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Formation of singularities for Hamilton-Jacobi equation with several space variables

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1. Introduction.

In this paper, we shall consider the Cauchy problem of Hamilton-Jacobi equation in $\boldsymbol{R}^{n+1}$:

$$(1.1)\qquad \begin{cases} u_t+f(u_x)=0 & \text{in} \quad D\equiv\{(t, x)\in\boldsymbol{R}^{n+1}; t>0\}, \\ u(0, x)=\phi(x) & \text{on} \quad \partial D, \end{cases}$$

where $f\in C^{\infty}(\boldsymbol{R}^n)$ is uniformly convex i.e. there exists $c>0$ such that

$$f''(p)\equiv[f_{p_i p_j}(p)]_{1\leqq i,j\leqq n}\geqq cI_n$$

and $\phi\in\mathcal{D}(\boldsymbol{R}^n)$.

Because of the nonlinearity of f, we cannot generally expect the global smoothness of solutions of (1.1). That is, singularities appear. Our purpose is to describe geometrically formation and propagation of their singularities. Hence we consider (1.1) in a weak sense. A generalized solution of (1.1) is a Lipschitz continuous function u satisfying (1.1) almost everywhere and having semi-concavity property: there exists $K\geqq 0$ such that

$$(1.2)\qquad u(t, x+y)+u(t, x-y)-2u(t, x)\leqq K|y|^2$$

for any t, x and y.

Global existence and uniqueness of generalized solutions for (1.1) was established by Douglis [4] and Kruzkov [6] for example. See also Benton [2]. But we are interested in the geometrical description of singularities of solutions. In this context, Tsuji [10], [11], [12] studied formation and propagation of their singularities explicitly in case $n=1$ and 2. He used the theory of singularities of mappings of the plane into the plane obtained by Whitney [13]. Our work is much inspired by his and aims an extension of his results to the case of general space dimensions. Similar results have been obtained by many authors for conservation laws. See, for example, Debeneix [3], Nakane [8] and Schaeffer [9].

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After [12], we shall construct concretely the solution of (1.1) by the method of characteristics. Generally speaking, it becomes multi-valued in finite time. By virtue of the singularity theory of C^∞-mappings in higher dimensional spaces, we shall clarify its structure as a multi-valued function. In fact, we shall show that under some assumptions, its graph has swallow's tail as singularities, as is pointed out in Arnold [1]. Thus it becomes triple-valued. See Figure 2. We can find which branch of three values to take in order to get a single-valued continuous solution. At the same time, we shall see that its singularities correspond to "Maxwell's line" in the graph. Thus we obtain the generalized solution of (1.1) and describe its singularities geometrically.

2. The method of characteristics.

Here, we shall construct the solution of (1.1) concretely by the method of characteristics. The characteristic line through $(0, y)$ associated with (1.1) is the solution of the following:

$$\left\{\begin{array}{ll} \dot{x} = f'(p), & x(0) = y, \\ \dot{p} = 0, & p(0) = \phi'(y), \\ \dot{v} = -f(p)+\langle p, f'(p)\rangle, & v(0) = \phi(y). \end{array}\right. \tag{2.1}$$

That is, it is written by

$$\left\{\begin{array}{l} x = x(t, y) = y+tf'(\phi'(y)), \\ p = p(t, y) = \phi'(y), \\ v = v(t, y) = \phi(y)+t(-f(\phi'(y))+\langle\phi'(y), f'(\phi'(y))\rangle). \end{array}\right. \tag{2.2}$$

Associated with this, we define C^∞-mappings $H_t: \boldsymbol{R}^n_y \to \boldsymbol{R}^n_x$ and $H: \boldsymbol{R}^{n+1}_{(t,y)} \to \boldsymbol{R}^{n+1}_{(t,x)}$ as follows:

$$H_t(y) = y+tf'(\phi'(y)), \qquad H(t, y) = (t, H_t(y)).$$

Then, the solution u of (1.1) is expressed by

$$u(t, x) = v(H^{-1}(t, x)),$$

which is C^∞ at (t, x) which is not the critical value of H. But near the critical values of H, u may become multi-valued. Hence, in the next section, we consider the singularities of H.

