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An ergodic control problem arising from the principal eigenfunction of an elliptic operator

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0. Introduction.

Let us consider the following second order quasi-linear partial differential equation:

$$-\frac{1}{2}\Delta v_\alpha + H(x, \nabla v_\alpha) + \alpha v_\alpha = 0 \tag{0.1}$$

with a quadratic growth nonlinear term $H(x, \nabla v_\alpha)$ on ∇v_α, where α is a positive constant. Such kinds of equations on bounded regions with periodic or Neumann boundary conditions have been studied by several authors (cf. Bensoussan-Frehse [3], Gimbert [6], Lasry [8], and Lions [9]) in connection with ergodic control problems, where the asymptotic behaviour of the solution v_α of (0.1) as α tends to 0 is investigated. The problems arise from stochastic control problem (cf. Bensoussan [2]). In those works important steps of the resolution of such problems are to deduce the estimates on the L^∞-norms of αv_α and ∇v_α by using the maximum principle and the Bernstein's method. But similar problems on the whole space have been out of consideration because the method does not work. We may say intuitively that main difficulty to treat such problems on the whole space lies in lack of uniform ergodicity of underlying diffusion processes and it seems to be necessary to employ completely different method.

In the present article we specialize the equation (0.1) to the case where

$$H(x, \nabla v_\alpha) = \frac{1}{2}|\nabla v_\alpha|^2 - V(x) \tag{0.2}$$

but treat it on whole Euclidean space $\boldsymbol{R}^n$. We notice the relationship between the equation (0.1) with (0.2) and the eigenvalue problem of a Schrödinger operator $-(1/2)\Delta + V$ in $L^2(\boldsymbol{R}^n)$:

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(0.3) $$-\frac{1}{2}\Delta\phi+V\phi=\lambda\phi.$$

More precisely, let us take the principal eigenvalue λ_1 of the operator and the corresponding normalized eigenfunction $\phi(x)$ and set

$$w=-\log\phi+\int\phi^2\log\phi\,dx,$$

then w satisfies the equation

(0.4) $$-\frac{1}{2}\Delta w+\frac{1}{2}|\nabla w|^2-V(x)+\lambda_1=0 \qquad \text{with} \quad \int w\phi^2dx=0.$$

We start with regarding (0.4) as a Bellman equation of ergodic control type and (0.1) with (0.2) as the corresponding equation of discounted type (cf. § 1).

Our theorems assert that under some conditions on $V(x)$ αv_α converges to λ_1, and $v_\alpha-\int v_\alpha\phi^2dx$ to w in a suitable function space as $\alpha\to 0$, where v_α is the positive solution of (0.1) with (0.2) (cf. § 3).

To study the equation (0.1) with (0.2) we take a transformation.

$$v_\alpha=-\log u_\alpha$$

and have the equation

(0.5) $$-\frac{1}{2}\Delta u_\alpha+Vu_\alpha=-\alpha u_\alpha\log u_\alpha, \qquad 0<u_\alpha\leqq 1.$$

For the proof of existence of the solutions of (0.5) we employ Tartar's methods which were useful for the study of quasi-variational inequalities (cf. § 2 and [**4**]).

We note that the relationship between ergodic control and the principal eigenvalue λ_1 of an elliptic operator has been studied by Karatzas [**7**] from a probabilistic view point in the case of $\boldsymbol{R}^1$ under rather stringent conditions on $V(x)$. But any results on convergence to w have not seen so far.

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1. Preliminaries.

1.1. Setting of the problem. Let $V(x)$ be a function on $\boldsymbol{R}^n$ such that

(1.1) $$V(x)\geqq 0, \quad \text{smooth}, \quad V(x)\longrightarrow\infty \qquad \text{as } |x|\to\infty.$$

Then the eigenvalue problem

(1.2) $$-\frac{1}{2}\Delta\phi+V\phi=\lambda\phi$$

in $L^2(\boldsymbol{R}^n)$ has been solved as follows (cf. [**10**], [**12**]). An operator $-(1/2)\Delta+V(x)$ on $C_0^\infty(\boldsymbol{R}^n)$ has a unique self-adjoint extension H in $L^2(\boldsymbol{R}^n)$ (as a sum of quadratic forms) and the resolvent operator $G_\gamma=(\gamma+H)^{-1}$, $\gamma>0$ is compact. Therefore the operator H has purely discrete spectrum:

$$0\leqq\lambda_1<\lambda_2\leqq\lambda_3\leqq\cdots$$

(cf. [10] Th. XIII. 67). It is also known that the principal eigenvalue λ_1 is simple and a corresponding eigenfunction $\phi(x)$ satisfies

$$0<\phi(x)\leqq Ae^{-B|x|} \tag{1.3}$$

for some positive constants A and B (cf. [10], Chap. XIII, [12] § C.1, § C.3 and [5]).

