2.3. *Let* $\mathring{\pi}_{M_+|X} : \mathring{T}^*_{M_+}X \to M_+$ *be the canonical projection. Then we have a sheaf isomorphism*

$$\Gamma_{M_+}(\mathcal{B}_M)|_N \xrightarrow{\sim} (\mathring{\pi}_{M_+|X})_* \mathcal{C}_{M_+|X}|_N .$$

Further $\mathcal{C}_{M_+|X}$ *satisfies the following:*

$$\mathcal{C}_{M_+|X} = \begin{cases} \mathcal{C}_{N|X} & \text{on } \mathring{T}^*_{M_+}X \cap \{t=0, \operatorname{Re}\theta>0\} \\ \mathcal{C}_M & \text{on } \mathring{T}^*_{M_+}X \cap \{t>0, \operatorname{Re}\theta=0\}, \end{cases}$$

and we have canonical injections

$$\mathcal{C}_{N|X} \rightarrowtail \mathcal{C}_{M_+|X} \rightarrowtail \Gamma_{\{t\geqq 0\}}(\mathcal{C}_M) \qquad \text{on } \mathring{T}^*_N X \cap \mathring{T}^*_M X .$$

PROPOSITION 2.4. *We take a complexification* Y *of* N *in* X. *Setting*

$$\bar{G}_+ = \{(t, x; \theta, \sqrt{-1}\eta) \in \mathring{T}^*_{M_+}X ; t=0, \operatorname{Re}\theta \geqq 0\},$$
$$T^*_Y X = \{(w, z; \theta dw + \zeta dz) \in T^*X ; w=0, \zeta=0\},$$

we define the canonical projection:

$$\iota : \mathring{T}^*_N X \setminus T^*_Y X \ni (x; \theta, \sqrt{-1}\eta) \longmapsto (x; \sqrt{-1}\eta) \in \sqrt{-1}\mathring{T}^*N .$$

Then the sections of $\mathcal{C}_{M_+|X}$ [*resp.* $\mathcal{C}_{N|X}$] *have unique continuation properties along the fibers of* $\iota|_{\bar{G}_+ \setminus T^*_Y X}$ [*resp.* ι].

Moreover $\mathcal{C}_{M_+|X}$ [resp. $\mathcal{C}_{N|X}$] is an $\mathcal{E}_X$ module on $\mathring{T}^*_{M_+}X$ [resp. $\mathring{T}^*_N X$]. The most important property of $\mathcal{C}_{N|X}$ as $\mathcal{E}_X$ modules is the division theorem due to Kashiwara-Kawai [**4, 5**]. Refer to Schapira [**20**] for a semiglobal version of the theorem and also to Kataoka [**11**] for the explicit calculation of projections.

THEOREM 2.5. *Let* K *be a compact subset of* $\iota^{-1}(p) \subset \mathring{T}^*_N X$ *with some* $p = (\mathring{x}; \sqrt{-1}\mathring{\eta}) \in \sqrt{-1}\mathring{T}^*N$, *and* $P(t, x, D_t, D_x)$ *be a microdifferential operator defined in a neighborhood of* K. *Assume that the principal symbol* $\sigma(P)$ *never vanishes on* $\partial K = K \setminus \operatorname{int}(K)$. *Then, putting*

$$s = (\textit{the number of zeros in } \operatorname{int}(K) \textit{ counting multiplicities}),$$

we have a direct decomposition of sections of $\mathcal{C}_{N|X}$:

$$\Gamma(K, \mathcal{C}_{N|X}) = P(t, x, D_t, D_x) \cdot \Gamma(K, \mathcal{C}_{N|X}) + \sum_{j=0}^{s-1} \mathcal{C}_N \cdot \delta^{(j)}(t) .$$

The sheaf $\mathcal{C}_{N|M_+}$ on $\sqrt{-1}\mathring{T}^*N$ of mild microfunctions introduced in § 1 is defined as a subsheaf of $\iota_*(\mathcal{C}_{M_+|X}|_{\bar{G}_+\setminus T_Y^*X})/\iota_*(\mathcal{C}_{N|X}|_{T_N^*X\setminus T_Y^*X})$ (see [11], [12]):

(2.10) $\hat{\mathcal{C}}_{N|M_+}|_p = \varinjlim_{R\to+\infty} [\{f(t, x)\in\iota_*(\mathcal{C}_{M_+|X}|_{\bar{G}_+\setminus T_Y^*X})_p$; for some open neighborhood V of p, $f(t, x)$ extends to $\iota^{-1}(V)\cap\{\theta\in\boldsymbol{C};\ |\theta|>R\}$ as a section of $\mathcal{C}_{N|X}\}/\iota_*(\mathcal{C}_{N|X}|_{T_N^*X\setminus T_Y^*X})_p]$

for any $p\in\sqrt{-1}\mathring{T}^*N$. Then "ext" in (1.10) of § 1 is an injective sheaf morphism (not $\mathcal{D}_X$-linear):

$$\text{ext}: \hat{\mathcal{C}}_{N|M_+}\ni f(t, x)\longmapsto f(t, x)Y(t)\in\iota_*(\mathcal{C}_{M_+|X}|_{\bar{G}_+\setminus T_Y^*X}). \tag{2.11}$$

That is, ext(f) is the canonical representative of f in the above equivalence class (see § 2.6). At the same time, setting $\Omega=M_+\setminus N$, Schapira-Zampieri [24] obtained another expression of $\hat{\mathcal{C}}_{N|M_+}$:

$$\hat{\mathcal{C}}_{N|M_+} = \boldsymbol{R}\iota_!(\mathcal{C}_{\Omega|X}|_{\bar{G}_-\setminus T_Y^*X}). \tag{2.12}$$

Here $\mathcal{C}_{\Omega|X}$ is a complex of sheaves on T^*X introduced by Schapira [22], which is supported by $(T_M^*X\cap\{t\geqq 0\})\cup\bar{G}_-$ with

$$\bar{G}_- = \{(0, x;\theta, \sqrt{-1}\eta)\in\mathring{T}_N^*X;\ \mathrm{Re}\,\theta\leqq 0\}.$$

Since $\mathcal{C}_{\Omega|X}$ is a complex of $\mathcal{E}_X$-modules, we have

PROPOSITION 2.6.

$\hat{\mathcal{C}}_{N|M_+}$ is an $\iota_*(\mathcal{E}_X|_{\bar{G}_-\setminus T_Y^*X})$-module.

REMARK. This is also proven by the technique of excision for the sections of $\mathcal{C}_{M_+|X}$. In fact, for any neighborhood V of $p\in\sqrt{-1}\mathring{T}^*N$, every section of

$$\Gamma(\bar{G}_+\cap\bar{G}_-\cap\iota^{-1}(V), \mathcal{C}_{M_+|X})\cap\Gamma(\bar{G}_-\cap\iota^{-1}(V)\cap\{\theta\in\boldsymbol{C};\ |\theta|>R\}, \mathcal{C}_{N|X})$$

is decomposed into a sum of

$$\Gamma(\bar{G}_+\cap\iota^{-1}(V), \mathcal{C}_{M_+|X})\cap\Gamma(\iota^{-1}(V)\cap\{\theta\in\boldsymbol{C};\ |\theta|>R\}, \mathcal{C}_{N|X})$$

and

$$\Gamma(\bar{G}_-\cap\iota^{-1}(V), \mathcal{C}_{N|X}).$$

Such a decomposition is explicitly given by the Cauchy integral concerning one holomorphic parameter because $\mathcal{C}_{N|X}$ is isomorphic to the sheaf of microfunctions with one holomorphic parameter (see [4, 5], [13]).

Fix a point $p=(\mathring{x};\sqrt{-1}\mathring{\eta})\in\sqrt{-1}\mathring{T}^*N$. Let $P(t, x, D_t, D_x)$ and $B_j(x, D_t, D_x)$ $\in\iota_*(\mathcal{E}_X|_{\mathring{T}_N^*X\setminus T_Y^*X})_p$ $(1\leqq j\leqq m_+)$ be microdifferential operators of order m and m_j $(<m)$ of the forms:

$$P(t, x, D_t, D_x) = D_t^m+\sum_{k=0}^{m-1}A_k(t, x, D_x)\cdot D_t^k,$$

$$B_j(x, D_t, D_x) = \sum_{k=0}^{m-1} B_{jk}(x, D_x)\cdot D_t^k \qquad (1 \leqq j \leqq m_+).$$

Then the microlocal well-posedness of elliptic boundary value problems for $(P; B_1, \cdots, B_{m_+})$ is formulated as follows.

THEOREM 2.7. *Suppose the following conditions* (C1) *and* (C2) *for some* m_+ $(0 \leqq m_+ \leqq m)$:

(C1) $H(\theta)=\sigma_m(P)(0, \mathring{x}, \theta, \sqrt{-1}\mathring{\eta})=0$ *has* $m-m_+$ *roots with respect to* θ *in* $\{\mathrm{Re}\,\theta>0\}$ *and* m_+ *roots in* $\{\mathrm{Re}\,\theta<0\}$.

(C2) *Under* (C1), *we decompose*

$$H(\theta) = H_+(\theta)\cdot H_-(\theta)$$

so that $H_+(\theta) \neq 0$ *on* $\{\mathrm{Re}\,\theta \leqq 0\}$ *and* $H_-(\theta) \neq 0$ *on* $\{\mathrm{Re}\,\theta \geqq 0\}$. *Define the Lopatinskii polynomial*

$$R_j(\mathring{x}, \sqrt{-1}\mathring{\eta})(\theta) = \sum_{k=0}^{m_+-1} \beta_{jk}(\mathring{x}, \sqrt{-1}\mathring{\eta})\cdot \theta^k$$
$$\equiv \sigma_{m_j}(B_j)(\mathring{x}, \theta, \sqrt{-1}\mathring{\eta}) \text{ modulo } H_-(\theta)$$

for $j=1, \cdots, m_+$ *and assume that the system of polynomials of* θ $\{R_j(\mathring{x}, \sqrt{-1}\mathring{\eta})(\theta); j=1, \cdots, m_+\}$ *is lineary independent over* $\boldsymbol{C}$; *or equivalently,*

$$\det(\beta_{j,k-1}(\mathring{x}, \sqrt{-1}\mathring{\eta}))_{1\leqq j,k\leqq m_+} \neq 0 .$$

Then for any $f \in \hat{\mathcal{C}}_{N|M_+}|_p$ *and any* $(v_j)_j \in \mathcal{C}_N^{m_+}|_p$, *the boundary value problem at* p

$$(2.13) \qquad \begin{cases} P(t, x, D_t, D_x)u = f(t, x) \\ B_j(x, D_t, D_x)u|_{t\to+0} = v_j(x) \qquad (1 \leqq j \leqq m_+) \end{cases}$$

has a unique solution $u(t, x) \in \hat{\mathcal{C}}_{N|M_+}|_p$.

PROOF. Under (C1), we can decompose P as

$$P = P_+(t, x, D_t, D_x)\cdot P_-(t, x, D_t, D_x)$$

by the Späth type theorem for microdifferential operators due to Sato et al. [17]. Here $P_\pm \in \iota_*(\mathcal{E}_X|_{\mathring{T}_N^*X \setminus T_Y^*X})|_p$, where P_+ and P_- are elliptic on $\overline{G}_- \cap \iota^{-1}(V)$ and $\overline{G}_+ \cap \iota^{-1}(V)$ respectively for some neighborhood V of p. Moreover P_- has the form

$$(2.14) \qquad P_- = D_t^{m_+} + \sum_{k=0}^{m_+-1} A'_k(t, x, D_x)\cdot D_t^k .$$

Then we can decompose B_j's as

$$(2.15) \qquad B_j = B'_j(t, x, D_t, D_x) + E_j(t, x, D_t, D_x)\cdot P_-(t, x, D_t, D_x)$$

for $j=1, \cdots, m_+$, where B'_j, $E_j \in \iota_*(\mathcal{E}_X|_{\mathring{T}^*_N X \setminus T^*_Y X})|_p$ and B'_j is written in the form:

$$B'_j = \sum_{k=0}^{m_+-1} B'_{jk}(t, x, D_x) \cdot D_t^k. \tag{2.16}$$

Therefore, noting Proposition 2.6, we can reduce the boundary value problem (2.13) to the following form:

$$\begin{cases} P_-(t, x, D_t, D_x)u = P_+(t, x, D_t, D_x)^{-1} f(t, x) \\ B'_j(0, x, D_t, D_x)u|_{t\to+0} = v_j(x) - (E_j \cdot P_+^{-1} f(t, x))_{t\to+0} \quad (1 \leqq j \leqq m_+). \end{cases}$$

Since $\sigma_{m_j-k}(B'_{jk})(0, \mathring{x}, \sqrt{-1}\mathring{\eta}) = \beta_{jk}(\mathring{x}, \sqrt{-1}\mathring{\eta})$, the second equations are uniquely solvable with respect to $\{D_t^j u(+0, x);\ j=0, \cdots, m_+-1\}$ under the condition (C2). On the other hand, the first equation is equivalent to

$$P_-(\mathrm{ext}(u)) = \mathrm{ext}(P_+^{-1} f) + \sum_{k=0}^{m_+-1} Q'_k \{D_t^k u(+0, x) \cdot \delta(t)\}, \tag{2.17}$$

where

$$Q'_k = D_t^{m_+-k-1} + \sum_{0 \leqq j+l \leqq m_+-k-2} (-)^l \binom{j+l}{l} \cdot (\partial/\partial t)^l A'_{j+l+k+1}(0, x, D_x) D_t^j.$$

Because P_- is elliptic on $\iota^{-1}(V) \cap (\overline{G}_+ \cup \{\theta \in \boldsymbol{C};\ |\theta| > R\})$ for some neighbornood V of p and some $R>0$, the solution "ext(u)" to (2.17) can be obtained uniquely as a section of

$$\varinjlim_{R\to+\infty} \iota_*(\mathcal{C}_{M_+|X}|_{\overline{G}_+ \setminus T^*_Y X}) \cap \iota_*(\mathcal{C}_{N|X}|_{\mathring{T}^*_N X \cap \{|\theta|>R\} \setminus T^*_Y X}).$$

This completes the proof. q.e.d.