3. Singularities of mapping H.

Let $J(H)$ (respectively $D(H)$) be the jacobian matrix (respectively the jacobian determinant) of mapping H. Then, by a direct calculation, we have

$$J(H_t) = I_n+tf''(\phi'(y))\phi''(y),$$

$$J(H)=\begin{pmatrix} 1 & 0 \\ * & J(H_t) \end{pmatrix},$$

$$D(H)=D(H_t)=\prod_{i=1}^{n}(1+t\lambda_i(y)).$$

Here $\lambda_i(y)$, $1\leqq i\leqq n$, are eigenvalues of $f''(\phi'(y))\phi''(y)$. We assume

(A.1) $\lambda_1(y)<\lambda_2(y)<\cdots<\lambda_n(y)$,

(A.2) $\min\limits_y \lambda_1(y)=\lambda_1(y^0)=-M<0$,

(A.3) the singularity of λ_1 is non-degenerate, i.e., $\lambda_1'(y)=0$ implies that $\lambda_1''(y)$ is invertible.

REMARK. (1) In (A.1), we have only to assume the simplicity of the minimum eigenvalue $\lambda_1(y)$.

(2) Set $t^0=1/M$. Then $D(H)>0$ if $t<t^0$ and $D(H)(t^0, y^0)=0$.

(3) (A.2)-(A.3) imply $\lambda_1''(y^0)>0$.

Under these assumptions, we shall show that the singularities of H must be fold or cusp near (t^0, y^0). In order that we give some lemmas.

LEMMA 1. *By any rotation around y^0, eigenvalues of $J(H)$ are kept invariant.*

PROOF. Let $x=A(X-y^0)+y^0$, where $A\in O(n)$ and $w(t, X)=u(t, x)$. Then w satisfies

$$(1.1)' \qquad w_t+g(w_X)=0, \qquad w(0, X)=\psi(X),$$

where $g(q)=f({}^tBq)$, $B=A^{-1}$ and $\psi(X)=\phi(x)$. A direct calculation shows $g''\psi''=Bf''\phi''A$. This completes the proof.

LEMMA 2. *By taking $A\in O(n)$ appropriately, we may assume that the n-th column vector of $C=J(H_{t^0})(y^0)$ is zero.*

PROOF. Note that $\operatorname{rank}C=n-1$. Let $\boldsymbol{a}_n$ be the unit eigenvector of C associated with the eigenvalue 0 and let $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$ be an orthonormal basis of $\boldsymbol{R}^n$ containing $\boldsymbol{a}_n$. Then $A=[\boldsymbol{a}_1 \cdots \boldsymbol{a}_n]$ is the desired matrix. This completes the proof.

LEMMA 3. *There exist at least one non-zero principal minors of C of order $n-1$.*

PROOF. Let $g(s, y)=s^nD(H_{1/s})(y)=\prod_{i=1}^n(s+\lambda_i(y))$. Then $g_s(-\lambda_1(y^0), y^0)=\prod_{i=2}^n(\lambda_i(y^0)-\lambda_1(y^0))>0$. On the other hand, by a direct calculation, it follows

$$g_s(-\lambda_1(y^0), y^0)=(-\lambda_1(y^0))^{n-1}$$
$$\times\{\text{sum of all the principal minors of } C \text{ of order } n-1\}.$$

These imply the assertion of this lemma.

From Lemmas 2 and 3, we have

LEMMA 4. *The following C^∞-mapping $h: \boldsymbol{R}^{n+1}_{(t,y)} \to \boldsymbol{R}^{n+1}_{(t,Y)}$ is a diffeomorphism near (t^0, y^0):*

$$h: \begin{cases} t = t, \\ Y_i = y_i + t f_{p_i}(\phi'(y)), \qquad 1 \leqq i \leqq n-1, \\ Y_n = y_n. \end{cases}$$

We write its inverse $h^{-1}: t=t,\ y_i = b_i(t, Y),\ 1 \leqq i \leqq n$. Then $b_n(t, Y) \equiv Y_n$ and b_i, $1 \leqq i \leqq n-1$, satisfy

$$Y_i = b_i(t, Y) + t f_{p_i}(\phi'(b(t, Y))). \tag{3.1}$$

Now, the mapping $H_1 = H \circ h^{-1}$ is expressed by

$$H_1: \begin{cases} t = t, \\ x' = Y', \\ x_n = Y_n + t f_{p_n}(\phi'(b(t, Y))). \end{cases}$$