Thus we are given a function ϕ satisfying

$$-\frac{1}{2}\Delta\phi+V\phi=\lambda_1\phi, \tag{1.4}$$

$$\frac{1}{2}\int|\nabla\phi|^2dx+\int V\phi^2dx+\int\phi^2dx<+\infty \tag{1.5}$$

and (1.3). We assume that $\phi(x)$ is normalized as $\int\phi^2(x)dx=1$. Let us set

$$w=-\log\phi+\int\phi^2\log\phi dx,$$

then w satisfies the equation

$$-\frac{1}{2}\Delta w+\frac{1}{2}|\nabla w|^2+\lambda_1=V \tag{1.6}$$

with $\int|\nabla w|^2\phi^2dx<+\infty$ and $\int w\phi^2=0$.

This equation looks like the Bellman equation of an ergodic control problem. Indeed (1.6) can be written as

$$-\frac{1}{2}\Delta w+\lambda_1=\inf_{z\in R^n}\left\{\sum_{i=1}^n z_i\frac{\partial w}{\partial x_i}+\frac{1}{2}|z|^2+V(x)\right\}. \tag{1.6$'$}$$

Therefore it is interesting to study the equation

$$-\frac{1}{2}\Delta v_\alpha+\frac{1}{2}|\nabla v_\alpha|^2+\alpha v_\alpha=V \tag{1.7}$$

and the limit of the solution v_α of (1.5) as $\alpha\to0$.

Indeed consider the following stochastic control problem

$$dy=z_tdt+db_t,\qquad y(0)=x$$

where b is a standard n dimensional Wiener process. The process z_t, the control is adapted to the family of σ-algebras $\mathcal{B}^t=\sigma(b(s), s\leqq t)$ generated by the Wiener process b_t. We want to minimize the cost function

$$J_x^\alpha(z(\cdot)) = E\int_0^\infty e^{-\alpha t}(V(y(t))+\frac{1}{2}|z(t)|^2)dt\,.$$

It is well known that the solution of (1.7) satisfies

$$v_\alpha(x) = \inf_{z(\cdot)} J_x^\alpha(z(\cdot))\,.$$

The ergodic control problem corresponds to the case $\alpha\to 0$. One expects $\alpha v_\alpha(x)$ to converge to a scalar λ_1 independent of x and $v_\alpha(x)-v_\alpha(x_0)$, to some w, which yields from (1.7)

$$-\frac{1}{2}\Delta w+\frac{1}{2}|\nabla w|^2+\lambda_1 = V$$

i. e. equation (1.6).

REMARK. $\left|\int\phi^2(x)\log\phi(x)dx\right|<+\infty$ since

$$0\wedge(-x\log A) \leqq -x\log x \leqq e^{-1-\xi}+\xi x \qquad \forall\xi,\ 0<x\leqq A,$$

in particular

$$0\wedge(-\phi\log A) \leqq -\phi\log\phi \leqq e^{-1-V}+V\phi$$

and we have (1.3) and (1.5).

1.2. Some function spaces and quadratic forms. Let us define a function space

$$H_V^1 = \left\{z \,\middle|\, \int|\nabla z|^2dx<+\infty,\ \int Vz^2dx<+\infty,\ \int z^2dx<+\infty\right\} \tag{1.8}$$

and a quadratic form

$$\varepsilon^V(u,v) = \frac{1}{2}\int\nabla u\cdot\nabla v\,dx+\int V\,uv\,dx, \qquad u,v\in H_V^1 \tag{1.9}$$

corresponding to the self-adjoint operator H:

$$\varepsilon^V(u,v) = (Hu,v), \qquad \forall u,v\in\mathcal{D}(H),$$

where (,) is ordinary $L^2(\boldsymbol{R}^n)$ inner product. H_V^1 is a Hilbert subspace of $L^2(\boldsymbol{R}^n)$ with inner product

$$(u,v)_1 = \varepsilon^V(u,v)+\int uv\,dx \tag{1.10}$$

and $C_0^\infty(\boldsymbol{R}^n)$ is dense in H_V^1 with respect to the norm $\|u\|_1=\sqrt{(u,u)_1}$. Moreover, following Carmona [5], we have

