

Examples of degenerations of Castelnuovo surfaces

By Tadashi ASHIKAGA and Kazuhiro KONNO

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Introduction.

Let $\rho: \mathcal{S} \rightarrow \Delta_\varepsilon$ be a proper flat morphism of a nonsingular threefold \mathcal{S} to a disk $\Delta_\varepsilon = \{t \in \mathbb{C}; |t| < \varepsilon\}$ with connected fibers. We call it a semi-stable degeneration if ρ is smooth over $\Delta_\varepsilon^* = \Delta_\varepsilon \setminus \{0\}$ and $S_0 = \rho^{-1}(0)$ is a reduced divisor with simple normal crossings. The divisor $S_t = \rho^{-1}(t)$ is called the singular fiber if $t=0$ and it is called a general fiber if $t \neq 0$. Let $\rho: \mathcal{S} \rightarrow \Delta_\varepsilon$ be a semi-stable degeneration and denote by $\rho^\circ: \mathcal{S}^* \rightarrow \Delta_\varepsilon^*$ its restriction to the punctured disk. Then the fundamental group of Δ_ε^* acts naturally on $H^2(S_t, \mathbb{Z})$, $t \neq 0$. Let N denote the logarithm of the monodromy action. We say the degeneration $\rho: \mathcal{S} \rightarrow \Delta_\varepsilon$ is a Type I (resp. Type II) degeneration if $N=0$ (resp. $N^2=0$). Type I degenerations are attractive, since one must study them if he want to make the period mapping proper.

In this article, we construct some semi-stable degenerations such that a general fiber is a *Castelnuovo surface*, that is, a minimal algebraic surface of general type with $c_1^2 = 3p_g - 7$ whose canonical map is birational onto its image.

In §1, we recall some fundamental results on Castelnuovo surfaces found in [1]. In §2, we construct a Type I degeneration. Note that a Castelnuovo surface with $p_g = 4$ is a quintic surface. Therefore, ours serves an explicit example of Type I degenerations of quintic surfaces whose existence was shown by Friedman [4] using Horikawa's family of deformations of a numerical quintic surface of type II_b [5]. We also refer the reader to [4] for further discussions on such degenerations. Friedman informed us that N. Shepherd-Barron constructed another Type I degeneration of quintic surfaces. The other examples of Type I degenerations of surfaces of general type can be found in [4], [11] and [12].

In §3, we extend Horikawa's canonical resolution of singularities on double coverings of surfaces [5] to the case of cyclic triple coverings. This is used in §4 in order to construct Type II degenerations. In our example, the singular fiber consists of a Castelnuovo surface Σ and a rational surface R , and the invariants of Σ are the "next" to those of a general fiber S_t on the line $c_1^2 = 3p_g - 7$, that is, $p_g(\Sigma) = p_g(S_t) - 1$ and $c_1^2(\Sigma) = c_1^2(S_t) - 3$. Thus we can descend

down along $c_1^2=3p_g-7$ by our Type II degenerations. An analogous phenomena was observed by Usui [12] on the Noether-Horikawa lines $c_1^2=2p_g-4$, $2p_g-3$.

1. Castelnuovo surfaces.

In this section, we recall some fundamental properties of Castelnuovo surfaces which can be found in [1].

1.1. Let a, b, c be integers satisfying

$$(1.1) \quad 0 \leq a \leq b \leq c, \quad a+b+c > 0.$$

We denote by $P_{a,b,c}$ the P^2 -bundle $P(\mathcal{O}_{P^1}(a) \oplus \mathcal{O}_{P^1}(b) \oplus \mathcal{O}_{P^1}(c))$ over the projective line P^1 . We denote by $\bar{\omega}$ the projection and put

T : the (relatively ample) tautological divisor,

F : a fiber of $\bar{\omega}: P_{a,b,c} \rightarrow P^1$.

We consider an irreducible member $S \in |4T - (a+b+c-2)F|$ which has at most rational double points (RDP's for short). Since the dualizing sheaf is given by $\omega_S = \mathcal{O}_S(T)$, the minimal resolution \tilde{S} of S is a surface whose canonical map is birational onto its image. Furthermore, its numerical invariants are given by

$$(1.2) \quad \begin{aligned} p_g(\tilde{S}) &= a+b+c+3, & q(\tilde{S}) &= 0, \\ c_1^2(\tilde{S}) &= T^2S = 3p_g(\tilde{S})-7. \end{aligned}$$

Conversely, almost all surfaces with $c_1^2=3p_g-7$ whose canonical map is birational arises in this way [1, §1]. On the other hand, *Castelnuovo's second inequality* ([2, p. 228]) says that a minimal surface of general type satisfies $c_1^2 \geq 3p_g-7$ if its canonical map is birational onto its image. Since \tilde{S} achieves the lower bound, we call it a *Castelnuovo surface* of type (a, b, c) . Note that the projection $\bar{\omega}$ induces on \tilde{S} a pencil of non-hyperelliptic curves of genus 3.

1.2. Put $a=b=0$ and $c=1$. Then (1.2) shows that \tilde{S} is a numerical quintic surface. It can be shown that the canonical image of \tilde{S} contains a line [1, Proposition 2.5]. Conversely, let S' be a quintic surface with at most RDP's and assume that it contains a line l . Then, by blowing up P^3 along l , we get $P_{0,0,1}$ and the minimal resolution of the proper transform of S' is a Castelnuovo surface of type $(0, 0, 1)$.

The following can be found in [1, §§3-4].

LEMMA 1.3. *Assume that $4a \geq p_g - 5$. Then a Castelnuovo surface of type (a, b, c) is simply connected. Further, if it is generic, the Kuranishi space is non-singular and the infinitesimal Torelli theorem holds.*

1.4. Returning to the situation of 1.1, we define a subclass of Castelnuovo surfaces. Among others, we use the following notation. For any nonnegative integer d , we denote by Σ_d the Hirzebruch surface of degree d . We let C_0 and f denote the section ($C_0^2=d$) and a fiber of $\Sigma_d \rightarrow \mathbf{P}^1$, respectively.

We choose sections X_0, X_1 and X_2 of $[T-aF], [T-bF]$ and $[T-cF]$, respectively, in such a way that they form a system of homogeneous coordinates on each fiber of $\mathbf{P}_{a,b,c} \rightarrow \mathbf{P}^1$. Assume that S contains the line Z defined by $X_1=X_2=0$. If $\nu: X \rightarrow \mathbf{P}_{a,b,c}$ denotes the blowing-up with center Z , X is isomorphic to $\mathbf{P}(\mathcal{O}_{\Sigma_{c-b}} \oplus \mathcal{O}_{\Sigma_{c-b}}(C_0+(b-a)f))$. Let $\pi: X \rightarrow \Sigma_{c-b}$ be the projection and H the tautological divisor on X . Then the proper transform S' of S by ν is linearly equivalent to $3H+\pi^*(C_0+(2a+2-c)f)$. If we denote by \tilde{S} the minimal resolution of S' , then the natural map $\tilde{S} \rightarrow \Sigma_{m-l}$ induced by π is clearly of degree 3. For this reason, we call such \tilde{S} a *trigonal Castelnuovo surface*.