In what follows, we consider them only near (t^0, y^0) or near $(t^0, Y^0) = h(t^0, y^0)$. Let $\Sigma^1 = \Sigma^1(H_1) = \{(t, Y) \in \boldsymbol{R}^{n+1}; \partial x_n/\partial Y_n = 0\}$ and $\Sigma^{1,1} = \Sigma^{1,1}(H_1) = \{(t, Y) \in \Sigma^1; \partial^2 x_n/\partial Y_n^2 = 0\}$. Then we have

PROPOSITION 5. *The point in $\Sigma^{1,1}$ is a cusp point of H_1.*

PROOF. We use the characterization of singularities obtained in Morin [7]. An easy calculation shows

$$\Sigma^1 = \{(t, Y) \in \boldsymbol{R}^{n+1}; 1 + t\lambda_1(b(t, Y)) = 0\},$$
$$\Sigma^{1,1} = \{(t, Y) \in \Sigma^1; \partial\lambda_1(b(t, Y))/\partial Y_n = 0\}.$$

We consider $\partial^3 x_n/\partial Y_n^3$ and $\partial^2 x_n/\partial t \partial Y_n$ on $\Sigma^{1,1}$. Then, on $\Sigma^{1,1}$

$$\partial^3 x_n/\partial Y_n^3 = D(h)^{-1} \prod_{i=2}^{n} (1 + t\lambda_i(b)) t \partial^2 \lambda_1/\partial Y_n^2,$$

$$\partial^2 \lambda_1/\partial Y_n^2 = \langle b_{Y_n}, \lambda_1'' b_{Y_n} \rangle + \langle \lambda_1', \partial^2 b/\partial Y_n^2 \rangle > 0,$$

$$\partial^2 x_n/\partial t \partial Y_n = D(h)^{-1} \prod_{i=2}^{n} (1 + t\lambda_i(b)) \lambda_1(b) \neq 0.$$

Thus, we have $\partial^3 x_n/\partial Y_n^3$, $\partial^2 x_n/\partial t \partial Y_n \neq 0$ on $\Sigma^{1,1}$. These imply that the point in $\Sigma^{1,1}$ is a cusp point of H_1.

LEMMA 6. *$\Sigma^{1,1}$ is a C^∞-submanifold of $\boldsymbol{R}^{n+1}$ of codimension 2 parametrized by Y'.*

PROOF. This lemma follows from the implicit function theorem and the fact:

$$\frac{\partial(1+t\lambda_1,\ \partial\lambda_1/\partial Y_n)}{\partial(t,\ Y_n)} = \det\begin{pmatrix} \lambda_1+t\sum_{i=1}^{n-1}\partial\lambda_1/\partial y_i\partial b_i/\partial t & \partial^2\lambda_1/\partial t\partial Y_n \\ t\partial\lambda_1/\partial Y_n & \partial^2\lambda_1/\partial Y_n^2 \end{pmatrix}$$

$$= \lambda_1\partial^2\lambda_1/\partial Y_n^2 \neq 0 \quad \text{at} \quad (t^0,\ Y^0).$$

The following lemma is easy to see.

LEMMA 7. *$H_1(\Sigma^{1,1})$ is a C^∞-submanifold of $\boldsymbol{R}^{n+1}$ of codimension 2 parametrized by x'.*

4. Singularities of the graph of the solution.

Now, we define C^∞-mappings which describe the graph of the solution of (1.1) and study its singularities. Let

$$H_2 : \boldsymbol{R}^{n+1}_{(t,y)} \longrightarrow \boldsymbol{R}^{n+2}_{(t,x,v)}$$

$$H_2 : \begin{cases} t = t, \\ x = y+tf'(\phi'(y)), \\ v = \phi(y)+t(-f(\phi'(y))+\langle\phi'(y),\ f'(\phi'(y))\rangle), \end{cases}$$

$$H_3 = H_2\circ h^{-1} : \boldsymbol{R}^{n+1}_{(t,Y)} \longrightarrow \boldsymbol{R}^{n+2}_{(t,x,v)}$$

$$H_3 : \begin{cases} t = t, \\ x' = Y', \\ x_n = Y_n+tf_{p_n}(\phi'(b)) \\ v = \phi(b)+t(-f(\phi'(b))+\langle\phi'(b),\ f'(\phi'(b))\rangle), \end{cases}$$