LEMMA 1.1. *$\{\phi f\,|\,f\in C_0^\infty(\boldsymbol{R}^n)\}$ is dense in H_V^1 with respect to the norm $\|\ \ \|_1$.*

PROOF. It suffices to prove that for each $f\in C_0^\infty(\boldsymbol{R}^n)$ there exists a sequence $f_j\in C_0^\infty(\boldsymbol{R}^n)$ such that $\|\phi f_j-f\|_1\to 0$ as $j\to\infty$. We first note that $f\phi^{-1}\in H_V^1$ for each $f\in C_0^\infty(\boldsymbol{R}^n)$, which is easily seen because $\phi(x)\geqq\delta>0$ on the support of f for a positive constant δ and $\int|\nabla\phi|^2dx<+\infty$. Therefore using a mollifier, $f\phi^{-1}$ can be approximated by a sequence of functions $f_j\in C_0^\infty(\boldsymbol{R}^n)$ such that

$$\int|\nabla(f\phi^{-1}-f_j)|^2dx+\int|f\phi^{-1}-f_j|^2dx \longrightarrow 0, \qquad j\to\infty,$$

supports of f_j and f are included in a compact set K, and $f_j\to f\phi^{-1}$ uniformly on K as $j\to\infty$.

Thus we see that

$$\begin{aligned}\|f-\phi f_j\|_1 &= \int|\nabla(f-\phi f_j)|^2dx+\int V(f-\phi f_j)^2dx+\int|f-\phi f_j|^2dx \\ &= \int_K|\nabla(f\phi^{-1}-f_j)|^2\phi^2dx+\int_K(V+1)(f\phi^{-1}-f_j)^2\phi^2dx+\int_K|\nabla\phi|^2(f\phi^{-1}-f_j)^2dx\end{aligned}$$

converges to 0 as $j\to\infty$ since we have (1.3) and (1.5). □

Let us set

$$H_\phi^1=\{f\in L_\phi^2 \mid |\nabla f|\in L_\phi^2\} \tag{1.11}$$

$$L_\phi^2=\left\{f \,\middle|\, \int f^2\phi^2dx<+\infty\right\} \tag{1.12}$$

Then H_ϕ^1 is a Hilbert space with inner product

$$(f,g)_\phi=\int\nabla f\cdot\nabla g\cdot\phi^2dx+\int fg\phi^2dx \tag{1.13}$$

and we see that $C_0^\infty(\boldsymbol{R}^n)$ is dense in H_ϕ^1 with respect to the norm $\|f\|_\phi=\sqrt{(f,f)_\phi}$ by standard arguments using a mollifier since $\phi(x)$ satisfies (1.3) and (1.5). Let us define a transformation from L_ϕ^2 to $L^2(\boldsymbol{R}^n)$ by $f\to\phi f$. This transformation is unitary from L_ϕ^2 to $L^2(\boldsymbol{R}^n)$ and it is useful to note that there are the following identities between the quadratic forms on H_ϕ^1 and H_V^1, which is noted by Albeverio-Høegh-Krohn-Streit [**1**] and Swanson [**13**].

LEMMA 1.2 ([**1**], [**5**], [**13**]). *One has*

$$\frac{1}{2}\int|\nabla f|^2\phi^2dx=\frac{1}{2}\int|\nabla(\phi f)|^2dx+\int V(\phi f)^2dx-\lambda_1\int(\phi f)^2dx, \qquad f\in H_\phi^1 \tag{1.14}$$

$$\frac{1}{2}\int\left|\nabla z-z\frac{\nabla\phi}{\phi}\right|^2dx=\frac{1}{2}\int|\nabla z|^2dx+\int Vz^2dx-\lambda_1\int z^2dx, \qquad z\in H_V^1. \tag{1.15}$$

