J. Math. Soc. Japan
Vol. 43, No. 1, 1991

Isometrical identities for the Bergman and the Szegö spaces on a sector

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(Received Oct. 9, 1989)
(Revised March 5, 1990)

1. Introduction.

Let $\Delta(\alpha)=\{z;|\arg z|<\alpha\}$. We consider the Bergman space

$$B_{\Delta(\alpha)}=\{F; F \text{ is analytic on } \Delta(\alpha),\ \|F\|_{B_{\Delta(\alpha)}}<\infty\},$$

where

$$\|F\|_{B_{\Delta(\alpha)}}=\left\{\iint_{\Delta(\alpha)}|F(x+iy)|^2dxdy\right\}^{1/2}.$$

In the case of $\alpha=\pi/4$ we showed that $\|F\|_{B_{\Delta(\alpha)}}^2$ is represented as a series of weighted square integrals of the derivatives of the trace of F on the positive real axis ([2]). The proof included two different ingredients: an integral transform and a heat equation on the positive real axis. Both of them required rather deep and lengthy arguments which worked only in the case of $\alpha=\pi/4$.

Here we present a general result for $0<\alpha<\pi/2$ by a completely different proof with minimum prerequisite knowledge. We shall show

THEOREM 1. *Let* $0<\alpha<\pi/2$. *If* $F\in B_{\Delta(\alpha)}$, *then*

$$\iint_{\Delta(\alpha)}|F(x+iy)|^2dxdy=\sin(2\alpha)\sum_{j=0}^{\infty}\frac{(2\sin\alpha)^{2j}}{(2j+1)!}\int_0^{\infty}x^{2j+1}|\partial^j f(x)|^2dx, \tag{1}$$

where f stands for the trace of F on the positive real axis. Conversely, if f is a smooth function on the positive real axis for which the right hand side of (1) *is finite, then f has an analytic continuation $F\in B_{\Delta(\alpha)}$ and* (1) *holds.*

It is natural to consider a counterpart of Theorem 1 for the Szegö space

$$S_{\Delta(\alpha)}=\left\{F; F \text{ is analytic on } \Delta(\alpha),\ \sup_{|\theta|<\alpha}\int_0^{\infty}|F(re^{i\theta})|^2dr<\infty\right\},$$

which is normed by the square root of $\int_{\partial\Delta(\alpha)}|F(z)|^2|dz|$ with $F(z)$ being the nontangential boundary values of F on $\partial\Delta(\alpha)$. We shall prove

THEOREM 2. *Let* $0<\alpha<\pi/2$. *If* $F\in S_{\Delta(\alpha)}$, *then*

$$\int_{\partial\Delta(\alpha)}|F(z)|^2|dz| = 2\cos\alpha\sum_{j=0}^{\infty}\frac{(2\sin\alpha)^{2j}}{(2j)!}\int_0^{\infty}x^{2j}|\partial^j f(x)|^2 dx\,, \tag{2}$$

where f stands for the trace of F on the positive real axis. Conversely, if f is a smooth function on the positive real axis for which the right hand side of (2) *is finite, then f has an analytic continuation $F\in S_{\Delta(\alpha)}$ and* (2) *holds.*

Let us note that there are results, corresponding to Theorems 1 and 2, for the Bergman and the Szegö spaces over a strip $S(\alpha)=\{w: |\mathrm{Im}\, w|<\alpha\}$, i. e., the square power of the Bergman or the Szegö norm of an analytic function G on $S(\alpha)$ is represented as a series of weighted square integrals of the derivatives of the trace of G on the real axis (see [3] and the following (3) and (6)). One might think that Theorems 1 and 2 can be deduced from those results by means of the conformal mapping $z=e^w$ in a straightforward fashion. However, it is not the case. Under the mapping, the derivatives of f are transformed into complicated forms (see [2; Section 5]), from which one can hardly imagine the right hand sides of (1) and (2).

We shall overcome this difficulty by making use of Mellin transform and certain expansions of $\sinh(2\alpha z)/z$ and $\cosh(2\alpha z)$ (see the following (5) and (8)). These expansions implicitly appear in fromulas for Gauss' hypergeometric series (cf. [1]). We shall, however, provide an elementary proof for the selfcontainedness. As remarked at the beginning, a special case of Theorem 1 was proved in conneciton with the heat equation. Since an approach from the heat equation parallel to that in [2] does not seem to work for the Szegö space even if $\alpha=\pi/4$, it may be interesting to consider reflections of the method in this paper to the heat equation conversely.

We would like to thank Professor K. Oikawa for giving a hint which led us to formulas for Gauss' hypergeometric series.