2. Type I degenerations.

The purpose of this section is to construct Type I degenerations of Castelnuovo surfaces. Our idea is to degenerate the pencil of nonhyperelliptic curves of genus 3 on a Castelnuovo surface to a pencil of hyperelliptic curves.

2.1. We keep the notation of §1. Assume that $3a+2 \geq b+c$. Then the linear system $|4T-(a+b+c-2)F|$ is free from base points. Take two integers α, β satisfying

$$(2.1) \quad a+b+c-2 = 2\alpha-\beta, \quad 2a \geq \alpha, \beta \geq 0.$$

We remark that $|M_\alpha|, M_\alpha=2T-\alpha F$ is free from base points. We choose $q \in H^0(\mathbf{P}_{a,b,c}, \mathcal{O}(M_\alpha))$ which defines an irreducible nonsingular divisor Q . Let ε be a sufficiently small positive number and put $\Delta_\varepsilon = \{z \in \mathbf{C}; |z| < \varepsilon\}$. We consider a family $\{S_t\}_{t \in \Delta_\varepsilon}$ of subvarieties of the \mathbf{P}^1 -bundle $Y = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(M_\alpha)) \rightarrow \mathbf{P}_{a,b,c}$ defined by

$$(2.2) \quad S_t : \begin{cases} a_0 Y_0^2 + a_1 Y_0 Y_1 + a_2 Y_1^2 = 0 \\ t Y_0 = q Y_1 \end{cases}, \quad t \in \Delta_\varepsilon$$

where (Y_0, Y_1) is a system of homogeneous coordinates on fibers of $Y \rightarrow \mathbf{P}_{a,b,c}$ and $a_j \in H^0(\mathbf{P}_{a,b,c}, \mathcal{O}(jM_\alpha + \beta F)), 0 \leq j \leq 2$. We assume that a_j are general. If $t \neq 0$, then S_t is biholomorphically equivalent to a surface in $\mathbf{P}_{a,b,c}$ defined by the equation

$$(2.3) \quad a_0 q^2 + t a_1 q + t^2 a_2 = 0.$$

By (2.1), we see that S_t is a Castelnuovo surface of type (a, b, c) . On the other hand, S_0 consists of $\beta+1$ components. One of them is a double covering of Q

and each of the other is isomorphic to P^2 .

LEMMA 2.2. For a generic choice of q and a_j , $0 \leq j \leq 2$, S_0 is a divisor with simple normal crossings consisting of a minimal surface Σ with $c_1^2 = 3p_g - 7 - \beta$ and β disjoint copies of P^2 . Furthermore, a double curve $\Sigma \cap P^2$ is a smooth conic (in P^2) and $p_g(\Sigma) = p_g(S_t)$, $t \neq 0$.

PROOF. Putting $t=0$ in the second equation of (2.2), we get $Y_1=0$ or $q=0$. If $Y_1=0$, then we get $a_0=0$ from the first equation. Since a_0 can be identified with a homogeneous form of degree β on P^1 , we may assume that its zeros are mutually distinct and give β fibers F_1, \dots, F_β on $P_{a,b,c}$. Thus $Y_0 = a_0 = 0$ defines a section \tilde{F}_j of the P^1 -bundle $Y|_{F_j} \rightarrow F_j$ for each j , $1 \leq j \leq \beta$ and therefore $\tilde{F}_j \cong P^2$. We next consider the case $q=0$. The first equation of (2.2) defines a double covering Σ of Q .

CLAIM. Σ is a regular minimal surface with $c_1^2(\Sigma) = 3p_g(\Sigma) - 7 - \beta$. Further, its geometric genus equals that of a general fiber S_t .

PROOF. We denote the restriction of T , etc. to Q by the same symbol if there is no danger of confusion. Then Σ is a divisor on $Y_Q := Y|_Q$ linearly equivalent to $2\mathcal{T} + \Pi^*(\beta F)$, where \mathcal{T} is a tautological divisor and Π is the projection map of Y_Q . It is easy to see that the linear system $|2\mathcal{T} + \Pi^*(\beta F)|$ is free from base points. Therefore, we can assume that Σ is irreducible and nonsingular. Since the canonical bundle of Y_Q is given by

$$K_{Y_Q} = \mathcal{O}_{Y_Q}(-2\mathcal{T} + \Pi^*(T - \beta F)),$$

we see that the canonical bundle of Σ is

$$K_\Sigma = (K_{Y_Q} + \Sigma)|_\Sigma = \mathcal{O}_\Sigma(\Pi^*T).$$

Since $\Pi|_\Sigma$ is of degree 2, we have

$$\begin{aligned} K_\Sigma^2 &= 2(T|_Q)^2 \quad (\text{on } Q) \\ &= 2T^2(2T - \alpha F) \quad (\text{on } P_{a,b,c}) \\ &= 3(a+b+c+3) - 7 - \beta. \end{aligned}$$

Consider the exact sequence

$$0 \longrightarrow \mathcal{O}(K_{Y_Q}) \longrightarrow \mathcal{O}(K_{Y_Q} + \Sigma) \longrightarrow \mathcal{O}_\Sigma(K_\Sigma) \longrightarrow 0.$$

Since Y_Q is rational, we have

$$H^p(Y_Q, \mathcal{O}(K_{Y_Q})) \cong \begin{cases} \mathbf{C} & \text{if } p=3, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for $p < 2$, we have

$$H^p(\Sigma, \mathcal{O}(K_\Sigma)) \cong H^p(Y_Q, \mathcal{O}(K_{Y_Q} + \Sigma)) \cong H^p(Q, \mathcal{O}(T)).$$

We consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}_{a,b,c}}(T-Q) \longrightarrow \mathcal{O}_{\mathbf{P}_{a,b,c}}(T) \longrightarrow \mathcal{O}_Q(T) \longrightarrow 0.$$

Since $T-Q \sim -T + \alpha F$, we have $H^p(\mathbf{P}_{a,b,c}, \mathcal{O}(T-Q))=0$ for any p . Therefore,

$$\begin{aligned} h^0(\Sigma, \mathcal{O}(K_\Sigma)) &= h^0(\mathbf{P}_{a,b,c}, \mathcal{O}(T)) = a+b+c+3, \\ h^1(\Sigma, \mathcal{O}) &= h^1(\Sigma, \mathcal{O}(K_\Sigma)) = 0. \end{aligned}$$

This gives the assertion.