PROOF. It is easy to see (1.14) holds for $f\in C_0^\infty(\boldsymbol{R}^n)$ and (1.15) holds for $z=\phi f$, $f\in C_0^\infty(\boldsymbol{R}^n)$. Since $C_0^\infty(\boldsymbol{R}^n)$ (resp. $\{\phi f\mid f\in C_0^\infty(\boldsymbol{R}^n)\}$) is dense in H_ϕ^1 (resp. H_V^1) we obtain (1.14) for $f\in H_\phi^1$ and (1.15) for $z\in H_V^1$. □

LEMMA 1.3. *One has the inequality*

$$\int (f-\bar{f})^2 \phi^2 dx \leqq \frac{1}{\lambda_2-\lambda_1} \frac{1}{2} \int |\nabla f|^2 \phi^2 dx, \qquad f \in H^1_\phi, \tag{1.16}$$

where $\bar{f}=\int f\phi^2 dx$.

PROOF. We first note that

$$\lambda_1 = \inf\left\{ \frac{(1/2)\int |\nabla u|^2 dx + \int V u^2 dx}{\int u^2 dx} \;\middle|\; u \in H^1_V \right\} \tag{1.17}$$

and

$$\lambda_2 = \inf\left\{ \frac{(1/2)\int |\nabla u|^2 dx + \int V u^2 dx}{\int u^2 dx} \;\middle|\; \int u\phi = 0,\ u \in H^1_V \right\} \tag{1.18}$$

by mini-max principle. Therefore from (1.14) it follows that

$$\inf\left\{ \frac{(1/2)\int |\nabla f|^2 \phi^2 dx}{\int f^2 \phi^2 dx} \;\middle|\; f \in H^1_\phi,\ \int f\phi^2 dx = 0 \right\}$$

$$= \inf\left\{ \frac{(1/2)\int |\nabla u|^2 dx + \int V u^2 dx - \lambda_1 \int u^2 dx}{\int u^2 dx} \;\middle|\; u \in H^1_V,\ \int u\phi = 0 \right\} = \lambda_2 - \lambda_1 .$$

Hence we have (1.16). □

We shall need in the following sections the function spaces with weights as follows. For $\mu \geqq 0$ let

$$\beta_\mu(x) = \exp\{-\mu(1+|x|^2)^{1/2}\} \tag{1.19}$$

and set

$$L^2_\mu = \{z \mid \beta_\mu z \in L^2\} \tag{1.20}$$

$$L^2_{V,\mu} = \left\{ z \in L^2_\mu \;\middle|\; \int V \beta_\mu^2 z^2 dx < +\infty \right\} \tag{1.21}$$

$$H^1_{V,\mu} = \left\{ z \in L^2_{V,\mu} \;\middle|\; \int |\nabla(\beta_\mu z)|^2 dx < +\infty \right\} \tag{1.22}$$

with the natural norm

$$\|z\|^2_{H^1_{V,\mu}} = \int z^2 \beta_\mu^2 V dx + \int z^2 \beta_\mu^2 dx + \int |\nabla(z\beta_\mu)|^2 dx . \tag{1.23}$$

We choose μ such that

$$0 < \mu^2 < 2\alpha$$

and consider the bilinear form on $H^1_{V,\mu}$

(1.24) $$a(z_1, z_2) = \frac{1}{2}\int \nabla z_1\Big(\nabla z_2\beta_\mu^2 - 2\mu\beta_\mu^2 \frac{x}{(1+|x|^2)^{1/2}} z_2\Big)dx + \int (V-\lambda_1+\alpha)z_1 z_2 \beta_\mu^2 dx.$$

LEMMA 1.4. *a is a continuous coercive form on $H^1_{V,\mu}$.*

PROOF. We first note that

$$a(z, z) = \frac{1}{2}\int \nabla z\Big(\nabla z - 2\mu\frac{x}{(1+|x|^2)^{1/2}}z\Big)\beta_\mu^2 dx + \int (V-\lambda_1+\alpha)z^2\beta_\mu^2 dx$$

$$= \frac{1}{2}\int |\nabla(z\beta_\mu)|^2 dx + \int (V-\lambda_1)z^2\beta_\mu^2 dx + \int\Big(\alpha - \frac{\mu^2|x|^2}{2(1+|x|^2)}\Big)z^2\beta_\mu^2 dx.$$