$$H_4 : \boldsymbol{R}^{n+2}_{(t,y,x_n)} \longrightarrow \boldsymbol{R}^{n+2}_{(t,x,w)}$$

$$H_4 : \begin{cases} t = t, \\ x_i = y_i+tf_{p_i}(\phi'(y)), \qquad 1\leqq i\leqq n-1, \\ x_n = x_n, \\ w = v+\phi_{y_n}(y)(x_n-y_n-tf_{p_n}(\phi'(y))), \end{cases}$$

$$H_5 = H_4\circ h_1^{-1} : \boldsymbol{R}^{n+2}_{(t,Y,x_n)} \longrightarrow \boldsymbol{R}^{n+2}_{(t,x,w)}$$

$$H_5 : \begin{cases} t = t, \\ x_i = Y_i, \qquad 1\leqq i\leqq n-1, \\ x_n = x_n, \\ w = v+\phi_{y_n}(b)(x_n-Y_n-tf_{p_n}(\phi'(b))), \end{cases}$$

where $h_1(t,\ y,\ x_n)=(h(t,\ y),\ x_n)$.
Then, it follows

LEMMA 8. $D(H_5) = w_{Y_n} = A(t,\ Y)(x_n-Y_n-tf_{p_n}(\phi'(b)))$, *where* $A(t,\ Y)=$

$\sum_{i=1}^{n}\partial b_i/\partial Y_n\cdot\partial^2\phi/\partial y_i\partial y_n<0$.

PROOF. By a direct calculation, we have

$$w_{Y_n}=A(t, y)(x_n-Y_n-tf_{p_n})+\sum_{i=1}^{n-1}\phi_{y_i}(b_{i,Y_n}+t\sum_{j,k=1}^{n}f_{p_ip_j}\phi_{y_jy_k}b_{k,Y_n}).$$

On the other hand, by differentiating (3.1) by Y_n, we have

$$0=b_{i,Y_n}+t\sum_{j,k=1}^{n}f_{p_ip_j}\phi_{y_jy_k}b_{k,Y_n}\quad(1\leqq i\leqq n-1). \tag{4.1}$$

Thus, the first part of the assertion is proved. Next, from (4.1), it follows

$$\sum_{i=1}^{n}b_{i,Y_n}\phi_{y_iy_n}=\phi_{y_ny_n}(1+t\sum_{j,k=1}^{n}f_{p_np_j}\phi_{y_jy_k}b_{k,Y_n})$$

$$-t\sum_{i,j,k=1}^{n}\phi_{y_ny_i}f_{p_ip_j}\phi_{y_jy_k}b_{k,Y_n}\equiv I_1-I_2.$$

Here,

$$I_1=\phi_{y_ny_n}\times\{n\text{-th element of }(I_n+tf''\phi'')b_{Y_n}\},$$

$$I_2=t\times(n\text{-th element of }\phi''f''\phi''b_{Y_n}).$$

From Lemma 2 and (4.1), it is easy to see that $b_{Y_n}={}^t(0,\cdots,0,1)$ at (t^0, Y^0), which implies $I_1=0$. On the other hand, at (t^0, Y^0), we have

$$I_2=t\langle\phi''f''\phi''b_{Y_n}, b_{Y_n}\rangle=t\langle f''\phi''b_{Y_n}, \phi''b_{Y_n}\rangle\geqq ct|\phi''b_{Y_n}|^2.$$

Here we use the uniform convexity of f. Now suppose $\phi''b_{Y_n}=0$. Then, $b_{Y_n}=-tf''\phi''b_{Y_n}=0$ at (t^0, Y^0), which is a contradiction. Thus I_2 must be positive. This proves the latter part of lemma.