To complete the proof of Lemma 2.2, we must study the intersection $\Sigma \cap (Y_1 = a_0 = 0)$. Since Q is a conic bundle, we can assume that $Q \cap F_j$ is a smooth conic for all j (viewed in $F_j \cong \mathbf{P}^2$). Putting $t=q=a_0=0$ in (2.2), we get

$$(a_1 Y_0 + a_2 Y_1) Y_1 = 0.$$

In this expression, $Y_1=0$ gives $\Sigma \cap \tilde{F}_j$. Thus if we choose a_1, a_2 generic, $\Sigma \cap \tilde{F}_j$ is mapped isomorphically to $Q \cap F_j$ via the projection $Y \rightarrow \mathbf{P}_{a,b,c}$.

Q. E. D. of Lemma 2.2.

The above consideration is summarized in the following :

THEOREM 2.3. *There exists a semi-stable degeneration $\rho : S \rightarrow \Delta_\varepsilon$ satisfying the following properties.*

- (1) *A general fiber S_t is a Castelnuovo surface.*
- (2) *The singular fiber $S_0 = \Sigma \cup (\beta \text{ disjoint copies of } \mathbf{P}^2)$, where Σ is a minimal surface of general type with $c_1^2 = 3p_g - 7 - \beta$, and $\Sigma \cap \mathbf{P}^2$ is a conic.*
- (3) *ρ is a Type I degeneration.*

PROOF. (1) and (2) follow from the construction. (3) is a consequence of the Clemens-Schmid exact sequence (see, e.g., [3] or [8]).

Furthermore, we can show the following. Since the proof follows easily from the arguments in [4], [7], we omit it.

PROPOSITION 2.4. *Let $\rho^\circ : S^* \rightarrow \Delta_\varepsilon^*$ denote the restriction of the family in Theorem 2.3 to the punctured disk $\Delta_\varepsilon^* = \Delta_\varepsilon - \{0\}$. Then the following hold.*

- (1) *No base change of it may be filled in smoothly, if $\beta > 0$.*
- (2) *Some power of the Picard-Lefschetz diffeomorphism is homotopic to the identity.*

REMARK 2.5. We see the following from our construction.

- (1) Put $(a, b, c) = (0, 0, 1)$ and $(\alpha, \beta) = (0, 1)$. Then we get an example of Type I degeneration of quintic surfaces. The main component Σ of the singular

fiber is of type (0) according to Horikawa's classification of surfaces with $(p_g, c_1^2) = (4, 4)$ ([6]). The existence of such a degeneration was shown by Friedman [4] using a complicated family of deformations of a numerical quintic surface (of type II_b) due to Horikawa [5].

(2) If $\beta = 0$, then we get a family of surfaces with $c_1^2 = 3p_g - 7$. The canonical map of the central fiber is of degree 2. On the other hand, that of a general fiber is birational. This example was already found in [1, §4].

2.6. (Variants) The construction in 2.1 is so simple that we can generalize it, for example, to the following set-up:

P : an irreducible nonsingular projective variety,

D_1, D_2 : divisors on P ,

$p: \mathbf{P}(\mathcal{E}) \rightarrow P$: the \mathbf{P}^2 -bundle associated with a locally free sheaf \mathcal{E} of rank 3 on P ,

T : the tautological divisor such that $\mathcal{O}(T) = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

Assume that

(1) general members of $|2T + p^*D_1|$ and $|4T + p^*(2D_1 + D_2)|$ are irreducible nonsingular, and

(2) general members of $|D_2|$ are nonsingular.

Then, as in 2.1, we can construct a semi-stable degeneration $\{Y_t\}_{t \in \Delta_\varepsilon}$ satisfying:

(3) a general fiber Y_t is a nonsingular member of $|4T + p^*(2D_1 + D_2)|$,

(4) the singular fiber $Y_0 = \Sigma \cup R$, where

(i) Σ is a double covering of a nonsingular $Q \in |2T + p^*D_1|$,

(ii) R is a \mathbf{P}^2 -bundle on a nonsingular $D \in |D_2|$, and

(iii) Σ and R intersect transversally along a conic bundle on D .

In particular, when P is a curve, one gets various examples of Type I degenerations of surfaces in this way. If we put harmlessly $D_2 = 0$, then we get a family of deformations such that, under a suitable assumption, the canonical maps of Y_t and Y_0 may have different flavors.

3. Canonical resolution for cyclic triple coverings.

In [5, §2], Horikawa gave a method, called the *canonical resolution*, in order to resolve singularities on double coverings of surfaces. In this section, we extend it to cyclic triple coverings of surfaces.

3.1. Let V be a nonsingular surface and L a line bundle on V . We denote by $\pi: X = \mathbf{P}(\mathcal{O}_V \oplus \mathcal{O}_V(L)) \rightarrow V$ the \mathbf{P}^1 -bundle associated with L . Let H be the tautological divisor on X and consider the complete linear system $|3H + \pi^*A|$, where A is a divisor on V . If we fix a system of fiber coordinates (Z_0, Z_1) on X , then any section $\phi \in H^0(V, \mathcal{O}_V(3H + \pi^*A))$ can be written as

$$\phi = \phi_A Z_0^3 + \phi_{A+L} Z_0^2 Z_1 + \phi_{A+2L} Z_0 Z_1^2 + \phi_{A+3L} Z_1^3,$$

where $\phi_{A+iL} \in H^0(V, \mathcal{O}_V(A+iL))$ for $0 \leq i \leq 3$. We set

$$|3H + \pi^*A|_C = \{(\phi) \in |3H + \pi^*A|; \phi = \phi_A Z_0^3 + \phi_{A+3L} Z_1^3\},$$

where (ϕ) is the divisor defined by ϕ . We call it the *cyclic subsystem* of $|3H + \pi^*A|$. For any member $(\phi) \in |3H + \pi^*A|_C$, we call the divisors $B_\phi = (\phi_{A+3L})$ and $B'_\phi = (\phi_A)$ the branch locus and the assistant branch locus of (ϕ) , respectively. With this notation, we define the *good-cyclic subsystem* $|3H + \pi^*A|_{GC}$ by the following two properties:

- (1) B_ϕ is reduced and B'_ϕ is nonsingular.
- (2) B'_ϕ passes through no singular points of B_ϕ , and they meet transversally.

Take a member $S = (\phi) \in |3H + \pi^*A|_{GC}$. We write $B = B_\phi$ and $B' = B'_\phi$ for the sake of simplicity. It follows from the definition of the good-cyclic system that S is an irreducible normal surface. Furthermore, it is easy to see that a point $P \in S$ is a singular point of S if and only if $\pi(P) \in \text{Sing}(B)$, the singular locus of B . Thus the local analytic equation of $P \in \text{Sing}(S)$ is of the form $\xi^3 + f(x, y) = 0$.

REMARK 3.2. By a result of Wavrik [13, §1], for a finite cyclic triple covering of surfaces $S \rightarrow V$, we can find a line bundle L on V so that S can be considered as a member of $|3H|_C$ on $X = \mathbf{P}(\mathcal{O}_V \oplus \mathcal{O}_V(L))$. This is a typical example of our situation.