We need the following proposition to calculate the cohomology group in the proof of Theorem 1.1. Let P and $p=(\mathring{x};\ \sqrt{-1}\mathring{\eta})$ be as above. Further let $\{S_k;\ k=0, \cdots, m-1\}$, $\{E_{jk};\ j=1, \cdots, m_+,\ k=0, \cdots, m-1\}$ be microdifferential operators belonging to $\iota_*(\mathcal{E}_X|_{\mathring{T}^*_N X \setminus T^*_Y X})_p$ such that

$$S_k = D_t^{m-k-1} + \sum_{l=0}^{m-k-2} S_{kl}(t, x, D_x) \cdot D_t^l \qquad (\mathrm{ord}\, S_k = m-k-1),$$

$$E_{jk} = E_{jk}(x, D_x) \qquad (\mathrm{ord}\, E_{jk} \leqq m_j - k),$$

where $0 \leqq m_+ \leqq m$ and $0 \leqq m_j < m$ $(j=1, \cdots, m_+)$. It is easy to find microdifferential operators $W_{kl}(x, D_x)$ $(k=1, \cdots, m-1,\ l=0, \cdots, k-1)$ at p of order $\leqq k-l$ such that the relations

$$g_k(x) = h_k(x) + \sum_{l=0}^{k-1} W_{kl}(x, D_x) h_l(x) \qquad (k=0, \cdots, m-1) \tag{2.18}$$

between $(g_k(x))$ and $(h_k(x)) \in \mathcal{C}_N^m|_p$ solve uniquely the equation

(2.19) $$\sum_{k=0}^{m-1} S_k(t, x, D_t, D_x)(g_k(x)\delta(t)) = \sum_{k=0}^{m-1} Q_k(x, D_t, D_x)(h_k(x)\delta(t)).$$

Here Q_k $(k=0, \cdots, m-1)$ are microdifferential operators associated to P and the boundary $\{t=0\}$:

$$Q_k = D_t^{m-k-1} + \sum_{0 \leq j+l \leq m-k-2} (-)^l \binom{j+l}{l} \cdot (\partial/\partial t)^l A_{j+l+k+1}(0, x, D_x) D_t^j$$

(see (1.3)). Under the preparation above, we have

PROPOSITION 2.8. *Setting*

$$B_j(x, D_t, D_x) = \sum_{k=0}^{m-1} E_{jk}(x, D_x) \cdot (D_t^k + \sum_{l=0}^{k-1} W_{kl}(x, D_x) D_t^l) \quad (j=1, \cdots, m_+),$$

we suppose the conditions (C1) *and* (C2) *in Theorem* 2.7 *for* $(P; B_1, \cdots, B_{m_+})$. *Then the problem*

(2.20) $$\begin{cases} Pg(t, x) - \sum_{k=0}^{m-1} S_k(g_k(x)\delta(t)) = f(t, x) \\ \sum_{k=0}^{m-1} E_{jk}(x, D_x) g_k(x) = e_j(x) \qquad (j=1, \cdots, m_+) \end{cases}$$

has a unique solution $(g(t, x), g_0(x), \cdots, g_{m-1}(x))$ *in*

$$\iota_*(\mathcal{C}_{M_+|X}|_{\bar{G}_+ \setminus T_Y^* X})_p \oplus \mathcal{C}_N^m|_p$$

for any $f(t, x) \in \iota_*(\mathcal{C}_{M_+|X}|_{\bar{G}_+ \setminus T_Y^* X})|_p$ *and any* $(e_j(x))_{j=1}^{m_+} \in \mathcal{C}_N^{m_+}|_p$.

PROOF. *Uniqueness*. Suppose $f=0$ and $e_j(x)=0$ for $j=1, \cdots, m_+$, then the first equation implies that $[g]$ modulo $\iota_*(\mathcal{C}_{N|X}|_{\mathring{T}_N^* X \setminus T_Y^* X})_p$ represents a germ of $\mathring{\mathcal{C}}_{N|M_+}$ at p, and that

$$\begin{cases} P[g] = 0 \\ g_k(x) = \left(D_t^k + \sum_{l=0}^{k-1} W_{kl}(x, D_x) D_t^l\right)[g(t, x)]|_{t \to +0} \qquad (k=0, \cdots, m-1). \end{cases}$$

(See (2.18).) Therefore by Theorem 2.7 we can conclude that $[g]=0$ in $\mathring{\mathcal{C}}_{N|M_+}$ at p. In particular $g_k=0$ for any k. Thus g must be zero.

Solvability. Take $f \in \iota_*(\mathcal{C}_{M_+|X}|_{\bar{G}_+ \setminus T_Y^* X})_p$. Since $\mathcal{C}_{M_+|X}$ coincides with $\mathcal{C}_{N|X}$ on $\bar{G}_+ \cap \{\mathrm{Re}\,\theta > 0\}$, we can decompose $f(t, x)$ by Theorem 2.5 as

$$f(t, x) = Ph(t, x) + \sum_{k=0}^{m-m_+-1} h_k(x) \cdot \delta^{(k)}(t),$$

where $h \in \iota_*(\mathcal{C}_{N|X}|_{\mathring{T}_N^* X \cap (\mathrm{Re}\,\theta > 0) \setminus T_Y^* X})_p$ and $(h_k) \in \mathcal{C}_N^{m-m_+}|_p$. Here h extends to a section of $\iota_*(\mathcal{C}_{M_+|X}|_{\bar{G}_+ \setminus T_Y^* X})_p$ because P is elliptic on $\bar{G}_+ \cap \{\mathrm{Re}\,\theta = 0\}$. Moreover, easily to see, we can find some $(g_k'(x))_k \in \mathcal{C}_N^m|_p$ satisfying

$$-\sum_{k=0}^{m-1} S_k(g_k'(x)\delta(t)) = \sum_{k=0}^{m-m_+-1} h_k(x)\delta^{(k)}(t).$$

Put $\tilde{g}=g-h$ and $\tilde{g}_k=g_k-g'_k$ ($k=0, \cdots, m-1$). Then (2.20) is reduced to the problem for $(\tilde{g}(t, x), \tilde{g}_0(x), \cdots, \tilde{g}_{m-1}(x))$:

$$(2.21)\qquad \begin{cases} P\tilde{g}(t, x) = \sum_{k=0}^{m-1} S_k(\tilde{g}_k(x)\delta(t)) \\ \sum_{k=0}^{m-1} E_{jk}(x, D_x)\tilde{g}_k(x) = e_j(x) - \sum_{k=0}^{m-1} E_{jk}(x, D_x)g'_k(x) \quad (j=1, \cdots, m_+). \end{cases}$$

Consider (2.18) and (2.19). We know that this is equivalent to the boundary value problem for $u \in \hat{C}_{N|M_+}$:

$$(2.22)\qquad \begin{cases} Pu = 0 \\ B_j(x, D_t, D_x)u(+0, x) = e_j(x) - \sum_{k=0}^{m-1} E_{jk}(x, D_x)g'_k(x) \quad (j=1, \cdots, m_+). \end{cases}$$

Finally we can solve (2.22) under (C1) and (C2) by Theorem 2.7. Then $\tilde{g}=\mathrm{ext}(u)$ and $\tilde{g}_k = \left(D_t^k + \sum_{l=0}^{k-1} W_{kl}(x, D_x)D_t^l\right)u|_{t\to+0}$ ($k=0, \cdots, m-1$) are the solution to (2.21). Thus the proof is completed. q.e.d.

2.5. Boundary values of hyperfunctions with holomorphic parameters. We characterize the boundary values of hyperfunctions with holomorphic parameters. We follow the notation prepared at the beginning of § 1; that is, let $M=(-T, T)\times N$ be an open subset of $\boldsymbol{R}_t \times \boldsymbol{R}_x^n$ with a complex neighorhood X in $\boldsymbol{C}_w \times \boldsymbol{C}_z^n$ with $z=x+\sqrt{-1}y$. We set

$$\tilde{M} = ((-T, T)\times \boldsymbol{C}^n)\cap X.$$

Then the sheaf $\mathcal{B}\mathcal{O}$ can be viewed as $\mathcal{H}om_{\mathcal{D}_{\tilde{X}}}(\mathcal{N}, \mathcal{B}_{\tilde{M}})$ where $\tilde{X}$ is a complex neighborhood of $\tilde{M}$ in $\boldsymbol{C}_w \times \boldsymbol{C}_z^n \times \boldsymbol{C}_{\bar{z}}^n$ and $\mathcal{N}$ is a coherent $\mathcal{D}_{\tilde{X}}$ module expressing the partial Cauchy-Riemann system

$$\mathcal{N}: \partial/\partial \bar{z}_j \cdot u = 0 \qquad (1 \leqq j \leqq n).$$

We take subsets Ω, F and L of $\tilde{M}$ as follows:

$$(2.23)\qquad \begin{aligned} &\Omega = \{(t, z)\in \tilde{M}; \phi(t, z, \bar{z}) = y_1 - \varphi(t, x, y') > 0\} \xrightarrow{\ j\ } \tilde{M} \\ &F = \{(t, z)\in \tilde{M}; \phi(t, z, \bar{z}) \leqq 0\} \longleftarrow L = \{(t, z)\in \tilde{M}; y_1 = \varphi(t, x, y')\}. \end{aligned}$$

Here $\varphi(t, x, y')$ is a real analytic function defined on $\tilde{M}$, and j is the inclusion map. Then by the theory of non-characteristic boundary value problems for systems of differential equations, we have an injective sheaf morphism

$$bv: (j_* j^{-1}\mathcal{B}\mathcal{O})|_L \cong (j_* j^{-1}\mathcal{H}om_{\mathcal{D}_{\tilde{X}}}(\mathcal{N}, \mathcal{B}_{\tilde{M}}))|_L \rangle\!\longrightarrow \mathcal{H}om_{\mathcal{D}_Z}(\mathcal{N}_Z, \mathcal{B}_L).$$

$$f(t, z) \longmapsto f(t, x_1 + i\varphi + i0, x' + iy')$$

Here, Z is a complexification of L. In fact this is obtained from the exact sequence

$$0 = \Gamma_{\bar{\Omega}}(\mathcal{B}\mathcal{O})|_L \longrightarrow (j_* j^{-1}\mathcal{B}\mathcal{O})|_L \longrightarrow \mathcal{H}_L^1(\mathcal{B}\mathcal{O}) \longrightarrow$$

and the isomorphism

$$\boldsymbol{R}\Gamma_L(\mathcal{B}\mathcal{O}) = \boldsymbol{R}\Gamma_L(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_{\tilde{X}}}(\mathcal{N}, \mathcal{B}_{\tilde{M}})) \xleftarrow{\sim} \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{N}_Z, \mathcal{B}_L)[-1]$$

due to M. Kashiwara [**31**] (cf. [**14**], [**19**], [**17**]).

The purpose of this §2.5 is to characterize the image of the above morphism. If we take coordinates of L as (t, x, y') with $y'=(y_2, \cdots, y_n)$ so that (t, x, y') expresses $(t, x, y_1=\varphi(t, x, y'), y')\in L$, $\mathcal{N}_Z$ is written explicitly as

$$\text{(2.25)} \qquad \mathcal{N}_Z: \left\{D_{x_j}+\sqrt{-1}D_{y_j}+2\cdot\frac{-(\sqrt{-1}+(\partial\varphi/\partial x_1))\cdot(\partial\varphi/\partial \bar{z}_j)}{1+(\partial\varphi/\partial x_1)^2}\cdot D_{x_1}\right\}f=0 \qquad (2\leqq j\leqq n).$$

To estimate the singular spectrum of the sections of $bv((j_*j^{-1}\mathcal{B}\mathcal{O})|_L)$, we take coordinates of T^*_LZ as $(t, x, y'; \sqrt{-1}(\tau dt+\xi dx+\eta' dy'))$ with $\tau\in\boldsymbol{R}$, $\xi=(\xi_1, \cdots, \xi_n)\in\boldsymbol{R}^n$ and $\eta'=(\eta_2, \cdots, \eta_n)\in\boldsymbol{R}^{n-1}$.