2. Proof of Theorem 1.

We collect several preliminary facts to be used in the proof. As in the introduction we let $S(\alpha)=\{z\,;\,|\mathrm{Im}\, z|<\alpha\}$ and

$$B_{S(\alpha)} = \{G\,; G \text{ is analytic on } S(\alpha),\ \|G\|_{B_{S(\alpha)}}<\infty\},$$

where $\|G\|_{B_{S(\alpha)}}=\left\{\iint_{S(\alpha)}|G(x+iy)|^2dxdy\right\}^{1/2}$. For $G\in B_{S(\alpha)}$ we write $g_y(x)=G(x+iy)$. If $y=0$, then we write g for g_0; this is the trace of G on the real axis. By Cauchy's theorem we have $\hat{g}_y(\xi)=e^{-y\xi}\hat{g}(\xi)$ (see e. g. [4; p. 99]), where $\hat{g}$ stands for the Fourier transform $\int_{-\infty}^{+\infty}e^{-ix\xi}g(x)dx$. Hence the Plancherel theorem and Fubini's theorem yield

(3) $$\|G\|_{B_{S(\alpha)}}^2=\frac{1}{2\pi}\int_{-\alpha}^{+\alpha}dy\int_{-\infty}^{+\infty}|e^{-y\xi}\hat{g}(\xi)|^2d\xi=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\sinh(2\alpha\xi)}{\xi}|\hat{g}(\xi)|^2d\xi.$$

Conversely, suppose g is a function on the real line for which the last integral of (3) is convergent. Then it is easy to see that this function

$$G(x+iy)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i(x+iy)\xi}\hat{g}(\xi)d\xi$$

is analytic on $S(\alpha)$ and belongs to $B_{S(\alpha)}$. We have (3) again in this case.

Next we recall some fundamental properties of the Mellin transform $\mathcal{M}f(\xi)=\int_0^\infty f(x)x^{\xi-1}dx$. The Parseval-Plancherel identity is

$$\int_0^\infty x^{2k-1}|f(x)|^2dx=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|\mathcal{M}f(k+i\eta)|^2d\eta,$$

where k is a real number. The Mellin transform of derivatives is calculated by means of integration by parts. If $\lim_{x\to0}x^{k+l-j}\partial^l f(x)=\lim_{x\to\infty}x^{k+l-j}\partial^l f(x)=0$ for $l=0,\cdots,j-1$, then $\mathcal{M}(\partial^j f)(k+i\eta)=(-1)^j(k+i\eta-1)\cdots(k+i\eta-j)\mathcal{M}f(k+i\eta-j)$. In particular, letting $k=j+1$, we obtain $|\mathcal{M}(\partial^j f)(j+1+i\eta)|^2=(j^2+\eta^2)\cdots(1^2+\eta^2)|\mathcal{M}f(1+i\eta)|^2$, and hence from the Parseval-Plancherel identity

(4) $$\int_0^\infty x^{2j+1}|\partial^j f(x)|^2dx=\frac{1}{2\pi}\int_{-\infty}^{+\infty}(j^2+\eta^2)\cdots(1^2+\eta^2)|\mathcal{M}f(1+i\eta)|^2d\eta,$$

provided $\lim_{x\to0}x^{l+1}\partial^l f(x)=\lim_{x\to\infty}x^{l+1}\partial^l f(x)=0$ for $l=0,\cdots,j-1$, where $(j^2+\eta^2)\cdots(1^2+\eta^2)$ is understood to be 1 if $j=0$.

Let us consider a Fréchet space $U=\{f\in C^\infty(0,\infty);\ \|f\|_{U_j}<\infty \text{ for } j=0,1,\cdots\}$, where

$$\|f\|_{U_j}^2=\int_0^\infty x^{2j+1}|\partial^j f(x)|^2dx.$$

We shall show that (4) holds for $f\in U$. Suppose $f\in U$. Take $\phi,\ \Psi\in C^\infty(0,\infty)$ such that

$$\phi(x)=\begin{cases}1 & \text{for } x\geqq1,\\ 0 & \text{for } 0<x\leqq1/2,\end{cases}\qquad \Psi(x)=\begin{cases}1 & \text{for } 0<x\leqq1.\\ 0 & \text{for } x\geqq2.\end{cases}$$

For $0<\varepsilon<1$ put

$$\varphi_\varepsilon(x)=\begin{cases}\phi\left(\dfrac{x}{\varepsilon}\right) & \text{for } 0<x\leqq1,\\ \Psi(\varepsilon x) & \text{for } x>1.\end{cases}$$

We observe that $\varphi_\varepsilon(x)=1$ for $\varepsilon\leqq x\leqq1/\varepsilon$ and $\varphi_\varepsilon\in C_0^\infty(0,\infty)$. In particular, φ_ε converges to 1 uniformly on every compact subset of $(0,\infty)$ as $\varepsilon\to0$; if $j\geqq1$, then $\partial^j\varphi_\varepsilon$ converges to 0 uniformly on every compact subset of $(0,\infty)$ as $\varepsilon\to0$.