Therefore a is a continuous form on $H^1_{V,\mu}$. We further remark that by (1.15)

$$\frac{1}{2}\int |\nabla(z\beta_\mu)|^2 dx + \int (V-\lambda_1)(z\beta_\mu)^2 dx \geqq 0$$

because $z\beta_\mu \in H^1_V$. Take $0<\theta<1$ such that

$$\theta\lambda_1 < \alpha - \frac{\mu^2}{2},$$

then we have

$$a(z, z) \geqq \frac{\theta}{2}\int |\nabla(z\beta_\mu)|^2 dx + \theta\int V z^2\beta_\mu^2 dx + \Big(\alpha - \frac{\mu^2}{2} - \theta\lambda_1\Big)\int z^2\beta_\mu^2 dx.$$

Hence a is coercive on $H^1_{V,\mu}$.

It is obvious that a is continuous on $H^1_{V,\mu}$. □

2. Study of the equation (1.7).

2.1. A transformation. We shall study (1.7) through the transformation

(2.1) $$v_\alpha = -\log u_\alpha, \qquad 0<u_\alpha\leqq 1,$$

and thus obtain

(2.2) $$-\frac{1}{2}\Delta u_\alpha + V u_\alpha = -\alpha u_\alpha \log u_\alpha, \qquad 0<u_\alpha\leqq 1.$$

Let us take a constant c such that $\sup_x c\phi(x)=1$. We set

(2.3) $$\phi_\alpha(x) = ce^{-\lambda_1/\alpha}\phi(x),$$

then $\phi_\alpha(x)$ is a subsolution of (2.2):

(2.4) $$-\frac{1}{2}\Delta\phi_\alpha + V\phi_\alpha \leqq -\alpha\phi_\alpha\log\phi_\alpha$$

because it follows that

$$\lambda_1\phi_\alpha \leqq -\alpha\phi_\alpha\log\phi_\alpha$$

from $\phi_\alpha \leqq e^{-\lambda_1/\alpha}$.

Now we introduce a supersolution of (2.1). Let us consider the following equation:

$$-\frac{1}{2}\Delta\chi_\alpha+(V-\lambda_1)\chi_\alpha+\alpha\chi_\alpha = \alpha e^{-\lambda_1/\alpha} \tag{2.5}$$

LEMMA 2.1. *For* $0<\mu^2/2<\alpha$, *there exists a unique solution* χ_α *of* (2.5) *in* $H^1_{V,\mu}$ *and it is a supersolution of* (2.2) *such that* $\chi_\alpha \geqq \phi_\alpha$.

PROOF. Because of Lemma 1.4 the bilinear form a is a continuous coercive form on $H^1_{V,\mu}$. Therefore

$$a(\chi_\alpha, z) = \int \alpha e^{-\lambda_1/\alpha} z\beta_\mu^2 dx \qquad \forall z \in H^1_{V,\mu}$$

has a unique solution χ_α in $H^1_{V,\mu}$. Now we set

$$\psi_\alpha = \chi_\alpha - \phi_\alpha \in H^1_{V,\mu},$$

then we have

$$-\frac{1}{2}\Delta\psi_\alpha+(V-\lambda_1)\psi_\alpha+\alpha\psi_\alpha = \alpha e^{-\lambda_1/\alpha} - \alpha\phi_\alpha \geqq -\lambda_1\phi_\alpha - \alpha\phi_\alpha \log\phi_\alpha \geqq 0.$$

Since

$$\alpha e^{-\lambda_1/\alpha}+(\lambda_1-\alpha)\phi_\alpha \geqq \inf_\xi [\alpha e^{-1-\xi/\alpha}+\xi\phi_\alpha] = -\alpha\phi_\alpha \log\phi_\alpha.$$

Thus we obtain $\psi_\alpha \geqq 0$, namely $\chi_\alpha \geqq \phi_\alpha$. Moreover we have

$$-\frac{1}{2}\Delta\chi_\alpha+V\chi_\alpha = \alpha e^{-\lambda_1/\alpha}+(\lambda_1-\alpha)\chi_\alpha \geqq \inf_\xi \{\alpha e^{-1-\xi/\alpha}+\xi\chi_\alpha\} = -\alpha\chi_\alpha \log\chi_\alpha$$

since $\chi_\alpha \geqq \phi_\alpha > 0$. Hence we see that χ_α is a supersolution of (2.2). □