We set $\Sigma^1=\Sigma^1(H_5)=\{(t, Y, x_n)\in\boldsymbol{R}^{n+2}; D(H_5)=0\}$. Then Lemma 8 says $\Sigma^1=\{x_n=Y_n+tf_{p_n}(\phi'(b))\}$. Hence $H_5|_{\Sigma^1}=H_3$. Furthermore, if we set $\Sigma^{1,1}=\Sigma^{1,1}(H_5)=\{\partial w/\partial Y_n=\partial^2w/\partial Y_n^2=0\}$ and $\Sigma^{1,1,1}=\Sigma^{1,1,1}(H_5)=\{\partial w/\partial Y_n=\partial^2w/\partial Y_n^2=\partial^3w/\partial Y_n^3=0\}$, then Proposition 5 says $\partial^4w/\partial Y_n^4$, $\partial^2w/\partial t\partial Y_n\neq0$ on $\Sigma^{1,1,1}$. Lemma 8 implies $\partial^2w/\partial x_n\partial Y_n=A(t, y)\neq0$. Again, by the results of [7], we have

PROPOSITION 9. *The point in $\Sigma^{1,1,1}$ is a swallow's tail of H_5.*

PROOF. We have only to show that, on $\Sigma^{1,1,1}$, the rank of the mapping $M: \boldsymbol{R}^{n+2}\to\boldsymbol{R}^3$ defined by $M(t, Y, x_n)=(\partial w/\partial Y_n, \partial^2w/\partial Y_n^2, \partial^3w/\partial Y_n^3)$ is equal to 3. A direct calculation shows

$$\frac{\partial M}{\partial(t, Y_n, x_n)}=-A(t, Y)^2\cdot\partial^4w/\partial Y_n^4\cdot\partial^3w/\partial t\partial Y_n^2\neq0$$

on $\Sigma^{1,1,1}$. This completes the proof.

Now, we shall find a system of coordinates which gives the canonical form of swallow's tail. Since $\partial^4w/\partial Y_n^4=-tA(t, Y)D(h)^{-1}\partial^2\lambda_1/\partial Y_n^2>0$, the unfolding theorem in [**9**] shows that there exist C^∞-functions $k=k(t, Y, x_n)$ and $a_i=a_i(t, Y', x_n)$, $i=1, 2, 3$ such that

$$w = k^4/4-a_1k^2/2-a_2k-a_3 \quad \text{and} \quad \partial k/\partial Y_n \neq 0.$$

Then, $\Sigma^{1,1,1}=\{(t, Y, x_n): k=a_1=a_2=0\}$. Define, for each Y' fixed, C^∞-mappings as follows:

$$M_1(t, x_n) = (a_1(t, Y', x_n), a_2(t, Y', x_n)),$$

$$M_2(t, Y_n, x_n) = (a_1(t, Y', x_n), a_2(t, Y', x_n), k(t, Y, x_n)).$$

The proof of Proposition 9 shows that these are diffeomorphisms. By the coordinate transformations:

$$P: \boldsymbol{R}^{n+2}_{(t,y,x_n)} \longrightarrow \boldsymbol{R}^{n+2}_{(T,Z,X_n)}$$

$$P: \begin{cases} T = a_1(t, Y', x_n), \\ Z' = Y', \\ Z_n = k(t, Y, x_n), \\ X_n = a_2(t, Y', x_n), \end{cases}$$

$$Q: \boldsymbol{R}^{n+2}_{(t,x,w)} \longrightarrow \boldsymbol{R}^{n+2}_{(T,Z',X_n,W)}$$

$$Q: \begin{cases} T = a_1(t, x), \\ Z' = x', \\ X_n = a_2(t, x), \\ W = w+a_3(t, x). \end{cases}$$

H_5 is transformed into $H_6=Q\circ H_5\circ P^{-1}: \boldsymbol{R}^{n+2}_{(T,Z,X_n)}\rightarrow \boldsymbol{R}^{n+2}_{(T,Z',X_n,W)}$ which is written by

$$H_6: \begin{cases} T = T, \\ Z' = Z', \\ X_n = X_n, \\ W = Z_n^4/4-TZ_n^2/2-X_nZ_n. \end{cases}$$

Thus we have obtained the canonical form of swallow's tail. The self-intersection submanifold, what we call Maxwell's line, is expressed by $\{W=-T^2/4, X_n=0, T>0\}$. In (t, x, w) space, it is expressed by $\{w=-a_1^2/4-a_3, a_2=0, a_1>0\}$, whose projection on (t, x) space is $\Gamma=\{a_2=0, a_1>0\}$. Since $\partial a_2/\partial x_n=-A\partial h/\partial Y_n\neq 0$, Γ is parametrized by (t, x'). We express it by $x_n=\varphi(t, x')$. The boundary $\partial\Gamma$ of Γ is expressed by $\{a_1=a_2=0\}$, which is parametrized by x'. It is easy to see that $\partial\Gamma=H_1(\Sigma^{1,1}(H_1))$. From the argument above, we can express the graph of the solution of (1.1). See Figure 2.