3.3. We now give a method to resolve singularities on S , which proceeds inductively as follows:

Step (1). Take a point $P_1 \in \text{Sing}(B)$ and denote by m_1 the multiplicity of B at P_1 . Let $\tau_1: V_1 \rightarrow V$ be the blow-up of V at P_1 . Then we have $\tau_1^*B = B_1 + m_1 E_1$, where B_1 is the proper transform of B by τ_1 and $E_1 = \tau_1^{-1}(P_1)$ is the exceptional curve. We define the line bundle $L(1)$ and divisors $B(1)$, $B'(1)$ and $A(1)$ on V_1 by

$$\begin{aligned} L(1) &= \tau_1^*L \otimes_{\mathcal{O}_{V_1}}(-[m_1/3]E_1), & B(1) &= \tau_1^*B - 3[m_1/3]E_1, \\ B'(1) &= \tau_1^*B' & \text{and } A(1) &= \tau_1^*A, \end{aligned}$$

where $[m_1/3]$ is the greatest integer not exceeding $m_1/3$. We put $X_1 = \mathbf{P}(\mathcal{O}_{V_1} \oplus \mathcal{O}_{V_1}(L(1)))$ and denote its projection map by π_1 . We let H_1 denote the tautological divisor on X_1 . Since we have $B(1) \sim 3L(1) + A(1)$ and $B'(1) \sim A(1)$, the pair $(B(1), B'(1))$ defines a member S_1 of $|3H_1 + \pi_1^*A(1)|_C$ on X_1 . The composite of the sheaf homomorphisms

$$\mathcal{O}_V(L) \xrightarrow{\tau_1^*} \mathcal{O}_{V_1}(\tau_1^*L) \longrightarrow \mathcal{O}_{V_1}(L(1)),$$

where the last map is obtained by tensoring $\mathcal{O}_{V_1}(-[m_1/3]E_1)$, induces the morphism $\tilde{\mu}_1: X_1 \rightarrow X$. From this, we get a birational morphism $\mu_1 = \tilde{\mu}_1|_{S_1}: S_1 \rightarrow S$. Here, we remark that S_1 is not necessarily normal, since $B(1)$ is not reduced if $m_1 \equiv 2 \pmod{3}$.

Step (i): Let P_i be a point of $Sing(B(i-1)_{red})$, where $B(i-1)_{red}$ denotes the reduced part of $B(i-1)$. We denote by m_i the multiplicity of $B(i-1)$ at P_i . Let $\tau_i: V_i \rightarrow V_{i-1}$ be the blow-up of V_{i-1} at P_i and set

$$\begin{aligned} L(i) &= \tau_i^*L(i-1) \otimes_{\mathcal{O}_{V_{i-1}}}(-[m_i/3]E_i), \\ B(i) &= \tau_i^*B(i-1) - 3[m_i/3]E_i, \quad B'(i) = \tau_i^*B'(i-1), \\ A(i) &= \tau_i^*A(i-1), \end{aligned}$$

where $E_i = \tau_i^{-1}(P_i)$. From the pair $(B(i), B'(i))$, we obtain a member $S_i \in |3H_i + \pi_i^*A(i)|_C$ on $X_i = \mathbf{P}(\mathcal{O}_{V_i} \oplus \mathcal{O}_{V_i}(L(i)))$, where H_i is the tautological divisor and π_i is the projection of X_i . Further, as before, we get a birational morphism $\mu_i: S_i \rightarrow S_{i-1}$.

LEMMA 3.4. *There exists a nonnegative integer n such that the curve $B(n)_{red}$ is nonsingular.*

PROOF. Clearly, there is a nonnegative integer r such that, after Step(r), the proper transform of B is nonsingular and $B(r)_{red}$ is a divisor with simple

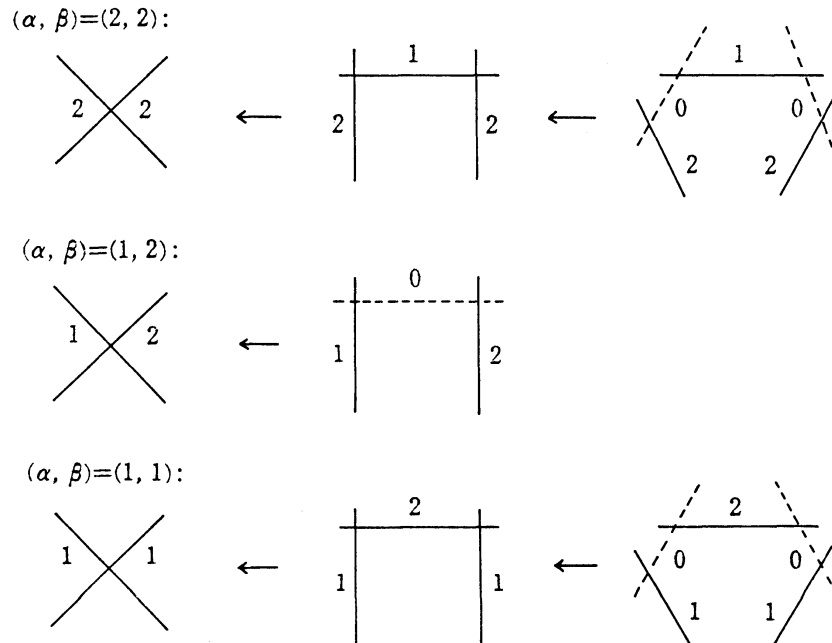


Figure 1.

normal crossings. Then the local analytic equation of a singular point P of $B(r)$ is of the form: $x^\alpha y^\beta = 0$, where $(\alpha, \beta) = (2, 2), (2, 1)$ or $(1, 1)$. Thus, if we proceed the steps as in Figure 1, we get the desired result.

3.5. Let n be the integer in Lemma 3.4. Then $B(n)_{red}$ is nonsingular and we have the diagram:

$$\begin{array}{ccccccc} S_n & \xrightarrow{\mu_n} & \cdots & \longrightarrow & S_1 & \xrightarrow{\mu_1} & S_0 = S \\ \downarrow \pi_n & & & & \downarrow \pi_1 & & \downarrow \pi_0 = \pi \\ V_n & \xrightarrow{\tau_n} & \cdots & \longrightarrow & V_1 & \xrightarrow{\tau_1} & V_0 = V. \end{array}$$

We set $\mu = \mu_1 \circ \cdots \circ \mu_n$ and $\tau = \tau_1 \circ \cdots \circ \tau_n$. Let B_n be the proper transform of B by μ . Then we can write $B(n)$ in the form

$$B(n) = B_n + 2 \sum_{j \in J} E_j + \sum_{k \in K} E_k,$$

where $\{E_j, E_k\}_{j \in J, k \in K}$ are the part of the reduced components of exceptional curves. Then the singular locus of S_n coincides with $\cup_{j \in J} \pi_n^{-1}(E_j)$ and the local analytic equation of $\text{Sing}(S_n)$ is $\xi^3 + x^2 = 0$. We may call it the *compound cusp* (see, Figure 2).