Let $G(t, z)$ be a germ of $j_*j^{-1}\mathcal{B}\mathcal{O}$ at $(\mathring{t}, \mathring{x}+\sqrt{-1}\mathring{y})\in L$. Then we note that the boundary value

$$\text{(2.26)} \qquad f(t, x, y') = G(t, x_1+\sqrt{-1}\varphi(t, x, y')+\sqrt{-1}0, x'+\sqrt{-1}y')$$

extends holomorphically to $\{\operatorname{Im}\tilde{x}_1>0\}$ with respect to the complexified variable $\tilde{x}_1$ for x_1. Therefore $f(t, x, y')$ satisfies

$$\text{(2.27)} \qquad \begin{aligned} SS(f) &\subset \{(t, x, y'; \sqrt{-1}(\tau, \xi, \eta'))\in\sqrt{-1}\mathring{T}^*L; \xi_1\geqq 0\}\cap\operatorname{char}(\mathcal{N}_Z) \\ &= \{(t, x, y'; \sqrt{-1}(\tau, \xi, \eta'))\in\sqrt{-1}\mathring{T}^*L; \xi_1\geqq 0, \\ &\quad \xi_j = \frac{-(\partial\varphi/\partial y_j)+(\partial\varphi/\partial x_1)\cdot(\partial\varphi/\partial x_j)}{1+(\partial\varphi/\partial x_1)^2}\cdot\xi_1 \quad (2\leqq j\leqq n), \\ &\quad \eta_j = \frac{(\partial\varphi/\partial x_j)+(\partial\varphi/\partial x_1)\cdot(\partial\varphi/\partial y_j)}{1+(\partial\varphi/\partial x_1)^2}\cdot\xi_1 \quad (2\leqq j\leqq n)\}. \end{aligned}$$

where $\operatorname{char}(\mathcal{N}_Z)$ $(\subset T^*Z)$ is the characteristic variety of $\mathcal{N}_Z$.

At the same time, putting a regular involutive submanifold Σ in $\sqrt{-1}T^*L$:

$$\text{(2.28)} \qquad \Sigma = \{(t, x, y'; \sqrt{-1}(\tau, \xi, \eta')); \xi=0, \eta'=0\},$$

we have

$$SS^2_\Sigma(f) \subset \{(t, x, y'; \pm\sqrt{-1}\tau dt; \sqrt{-1}(x_1^* dx_1+\sum_{j=2}^{n}(x_j^* dx_j+y_j^* dy_j))); \tau>0, x_1^*\geqq 0\}.$$

(Recall the characterization of $SS^2_\Sigma(\cdot)$ by the expression using boundary values of microfunctions with holomorphic parameters.)

On the other hand, the micro-characteristic variety of $\mathcal{N}_Z$ along Σ^C (a complexification of Σ in T^*Z) is obtained in a similar way to (2.27). (Refer to

Bony-Schapira [1] and Kashiwara-Schapira [8] for the micro-characteristic variety.) Hence we get the estimate:

$$SS^2_{\Sigma}(f)\subset\Big\{\Big(t,\ x,\ y'\ ;\ \pm\tau\sqrt{-1}dt\ ;\ \sqrt{-1}\lambda\Big(dx_1+\sum_{j=2}^{n}\frac{-(\partial\varphi/\partial y_j)+(\partial\varphi/\partial x_1)(\partial\varphi/\partial x_j)}{1+(\partial\varphi/\partial x_1)^2}dx_j+\sum_{j=2}^{n}\frac{(\partial\varphi/\partial x_j)+(\partial\varphi/\partial x_1)(\partial\varphi/\partial y_j)}{1+(\partial\varphi/\partial x_1)^2}dy_j\Big)\Big)\ ;\ \tau,\ \lambda>0\Big\}. \tag{2.29}$$

Remark that it is also possible to derive the estimate (2.29) from Tose [29] and Tose-Uchida [30].

Consequently, we have the characterization for $bv(j_*j^{-1}\mathcal{B}\mathcal{O}|_L)$:

Proposition 2.9. *For* $u\in\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{N}_Z,\ \mathcal{B}_L)$, *$u$ belongs to* $bv(j_*j^{-1}\mathcal{B}\mathcal{O}|_L)$ *if and only if the conditions* (2.27) *for* $SS(f)$ *and* (2.29) *for* $SS^2_{\Sigma}(f)$ *are satisfied.*

Proof. We have only to prove the "if"-part.

Let $u(t,\ x,\ y')$ be any hyperfunction at $p=(\mathring{t},\ \mathring{x},\ \mathring{y})\in L$ satisfying the equations $\mathcal{N}_Z$ and conditions (2.27) and (2.29). Then our problem is reduced to finding a hyperfunction $U(s,\ t,\ x,\ y')$ on

$$\{0<s<\delta,\ |t-\mathring{t}|+|x-\mathring{x}|+|y'-\mathring{y}'|<\delta\}$$

such that

$$\begin{cases}\Big\{D_s-\dfrac{(\partial\varphi/\partial x_1)+\sqrt{-1}}{(\partial\varphi/\partial x_1)^2+1}\cdot D_{x_1}\Big\}U(s,\ t,\ x,\ y')=0\\ U(+0,\ t,\ x,\ y)=u(t,\ x,\ y')\end{cases} \tag{2.30}$$

for some $\delta>0$. Here the operator in (2.30) is just the operator $(\sqrt{-1}-\varphi_{x_1})^{-1}\cdot(D_{\hat{x}_1}+\sqrt{-1}D_{\hat{y}_1})$ in the coordinates $(\hat{t},\ \hat{x},\ \hat{y})$ with

$$\hat{t}=t,\qquad \hat{x}=x,\qquad \hat{y}_1=\varphi(t,\ x,\ y')+s,\qquad \hat{y}'=y'. \tag{2.31}$$

In fact, if such U exists, we easily see that $U(y_1-\varphi(t,\ x,\ y'),\ t,\ x,\ y')\in j_*j^{-1}\mathcal{B}\mathcal{O}|_p$ and $u=bv(U)$. Moreover, by the microlocal uniqueness of solutions of (2.30), we have only to solve the boundary value problem (2.30) microlocally at each point of

$$K=\{(p\ ;\ \sqrt{-1}(\tau dt+\xi dx+\eta' dy'))\in\sqrt{-1}\mathring{T}^*_p(\boldsymbol{R}_t\times\boldsymbol{R}^n_x\times\boldsymbol{R}^{n-1}_{y'})\}.$$

Noting the condition (2.27) for the singular spectrum of u, we can solve (2.30) by Theorem 2.7 at any point of K except for $(p\ ;\ \pm\sqrt{-1}dt)$. On the other hand, at $(p\ ;\ \pm\sqrt{-1}dt)$ we have the condition (2.29) for the second spectrum of u along $\Sigma=\{\xi=0,\ \eta'=0\}$.

Hereafter we denote by p^* one of $(p\ ;\ \pm\sqrt{-1}dt)$. Hence, by replacing Σ

with a larger involutive submanifold

$$\Sigma' = \{\xi_1=0\},$$

we obtain the estimate

$$SS^2_{\Sigma'}(u) \subset \{(t, x, y'; \sqrt{-1}(\tau, \xi_1=0, \xi', \eta'); \sqrt{-1}\lambda dx_1); \lambda>0\} \tag{2.32}$$

in some neighborhood of p^*. We claim that $u(t, x, y')$ extends holomorphically with respect to x_1 to $\{\mathrm{Im}\, x_1>0\}$. In fact, after cutting the support of u, by using the Cauchy kernel we can decompose $u(t, x, y')$ into a sum

$$u(t, x, y') = F_+(t, x_1+\sqrt{-1}0, x', y')+F_-(t, x_1-\sqrt{-1}0, x', y')$$

in a neighborhood of p. Here $F_\pm(t, z_1, x', y')$ are hyperfunctions with holomorphic parameter z_1 defined on

$$\{\pm\mathrm{Im}\, z_1>0,\ |t-\mathring{t}|+|z_1-\mathring{x}_1|+|x'-\mathring{x}'|+|y'-\mathring{y}'|<\delta\}$$

respectively for some $\delta>0$. By the definition of $SS^2_{\Sigma'}(\cdot)$, we know that

$$(p^*; -\sqrt{-1}\varepsilon dx_1) \notin SS^2_{\Sigma'}(F_\varepsilon(t, x_1+\sqrt{-1}\varepsilon 0, x', y'))$$

for $\varepsilon=\pm 1$. Thus by (2.32), we conclude that $F_-(t, x_1-\sqrt{-1}0, x', y')$ must belong to $\mathcal{C}_{\tilde{\Sigma}'}$ as a microfunction at p^*, where $\tilde{\Sigma}'$ is the partial complexification of Σ'.

Combining these facts, we can find a hyperfunction $\tilde{u}(t, z_1, x', y')$ with holomorphic parameter z_1 defined in

$$\{\mathrm{Im}\, z_1>0,\ |t-\mathring{t}|+|z_1-\mathring{x}_1|+|x'-\mathring{x}'|+|y'-\mathring{y}'|<\delta\}$$

for some $\delta>0$ such that

$$p^* \notin SS(\tilde{u}(t, x_1+\sqrt{-1}0, x', y')-u(t, x, y')).$$

Hence we can replace u by $\tilde{u}(t, x_1+\sqrt{-1}0, x', y')$ in the problem (2.30) at p^*.

Recalling (2.26), we consider the coordinate transformation:

$$\begin{aligned}&(\tilde{t}, \tilde{x}_1, \tilde{x}', \tilde{y}_1, \tilde{y}')\\ &= (t, x_1-\mathrm{Im}\,\varphi(t, x_1+\sqrt{-1}y_1, x', y'), x', y_1+\mathrm{Re}\,\varphi(t, x_1+\sqrt{-1}y_1, x', y'), y')\end{aligned}$$

with a non vanishing Jacobian $1+(\partial\varphi/\partial x_1)^2$ on $\{y_1=0\}$. Note that

$$\tilde{y}_1-\varphi(\tilde{t}, \tilde{x}, \tilde{y}') = \{1+(\partial\varphi/\partial x_1)^2(t, x, y')\}(y_1+O(y_1^2)) \quad \text{as } y_1\to 0.$$

Therefore $U(\tilde{t}, \tilde{x}, \tilde{y})=\tilde{u}(t, x_1+\sqrt{-1}y_1, x', y')$ is a hyperfunction on

$$\{\tilde{y}_1>\varphi(\tilde{t}, \tilde{x}, \tilde{y}'),\ |\tilde{t}-\mathring{t}|+|\tilde{x}-\mathring{x}|+|\tilde{y}'-\mathring{y}'|+|\tilde{y}_1-\varphi(\mathring{t}, \mathring{x}, \mathring{y}')|<\delta\}$$

for a small $\delta>0$. It is easy to see that $U(t, x, s+\varphi(t, x, y'), y')$ is a hyperfunction solution to (2.30) for $\tilde{u}(t, x_1+\sqrt{-1}0, x', y')$. This completes the proof. q. e. d.

Further we obtain the following characterization for $bv(\mathcal{BO}|_L)$ by easier arguments.

PROPOSITION 2.10. *For any* $u\in\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{N}_Z, \mathcal{B}_L)$, *u belongs to* $bv(\mathcal{BO}|_L)$ *if and only if*

$$SS(u)\subset\{(t, x, y'; \pm\sqrt{-1}\tau dt); \tau>0\}=\Sigma, \text{ and } SS^2_{\Sigma}(u)=\emptyset .$$

Concerning the operation of $\mathcal{D}_X|_L$ on $bv(j_*j^{-1}\mathcal{BO}|_L)$, we have the following lemma. Here we use another coordinate system (s, t, x, y') of $\tilde{M}$ having the correspondence

$$(t, x, s+\varphi(t, x, y'), y')\in\tilde{M}$$

with the original coordinates. Noting that, the operator $\partial/\partial\bar{z}_1$ in the original coordinates is written as

$$P_0=1/2\cdot\{D_{x_1}-(\partial\varphi/\partial x_1-\sqrt{-1})D_s\}, \tag{2.33}$$

we set the subsheaf

$$\mathfrak{C}_{P_0}=\{P\in\mathcal{D}_{\tilde{X}}|_Z; PP_0=P_0P\}$$

of ring of $\mathcal{D}_{\tilde{X}}|_Z$, which includes $\mathcal{D}_X|_Z$. Then we can define a morphism γ as a composition of division by P_0 and restriction on Z as follows:

LEMMA 2.11. *There is an algebra-homomorphism* $\gamma:\mathfrak{C}_{P_0}\to\mathcal{D}_Z$ *such that, for the generators of* $\mathcal{D}_X|_Z$, *we have*

$$\gamma(t)=t,\quad \gamma(z_1)=x_1+\sqrt{-1}\varphi(t, x, y'),\quad \gamma(z')=x'+\sqrt{-1}y',$$
$$\gamma(D_t)=D_t+\frac{\varphi_t}{\sqrt{-1}-\varphi_{x_1}}D_{x_1},\quad \gamma(D_{z_1})=\frac{\sqrt{-1}}{\sqrt{-1}-\varphi_{x_1}}D_{x_1}, \tag{2.34}$$
$$\gamma(D_{z'})=D_{z'}+\frac{\varphi_{z'}}{\sqrt{-1}-\varphi_{x_1}}\cdot D_{x_1}.$$

In particular every operator in $\gamma(\mathcal{D}_X|_L)$ *commutes with the tangential Cauchy-Riemann operators in* (2.25). *Further, for any operator P in* $\mathcal{D}_X|_L$, γ *is compatible with the morphism bv concerning operation on* $j_*j^{-1}(\mathcal{BO})|_L$.