EXAMPLE (See Arnold [1]). Let $n=1$, $f(p)=p^2/2$, $\phi(x)=-3x^{4/3}/4$. Then the solution of the characteristic equation is:

$$x = y - t y^{1/3}, \qquad v = -3y^{4/3}/4 + t y^{2/3}/2.$$

Let $Y = y^{1/3}$. Then the mappings H_2 and H_4 are written by

$$H_2 : \begin{cases} t = t, \\ x = Y^3 - tY, \\ v = -3Y^4/4 + tY^2/2, \end{cases}$$

$$H_4 : \begin{cases} t = t, \\ x = x, \\ w = Y^4/4 - tY^2/2 - xY. \end{cases}$$

Thus H_4 is the canonical form of swallow's tail. Note that the above ϕ does not belong to $\mathscr{D}(\boldsymbol{R})$. Nevertheless, this example expresses the essential part of the formation of singularities.

5. Structure of u as a multi-valued function.

In the previous section, we have got a geometrical structure of the solution u of (1.1) as a multi-valued function. In this section, we study analytically its structure and get a single-valued continuous solution.

Proposition 5 says that H_1^{-1} is triple-valued in Ω, which is the domain surrounded by $H_1(\Sigma^1(H_1))$. See Figure 1.

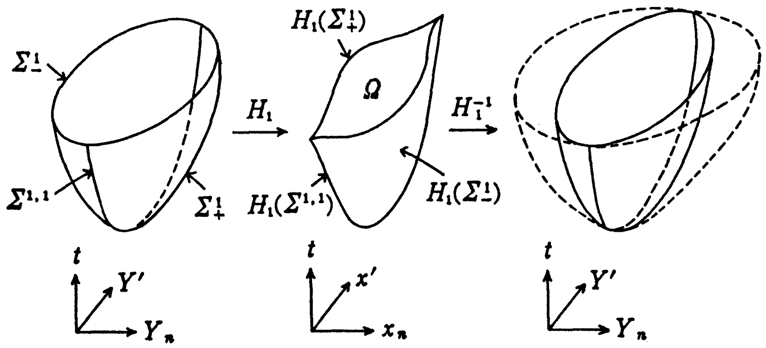

Figure 1.

We write the three branches of H_1^{-1} by

$$H_1^{-1} : \begin{cases} t = t, \\ Y' = x', \\ Y_n = G_n^{(i)}(t, x), \qquad 1 \leq i \leq 3, \end{cases}$$

where $G_n^{(1)} < G_n^{(2)} < G_n^{(3)}$ in Ω. If we set

$$u_i(t, x) = (v \circ h^{-1})(t, x', G^{(i)}(t, x)) \qquad 1 \leqq i \leqq 3,$$
$$g^{(i)}(t, x) = b(t, x', G^{(i)}(t, x)) \qquad 1 \leqq i \leqq 3,$$

then $u_i(t, x) = v(g^{(i)}(t, x))$ $1 \leqq i \leqq 3$. By a direct calculation, we have

LEMMA 10. $u_{i, x_j}(t, x) = \phi_{y_j}(g^{(i)}(t, x))$ *for all* i *and* j.

LEMMA 11. $\langle g^{(i)}(t, x) - g^{(j)}(t, x), \phi'(g^{(i)}(t, x)) - \phi'(g^{(j)}(t, x))\rangle < 0$ *for* $i \neq j$.

PROOF. From the definition, we have

$$g^{(i)} + tf'(\phi'(g^{(i)})) = g^{(j)} + tf'(\phi'(g^{(j)})).$$

Then $\phi'(g^{(i)}) \neq \phi'(g^{(j)})$ for $i \neq j$, since $g^{(i)} \neq g^{(j)}$. Hence,

$$\begin{aligned}
&\langle g^{(i)} - g^{(j)}, \phi'(g^{(i)}) - \phi'(g^{(j)})\rangle \\
&= -t\langle f'(\phi'(g^{(i)})) - f'(\phi'(g^{(j)})), \phi'(g^{(i)}) - \phi'(g^{(j)})\rangle \\
&= -t\langle f''(*)(\phi'(g^{(i)}) - \phi'(g^{(j)})), \phi'(g^{(i)}) - \phi'(g^{(j)})\rangle \\
&\leqq -ct|\phi'(g^{(i)}) - \phi'(g^{(j)})|^2 \\
&< 0.
\end{aligned}$$

This completes the proof.