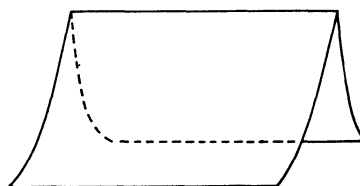


Figure 2.

If $\bar{\mu}: S^* \rightarrow S_n$ denotes the normalization of S_n , then S^* is nonsingular. We call $\tilde{\mu} = \bar{\mu} \circ \mu: S^* \rightarrow S$ the *canonical resolution*. In general, this is not the minimal resolution. We get the minimal resolution \tilde{S} of S by contracting all (possibly infinitely near) (-1) curves on S^* .

We give the formula for calculating the difference between $(\chi(\mathcal{O}_S), \omega_S^2)$ and $(\chi(\mathcal{O}_{S^*}), \omega_{S^*}^2)$.

PROPOSITION 3.6. Let $\tilde{\mu}: S^* \rightarrow S$ be the canonical resolution as above and n the integer as in Lemma 3.4. Then,

- (1) $\chi(\mathcal{O}_{S^*}) = \chi(\mathcal{O}_S) - \frac{1}{2} \sum_{i \in I} [m_i/3] (5[m_i/3] - 3) + \sum_{j \in J} (1 - C_j^2),$
- (2) $\omega_{S^*}^2 = \omega_S^2 - 3 \sum_{i \in I} (2[m_i/3] - 1)^2 + \sum_{j \in J} 8,$

where m_i is the multiplicity of $B(i-1)$ at the center of the blow-up $\tau_i: V_i \rightarrow V_{i-1}$, C_j^2 is the self-intersection number in S^* of the nonsingular curve C_j which is a

reduced component of the pull-back by μ of the singular locus of S_n .

PROOF. First, we shall measure the difference between the invariants $(\chi(\mathcal{O}_S), \omega_S^2)$ and $(\chi(\mathcal{O}_{S_n}), \omega_{S_n}^2)$. Since $K_X+S \sim H+\pi^*(K_V+L+A)$, by using the exact sequence

$$0 \longrightarrow \mathcal{O}(K_X) \longrightarrow \mathcal{O}(K_X+S) \longrightarrow \omega_S \longrightarrow 0$$

and the fact that

$$\pi_*\mathcal{O}_X(H) \cong \mathcal{O}_V \oplus \mathcal{O}_V(L), \quad R^p\pi_*\mathcal{O}_X(H) = 0 \quad (p > 0),$$

we get

$$(3.1) \quad \begin{aligned} \chi(\mathcal{O}_S) &= \chi(\omega_S) = \chi(K_X+S) - \chi(K_X) \\ &= \chi(K_V+L+A) + \chi(K_V+2L+A) - \chi(\mathcal{O}_V). \end{aligned}$$

Since, in the process of the canonical resolution, we have

$$K_{V_i} \sim \tau_i^*K_{V_{i-1}} + E_i, \quad L(i) = \tau_i^*L(i-1) - [m_i/3]E_i, \quad A(i) = \tau_i^*A(i-1),$$

we get

$$(3.2) \quad \begin{aligned} K_{V_i} + L(i) + A(i) &\sim \tau_i^*(K_{V_{i-1}} + L(i-1) + A(i-1)) + (1 - [m_i/3])E_i, \\ K_{V_i} + 2L(i) + A(i) &\sim \tau_i^*(K_{V_{i-1}} + 2L(i-1) + A(i-1)) + (1 - 2[m_i/3])E_i. \end{aligned}$$

CLAIM. For a divisor D on V_{i-1} and an integer $k \geq -1$, set $D_k = \tau_i^*D - kE_i$. Then,

$$\chi(V_i, \mathcal{O}(D_k)) = \chi(V_{i-1}, \mathcal{O}(D)) - k(k+1)/2.$$

PROOF. If $k = -1$, then this is easy. If $k \geq 0$, then we consider the exact sequence

$$0 \longrightarrow \mathcal{O}(D_k) \longrightarrow \mathcal{O}(D_{k-1}) \longrightarrow \mathcal{O}_{E_i}(k-1) \longrightarrow 0$$

to conclude that $\chi(\mathcal{O}(D_k)) = \chi(\mathcal{O}(D_{k-1})) - k$. Therefore, we get the desired result inductively.

By this, (3.1) and (3.2), a calculation shows

$$(3.3) \quad \chi(\mathcal{O}_{S_n}) = \chi(\mathcal{O}_S) - \frac{1}{2} \sum_i [m_i/3] (5[m_i/3] - 3).$$

Next, we compute $\omega_{S_n}^2$. Since

$$H_i^2 = c_1(\mathcal{O}_{V_i} \oplus \mathcal{O}_{V_i}(L(i))) \cdot H_i,$$

we have

$$\begin{aligned} \omega_{S_i}^2 &= (K_{X_i} + S_i)^2 \cdot S_i \\ &= (H_i + \pi_i^*(K_{V_i} + L(i) + A(i)))^2 (3H_i + \pi_i^*A(i)) \\ &= 3(K_{V_i} + 2L(i) + A(i))^2. \end{aligned}$$

It follows from (3.2) that

$$\begin{aligned} \omega_{S_i}^2 &= 3((K_{V_{i-1}}+2L(i-1)+A(i-1))^2-(2[m_i/3]-1)^2) \\ &= \omega_{S_{i-1}}^2-3(2[m_i/3]-1)^2. \end{aligned}$$

Thus we get the following formula.

$$(3.4) \quad \omega_{S_n}^2 = \omega_{S_3}^2 - 3 \sum_i (2[m_i/3]-1)^2.$$

In order to get the formulae in Proposition 3.6, we have to measure the contribution of the compound cusps. We remark that the singular locus of S_n is the disjoint union of curves: $\cup_{j \in J} C_j$. Put $\bar{\mu}^* C_j = \tilde{C}_j$. Then $(\tilde{C}_j)_{red}$ is isomorphic to C_j and, identifying these, we have $\tilde{C}_j = 2C_j$. Let $\sigma: \tilde{X} \rightarrow X_n$ be the blow-up of X_n with center $\cup_{j \in J} C_j$, and let $\mathcal{E}_j = \sigma^{-1}(C_j)$ be the exceptional divisors. Then S^* is the proper transform of S_n by σ and $\bar{\mu} = \sigma|_{S^*}$. Thus we have $S^* \sim \sigma^* S_n - 2 \sum_{j \in J} \mathcal{E}_j$. From the exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}(-\sigma^* S_n) \longrightarrow \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_{\sigma^* S_n} \longrightarrow 0,$$

it follows that

$$\begin{aligned} \chi(\mathcal{O}_{\sigma^* S_n}) &= \chi(\mathcal{O}_{\tilde{X}}) - \chi(\mathcal{O}_{\tilde{X}}(-\sigma^* S_n)) \\ &= \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-S_n)) = \chi(\mathcal{O}_{S_n}). \end{aligned}$$