PROOF. Define the subsheaf $\mathfrak{C}_s$ of $\mathcal{D}_{\tilde{X}}|_Z$ by

$$\mathfrak{C}_s=\{Q\in\mathcal{D}_{\tilde{X}}|_Z; Qs=sQ\}.$$

Then $\mathfrak{C}_s$ is a sheaf of rings, and there is a natural identification

$$\alpha:\mathfrak{C}_s/s\mathfrak{C}_s \xrightarrow{\sim}\mathcal{D}_Z .$$

On the other hand, the natural morphisms $\mathfrak{C}_s \xrightarrow{\beta} (\mathcal{D}_{\tilde{X}}/\mathcal{D}_{\tilde{X}}P_0)|_Z$ becomes an isomorphism of sheaves because any equivalence class of $(\mathcal{D}_{\tilde{X}}/\mathcal{D}_{\tilde{X}}P_0)_p$ has a unique representative in $\mathfrak{C}_s$ for any $p \in Z$. Hence, we obtain a natural morphism

$$\delta : \mathfrak{C}_{P_0} \longrightarrow \mathfrak{C}_s$$

as the composite of morphisms

$$\mathfrak{C}_{P_0} \longrightarrow (\mathcal{D}_{\tilde{X}}/\mathcal{D}_{\tilde{X}}P_0)|_Z \xrightarrow{\beta^{-1}} \mathfrak{C}_s .$$

Then we can easily find that δ is a ring homomorphism. Therefore the composite of δ and the ring homomorphism

$$\mathfrak{C}_s \longrightarrow \mathfrak{C}_s/s\mathfrak{C}_s \xrightarrow{\sim} \mathcal{D}_Z .$$

gives the desired algebra homomorphism γ.

To show the compatibility of γ with bv, we take $p \in L$, $f(t, z) \in (j_* j^{-1}\mathcal{B}\mathcal{O})_p$ and $P \in \mathfrak{C}_{P_0, p}$. Then $bv(f) = g(t, x, y')$ is characterized by the property

$$P_0\tilde{f} - \frac{\sqrt{-1}-\varphi_{x_1}}{2} g(t, x, y')\delta(s) \in P_0\Gamma_L(\mathcal{B}_{\tilde{M}})$$

for any flabby extension $\tilde{f} \in \Gamma_{\bar{\Omega}}(\mathcal{B}_{\tilde{M}})$ of f. If $\tilde{f}$ is the canonical flabby extension of f, we have

$$P_0\tilde{f} = \frac{\sqrt{-1}-\varphi_{x_1}}{2} g(t, x, y')\delta(s).$$

We decompose P as

$$P = Q \cdot P_0 + R(t, s, x, y', D_t, D_x, D_{y'})$$

by some $Q, R \in \mathcal{D}_{\tilde{X}, p}$ with $[R, s] = 0$. Then we have

$$\left[R, \frac{2}{\sqrt{-1}-\varphi_{x_1}} P_0\right] = 0 .$$

(This can be seen from the two facts $\left[\left[R, \dfrac{2}{\sqrt{-1}-\varphi_{x_1}} P_0\right], s\right] = 0$ and

$$\left[R, \frac{2}{\sqrt{-1}-\varphi_{x_1}} P_0\right] = \left[P - Q \cdot P_0, \frac{2P_0}{\sqrt{-1}-\varphi_{x_1}}\right] \in \mathcal{D}_{\tilde{X}, p} \cdot P_0 .)$$

Hence we have, for the canonical extension $\tilde{f}$ of f,

$$\begin{aligned} P_0(P\tilde{f}) &= P_0(Q \cdot P_0 + R)\tilde{f} \\ &= \left\{P_0QP_0 + \frac{\sqrt{-1}-\varphi_{x_1}}{2} R \cdot \left(\frac{\sqrt{-1}-\varphi_{x_1}}{2}\right)^{-1} \cdot P_0\right\}\tilde{f} \\ &= \left\{P_0Q + \frac{\sqrt{-1}-\varphi_{x_1}}{2} R \cdot \left(\frac{\sqrt{-1}-\varphi_{x_1}}{2}\right)^{-1}\right\}\left(\frac{\sqrt{-1}-\varphi_{x_1}}{2} g \cdot \delta(s)\right). \end{aligned}$$

Thus

$$P_0 P\tilde{f} - \frac{\sqrt{-1}-\varphi_{x_1}}{2}(R_0 g)\delta(s) \in P_0(\Gamma_L(\mathscr{B}_{\tilde{M}})_p)$$

where $R_0 = R(t, 0, x, y', D_t, D_x, D_{y'}) = \gamma(P)$. Since $P\tilde{f}$ is a flabby extension of Pf, we have

$$bv(Pf) = R_0 g = \gamma(P)(bv(f)).$$

q. e. d.

2.6. Mild microfunctions. First of all we study the microlocalization $\mu_M(\Gamma_{\tilde{M}_+}(\mathscr{B}\mathcal{O}))$ of $\Gamma_{\tilde{M}_+}(\mathscr{B}\mathcal{O})$ along M. We can show that the complex $\mu_M(\Gamma_{\tilde{M}_+}(\mathscr{B}\mathcal{O}))[n]$ is concentrated in degree 0 if we consult with an abstract vanishing theorem in Kashiwara-Laurent [7]. Moreover we have

LEMMA 2.12. *The sheaf* $\mu_M(\Gamma_{\tilde{M}_+}(\mathscr{B}\mathcal{O}))[n]$ *is conically flabby. Especially, the complex* $R\Gamma_K(\mu_M(\Gamma_{\tilde{M}_+}(\mathscr{B}\mathcal{O}))[n]$ *is concentrated in degree* 0 *for any closed conic subset* K *of* $\mathring{T}^*_M\tilde{M}$.

PROOF. We can prove the flabbiness in the same way as the proof of that of $\mathcal{C}_M$ in Sato et al. [17] if we show that

$$H^j(V, \Gamma_{\{t\geq 0\}}(\mathscr{B}_t\mathcal{O}_w)|_{\{\mathrm{Im}\, w=0\}}) = 0 \qquad (j \geqq 1)$$

for any open subset V of $\boldsymbol{R}_t \times \boldsymbol{R}_u^{2n-1}$. Here $\mathscr{B}_t\mathcal{O}_w$ denotes the sheaf of hyperfunctions with holomorphic parameter $w = u + \sqrt{-1}v \in \boldsymbol{C}^{2n-1}$. Following P. Schapira [18] consider the complex

$$0 \longrightarrow \Gamma_{\{t\geq 0\}}(\mathscr{B}_{(t,u,u_{2n})}) \xrightarrow{\Delta} \Gamma_{\{t\geq 0\}}(\mathscr{B}_{(t,u,u_{2n})}) \longrightarrow 0. \tag{2.35}$$

where the operator $\Delta = (\partial/\partial u_1)^2 + \cdots + (\partial/\partial u_{2n})^2$. Here we find by using the elementary solution of Δ and the flabbiness of $\mathscr{B}$ that the morphism Δ in (2.35) is globally surjective on any open subset of $\boldsymbol{R}_t \times \boldsymbol{R}^{2n}_{(u,u_{2n})}$. Moreover by partial ellipticity of Δ, the complex (2.35) is quasi-isomorphic to

$$0 \longrightarrow \Gamma_{\{t\geq 0\}}(\mathscr{B}_t\mathcal{O}_{(w,w_{2n})})|_H \xrightarrow{\Delta} \Gamma_{\{t\geq 0\}}(\mathscr{B}_t\mathcal{O}_{(w,w_{2n})})|_H \longrightarrow 0, \tag{2.36}$$

where $H = \{\mathrm{Im}\,(w, w_{2n}) = 0\}$. On the other hand, the kernel of Δ in (2.36) is isomorphic to $\Gamma_{\{t\geq 0\}}(\mathscr{B}_t\mathcal{O}_w)^2$ on $H \cap \{w_{2n} = 0\} = \boldsymbol{R}_t \times \boldsymbol{R}_u^{2n-1}$. (As for partially elliptic operators, refer to Bony-Schapira [1] and Kashiwara-Schapira [8].) This completes the proof. q. e. d.

We can show from the definitions (1.12), (2.10) of $\hat{\mathcal{C}}_{R_+\times N}$ and $\hat{\mathcal{C}}_{N|M_+}$ that there exists a natural injective sheaf morphism

$$\Phi: \hat{\mathcal{C}}_{R_+\times N} \longrightarrow \hat{\iota}_*(\mathcal{C}_{M_+|X})/\hat{\iota}_*(\mathcal{C}_{N|X}) \tag{2.37}$$

on $\mathring{T}^*_M\tilde{M}$. Here we extend the projection $\hat{\iota}$ in §1 to

$$(2.38)\qquad \hat{\imath}: T^*_M X \cup T^*_N X \longrightarrow T^*_M \tilde{M}.$$
$$\quad\Downarrow\qquad\qquad\qquad\Downarrow$$
$$(t, x; \theta dw+\sqrt{-1}\eta dx) \longmapsto (t, x; \sqrt{-1}\eta dx).$$

Moreover the sheaf morphism ext in (2.11) induces an injective sheaf morphism

$$(2.39)\qquad \text{ext}: \hat{C}_{R_+\times N} \longrightarrow \hat{\imath}_*(C_{M_+|X})$$

on $\mathring{T}^*_M\tilde{M}$, which gives a splitting of Φ. Here we remark that ext is not $\mathcal{D}_X$-linear on $\{t=0\}$.

In this situation, the main aim of this section is to construct a splitting of ext in (2.39) through $\mu_M(\Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O}))[n]$. For this purpose, we rewrite the sheaves in (2.39) functorially:

$$(2.40)\qquad \hat{C}_{R_+\times N} = \boldsymbol{R}\hat{\imath}_!(C_{\Omega|X}),$$

$$(2.41)\qquad \hat{\imath}_*(C_{M_+|X}) = \boldsymbol{R}\hat{\imath}_*(C_{M_+|X}),$$

where

$$C_{\Omega|X} = \mu\,\text{hom}(\boldsymbol{Z}_\Omega, \mathcal{O}_X)[n+1], \qquad C_{M_+|X} = \mu\,\text{hom}(\boldsymbol{Z}_{M_+}, \mathcal{O}_X)[n+1]$$

with $\Omega=\text{int}\,M_+$. Remark that the restriction of right hand side of (2.40) to $\{t=0\}$ is nothing but $\hat{C}_{N|M_+}$ by (2.12), and that $C_{M_+|X}$ is cohomologically trivial. These remarks justify (2.40) and (2.41). Indeed the conically cohomological triviality of $C_{M_+|X}$ comes from that of sheaves $C_{N|X}$ and $C_{\Omega|X}|_{T^*_M X\cap T^*_N X}$ $(=(C_{M_+|X}/C_{N|X})|_{T^*_N X\cap T^*_M X})$ due to Schapira-Zampieri [**24**]. Therefore the morphism "ext" in (2.11) gives a morphism

$$(2.42)\qquad \text{ext}^\vee: (\boldsymbol{R}\hat{\imath}_!(C_{\Omega|X}))^\vee[-n] \longrightarrow (\boldsymbol{R}\hat{\imath}_*(C_{M_+|X}))^\vee[-n]$$

on $T_M\tilde{M}$, where $\vee$ denotes the inverse Fourier-Sato transformation

$$\vee: D^+_{\text{con}}(T^*_M\tilde{M}) \longrightarrow D^+_{\text{con}}(T_M\tilde{M}).$$

Refer to [**9**] for more details about $\vee$. Hence we have only to give a splitting of $\text{ext}^\vee$ in (2.42) through $\mu_M(\Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O}))^\vee=\nu_M(\Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O}))$. Here $\nu_M(*)$ is the functor of specialization along M defined in Kashiwara-Schapira [**9**], which is equivalent outside the zero-section to the real monoidal transformation with center M by Sato et. al. [**17**].