Moreover,

$$x^j\partial^j\varphi_\varepsilon(x)=\begin{cases}\left(\dfrac{x}{\varepsilon}\right)^j\partial^j\phi\left(\dfrac{x}{\varepsilon}\right) & \text{for } 0<x\leqq 1\,,\\ (\varepsilon x)^j\partial^j\Psi(\varepsilon x) & \text{for } x\geqq 1\,,\end{cases}$$

and so

$$|x^j\partial^j\varphi_\varepsilon(x)|\leqq \sup_{0<t<\infty} t^j|\partial^j\phi(t)|+\sup_{0<t<\infty} t^j|\partial^j\Psi(t)|<\infty\,.$$

Therefore, letting $f_\varepsilon=\varphi_\varepsilon f$, we obtain from Leibniz's formula and the dominated convergence theorem that $\|f_\varepsilon-f\|_{U_j}\to 0$ for $j=0, 1, \cdots$, as $\varepsilon\to 0$. Thus $C_0^\infty(0, \infty)$ is dense in U. For $f\in U$ we can define the Mellin transform $\mathcal{M}f(1+i\eta)$ in the sense of mean convergence, i. e.

$$\mathcal{M}f(1+i\eta)=\underset{\varepsilon\to 0}{\text{l. i. m.}}\,\mathcal{M}f_\varepsilon(1+i\eta)\,.$$

Since (4) applies to f_ε, it follows from Minkowski's inequality and Fatou's lemma that

$$\left|\left(\int_0^\infty x^{2j+1}|\partial^j f_\varepsilon(x)|^2dx\right)^{1/2}-\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty}(j^2+\eta^2)\cdots(1^2+\eta^2)|\mathcal{M}f(1+i\eta)|^2d\eta\right)^{1/2}\right|$$

$$\leqq\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty}(j^2+\eta^2)\cdots(1^2+\eta^2)|\mathcal{M}f_\varepsilon(1+i\eta)-\mathcal{M}f(1+i\eta)|^2d\eta\right)^{1/2}$$

$$\leqq\liminf_{\varepsilon'\to 0}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty}(j^2+\eta^2)\cdots(1^2+\eta^2)|\mathcal{M}f_\varepsilon(1+i\eta)-\mathcal{M}f_{\varepsilon'}(1+i\eta)|^2d\eta\right)^{1/2}$$

$$=\liminf_{\varepsilon'\to 0}\|f_\varepsilon-f_{\varepsilon'}\|_{U_j}=\|f_\varepsilon-f\|_{U_j}\,.$$

Hence, letting $\varepsilon\to 0$, we obtain (4).

Finally we present an expansion of $\sinh(2\alpha z)/z$, $0<\alpha<\pi/2$, into successive polynomials $(j^2+z^2)\cdots(1^2+z^2)$, $j=1, 2, \cdots$, which may be of interest on its own. We have for $0<\alpha<\pi/2$

$$\text{(5)}\qquad \frac{\sinh(2\alpha z)}{z}=\sin(2\alpha)\left\{1+\sum_{j=1}^{\infty}\frac{(2\sin\alpha)^{2j}}{(2j+1)!}(j^2+z^2)\cdots(1^2+z^2)\right\}.$$

Prof. K. Oikawa suggested a nice trick—replacement of arguments and parameters—, by which we can reduce (5) to a well-known formula for Gauss' hypergeometric series. We denote by $F(\alpha, \beta; \gamma; z)$ Gauss' hypergeometric series

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{j=0}^{\infty}\frac{\Gamma(\alpha+j)\Gamma(\beta+j)}{\Gamma(\gamma+j)}\frac{z^j}{j!}.$$

It is known (e. g. [**1**; 15.1.16]) that

$$F\left(1+z, 1-z; \frac{3}{2}; \sin^2\alpha\right)=\frac{\sin(2\alpha z)}{z\sin(2\alpha)}.$$

Replacing z by iz and developing the left hand side into the series as above, we obtain (5). However, one should note that, in many books, $F(\alpha, \beta; \gamma; z)$ is taken as the analytic continuation, which is not represented by the original series in general. In fact, (5) does not hold if $\alpha \geqq \pi/2$. Therefore it is worthwhile to give an elementary proof of (5) (see Appendix).