On the other hand, from the exact sequences

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{2 \sum \mathcal{E}_j}(-S^*) \longrightarrow \mathcal{O}_{\sigma^* S_n} \longrightarrow \mathcal{O}_{S^*} \longrightarrow 0, \\ 0 &\longrightarrow \bigoplus_j \mathcal{O}_{\mathcal{E}_j}(-S^* - \mathcal{E}_j) \longrightarrow \mathcal{O}_{2 \sum \mathcal{E}_j}(-S^*) \longrightarrow \bigoplus_j \mathcal{O}_{\mathcal{E}_j}(-S^*) \longrightarrow 0, \end{aligned}$$

we get

$$\begin{aligned} \chi(\mathcal{O}_{\sigma^* S_n}) &= \chi(\mathcal{O}_{S^*}) + \chi(\mathcal{O}_{2 \sum \mathcal{E}_j}(-S^*)) \\ &= \chi(\mathcal{O}_{S^*}) + \sum_j \{ \chi(\mathcal{O}_{\mathcal{E}_j}(-S^* - \mathcal{E}_j)) + \chi(\mathcal{O}_{\mathcal{E}_j}(-S^*)) \}. \end{aligned}$$

Since $-S^* - \sum \mathcal{E}_j \sim \sum \mathcal{E}_j - \sigma^* S_n$ and since \mathcal{E}_j is an exceptional divisor which can be identified with a P^1 -bundle on C_j , we have

$$\chi(\mathcal{O}_{\mathcal{E}_j}(-S^* - \mathcal{E}_j)) = \chi(\mathcal{O}_{\mathcal{E}_j}(\mathcal{E}_j - \sigma^* S_n)) = 0.$$

Note that we have $\mathcal{O}_{\mathcal{E}_j}(-S^*) = \mathcal{O}_{\mathcal{E}_j}(-2C_j)$. Therefore, the exact sequences

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{\mathcal{E}_j}(-2C_j) \longrightarrow \mathcal{O}_{\mathcal{E}_j} \longrightarrow \mathcal{O}_{2C_j} \longrightarrow 0, \\ 0 &\longrightarrow \mathcal{O}_{C_j}(-C_j) \longrightarrow \mathcal{O}_{2C_j} \longrightarrow \mathcal{O}_{C_j} \longrightarrow 0 \end{aligned}$$

show

$$\begin{aligned} \chi(\mathcal{O}_{\mathcal{E}_j}(-S^*)) &= \chi(\mathcal{O}_{\mathcal{E}_j}) - \chi(\mathcal{O}_{2C_j}) \\ &= \chi(\mathcal{O}_{\mathcal{E}_j}) - \chi(\mathcal{O}_{C_j}) - \chi(\mathcal{O}_{C_j}(-C_j)) \\ &= -\chi(\mathcal{O}_{C_j}) + \deg([C_j]|_{C_j}). \end{aligned}$$

Since C_j is isomorphic to P^1 , by summing up, we get

$$\chi(\mathcal{O}_{S^*}) = \chi(\mathcal{O}_{S_n}) + \sum_j (1 - C_j^2).$$

On the other hand, since we have

$$\bar{\mu}^* \omega_{S_n} \sim \omega_{S^*} + \sum_j \tilde{C}_j \sim \omega_{S^*} + 2 \sum_j C_j,$$

we get

$$\begin{aligned} \omega_{S_n}^2 &= (\bar{\mu}^* \omega_{S_n})^2 = \omega_{S^*}^2 + 4 \sum_j (\omega_{S^*} \cdot C_j + C_j^2) \\ &= \omega_{S^*}^2 + \sum_j 8. \end{aligned}$$

This completes the proof of Proposition 3.6.

EXAMPLE 3.7. By means of our canonical resolution, we resolve some isolated singularities on a surface S . Let P be an isolated singularity on S whose local analytic equation is of the form $\xi^3 + f(x, y) = 0$. As in [1, §5], we define the type of singularity P by $(\chi(\mathcal{O}_S) - \chi(\mathcal{O}_{\tilde{S}}) : \omega_{\tilde{S}}^2 - \omega_{S^*}^2)$, where \tilde{S} is the minimal resolution of $P \in S$. If S^* denotes the canonical resolution, we can calculate it by Proposition 3.6, since we have $\chi(\mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{S^*})$ and $\omega_{\tilde{S}}^2 = \omega_{S^*}^2 + (\text{the number of contractions})$. For convenience, we explain by drawing figures. The types of the singularities are also indicated.

We use the following notation in the figures: A line stands for a curve and the number near it is its self-intersection number.

On V_i : a solid line means a component of the reduced brunch locus, a double line means a component of the double branch locus and a broken line means an exceptional curve which is not a component of the branch locus.

On S^* and \tilde{S} : a solid line means an exceptional rational curve, a double line means an exceptional elliptic curve and a broken line means the inverse image of the branch locus.

EXAMPLE 1. $\xi^3 + xy = 0$ (a RDP of type A_2); Type (0:0).

EXAMPLE 2. $\xi^3 + x^3 + y^3 = 0$ (a simple elliptic singularity of type \tilde{E}_6 , see [9]; Type (1:3).

EXAMPLE 3. $\xi^3 + x^2 + y^6 = 0$ (a simple elliptic singularity of type \tilde{E}_8), see [9]; Type (1:1).

EXAMPLE 4. $\xi^3 + x^5 + y^5 = 0$; Type (3:10). Compare this with Tomari [10 (2.9)].

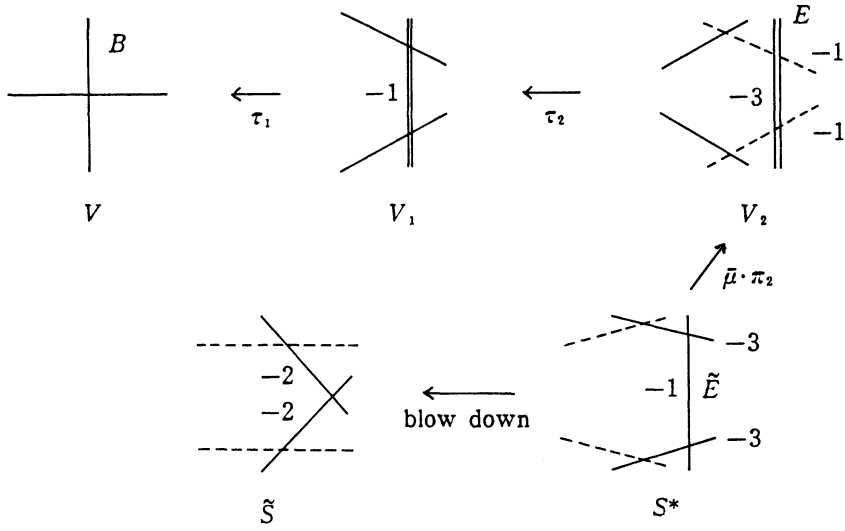


Figure 3.