Let us introduce sheaves $\tilde{A}_{R_+\times N}$, $\tilde{B}_{R_+\times N}$ and $\tilde{B}'_{R_+\times N}$ on $T_M\tilde{M}$. We take local coordinates $(w, z=x+\sqrt{-1}y)\in \boldsymbol{C}\times\boldsymbol{C}^n$ and $(t, x; \sqrt{-1}v\cdot\partial/\partial y)\in \boldsymbol{R}\times\sqrt{-1}T\boldsymbol{R}^n$ for X and $T_M\tilde{M}$ respectively. Set

$$D_\delta(\mathring{t}, \mathring{x}; \mathring{v}) = \{(w, z)\in\boldsymbol{C}\times\boldsymbol{C}^n;\ |w-\mathring{t}|+|z-\mathring{x}|<\delta,$$
$$\delta\cdot\text{Im}\,z\cdot v>\{|\text{Im}\,z|^2-(\text{Im}\,z\cdot\mathring{v}/|\mathring{v}|)^2\}^{1/2}+|\text{Im}\,w|+(-\text{Re}\,w)_+\}$$
$$\text{(in case } \mathring{v}\neq 0),$$

$$D_\delta(\mathring{t}, \mathring{x}; 0) = \{(w, z) \in \boldsymbol{C} \times \boldsymbol{C}^n; |w - \mathring{t}| + |z - \mathring{x}| < \delta\} \quad \text{(in case } \mathring{v} = 0),$$

$$S_h = \{(w, z) \in \boldsymbol{C} \times \boldsymbol{C}^n; |\mathrm{Im}\, w| + (-\mathrm{Re}\, w)_+ \leqq h(|\mathrm{Im}\, z|)\},$$

for $\delta > 0$, $(\mathring{t}, \mathring{x}; \sqrt{-1}\mathring{v}) \in T_M\tilde{M}$ and $h \in \Theta$. Here $(t)_+ = t$ for $t \geqq 0$, and $(t)_+ = 0$ for $t \leqq 0$, and Θ is a directed set:

$$\Theta = \{h(t) \in C^1([0, +\infty)); h(t) \geqq 0, h'(t) \geqq 0, h(0) = h'(0) = 0\}$$

with the natural order

$$h_1 \leqq h_2 \quad \text{if and only if } h_1(t) \leqq h_2(t) \text{ for any } t \in [0, +\infty).$$

Then we can construct the sheaves $\tilde{A}_{R_+ \times N}$, $\tilde{B}'_{R_+ \times N}$ and $\tilde{B}_{R_+ \times N}$ supported by $\{t \geqq 0\}$ with stalks

$$\tilde{A}_{R_+ \times N}|_q = \varinjlim_{\delta \to +0} \Gamma(D_\delta(q), \mathcal{O}_X), \tag{2.43}$$

$$\tilde{B}'_{R_+ \times N}|_q = \varinjlim_{\delta \to +0} H^1_{\tilde{M}_+}(D_\delta(q), \mathcal{O}_X), \tag{2.44}$$

$$\tilde{B}_{R_+ \times N}|_q = \varinjlim_{\substack{\delta \to +0 \\ h \in \Theta}} H^1_{S_h}(D_\delta(q), \mathcal{O}_X) \tag{2.45}$$

for any $q = (t, x; \sqrt{-1}v) \in T_M\tilde{M}$ with $t \geqq 0$. It is easy to see that

$$\tilde{B}'_{R_+ \times N} \xrightarrow{\sim} \nu_M(\Gamma_{\tilde{M}_+}(\mathcal{BO})). \tag{2.46}$$

For the concentration of the right hand side comes from the pseudo-convexity of $D_\delta(q) \setminus \tilde{M}_+$.

REMARK. i) We omitted that the orientation sheaves in the above definitions.

ii) The sheaf $\tilde{A}_{R_+ \times N}$ is equal to $\nu_M(\mathcal{O}_X)|_{T_M\tilde{M}}$ on $T_M\tilde{M} \cap \{t > 0\}$. Moreover the stalk of $\nu_M(\mathcal{O}_X)$ at any point of $\{t = 0\} \cap T_M\tilde{M}$ is properly embedded in that of $\tilde{A}_{R_+ \times N}$. Indeed $\tilde{A}_{R_+ \times N}|_{\{t=0\}}$ is nothing but the sheaf $\tilde{A}_{M_+}$ introduced by Kataoka in Definition 2.1.15 of [**11**].

iii) The sheaves given in (2.43)$\sim$(2.45) depend only on the product structure $\boldsymbol{R} \times N$ of M, and in this sense they are coordinate-invariant. Similarly to (2.46), it is possible to define $\tilde{A}_{R_+ \times N}$ and $\tilde{B}_{R_+ \times N}$ functorially. In fact, Kataoka defined $\tilde{A}_{M_+}$ and $\tilde{B}_{M_+}$ (a variant of $\tilde{B}_{R_+ \times N}$) functorially from $\mathcal{O}_X$ in [**11**] by employing the real monoidal transformation of X with center M_+, which is a generalization of real monoidal transformaitons due to Sato et al. [**17**] (see Definition 2.1.13 of [**11**]). By a method similar to $\tilde{A}_{M_+}$ and $\tilde{B}_{M_+}$, we can define $\tilde{A}_{R_+ \times N}$ and $\tilde{B}_{R_+ \times N}$ functorially from the sheaf $\mathcal{O}_X$.

LEMMA 2.13. *For any* $q = (\mathring{t}, \mathring{x}; \sqrt{-1}\mathring{v}) \in T_M\tilde{M}$, *we have*

$$\tilde{B}_{R_+\times N}|_q \xrightarrow{\sim} \{f(t, x)\in\Gamma_{M_+}(\mathcal{B}_M)|_{(\mathring{t},\mathring{x})}\,;$$
$$SS(f)\cap\{|t-\mathring{t}|+|x-\mathring{x}|<\varepsilon\}\subset\{\sqrt{-1}\tau dt+\sqrt{-1}\eta dx\,;\,\mathring{v}\cdot\eta\geqq\varepsilon|\eta|\}$$
$$\textit{for some } \varepsilon>0\}$$
$$=\varinjlim_{\varepsilon\to+0}\Gamma_{\{\mathring{v}\cdot\eta\geqq\varepsilon|\eta|,\,\theta\in C\}}(\{|t-\mathring{t}|+|x-\mathring{x}|<\varepsilon\},\, C_{M_+|X}).$$

where $(t, x;\, \theta, \sqrt{-1}\eta)$ *is a coordinate system for* $T^*_{M_+}X$ *introduced in* (2.9).

PROOF. The last equality is obvious from Proposition 2.3. Further, in a similar way to Proposition 2.1.27 of Kataoka [**11**], we can show that

$$\varinjlim_{\varepsilon\to+0}\Gamma_{\{\mathring{v}\cdot\eta\geqq\varepsilon|\eta|,\,\theta\in C\}}(\{|t-\mathring{t}|+|x-\mathring{x}|<\varepsilon\},\, C_{M_+|X})$$
$$\xleftarrow{\sim}\varinjlim_{\substack{\delta\to+0\\ h\in\Theta}} H^1_{S_h}(D_\delta(q),\, \mathcal{O}_X)=\tilde{B}_{R_+\times N}|_q\,.$$

REMARK. We can prove Lemma 2.13 by a variation of Theorem 1.1 of Schapira-Zampieri [**24**], and a Mayer-Vietoris exact sequence. But the argument is complicated, and it is sufficient for our aim to see that there is a natural morphism

$$\tilde{B}'_{R_+\times N}|_q\longrightarrow\varinjlim_{\varepsilon\to+0}\Gamma_{\{\mathring{v}\cdot\eta\geqq\varepsilon|\eta|,\,\theta\in C\}}(\{|t-\mathring{t}|+|x-\mathring{x}|<\varepsilon\},\, C_{M_+|X}).$$

LEMMA 2.14. *We have the following isomorphisms*

$$\boldsymbol{R}\hat{\imath}_!(C_{\Omega|X})^\vee[-n]\xleftarrow{\sim}\tilde{A}_{R_+\times N}\,, \tag{2.47}$$

$$H^l(\boldsymbol{R}\hat{\imath}_*(C_{M_+|X})^\vee[-n])=\begin{cases}0 & (l<0)\\ \tilde{B}_{R_+\times N} & (l=0).\end{cases} \tag{2.48}$$

PROOF. We have only to prove (2.47) and (2.48) for each stalk at $q=(\mathring{t},\,\mathring{x}\,;\,\sqrt{-1}\mathring{v})\in T_M\tilde{M}$. Then by Definition 2.1.2 and Proposition 2.1.4 of Kashiwara-Schapira [**9**], we have

$$F^\vee[-n]|_q=\varinjlim_{q\in U}\boldsymbol{R}\Gamma_{U^{\circ a}}(\pi^{-1}(\tau(U)),\, F) \tag{2.49}$$

for any $F\in D^+_{\mathrm{con}}(T^*_M\tilde{M})$. Here U runs over a neighborhood system of q in $T_M\tilde{M}$, and $\tau: T_M\tilde{M}\to M$, $\pi: T^*_M\tilde{M}\to M$ are natural projections, and

$$U^{\circ a}=\{(t,\, x\,;\, -\sqrt{-1}\eta)\in T^*_M\tilde{M}\,;\,(\sqrt{-1}v)\cdot(\sqrt{-1}\eta)\geqq 0$$
$$\text{for any } (t,\, x\,;\, \sqrt{-1}v)\in U\}.$$

Apply (2.49) to $F=\boldsymbol{R}\hat{\imath}_!(C_{\Omega|X})$. Then

$$H^l(\boldsymbol{R}\hat{\imath}_!(C_{\Omega|X})^\vee[-n])_q$$
$$=\varinjlim_{\varepsilon\to+0}H^l_{\{\mathring{v}\cdot\eta\geqq\varepsilon|\eta|,\,|\theta|\leqq\varepsilon^{-1}\mathring{v}\cdot\eta\}}(\{|t-\mathring{t}|+|x-\mathring{x}|<\varepsilon\},\, C_{\Omega|X}).$$

Since $\operatorname{supp}(C_{\Omega|X}) \subset T_N^*X \cup T_M^*X$, Theorem 1.1 of Schapira-Zampieri [24] implies

$$H^l(\boldsymbol{R}\hat{\iota}(C_{\Omega|X})^\vee[-n])_q = \varinjlim_{\varepsilon\to+0} \varinjlim_{V_\varepsilon} H^l(V_\varepsilon, \mathcal{O}_X). \tag{2.50}$$

Here V_ε ranges through the family of open subsets of X such that

$$C(X \setminus V_\varepsilon, M_+) \cap \left\{(\mathring{v}^2-\varepsilon^2)^{1/2} \cdot \left|\tilde{y} - \frac{|\tilde{w}|}{\varepsilon}\mathring{v}\right| + \frac{|\tilde{w}|}{\varepsilon}\mathring{v}^2 \leqq \tilde{y}\cdot\mathring{v}\right\} = \emptyset .$$

Then, since V_ε can be chosen convex, the cohomologies of (2.50) vanish for the degree $\neq 0$, and the 0-th cohomology is equal to $\tilde{A}_{\boldsymbol{R}_+\times N}|_q$. Thus we have proved (2.47).

Next apply (2.49) to $F = \boldsymbol{R}\hat{\iota}_*(C_{M_+|X})$. Then

$$\begin{aligned} &H^l(\boldsymbol{R}\hat{\iota}_*(C_{M_+|X})^\vee[-n])_q \\ &\quad = \varinjlim_{\varepsilon\to+0} H^l_{\{\mathring{v}\cdot\eta \geqq \varepsilon|\eta|,\, \theta\in C\}}(\{|t-\mathring{t}|+|x-\mathring{x}|<\varepsilon\}, C_{M_+|X}). \end{aligned} \tag{2.51}$$

Since $C_{M_+|X}$ is concentrated in degree 0, we have the vanishing of (2.51) for $l<0$. Further, on account of Lemma 2.13, we find that the 0-th cohomology group of (2.51) is equal to $\tilde{B}_{\boldsymbol{R}_+\times N}$. Thus we obtained (2.48). q. e. d.

Note that there exist natural morphisms of sheaves

$$\alpha^\vee : \tilde{A}_{\boldsymbol{R}_+\times N} \ni f(w, z) \longrightarrow [f(t, z)\cdot Y(t)] \in \tilde{B}'_{\boldsymbol{R}_+\times N}$$
$$\beta^\vee : \tilde{B}'_{\boldsymbol{R}_+\times N} \longrightarrow \tilde{B}_{\boldsymbol{R}_+\times N} .$$

Here $\beta^\vee$ is $\mathcal{D}_X$-linear. Moreover taking into account of the original definition of ext by Kataoka (Proposition 2.1.18 and Corollary 2.1.24 of [11]), we find that $\beta^\vee\alpha^\vee$ is just equal to $\mathrm{ext}^\vee$. Hence by Lemma 2.14, we get

LEMMA 2.15. *There exist canonical sheaf morphisms α and β on $\mathring{T}_{\tilde{M}}^*\tilde{M}$:*

$$\alpha : \mathring{C}_{\boldsymbol{R}_+\times N} \longrightarrow \mu_M(\Gamma_{\tilde{M}_+}(\mathcal{BO}))[n] , \tag{2.52}$$

$$\beta : \mu_M(\Gamma_{\tilde{M}_+}(\mathcal{BO}))[n] \longrightarrow \hat{\iota}_*(C_{M_+|X}) . \tag{2.53}$$

Here β is $\mathcal{D}_X$-linear, and for any section f of $\mathring{C}_{\boldsymbol{R}_+\times N}$ we have

$$\beta(\alpha(f)) = \mathrm{ext}(f) .$$

REMARK. There is also a natural morphism

$$\alpha' : C_N \ni f(x) \longmapsto f(x)\cdot\delta(t) \in \mu_M(\Gamma_{\tilde{M}_+}(\mathcal{BO}))[n] ,$$

which splits to a natural morphism

$$C_N \ni f(x) \longmapsto f(x)\cdot\delta(t) \in \hat{\iota}_*(C_{M_+|X}) .$$

In fact, j_*C_N with $j : T_N^*Y \simeq \{t=0\} \cap T_{\tilde{M}}^*\tilde{M} \to T_{\tilde{M}}^*\tilde{M}$ can be imbedded in $\mathring{C}_{\boldsymbol{R}_+\times N}$, and thus we can write

$$\alpha'(f(x)) = D_t(\alpha(f(x)))\,.$$

3. Proof of the main theorem.

We prepare the following lemmas.