LEMMA 12. $u_{1, x_n} > u_{2, x_n} > u_{3, x_n}$ *in* Ω.

PROOF. Let $I = \langle g^{(i)} - g^{(j)}, \phi'(g^{(i)}) - \phi'(g^{(j)})\rangle$. Then

$$\begin{aligned}
I &= (G_n^{(i)} - G_n^{(j)})\langle b_{Y_n}(t, x', *), \phi'(g^{(i)}) - \phi'(g^{(j)})\rangle \\
&= (G_n^{(i)} - G_n^{(j)})(\phi_{y_n}(g^{(i)}) - \phi_{y_n}(g^{(j)})) \\
&\quad + (G_n^{(i)} - G_n^{(j)})\langle b'_{Y_n}, \phi_{Y'}(g^{(i)}) - \phi_{Y'}(g^{(j)})\rangle \\
&\equiv I_1 + I_2.
\end{aligned}$$

Since $b'_{Y_n}(t^0, Y^0) = 0$, for any $\varepsilon > 0$, $|b'_{Y_n}| < \varepsilon$ in a sufficiently small neighborhood of (t^0, Y^0). Hence

$$\begin{aligned}
|I_2| &\leqq \varepsilon |G_n^{(i)} - G_n^{(j)}|\,|\phi'(g^{(i)}) - \phi'(g^{(j)})| \\
&\leqq \varepsilon t |f_{p_n}(\phi'(g^{(i)})) - f_{p_n}(\phi'(g^{(j)}))|\,|\phi'(g^{(i)}) - \phi'(g^{(j)})| \\
&\leqq \varepsilon M t |\phi'(g^{(i)}) - \phi'(g^{(j)})|^2
\end{aligned}$$

for some $M > 0$. On the other hand, Lemma 11 says

$$I_1 + I_2 \leqq -ct|\phi'(g^{(i)}) - \phi'(g^{(j)})|^2.$$

By taking ε small, we have $I_1 < 0$. Then, it follows from Lemma 10

$$(G_n^{(i)} - G_n^{(j)})(u_{i, x_n} - u_{j, x_n}) < 0.$$

Since $G_n^{(1)} < G_n^{(2)} < G_n^{(3)}$, the assertion is proved.

Note that $u_1 = u_2$ on $H_1(\Sigma_-^1)$ and $u_2 = u_3$ on $H_1(\Sigma_+^1)$ (see Figure 1). Thus we have

LEMMA 13. *$u_1<u_2$ and $u_2>u_3$ in Ω.*

The graph of u as a multi-valued function is as follows:

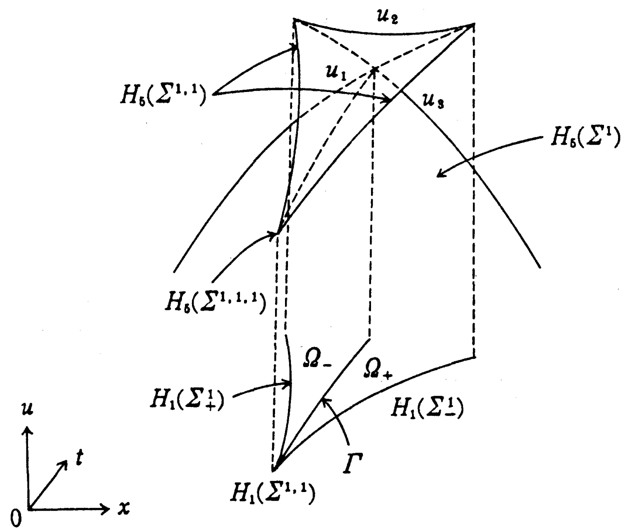

Figure 2. Graph of u.

This graph corresponds to the image of $\Sigma^1(H_5)$ by H_5.