Now we have the formula

$$\chi_\alpha(x) = \alpha e^{-\lambda_1/\alpha}\int_0^\infty e^{-\alpha t}u(x, t)dt, \tag{2.6}$$

where $u(x, t)$ is the solution of

$$\frac{\partial u}{\partial t} - \frac{1}{2}\Delta u+(V-\lambda_1)u = 0, \qquad u(x, 0)=1. \tag{2.7}$$

We shall use the following estimate to know a majoration of χ_α. Its probabilistic counterpart has been shown by Simon [**11**]. But for completeness we will give the proof of an analytical version.

LEMMA 2.2. *One has the estimate* (Simon [**11**])

$$u(x, t) \leqq \begin{cases} e^{\lambda_1 t} & \text{for } t \leqq 1 \\ c_n t^{n/2} & \text{for } t \geqq 1. \end{cases} \tag{2.8}$$

PROOF. Consider the equation

$$\frac{\partial z}{\partial t}-\frac{1}{2}\Delta z+(V-\lambda_1)z=0, \qquad z(x,0)=f(x) \tag{2.9}$$

with $f\in L^2(\boldsymbol{R}^n)$, then one has

$$|z(x,1)| \leqq \frac{e^{\lambda_1}}{2^{n/2}\pi^{n/2}}|f|_{L^2} \qquad \forall x. \tag{2.10}$$

Indeed we may assume $f\geqq 0$ without loss of generality since we can check by comparison arguments

$$|z(x,1)| \leqq \eta(x,1).$$

Where η corresponds to (2.9) with $f(x)$ replaced by $|f(x)|$. Now for $f\geqq 0$ one has $z\geqq 0$. Hence

$$z(x,1)\leqq \zeta(x,1). \tag{2.11}$$

Where ζ is the solution of

$$\frac{\partial \zeta}{\partial t}-\frac{1}{2}\Delta\zeta-\lambda_1\zeta=0, \qquad \zeta(x,0)=f(x). \tag{2.12}$$

But

$$\zeta(x,1)=\frac{e^{\lambda_1}}{(2\pi)^{n/2}}\int_{R^n}f(y)e^{-1/2|y-x|^2}dy \leqq \frac{e^{\lambda_1}|f|_{L^2}}{(2\pi)^{n/2}}\left(\int_{R^n}e^{-|y-x|^2}dy\right)^{1/2}=\frac{e^{\lambda_1}|f|_{L^2}}{2^{n/2}\pi^{n/4}}.$$

On the other hand one has the energy estimate

$$\frac{1}{2}|z(t)|_{L^2}^2+\int_0^t\int_{R^n}\left(\frac{1}{2}|\nabla z|^2+(V-\lambda_1)z^2\right)dx\,ds=\frac{1}{2}|f|_{L^2}^2 \tag{2.13}$$

and from (1.15) it follows that

$$|z(t)|_{L^2}\leqq |f|_{L^2}. \tag{2.14}$$

Thus we have

$$|z(x,t)| \leqq \frac{e^{\lambda_1}}{2^{n/2}\pi^{n/4}}|f|_{L^2} \qquad \forall x,\ \forall t\geqq 1. \tag{2.15}$$

Indeed, since $z(x,t)$ for $t\geqq 1$ can be considered as the value at time 1 of the solution of (2.9) with initial value $z(x,t-1)$, we have

$$|z(x,t)| \leqq \frac{e^{\lambda_1}}{2^{n/2}\pi^{n/4}}|z(\cdot,t-1)|_{L^2}\leqq \frac{e^{\lambda_1}}{2^{n/2}\pi^{n/4}}|f|_{L^2}^2.$$

Let us turn to (2.7). One first has

$$u(x,t)\leqq e^{\lambda_1 t} \qquad \text{for } t\leqq 1. \tag{2.16}$$

Next for $t\geqq 1$ we can write

$$u=z+\zeta,$$

where z is the solution of (2.9) with

$$f(x)=1_{(|x-x_0|\leq Rt_0)},\qquad x_0, t_0 \text{ fixed},\quad R\geqq 2\sqrt{\lambda_1}$$

and ζ the solution of

$$\frac{\partial\zeta}{\partial t}-\frac{1}{2}\Delta\zeta+(V-\lambda_1)\zeta=0,\qquad \zeta(x, 0)=1_{(|x-x_0|>Rt_0)}.$$