PROOF OF THEOREM 1. Let $F \in B_{\Delta(\alpha)}$ and let f be its trace on the positive real axis. In view of Cauchy's integral formula $|\partial^j f(x)| \leqq (j!/2\pi r^j)\int_0^{2\pi} |F(x+re^{i\theta})| d\theta$ for $0<r<x \sin\alpha$. Hence

$$|\partial^j f(x)| \leqq \frac{2}{x \sin\alpha} \frac{j!}{2\pi} \int_{(1/2)x \sin\alpha}^{x \sin\alpha} \frac{dr}{r^j} \int_0^{2\pi} |F(x+re^{i\theta})| d\theta$$

$$\leqq \frac{j!}{\sqrt{\pi}} \left(\frac{2}{x \sin\alpha}\right)^{j+1} \left(\iint_{|u+iv-x|<x \sin\alpha} |F(u+iv)|^2 du dv\right)^{1/2}$$

by the Schwarz inequality. Since $|u+iv-x|<x \sin\alpha$ implies $|u+iv|/(1+\sin\alpha) <x<|u+iv|/(1-\sin\alpha)$, it follows from Fubini's theorem that

$$\int_0^\infty x^{2j+1} |\partial^j f(x)|^2 dx \leqq \frac{(j!)^2}{\pi} \left(\frac{2}{\sin\alpha}\right)^{2j+2} \int_0^\infty \frac{dx}{x} \iint_{|u+iv-x|<x \sin\alpha} |F(u+iv)|^2 du dv$$

$$\leqq \frac{(j!)^2}{\pi} \left(\frac{2}{\sin\alpha}\right)^{2j+2} \log \frac{1+\sin\alpha}{1-\sin\alpha} \iint_{\Delta(\alpha)} |F(u+iv)|^2 du dv .$$

Therefore $f \in U$ and (4) holds.

Put $G(z)=e^z F(e^z)$. Then a simple calculation shows $G \in B_{S(\alpha)}$ and $\hat{g}(\xi)= \mathcal{M}f(1-i\xi)$, where g is the trace of G on the real axis. Hence (3), after the change of the variable $\xi=-\eta$, leads to

$$\iint_{\Delta(\alpha)} |F(x+iy)|^2 dx dy = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\sinh(2\alpha\eta)}{\eta} |\mathcal{M}f(1+i\eta)|^2 d\eta .$$

By (4) and (5) we see that the right hand side is equal to that of (1).

Conversely, suppose f is a smooth function on the positive real axis for which the right hand side of (1) is finite. Then $f \in U$ and (4) holds. Define the smooth function g on the real line by $g(x)=e^x f(e^x)$. Since $\hat{g}(\xi)=\mathcal{M}f(1-i\xi)$, it follows from (4) that the last integral of (3) is equal to the right hand side of (1), and in particular, is convergent. Hence g has an analytic continuation G to $S(\alpha)$ and (3) holds. By $\log z$ we denote the single valued branch of the logarithm of z on the sector $\Delta(\alpha)$ which assumes real values on the positive real axis. We see that the function $F(z)=G(\log z)/z$ is the analytic continuation of f to $\Delta(\alpha)$ and (1) holds. The theorem is proved.

3. Proof of Theorem 2.

Since the proof of Theorem 2 can be carried out in a way parallel to that of Theorem 1, we shall give only a sketch.

SKETCH OF PROOF OF THEOREM 2. Let us condider the Szegö space $S_{S(\alpha)} = \{G; G \text{ is analytic on } S(\alpha), \sup_{|y|<\alpha} \|G(\cdot+iy)\|_2 < \infty\}$ on $S(\alpha)$. We have

$$\int_{\partial S(\alpha)} |G(z)|^2 |dz| = \frac{1}{\pi}\int_{-\infty}^{+\infty} \cosh(2\alpha\xi) |\hat{g}(\xi)|^2 d\xi, \tag{6}$$

where g is the trace of G on the real axis. Let

$$V = \left\{ f \in C^\infty(0, \infty); \int_0^\infty x^{2j} |\partial^j f(x)|^2 dx < \infty \text{ for } j=0, 1, \cdots \right\}.$$