In order to get a single-valued continuous solution, we must pass from u_1 to u_3 on Γ. Let $\Omega_\pm=\{(t, x)\in\Omega\,;\, x_n\gtrless\varphi(t, x')\}$ and

$$u(t, x)=\begin{cases} u_1(t, x) & \text{in } \Omega_-, \\ u_3(t, x) & \text{in } \Omega_+. \end{cases}$$

Then this function is the desired solution. It is C^∞ outside Γ. Thus, its singularities propagate along Γ.

6. Semi-concavity of u.

Now, we shall prove the semi-concavity of u constructed above. Since C^2-function is automatically semi-concave, we have only to consider u on Γ.

Let $\Gamma_t=\{x\,;\, x_n=\varphi(t, x')\}$ and $u_x(t, x\pm 0y)=\lim_{\varepsilon\to+0} u_x(t, x\pm\varepsilon y)$ for $x\in\Gamma_t$. Then, we have

$$\begin{aligned} &u(t, x+y)+u(t, x-y)-2u(t, x) \\ &=\int_0^1\langle u_x(t, x+sy)-u_x(t, x+0y), y\rangle ds+\int_0^1\langle u_x(t, x-0y)-u_x(t, x-sy), y\rangle ds \\ &\quad+\langle u_x(t, x+0y)-u_x(t, x-0y), y\rangle \end{aligned}$$

$$= \int_0^1 \langle u_{1,x}(t, x+sy) - u_{1,x}(t, x), y\rangle ds + \int_0^1 \langle u_{3,x}(t, x) - u_{3,x}(t, x-sy), y\rangle ds$$
$$+ \langle u_{1,x}(t, x) - u_{3,x}(t, x), y\rangle$$
$$\equiv I_1 + I_2 + I_3 .$$

First, we estimate I_1. The same holds for I_2. Since $u=u_i$ are not C^∞ on $H_1(\Sigma^1(H_1))$, we must pay attention to the behaviour of u'' near $H_1(\Sigma^1(H_1))$. We use the canonical form of swallow's tail described in section 4. Then, u is expressed by

$$u = (W + a_3)|_{\Sigma^1(H_5)} = a_3 - 3Z_n^4/4 + TZ_n^2/2 ,$$

where $T=a_1$, $X_n=a_2=Z_n^3-TZ_n$ and a_i are C^∞-functions. Hence,

$$u_{x_i} = a_{3,x_i} + (-3Z_n^2+T)Z_n Z_{n,x_i} + Z_n^2 T_{x_i}/2 .$$

Since $X_{n,x_i}=(3Z_n^2-T)Z_{n,x_i}-Z_n T_{x_i}$, it follows

$$u_{x_i} = a_{3,x_i} - Z_n X_{n,x_i} - Z_n^2 T_{x_i}/2 ,$$
$$u_{x_i x_j} = (T-3Z_n^2)^{-1}(X_{n,x_i}+Z_n T_{x_i})(X_{n,x_j}+Z_n T_{x_j}) + a_{3,x_i x_j} - Z_n a_{2,x_i x_j} Z_n^2 a_{1,x_i x_j}/2 .$$

Since $T-3Z_n^2<0$ in the domain we consider, the first term of $u_{x_i x_j}$ above yields negative semi-definite part of u''. On the other hand, the other terms are continuous. Hence, these imply

$$I_1 + I_2 \leqq K|y|^2 \qquad \text{for some } K>0 .$$

As for I_3, we have only to show in case $y=\boldsymbol{n}=\boldsymbol{n}(t, x)$ is a unit normal of Γ_t at x oriented from Ω_- to Ω_+. Since $u_1-u_3 \lessgtr 0$ in $\Omega_\mp$,

$$\langle u_{1,x} - u_{3,x}, \boldsymbol{n}\rangle = \frac{d}{ds}(u_1(t, x+s\boldsymbol{n}) - u_3(t, x+s\boldsymbol{n}))|_{s=0} \leqq 0 .$$

Thus we have shown the semi-concavity of u.

THEOREM 14. *Assume* (A.1)-(A.3). *Then the solution of* (1.1) *obtained by the method of characteristics becomes triple-valued near* $(t^0, x^0)=H_1(t^0, y^0)$. *Its graph has swallow's tail as singularities. From it, we can obtain the single-valued generalized solution of* (1.1), *whose singularities correspond to Maxwell's line of swallow's tail.*

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Note added in proof. The author has been informed that Prof. G. Kossioris has obtained similar results for viscosity solutions of Hamilton-Jacobi equation with several space variables.