Therefore from (3.10)

$$|z(x, t)|\leqq\frac{e^{\lambda_1}}{2^{n/2}\pi^{n/4}}R^{n/2}t_0^{n/2}|B_1|^{1/2}$$

where $|B_1|$ is the volume of the unit ball in $\boldsymbol{R}^n$. In particular

$$|z(x_0, t_0)|\leqq\frac{e^{\lambda_1}}{2^{n/2}\pi^{n/2}}R^{n/2}t_0^{n/2}|B_1|^{1/2}. \tag{2.17}$$

Next

$$\zeta(x, t)\leqq\frac{e^{\lambda_1 t}}{(2\pi t)^{n/2}}\int_{|y-x_0|\geq Rt_0}e^{-(1/(2t))|y-x|^2}dy$$

and particularly

$$\zeta(x_0, t_0)\leqq\frac{e^{\lambda_1 t_0}}{(2\pi t_0)^{n/2}}\int_{|y-x_0|\geq Rt_0}e^{-(1/(2t_0))|y-x_0|^2}dy$$

$$=\frac{e^{\lambda_1 t_0}}{(2\pi)^{n/2}}\int_{|\xi|\geq Rt_0^{1/2}}e^{-1/2|\xi|^2}d\xi\leqq 2^{n/2}e^{(\lambda_1-R^2/4)t_0}.$$

Therefore we have proved that

$$u(x, t)\leqq e^{\lambda_1}\frac{\lambda_1^{n/4}}{\pi^{n/4}}t^{n/2}|B_1|+2^{n/2},\qquad t\geqq 1, \tag{2.18}$$

which completes the proof of the desired result. □

It follows from Lemma 2.2. that

$$\chi_\alpha(x)\leqq e^{\lambda_1+\alpha}e^{-\lambda_1/\alpha}+c_n e^{-\lambda_1/\alpha}\alpha^{-n/2}\Gamma\left(\frac{n}{2}+1\right) \tag{2.19}$$

hence

$$-\alpha\log\chi_\alpha(x)\geqq\lambda_1+\frac{n}{2}\alpha\log\alpha-\alpha\log K_n\qquad\text{for}\quad 0<\alpha\leqq 1, \tag{2.20}$$

where $K_n=e^{\lambda_1+1}+c_n\Gamma(n/2+1)$.

2.2. Definition of a monotone map. We begin with

LEMMA 2.3. *If $z\in L^2_{V,\mu}$ with $0<z\leqq\chi_\alpha$ and γ is sufficiently large, then $\gamma z-\alpha z\log z\in L^2_\mu$ and the map $z\to\gamma z-\alpha z\log z$ is monotone increasing.*

PROOF. Since $z\leqq\chi_\alpha$ one has

$$\gamma-\alpha\log z-\alpha\geqq\gamma-\alpha-\alpha\log\chi_\alpha\geqq\gamma-\alpha+\lambda+\frac{n}{2}\alpha\log\alpha-\alpha\log K_n$$

$$\geqq \gamma+\lambda-\alpha(1+\log K_n)-\frac{n}{2e}>0$$

if γ is sufficiently large. Therefore the map

$$z \to \gamma z-\alpha z\log z$$

is monotone increasing. Moreover

$$0\leqq \gamma z-\alpha z\log z\leqq \alpha\exp\Big(\frac{\gamma-\alpha}{\alpha}-\frac{\xi}{\alpha}\Big)+\xi z \qquad \forall \xi\in \boldsymbol{R}$$

hence in particular

$$0\leqq \gamma z-\alpha z\log z\leqq \alpha\exp\Big(\frac{\gamma-\alpha}{\alpha}-\frac{\sqrt{V}}{\alpha}\Big)+z\sqrt{V}\,.$$