If $F \in S_{\Delta(\alpha)}$, then

$$\iint_{\Delta(\alpha)} |F(u+iv)|^2 \frac{dudv}{|u+iv|} \leqq 2\alpha \sup_{|\theta|<\alpha} \int_0^\infty |F(re^{i\theta})|^2 dr < \infty,$$

so that it follows from Cauchy's integral formula that the trace f of F on the positive real axis belongs to V. The Parseval-Plancherel identity corresponding to (4) is

$$\int_0^\infty x^{2j} |\partial^j f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\left(j-\frac{1}{2}\right)^2 + \eta^2\right) \cdots \left(\left(\frac{1}{2}\right)^2 + \eta^2\right) \left| \mathcal{M}f\left(\frac{1}{2}+i\eta\right)\right|^2 d\eta \tag{7}$$

for $f \in V$, where $((j-1/2)^2+\eta^2) \cdots ((1/2)^2+\eta^2)$ is understood to be 1 if $j=0$. We expand $\cosh(2\alpha z)$, $0<\alpha<\pi/2$, into successive polynomials $((j-1/2)^2+z^2) \cdots ((1/2)^2+z^2)$ to obtain

$$\cosh(2\alpha z) = \cos\alpha \left\{ 1 + \sum_{i=1}^{\infty} \frac{(2\sin\alpha)^{2j}}{(2j)!} \left(\left(j-\frac{1}{2}\right)^2 + z^2\right) \cdots \left(\left(\frac{1}{2}\right)^2 + z^2\right) \right\} \tag{8}$$

(see [**1**; 15.1.18] or Appendix). Since a member in a Szegö space is a "half order differential", the transform $G(z) = e^{z/2} F(e^z)$ gives an isometry between $F \in S_{\Delta(\alpha)}$ and $G \in S_{S(\alpha)}$. Hence

$$\int_{\partial S(\alpha)} |G(z)|^2 |dz| = \int_{\partial \Delta(\alpha)} |F(z)|^2 |dz|,$$

and $\hat{g}(\xi) = \mathcal{M}f(1/2 - i\xi)$, where f and g are the traces of F and G on the positive real axis and on the real axis, respectively. Therefore (6)-(8) altogether yield (2). The converse part of the theorem can be proved easily. The proof is complete.

4. Appendix.

In this section we give an elementary proof of (5) and (8). Since they can be proved similarly, we show only (5).

PROOF OF (5). Replacing z by iz, we reformulate (5) as

$$\frac{\sin(2\alpha z)}{z} = \sin(2\alpha) \left\{ 1 + \sum_{j=1}^{\infty} \frac{(2\sin\alpha)^{2j}}{(2j+1)!} (j^2 - z^2) \cdots (1^2 - z^2) \right\}. \tag{9}$$

Since the right hand side converges on the whole z-plane, it is sufficient to show the equality on the strip $-1<\mathrm{Re}\,z<1$. Hereafter we let $-1<\mathrm{Re}\,z<1$. Using the Gamma function, we rewrite the right hand side as

$$\text{(10)}\qquad \frac{\sin(2\alpha)}{\Gamma(1-z)\Gamma(1+z)}\sum_{j=0}^{\infty}(2\sin\alpha)^{2j}\frac{\Gamma(j+1-z)\Gamma(j+1+z)}{\Gamma(2j+2)}$$
$$=\frac{\sin(2\alpha)\sin(\pi z)}{\pi z}\sum_{j=0}^{\infty}(2\sin\alpha)^{2j}B(j+1-z,\ j+1+z).$$

By definition

$$B(j+1-z,\ j+1+z)=\int_0^1 t^{j-z}(1-t)^{j+z}dt=\int_0^1\{t(1-t)\}^j\left(\frac{1-t}{t}\right)^z dt.$$

Since $0<\alpha<\pi/2$, it follows that

$$\sum_{j=0}^{\infty}(2\sin\alpha)^{2j}\{t(1-t)\}^j=\frac{1}{1-(2\sin\alpha)^2 t(1-t)},$$

where the series converges absolutely and uniformly for $0\leqq t\leqq 1$. Hence interchanging the integral and the summation, we obtain

$$\sum_{j=0}^{\infty}(2\sin\alpha)^{2j}B(j+1-z,\ j+1+z)=\int_0^1\frac{1}{1-(2\sin\alpha)^2 t(1-t)}\left(\frac{1-t}{t}\right)^z dt.$$

Changing the variable by $s=(1-t)/t$, and then applying the residue theorem, we find that the right hand side is equal to

$$\int_0^{\infty}\frac{s^z}{s^2+2s\cos(2\alpha)+1}ds=\frac{\pi}{\sin(2\alpha)}\frac{\sin(2\alpha z)}{\sin(\pi z)}$$

for $0<\alpha<\pi/2$. This, together with (10), shows (9).

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