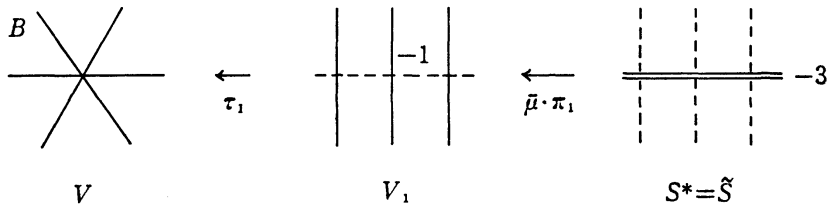


Figure 4.

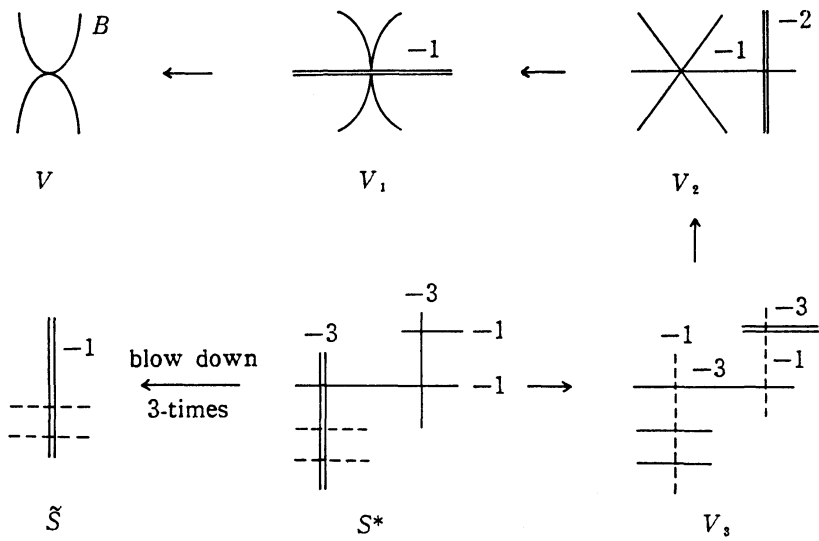


Figure 5.

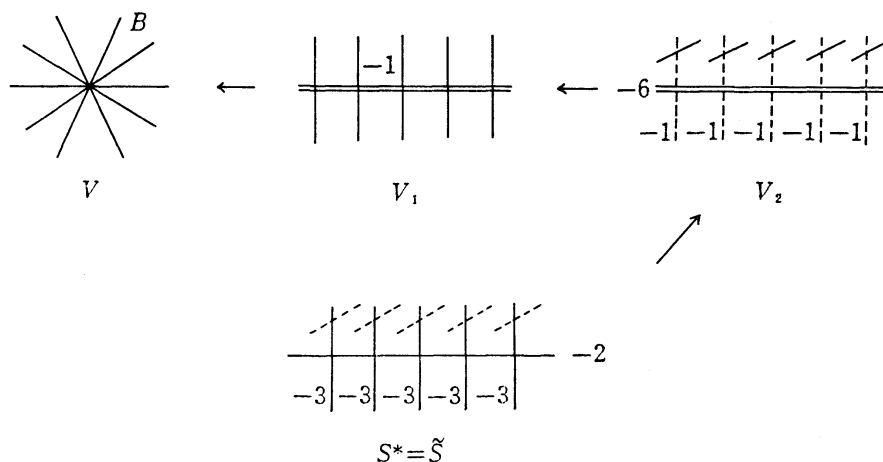


Figure 6.

4. Type II degenerations.

In this section, we construct Type II degenerations of trigonal Castelnuovo surfaces by using the canonical resolution given in the previous section. The construction begun in 4.2 will show the following :

THEOREM 4.1. *Let x and y be integers satisfying $y=3x-7$ and $x \geq 5$. Then there exists a semi-stable degeneration $\rho : \mathcal{S} \rightarrow \Delta_s$ of surfaces such that*

- (1) *a general fiber \mathcal{S}_t is a trigonal Castelnuovo surface with $p_g(\mathcal{S}_t)=x$ and $c_1(\mathcal{S}_t)^2=y$,*
- (2) *the singular fiber \mathcal{S}_0 consists of two reduced components Σ and R meeting transversally, which satisfy*
 - (i) *Σ is a trigonal Castelnuovo surface with $p_g(\Sigma)=x-1$ and $c_1(\Sigma)^2=y-3$,*
 - (ii) *R is a nonsingular rational surface,*
 - (iii) *the double curve $C=\Sigma \cap R$ is a nonsingular elliptic curve,*
- (3) *ρ is a Type II degeneration.*

4.2. We fix a fiber \tilde{f} on $V=\Sigma_{m-1}$ and take a point P on \tilde{f} . Assuming that

$$(4.1) \quad 3b+2 > a+c,$$

we take reduced members $\tilde{B} \in |4C_0+(3b+1-a-c)f|$ and $B' \in |C_0+(2a+2-c)f|$ on V which satisfy the following conditions :

- (1) *The singularities of \tilde{B} are at most ordinary double points and P is a singular point of \tilde{B} .*
- (2) *B' is nonsingular and meets transversally with \tilde{B} (at nonsingular points).*

We take a moving fiber f_t which depends holomorphically on the parameter $t \in \Delta_s$ and satisfies $f_0=\tilde{f}$. We assume that the divisor $B_t=\tilde{B}+f_t$ satisfies the following conditions :

- (3) P is an ordinary triple point of B_0 .
- (4) $B_t, t \neq 0$, has at most ordinary double points.
- (5) B meets B_t transversally at nonsingular points.

Then, for each $t \in \Delta_s$, the pair (B_t, B') defines a member $S_t \in |3H + \pi^*(C_0 + (2a + 2 - c)f)|_{GC}$ on $X = \mathbf{P}(\mathcal{O}_V \oplus \mathcal{O}_V(C_0 + (b - a)f))$. We remark that $S_t, t \neq 0$, has at most RDP's of type A_2 whereas S_0 has a unique elliptic singularity of type \tilde{E}_6 in addition to RDP's of type A_2 .

4.3. The collection $\{S_t\}_{t \in \Delta_s}$ defines a threefold \mathcal{S}' on $X \times \Delta_s$. By the canonical resolution given in §3, we can resolve \mathcal{S}' in the following way: In the figures below, the wavy line means the assistant branch locus. For the other notation, see 3.7.

Step 1. The branch locus and the assistant branch locus on $V \times \Delta_s$ near $f \times \Delta_s$ are as in Figure 7.

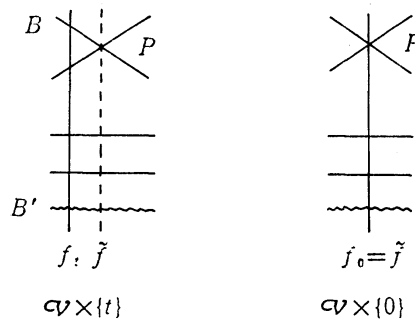


Figure 7.