LEMMA 3.1. *We follow the notation of* § 2.5. *Suppose that* $\varphi(t, x, y')$ *is a real analytic and convex function. Setting*

$$\tilde{M}_+ = \{(t, z) \in \tilde{M}\,;\, t \geqq 0\} \text{ and } L_+ = L \cap \tilde{M}_+ = \{y_1 - \varphi(t, x, y') = 0,\ t \geqq 0\},$$

we define the sheaves $\mathcal{G}$ *and* $\mathcal{G}_0$ *on* L *as follows: For any open subset* V *of* L

$$\Gamma(V, \mathcal{G}) = \{f(t, x, y') \in \Gamma_{L_+}(V, \mathcal{B}_L)\,;\ f \text{ satisfies the equations } \mathcal{N}_Z \text{ in (2.25), and the estimates (2.27) for } SS(f) \text{ and (2.29) for } SS^{\frac{2}{2}}_{\Sigma}(f)\},$$

$$\Gamma(V, \mathcal{G}_0) = \{f(t, x, y') \in \Gamma(V, \mathcal{G})\,;\ SS(f) \subset \Sigma \text{ and } SS^{\frac{2}{2}}_{\Sigma}(f) = \emptyset\}.$$

Then we have a quasi-isomorphism as $\mathcal{D}_X$ *modules on* L:

$$\boldsymbol{R}\Gamma_{\tilde{M}_+ \cap F}(\mathcal{B}\mathcal{O})|_L \xrightarrow{\sim} (j_* j^{-1} \Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O}))|_L / \Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O})|_L[-1] \tag{3.1}$$

and isomorphisms

$$(j_* j^{-1} \Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O}))|_L \underset{\sim}{\overset{bv}{\longrightarrow}} \mathcal{G} \quad \text{and} \quad \Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O})|_L \underset{\sim}{\overset{bv}{\longrightarrow}} \mathcal{G}_0\,. \tag{3.2}$$

PROOF. The vanishing of 0-th cohomology group of the left hand side of (3.1) follows from the unique continuation property of $\mathcal{B}\mathcal{O}$. Further, noting the triangle

$$\longrightarrow \boldsymbol{R}\Gamma_{\tilde{M}_+ \cap F}(\mathcal{B}\mathcal{O}) = \boldsymbol{R}\Gamma_F \Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O}) \longrightarrow \Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O}) \longrightarrow \boldsymbol{R}j_* j^{-1} \Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O}) \overset{+1}{\longrightarrow}\,,$$

we obtain

$$\boldsymbol{R}^1\Gamma_{\tilde{M}_+ \cap F}(\mathcal{B}\mathcal{O})|_L \xrightarrow{\sim} (j_* j^{-1} \Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O}))|_L / \Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O})|_L\,.$$

On the other hand, since the support of any section of $\mathcal{B}\mathcal{O}$ has a fibre structure with respect to the map $(t, z) \mapsto t$, Propositions 2.9 and 2.10 imply

$$(j_* j^{-1} \Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O}))|_L \underset{\sim}{\overset{bv}{\longrightarrow}} \mathcal{G} \quad \text{and} \quad \Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O})|_L \underset{\sim}{\overset{bv}{\longrightarrow}} \mathcal{G}_0\,.$$

Hence the problem left to us is to show $\boldsymbol{R}^q\Gamma_{\tilde{M}_+ \cap F}(\mathcal{B}\mathcal{O})|_L = 0$ for any $q \geqq 2$. First we remark that

$$\boldsymbol{R}^q\Gamma_{\tilde{M}_+ \cap F}(\mathcal{B}\mathcal{O})|_L = \boldsymbol{R}^{q+1}\Gamma_{\tilde{M}_+ \cap F}(\mathcal{O}_X)|_L\,. \tag{3.3}$$

Note that $X \setminus (\tilde{M}_+ \cap F)$ is a union of two Stein open subsets:

$$\{(w, z) \in X\,;\, w \notin [0, +\infty)\} \cup \{(w, z) \in X\,;\, \mathrm{Im}\, z_1 > \varphi(\mathrm{Re}\, w, \mathrm{Re}\, z, \mathrm{Im}\, z')\}$$

if X itself is a Stein open subset. Hence by the Mayer-Vietoris exact sequence, we conclude that the right hand side of (3.3) vanishes in case $q\geqq 2$. This completes the proof. q.e.d.

LEMMA 3.2. *Under the assumptions of Theorem* 1.1, *the following sequence is exact in a neighborhood of* $p=(\mathring{t}, \mathring{z})$:

$$(3.4)\qquad 0\longrightarrow \mathcal{D}_X^{m_+}\xrightarrow{h_1}\mathcal{D}_X\oplus\mathcal{D}_X^{m_+}\oplus\mathcal{D}_X^{m}\xrightarrow{h_0}\mathcal{D}_X\oplus\mathcal{D}_X^{m}\longrightarrow\mathfrak{M}\longrightarrow 0,$$

where

$$(3.5)\qquad \begin{aligned} h_0(q\oplus(q_j)_{j=1}^{m_+}\oplus(r_k)_{k=0}^{m-1}) &= qP\oplus\Big(-qQ_k+\sum_{j=1}^{m_+}q_jB_{jk}+r_kt\Big)_{k=0}^{m-1},\\ h_1((s_j)_{j=1}^{m_+}) &= 0\oplus(s_jt)_{j=1}^{m_+}\oplus\Big(-\sum_{j=1}^{m_+}s_jB_{jk}\Big)_{k=0}^{m-1}. \end{aligned}$$

PROOF. On account of coherency of $\mathcal{D}_X$, we have only to show the exactness of (3.4) for its stalk at $p=(\mathring{t}, \mathring{z})$. The exactness of (3.4) at $\mathcal{D}_X^{m_+}$ and at $\mathcal{D}_X\oplus\mathcal{D}_X^m$ is trivial to see. To verify the exactness at $\mathcal{D}_X\oplus\mathcal{D}_X^{m_+}\oplus\mathcal{D}_X^m$, we take any

$$\alpha = q\oplus(q_j)_{j=0}^{m_+}\oplus(r_k)_{k=0}^{m-1}\in(\mathrm{Ker}\,(h_0))_p.$$

Then we find that $q=0$ and that there exist $(q'_j)_j$ and $(q''_j)_j$ in $\mathcal{D}_{X,p}^{m_+}$ satisfying the equations

$$(3.6)\qquad q_j=q'_j\cdot t+q''_j,\qquad [q''_j, D_t]=0\qquad (j=1,\cdots,m_+)$$

and

$$(3.7)\qquad r_k=-\sum_{j=1}^{m_+}q'_j\cdot B_{jk},\qquad \sum_{j=1}^{m_+}q''_j\cdot B_{jk}=0\qquad (k=0,\cdots,m-1).$$

In case $\mathring{t}\neq 0$,

$$\alpha = h_1\Big(\Big(q'_j+q''_j\cdot\frac{1}{t}\Big)_{j=1}^{m_+}\Big)\in(\mathrm{Im}\,h_1)_p.$$

In case $\mathring{t}=0$, we recall the assumption (A2). Then we know in particular that the matrix

$$(\sigma_{m_j-k}(B_{jk}))_{1\leq j\leq m_+,\,0\leq k\leq m-1}$$

is of the maximal rank m_+, (In fact, the polynomials in θ

$$\Big\{\sum_{k=0}^{m_j}\sigma_{m_j-k}(B_{jk})(\mathring{z}, \mathring{\zeta})\cdot(\theta+\mathring{\tau})^k;\ j=1,\cdots,m_+\Big\}$$

are $\boldsymbol{C}$-linearly independent. For otherwise

$$\Big\{\sum_{k=0}^{m_+-1}\beta_{jk}(\mathring{z}, \mathring{\tau}, \mathring{\zeta})\cdot(\theta+\mathring{\tau})^k;\ j=1,\cdots,m_+\Big\}$$

become $\boldsymbol{C}$-linearly dependent.) Taking into account of this fact, we obtain, from the second equation of (3.7), $(q_j'')_{j=1}^{m_+}=0$ in $\mathcal{E}_{X,(0,\mathring{z};\mathring{\tau},\mathring{\zeta})}^{m_+}$. This implies $(q_j'')_{j=0}^{m_+}=0$ in $\mathcal{D}_{X,p}^{m_+}$. Thus by (3.6) and (3.7), we show $\alpha\in(\mathrm{Im}\, h_1)_p$. This completes the proof. q. e. d.

PROOF OF THEOREM 1.1. Without loss of generality we may assume $\mathring{t}\geqq 0$ and $\mathrm{Im}\,\mathring{\zeta}_1>0$ because $\mathring{\zeta}=\mathring{\xi}+\sqrt{-1}\mathring{\eta}\neq 0$. We put $p=(\mathring{t}, \mathring{z})$. Since our conditions (A1) and (A2) are of open properties, we have only to show

$$\boldsymbol{R}\Gamma_F(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O})))|_p = 0. \tag{3.8}$$

Here

$$F=\{(t, z)\in\tilde{M};\ -y_1+\varphi(t, x, y')\geqq 0\}$$

with a real analytic and strictly convex function φ defined in a neighborhood of $(\mathring{t}, \mathring{x}, \mathring{y}')$ satisfying

$$\mathring{y}_1=\varphi(\mathring{t}, \mathring{x}, \mathring{y}') \quad\text{and}\quad (\varphi_t, \varphi_x, \varphi_{y'})=(\mathring{\tau}/\mathring{\eta}_1, \mathring{\xi}/\mathring{\eta}_1, -\mathring{\eta}'/\mathring{\eta}_1) \text{ at } (\mathring{t}, \mathring{x}, \mathring{y}').$$

Hence, by Lemma 3.1, we can reduce (3.8) to

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, j_*j^{-1}\Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O})|_L/\Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O})|_L)=0 \text{ at } p. \tag{3.9}$$

Therefore we have only to show the vanishing of 0-th, 1-st and 2-nd cohomology groups of (3.9) by using the resolution (3.4) for $\mathfrak{M}$.

0-th cohomology group. Let $U, U_0, \cdots, U_{m-1}$ be given germs of $(j_*j^{-1}\Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O}))_p$ satisfying

$$\begin{cases} PU \equiv Q_0U_0+\cdots+Q_{m-1}U_{m-1} \\ \sum_{k=0}^{m-1} B_{jk}U_k \equiv 0 \qquad (j=1, \cdots, m_+) \\ t\cdot U_k \equiv 0 \qquad (k=0, \cdots, m-1) \end{cases}, \tag{3.10}$$

modulo $\Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O})|_p$. Setting $\tilde{u}=bv(U)$, $v_0=bv(U_0)$, $\cdots$, and $v_{m-1}=bv(U_{m-1})$, we take the boundary values of (3.10) on L. Then by Lemma 2.11 we have some differential operators $P'=\gamma(P)$, $B_{jk}'=\gamma(B_{jk})$ $(1\leqq j\leqq m_+,\ 0\leqq k\leqq m-1)$, $Q_k'=\gamma(Q_k)$ $(0\leqq k\leqq m-1)$ of the forms

$$\begin{aligned} P' &= D_t^m+\sum_{k=0}^{m-1} A_k'(t, x, y', D_x, D_{y'})\cdot D_t^k \\ B_{jk}' &= B_{jk}'(t, x, y', D_x, D_{y'}) \\ Q_k' &= D_t^{m-k-1}+\sum_{l=0}^{m-k-2} Q_{kl}'(t, x, y', D_x, D_{y'})D_t^l \end{aligned} \tag{3.11}$$

such that

$$\mathrm{order}(A_k')\leqq m-k\,,\quad \mathrm{order}(B_{jk}')\leqq m_j-k \quad\text{and}\quad \mathrm{order}(Q_k')=m-k-1.$$

Hence the germs $\tilde{u}, v_0, \cdots, v_{m-1}$ of $\mathcal{Q}$ satisfy the equations:

$$(3.12)\qquad \begin{cases} P'\tilde{u} \equiv Q'_0 v_0 + \cdots + Q'_{m-1} v_{m-1} \\ \sum_{k=0}^{m-1} B'_{jk} v_k \equiv 0 \qquad (j=1, \cdots, m_+) \\ t\cdot v_k \equiv 0 \qquad (k=0, \cdots, m-1) \end{cases}$$

modulo $\mathcal{G}_0$.