Step 2. Let $\mathcal{C}_V^{(1)}$ be the blow-up of $V \times \Delta_s$ with center $P \times \Delta_s$ and $E_1 \times \Delta_s$ the exceptional set.

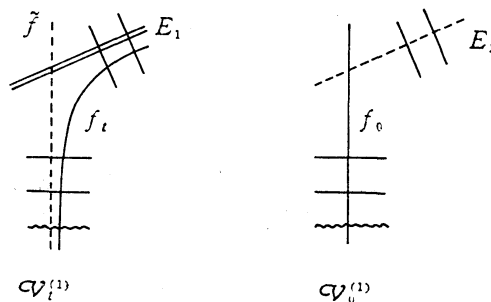


Figure 8.

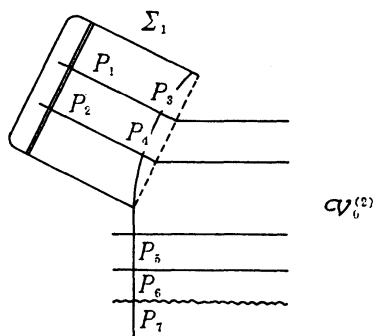


Figure 9.

Step 3. Let $\mathcal{CV}^{(2)}$ be the blow-up of $\mathcal{CV}^{(1)}$ with center $E_1 \times \{0\}$. The exceptional set is Σ_1 , the Hirzebruch surface of degree 1.

Step 4. To get $\mathcal{CV}^{(3)}$, blow up $\mathcal{CV}^{(2)}$ along $P_1 \times \Delta_\varepsilon$ and $P_2 \times \Delta_\varepsilon$ one time, along $P_i \times \Delta_\varepsilon$ ($3 \leq i \leq 6$) three times, and along the other (compound) double points of the branch locus similarly.

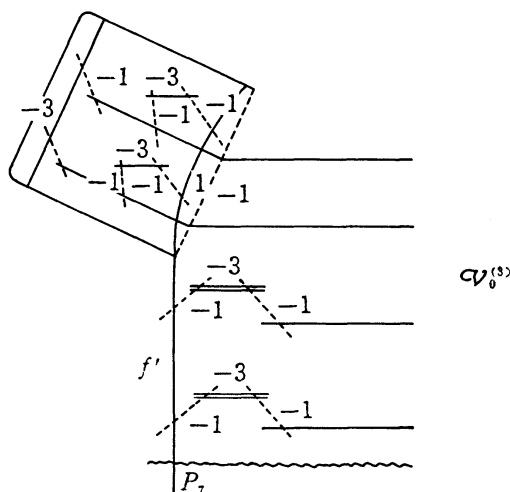


Figure 10.

Step 5. Take the cyclic triple covering of $\mathcal{CV}^{(3)}$ with the assigned branch locus and the normalization $\mathcal{S}^{(1)}$ of it. Let $\tilde{\mu}: \mathcal{S}^{(1)} \rightarrow \mathcal{CV}^{(3)}$ be the natural map.

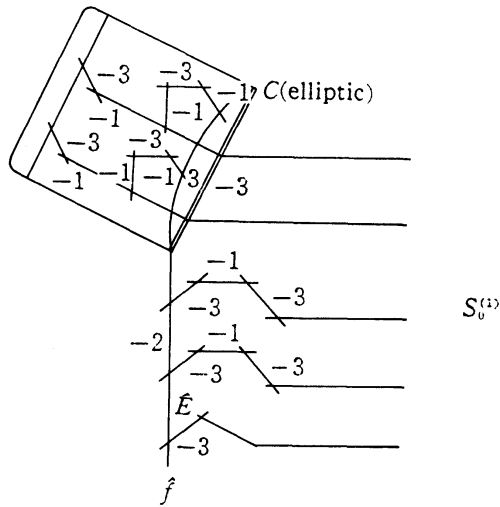


Figure 11.

We consider \hat{E} and \hat{f} in Figure 11. Let U be an open neighbourhood of $P_7 \in V_0^{(3)}$ in Figure 10. If $(x, y) \times (\xi_0 : \xi_1)$ is a system of coordinates on $U \times \mathbf{P}^1$, then the equation of $S_0^{(1)}$ over U is of the form $x\xi_0^3 + y\xi_1^3 = 0$. This implies that

$$0 = (\bar{\mu}^* f') \hat{E} = (3\hat{f} + \hat{E}) \hat{E} = 3 + \hat{E}^2,$$

$$-15 = (\bar{\mu}^* f')^2 = (3\hat{f} + \hat{E})^2 = 9\hat{f}^2 + 3.$$

Thus we get $\hat{E}^2 = -3$ and $\hat{f}^2 = -2$.

Step 6. Let $\mathcal{S}^{(1)} \rightarrow \mathcal{S}$ denote the nonsingular contractions. The main component Σ of the singular fiber is minimal, since its canonical linear system is free from fixed components as is easily seen.

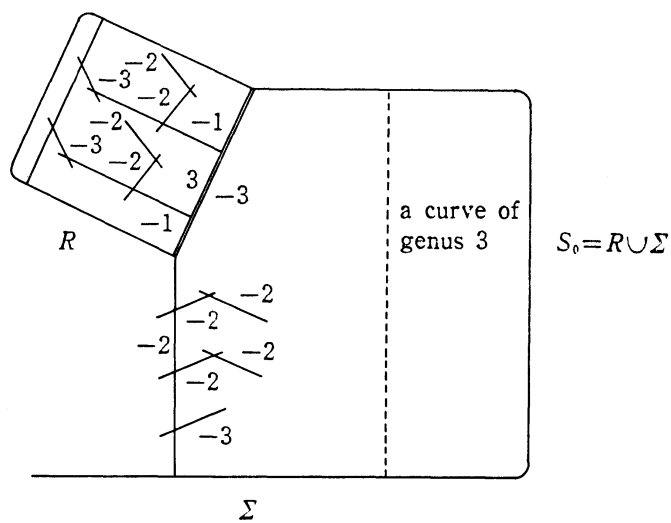


Figure 12.

PROOF OF THEOREM 4.1. For any integers x, y with $y=3x-7$ and $x \geq 5$, we can find integers a, b and c satisfying (1.1), (4.1) and $(p_g(\mathcal{S}_t), c_1(\mathcal{S}_t)^2)=(x, y)$ for $t \neq 0$ (see, (1.2)). Thus we get (1). The assertion (2) follows from the construction and the consideration in 3.8. The assertion (3) follows from the well-known criterion (see, [8]). Thus we get Theorem 4.1.

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Tadashi ASHIKAGA

Department of Applied Physics
Faculty of Engineering
Tôhoku Gakuin University
Tagajou 985
Japan

Kazuhiro KONNO

Department of Mathematics
College of General Education
Kyushu University
Ropponmatsu, Fukuoka 810
Japan