Further, from (2.34) we know that the values of $\sigma_m(P')$, $\sigma_{m_j-k}(B'_{jk})$ and $\sigma_{m-k-1}(Q'_k)$ at $(t, x, y'; \theta, \sqrt{-1}\xi, \sqrt{-1}\eta')$ coincide with those of $\sigma_m(P)$, $\sigma_{m_j-k}(B_{jk})$ and $\sigma_{m-k-1}(Q_k)$ at

$$(3.13)\qquad \Big(t,\ x_1+\sqrt{-1}\varphi,\ x'+\sqrt{-1}y';\quad \theta+\frac{\sqrt{-1}\varphi_t}{\sqrt{-1}-\varphi_{x_1}}\xi_1,\ \frac{-\xi_1}{\sqrt{-1}-\varphi_{x_1}},\ \frac{\sqrt{-1}}{2}\Big\{\xi'-\sqrt{-1}\eta'+\frac{\varphi_{x'}-\sqrt{-1}\varphi_{y'}}{\sqrt{-1}-\varphi_{x_1}}\xi_1\Big\}\Big).$$

From now on, we consider $\tilde{u}, v_0, \cdots, v_{m-1}$ as sections of $C_{L_+|Z}$. Then they satisfy equations (3.12) microlocally on

$$(3.14)\qquad H=\{(t, x, y'; \theta, \sqrt{-1}\xi, \sqrt{-1}\eta')\in T^*Z; (\xi, \eta')\neq 0, t\geq 0, \theta\in \boldsymbol{C}, \operatorname{Re}\theta\geq 0,\ x, y', \xi \text{ and } \eta' \text{ are real}, t\cdot(\operatorname{Re}\theta)=0, |t-\mathring{t}|+|x-\mathring{x}|+|y'-\mathring{y}'|<\delta\}$$

for some $\delta>0$; we will take another $\delta>0$ small enough if necessary. On the other hand, by the condition (2.27) for the estimates of singular spectrums for sections of $\mathcal{G}$, we have only to consider $\tilde{u}, v_0, \cdots, v_{m-1}$ on

$$(3.15)\qquad H'=\{(t, x, y'; \theta, \sqrt{-1}\xi, \sqrt{-1}\eta')\in H; \xi_1>0,\ \xi_j=\frac{-\varphi_{y_j}+\varphi_{x_1}\varphi_{x_j}}{1+\{\varphi_{x_1}\}^2}\cdot\xi_1,\ \eta_j=\frac{\varphi_{x_j}+\varphi_{x_1}\varphi_{y_j}}{1+\{\varphi_{x_1}\}^2}\cdot\xi_1 \quad \text{for } j=2, \cdots, n\}.$$

Then for any $(t, x, y'; \theta, \sqrt{-1}\xi, \sqrt{-1}\eta')\in H'$, the corresponding point in (3.13) becomes

$$(3.16)\qquad \Big(t,\ x_1+\sqrt{-1}\varphi,\ x'+\sqrt{-1}y';\quad \frac{\xi_1}{1+\{\varphi_{x_1}\}^2}\Big((1+(\varphi_{x_1})^2)\frac{\theta}{\xi_1}+\varphi_t-\sqrt{-1}\varphi_{x_1}\varphi_t,\ \varphi_{x_1}+\sqrt{-1},\ \varphi_{x'}-\sqrt{-1}\varphi_{y'}\Big)\Big).$$

It follows from the second and the third systems in (3.12) that

$$(3.17)\qquad v_k(t, x, y')=u_k(x, y')\delta(t) \qquad (k=0, \cdots, m-1)$$

with some microfunctions $u_k(x, y')$'s satisfying

$$(3.18)\qquad \sum_{k=0}^{m-1} B'_{jk}(0, x, y', D_x, D_{y'})u_k=0 \qquad (j=1, \cdots, m_+).$$

These facts combined with (A1) assure that $\tilde{u}$ vanishes on $H'\cap\{t>0\}$. For P' is elliptic there.

In case $H'\cap\{t=0\}\neq\emptyset$, we may assume $\mathring{t}=0$, and then we can use the additional condition (A2). In fact, $\tilde{u}$ defines a $\mathring{C}_{L_0|L_+}$ solution $u(t, x, y')$ on $\iota(\{t=0\}\cap H')$ because of the equation (3.12) and the relations (3.17) where $L_0=L\cap\{t=0\}$. Then it is easy to see that

$$u_k(x, y') := R_k(t, x, y', D_t, D_x, D_{y'})u|_{t\to+0} \qquad (k=0, \cdots, m-1) \tag{3.19}$$

where $R_k=\gamma(D_t^k)$ is a differential operator of order k with symbol

$$\sigma_k(R_k)(t, x, y', \theta, \sqrt{-1}\xi, \sqrt{-1}\eta')=\left(\theta+\frac{\varphi_t-\sqrt{-1}\varphi_t\varphi_{x_1}}{1+(\varphi_{x_1})^2}\xi_1\right)^k. \tag{3.20}$$

Thus u satisfies the boundary value problem

$$\begin{cases} P'u=0 \\ \sum_{k=0}^{m-1} B'_{jk}R_k u|_{t\to+0}=0 \qquad (j=1, \cdots, m_+) \end{cases}$$

as a section of $\mathring{C}_{L_0|L_+}$. Therefore, under (A1) and (A2), it becomes a microlocal boundary value problem satisfying the conditions of Theorem 2.7. Hence we deduce that $u=0$, and thus $\tilde{u}=\mathrm{ext}(u)=0$, $v_0=\cdots=v_{m-1}=0$ on $H'\cap\{t=0\}$. Consequently we conclude that

$$SS(\tilde{u})\cup SS(v_0)\cup\cdots\cup SS(v_{m-1})\subset\Sigma.$$

The remaining problem is to show the vanishing of SS_Σ^2 for $\tilde{u}, v_0, \cdots$ and v_{m-1}. Since t is a solvable operator for $\mathcal{G}_0$, $tv_k\in\mathcal{G}_0$ implies that $v_k-u_k\cdot\delta(t)\in\mathcal{G}_0$ for some hyperfunction $u_k(x, y')$. On the other hand, it follows from $SS(v_k)\subset\Sigma$ that $u_k(x, y')$ is analytic. Hence $SS_\Sigma^2(v_k)=\emptyset$ for $k=0, 1, \cdots, m-1$, and also by the first equation of (3.12), we have $SS_\Sigma^2(\tilde{u})=\emptyset$. That is, $\tilde{u}, v_0, \cdots, v_{m-1}$ belong to $\mathcal{G}_0$. This shows the vanishing of the 0-th cohomology group.

The 1*st cohomology group.* Let P', Q'_k and B'_{jk} be operators defined in (3.11). Then it is sufficient to show the following assertion: If germs E, E_j $(j=1, \cdots, m_+)$, I_k $(k=0, \cdots, m-1)$ of $\mathcal{G}$ at p satisfy the equations

$$t\cdot E_j-\sum_{k=0}^{m-1} B'_{jk}I_k\equiv 0 \qquad (j=1, \cdots, m_+) \qquad \text{modulo } \mathcal{G}_0, \tag{3.21}$$

there exist some germs U, U_k $(k=0, \cdots, m-1)$ of $\mathcal{G}$ at p such that

$$\begin{cases} P'U-\sum_{k=0}^{m-1} Q'_kU_k\equiv E \\ \sum_{k=0}^{m-1} B'_{jk}U_k\equiv E_j \qquad (j=1, \cdots, m_+) \\ t\cdot U_k\equiv I_k \qquad (k=0, \cdots, m-1) \end{cases} \tag{3.22}$$

modulo $\mathcal{G}_0$. Since the morphisms

$$t\cdot : j_*j^{-1}\Gamma_{\tilde{M}_+}(\mathcal{BO})|_p \hookleftarrow \quad \text{and} \quad t\cdot : \Gamma_{\tilde{M}_+}(\mathcal{BO})|_p \hookleftarrow$$

are surjective, we may assume $I_k=0$ $(k=0, \cdots, m-1)$ and that the equations (3.21) hold as hyperfunctions. Under the additional assumption $(I_k)_{k=0}^{m-1}=0$, we will show that (3.22) has a unique solution $(U, U_0, \cdots, U_{m-1})$ as a section of $\hat{\iota}_*(\mathcal{C}_{L_+|Z}|_H)^{m+1}$ in a neighborhood of $\hat{\iota}(H')$ (see (3.14), (3.15)), where $\hat{\iota}$ is the projection

$$\hat{\iota}: H \ni (t, x, y'; \theta, \sqrt{-1}\xi, \sqrt{-1}\eta') \longmapsto (t, x, y'; \sqrt{-1}\xi, \sqrt{-1}\eta') \in \boldsymbol{R}_t \times \sqrt{-1}\mathring{T}^*(\boldsymbol{R}_x^n \times \boldsymbol{R}_{y'}^{n-1}).$$

For the moment, we consider $U, U_0, \cdots, U_{m-1}, E, E_1, \cdots, E_{m_+}$ as sections of $\mathcal{C}_{L_+|Z}$. Then we can write

$$(3.23) \qquad \begin{aligned} E_j(t, x, y') &= e_j(x, y')\cdot\delta(t) \qquad (j=1, \cdots, m_+), \\ U_k(t, x, y') &= u_k(x, y')\cdot\delta(t) \qquad (k=0, \cdots, m-1) \end{aligned}$$

with some microfunctions $e_j(x, y')$ $(j=1, \cdots, m_+)$, $u_k(x, y')$ $(k=0, \cdots, m-1)$ satisfying

$$(3.24) \qquad \sum_{k=0}^{m-1} B'_{jk}(0, x, y', D_x, D_{y'})u_k = e_j \qquad (j=1, \cdots, m_+).$$

Hence our claim for the part $H'\cap\{t>0\}$ follows from the ellipticity of P' by (A1). In case $H\cap\{t=0\}\neq\emptyset$, we may assume $\mathring{t}=0$, and we can utilize the additional condition (A2). Therefore, we can apply Proposition 2.8 to find $(U, u_0, \cdots, u_{m-1})$ by putting $P=P'$, $S_k=Q'_k$ $(k=0, \cdots, m-1)$ and $E_{jk}=B'_{jk}$ $(j=1, \cdots, m_+, k=0, \cdots, m-1)$ in (2.20). In fact, since the operator $D_t^k+\sum_{l=0}^{k-1}W_{kl}(x, D_x)D_t^l$ corresponds to $R_k(0, x, y', D_t, D_x, D_{y'})=\gamma(D_t^k)|_{t=0}$ in this situation, the conditions (A1) and (A2) imply (C1) and (C2) for the pair $(P', \sum_{k=0}^{m-1}B'_{jk}R_k|_{t=0})$ in a neighborhood of H' (see (3.13), (3.16) and (3.20)). Thus our claim has been verified. Moreover, it follows at the same time from the uniqueness in the above claim that this solution $(U, U_0, \cdots, U_{m-1})$ extends to H as a section of $(\Gamma_{H'}(\mathcal{C}_{L_+|Z}))^{m+1}$ and that it satisfies the equations $\mathcal{N}_Z$ of (2.25) there. The latter part is because all operators in (3.22) commute with any operator in $\mathcal{N}_Z$ (see Lemma 2.11). After all, since P' is elliptic on $\Sigma=\{\xi=0, \eta'=0\}$, there exist hyperfunctions $V(t, x, y')\in\Gamma_{L_+}(\mathcal{B}_L)|_p$ and $V_0(x, y'), \cdots, V_{m-1}(x, y')\in\mathcal{B}_{L_0}|_{(\mathring{x}, \mathring{y}')}$ such that

$$(3.25) \qquad \begin{cases} P'V-\displaystyle\sum_{k=0}^{m-1} Q'_k(V_k\cdot\delta(t))=E \qquad \text{at } p, \\ \displaystyle\sum_{k=0}^{m-1} B'_{jk}(V_k\cdot\delta(t))=E_j \qquad \text{at } p \text{ for } j=1, \cdots, m_+, \end{cases}$$

and that

$$SS(V)\cup SS(V_0\delta(t))\cup\cdots\cup SS(V_{m-1}\delta(t))\subset H'\cup\Sigma,$$
$$(3.26)\qquad SS(A_lV)\subset\Sigma\qquad(l=2,\cdots,n),$$
$$\text{and}\quad SS((A_l|_{t=0})V_k)=\emptyset\qquad(l=2,\cdots,n,\ k=0,\cdots,m-1).$$

Here $A_2,\cdots,A_n$ are tangential Cauchy-Riemann operators in (2.25). Let $Z_0=Z\cap\{w=0\}$ $(\subset\tilde{X})$ be a complexification of $L_0=L\cap\{t=0\}$ with the embedding $Z_0\to\tilde{Y}=\tilde{X}\cap\{w=0\}$ induced from the embedding

$$L_0\ni(x,y')\longmapsto(0,x,\varphi(0,x,y'),y')\in\tilde{M}\cap\{t=0\}.$$

Then we have following quasi-isomorphism of Cauchy-Kowalevsky's type due to M. Kashiwara [**31**] (see also [**17**]).

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_{\tilde{Y}}}(\mathcal{N}',\mathcal{O}_{\tilde{Y}})|\xrightarrow{\sim}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_{Z_0}}(\mathcal{N}'_{Z_0},\mathcal{O}_{Z_0}),$$

where $\mathcal{N}'=\mathcal{D}_{\tilde{Y}}/\sum_{l=1}^{n}\mathcal{D}_{\tilde{Y}}D_{\bar{Z}_l}$ and $\mathcal{N}'_{Z_0}$ is the $\mathcal{D}_{Z_0}$ module induced on Z_0 from $\mathcal{N}'$. Therefore $\mathcal{E}xt^1_{\mathcal{D}_{Z_0}}(\mathcal{N}'_{Z_0},\mathcal{O}_{Z_0})=\mathcal{E}xt^1_{\mathcal{D}_{\tilde{Y}}}(\mathcal{N}',\mathcal{O}_{\tilde{Y}})|_{Z_0}=0$. Hence we can modify $V_0,\cdots,V_{m-1}$ by subtracting suitable analytic functions so that

$$(A_l|_{t=0})V_k=0\qquad\text{at }(\mathring{x},\mathring{y}')\text{ for }l=2,\cdots,n\text{ and }k=0,\cdots,m-1.$$

Combined with (3.26), this implies $V_0\delta(t),\cdots,V_{m-1}\delta(t)\in\mathcal{G}|_p$ because the estimates for $SS^2_\Sigma(V_k\delta(t))$ follow from those for $SS(V_k)$. Then after modifying V on Σ as a section of $C_{L_+|Z}$, we reobtain

$$(3.27)\qquad P'V-\sum_{k=0}^{m-1}Q'_k(V_k\delta(t))=E\ \text{at}\ p$$

for modified $V_0,\cdots,V_{m-1}$. In case $\mathring{t}=0$, this implies

$$(3.28)\qquad A_lV=0\qquad(l=2,\cdots,n)$$

at p because $P'(A_lV)=0$ at p $(l=2,\cdots,n)$. In case $\mathring{t}>0$, after some modification of V as those for V_k, we have (3.28). Moreover, we obtain from (3.27) a proper estimate for $SS^2_\Sigma(V)$ since P' is elliptic on Σ. Combining these estimates with (3.26), we derive the fact $V\in\mathcal{G}|_p$. Consequently $(V,V_0\delta(t),\cdots,V_{m-1}\delta(t))\in(\mathcal{G}|_p)^{m+1}$ solves (3.22). Thus we have shown the vanishing of the 1st cohomology group.

The second cohomology group. Let $W_1,\cdots,W_{m_+}$ be arbitrary germs of $\mathcal{G}$ at p. Then we have only to find some germs $V_1,\cdots,V_{m_+},U_0,\cdots,U_{m-1}$ of $\mathcal{G}$ at p satisfying

$$W_j\equiv t\cdot V_j-\sum_{k=0}^{m-1}B_{jk}U_k\qquad(j=1,\cdots,m_+)$$

modulo $\mathcal{G}_0$. In fact, it is possible even when $U_0=U_1=\cdots=U_{m-1}=0$ because

$$t\cdot:j_*j^{-1}\Gamma_{\tilde{M}_+}(\mathcal{BO})|_p\hookleftarrow$$

is surjective.

Finally our long proof of Theorem 1.1 is completed. q.e.d.

PROOF OF THEOREM 1.2. By applying Kashiwara's theorem to $1/\sigma_m(t^2, z, \theta, \zeta)$, we know that (H1) is equivalent to the following:

(H1)′ The equation $\sigma_m(P)(t, z, \theta+\tau, \zeta)=0$ has no purely imaginary roots with respect to θ on

$$W=\bigcup_{0<\varepsilon<\delta}\{(t, z; \tau dt+\mathrm{Re}(\zeta dz)\in \mathring{T}^*\tilde{M}; 0\leqq t<\delta, |\mathrm{Re}\, z-\mathring{x}|+|\mathrm{Im}\,\zeta|<\delta,$$
$$|\tau|+|\mathrm{Im}\, z|+|\mathrm{Re}\,\zeta-\varepsilon\mathring{x}^*|<\varepsilon\delta\}$$

for some small $\delta>0$.

Therefore, since the number of roots in $\{\theta\in \boldsymbol{C}; \mathrm{Re}\,\theta>0\}$ is constant on W, the assumption (H1)′, (H2) implies the assumption (A1), (A2) for every $(t, z; \tau, \zeta)\in W$. Hence, by Theorem 1.1, we have for the system $\mathfrak{M}$ in (1.4) associated with $(P, B_1, \cdots, B_{m_+})$

$$W\cap SS(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O})))=\emptyset .$$

Therefore by the formula (2.6),

$$(0, \mathring{x}; dx_1)\notin SS(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{M_+}(\mathcal{B}_M))).$$

For we have

$$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{M_+}(\mathcal{B}_M))=\boldsymbol{R}\Gamma_M(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O})))[n].$$

Hence we obtain

$$\boldsymbol{R}\Gamma_{\{x_1\geqq \mathring{x}_1\}}(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{M_+}(\mathcal{B}_M)))|_{(0,\mathring{x})}=0. \tag{3.29}$$

In particular

$$\varinjlim_U \Gamma(U, \mathcal{F})\xrightarrow{\sim}\varinjlim_U \Gamma(U\cap\{x_1<\mathring{x}_1\}, \mathcal{F})$$

with $\mathcal{F}=\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{M_+}(\mathcal{B}_M))$, where U runs over a neighborhood system of $(0, \mathring{x})$. Combined with the resolution (3.4), the observation in (1.1)~(1.5) implies Theorem 1.2. q.e.d.

PROOF OF THEOREM 1.3. The uniqueness part is contained in Theorem 1.2. Thus we have only to prove the existence part. By (3.29), we have

$$\mathcal{E}xt^1_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{M_+\cap\{x_1\geqq\mathring{x}_1\}}(\mathcal{B}_M))_{(0,\mathring{x})}=0.$$

Consider the resolution (3.4) of $\mathfrak{M}$. Then by the vanishing of the above cohomology group, we can find a solution $(U, U_0, \cdots, U_{m-1})$ with value in $\Gamma_{M_+\cap\{x_1\geqq\mathring{x}_1\}}(\mathcal{B}_M)$ at $(0, \mathring{x})$ satisfying

$$(3.30)\quad \begin{cases} PU-\sum_{k=0}^{m-1} Q_k U_k = \mathrm{ext}(f) \\ \sum_{k=0}^{m-1} B_{jk}U_k = g_j(x)\cdot\delta(t) \qquad (1\leqq j\leqq m_+) \\ t\cdot U_k = 0 \qquad (k=0, \cdots, m-1). \end{cases}$$

Hence U_k has a form $u_k(x)\cdot\delta(t)$ for every k. Let $u(t, x)$ be the restriction of $U(t, x)$ to $\{t>0\}$. Then it is clear from the observation (1.1)~(1.5) that $u(t, x)$ satisfies (1.11). q.e.d.

PROOF OF THEOREM 1.4. Let u be a section of $\hat{C}_{R_+\times N}$ over

$$W=\{(t, x; \sqrt{-1}\eta)\in \mathring{T}_M^*\tilde{M}; 0\leqq t<\delta, |x-\mathring{x}|<\delta, |\eta-\mathring{\eta}|<\delta, \psi(t, x, \eta)<0\}$$

satisfying (1.15) on W. Therefore

$$u^*=(\alpha(u), u(+0, x)\cdot\delta(t), \cdots, D_t^{m-1}u(+0, x)\cdot\delta(t))$$

expresses a section of $\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mu_M(\Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O}))[n])$ over W, where α is defined in (2.52) and $\mathfrak{M}$ is the coherent $\mathcal{D}_X$ module defined in (1.4).

Under the assumption (S1) and (S2), we claim that

$$(3.31)\qquad (0, \mathring{x}, \mathring{\eta}; d\psi(0, \mathring{x}, \mathring{\eta}))\notin SS(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mu_M(\Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O}))[n])).$$

Indeed, in the same way as in the proof of Theorem 1.2, we can deduce from the assumption (S1) and (S2), the condition (A1) and (A2) for any $(t, z; \tau, \zeta)\in V$ with

$$V=\bigcup_{0<\varepsilon<\delta}\{(t, z; \tau, \mathrm{Re}(\zeta dz))\in T^*\tilde{M}; 0\leqq t<\delta, |\mathrm{Re}\,z-\mathring{x}|+|\mathrm{Im}\,\zeta-\mathring{\eta}|<\delta,$$
$$|\tau-\varepsilon\mathring{t}^*|+|\mathrm{Im}\,z-\varepsilon\mathring{\eta}^*|+|\mathrm{Re}\,\zeta-\varepsilon\mathring{x}^*|<\varepsilon\delta\}$$

for some small $\delta>0$. Hence by Theorem 1.1 and the formula (2.7), we obtain (3.31). In particular, we have

$$\varinjlim_U \Gamma(U, \mathcal{F}) \xrightarrow{\sim} \varinjlim_U \Gamma(U\cap\{\psi(t, x, \eta)<0\}, \mathcal{F})$$

with $\mathcal{F}=\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mu_M(\Gamma_{M_+}(\mathcal{B}\mathcal{O}))[n])$. Here U moves over a neighborhood system of $(0, \mathring{x}, \mathring{\eta})$. Hence u^* extends uniquely to $(0, \mathring{x}; \sqrt{-1}\mathring{\eta})$ as a section of $\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mu_M(\Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O}))[n])$. Let $u^{*\sim}=(U, U_0, \cdots, U_{m-1})$ be the unique extension, and consider $\beta(U)$, where β is defined in (2.53). Then since $\sigma(P)\neq 0$ on $\{(0, \mathring{x}; \theta dw); \theta\neq 0\}$, we can extend $\beta(U)$ as a section of $C_{N|X}$ to

$$\{(x; \theta dw+\sqrt{-1}\eta dt)\in T_N^*X; |x-\mathring{x}|+|\eta-\mathring{\eta}|<\delta, |\theta|>\delta^{-1}\}$$

with some $\delta>0$ by using (1.4). Therefore $[\beta(U)]$ modulo $\iota_*(C_{N|X})$ becomes a germ of $\hat{C}_{R_+\times N}$ at $(0, \mathring{x}; \sqrt{-1}\mathring{\eta})$. Then by Lemma 2.15 we find that $[\beta(U)]$ gives a unique extension of u to $(0, \mathring{x}; \sqrt{-1}\mathring{\eta})$ as a $\hat{C}_{R_+\times N}$-solution of (1.15).

q.e.d.

PROOF OF THEOREM 1.5. The uniqueness part is included in Theorem 1.4. Thus it is sufficient to show the existence of the solutions. By (3.31) and Lemma 2.12, we have

$$\mathcal{E}xt^1_{\mathcal{D}_X}(\mathfrak{M}, \Gamma_{\{\psi\geqq 0\}}(\mu_M(\Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O}))[n]))_{(0,\mathring{x},\mathring{\eta})}=0.$$

Thus in the same way as in the proof of Theorem 1.3, we can find for any $f(t, x)$ and $(g_j(x))_j$ in Theorem 1.5, a solution $(U, U_0, \cdots, U_{m-1})$ with value in $\Gamma_{\{\psi\geqq 0\}}(\mu_M(\Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O}))[n])$ at $(0, \mathring{x}, \mathring{\eta})$ satisfying

$$(3.32)\qquad \begin{cases} PU-\sum_{k=0}^{m-1} Q_k U_k = \alpha(f) \\ \sum_{k=0}^{m-1} B_{jk}U_k = g_j(x)\delta(t) \qquad (1\leqq j\leqq m_+) \\ t\cdot U_k = 0 \qquad (k=0, \cdots, m-1) \end{cases}$$

as germs of $\mu_M(\Gamma_{\tilde{M}_+}(\mathcal{B}\mathcal{O}))[n]$. Then by considering $(\beta(U), \beta(U_0), \cdots, \beta(U_{m-1}))$ as sections of $\mathcal{C}_{M_+|X}$ with support in $\hat{\imath}^{-1}(\{\psi\geqq 0\})$, we can employ the argument similar to the proof of Theorem 1.3. In fact, $\beta(U_k)(t, x)$ has the form $u_k(x)\cdot\delta(t)$ with some microfunction $u_k(x)\in\mathcal{C}_N|_{(\mathring{x};\sqrt{-1}\mathring{\eta})}$ for any k, and thus $\beta(U)(t, x)$ satisfies the equation

$$(3.33)\qquad P\beta(U)=\sum_{k=0}^{m-1} Q_k(u_k(x)\delta(t))+\mathrm{ext}(f)$$

as a section of $\mathcal{C}_{M_+|X}$. Hence by the same argument as in the proof of Theorem 1.4, we conclude $[\beta(U)] \bmod \hat{\imath}_*(\mathcal{C}_{N|X})$ is a section of $\mathring{\mathcal{C}}_{R_+\times N}$ in a neighborhood of $(0, \mathring{x}, \mathring{\eta})$. Then it is easy to see that $[\beta(U)]$ satisfies (1.16). q.e.d.

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