Therefore we have

$$\int\beta_\mu^2(\gamma z-\alpha z\log z)^2dx\leqq 2\int\beta_\mu^2 z^2 V\,dx+2\alpha^2e^{2(\gamma-\alpha)/\alpha}\int\beta_\mu^2 dx,$$

which is finite, hence the desired result is obtained. □

Define

$$K_\alpha^0=\{z\in L_{V,\mu}^2 \mid 0<z\leqq\zeta_\alpha\},\qquad \zeta_\alpha=\chi_\alpha\wedge 1,$$

and the operator $T_{\gamma,\alpha}$ defined on K_α^0 by

$$-\frac{1}{2}\Delta\zeta+V\zeta+\gamma\zeta=\gamma z-\alpha z\log z,\qquad \zeta\in H_{V,\mu}^1 \tag{2.21}$$

$$\zeta=T_{\gamma,\alpha}z,\qquad z\in K_\alpha^0. \tag{2.22}$$

(2.21) is solved as follows. Let

$$b(z_1,z_2)=\frac{1}{2}\int\nabla z_1\Big(\nabla z_2\beta_\mu^2-2\mu\beta_\mu^2\frac{x}{(1+|x|^2)^{1/2}}z_2\Big)dx+\int(V+\gamma)z_1z_2\beta_\mu^2dx.$$

Since

$$b(z,z)=\frac{1}{2}\int|\nabla(z\beta_\mu)|^2dx+\int Vz^2\beta_\mu^2dx+\int\Big\{\gamma-\frac{\mu^2|x|^2}{2(1+|x|^2)}\Big\}z^2\beta_\mu^2dx$$

b is a continuous coercive form on $H_{V,\mu}^1$ for $\gamma>\mu^2/2$. Hence

$$b(\zeta,\xi)=\int(\gamma z-\alpha z\log z)\xi\beta_\mu^2dx\qquad \forall\xi\in H_{V,\mu}^1 \tag{2.24}$$

has a unique solution for $z\in L_{V,\mu}^2$ with $z>0$. Since $z\in L_{V,\mu}^2$ with $z>0$ implies $\gamma z-\alpha z\log z\in L_\mu^2$.

Let us set

$$K_\alpha=\{z\in L_{V,\mu}^2 \mid \phi_\alpha\leqq z\leqq\zeta_\alpha\},$$

then we have

LEMMA 2.4. *The operator $T_{\gamma,\alpha}$ maps K_α into itself.*

PROOF. As noted above (2.21) defines a unique ζ in $H^1_{V,\mu}$. Let us check that $\zeta\in K_\alpha$. Indeed, let

$$\phi=\zeta-\phi_\alpha\in H^1_{V,\mu}$$

and from (2.4) and (2.21) it follows that

$$-\frac{1}{2}\Delta\phi+V\phi+\gamma\phi\geqq\gamma z-\alpha z\log z-(\gamma\phi_\alpha-\alpha\phi_\alpha\log\phi_\alpha)\geqq 0$$

since $z\geqq\phi_\alpha$. This implies $\phi\geqq 0$, namely $\zeta\geqq\phi_\alpha$.

Similarly let us set

$$\xi=\chi_\alpha-\zeta,$$

then we have

$$-\frac{1}{2}\Delta\xi+V\xi+\gamma\xi\geqq\gamma\chi_\alpha-\alpha\chi_\alpha\log\chi_\alpha-(\gamma z-\alpha z\log z)\geqq 0$$

Since $\chi_\alpha\geqq z$. Therefore, noting that $\xi\in H^1_{V,\mu}$, we obtain $\xi\geqq 0$, hence $\chi_\alpha\geqq\zeta$.

Consider next $(\zeta-1)^+$ which belongs to $H^1_{V,\mu}$. We have

$$b(\zeta,(\zeta-1)^+)=\int(\gamma z-\alpha z\log z)(\zeta-1)^+dx.$$

Since $\zeta=(\zeta-1)^++\zeta\wedge 1$ we deduce

$$\frac{1}{2}\int\nabla(\zeta-1)^+\Big(\nabla(\zeta-1)^+\beta_\mu^2-2\mu\beta_\mu^2\frac{x}{(1+|x|^2)^{1/2}}(\zeta-1)^+\Big)dx+\int V(\zeta-1)^{+2}\beta_\mu^2dx$$

$$+\int\gamma((\zeta-1)^{+2}dx+\int V(\zeta\wedge 1)(\zeta-1)^+\beta_\mu^2dx$$

$$=\int(\gamma z-\alpha z\log z-\gamma(\zeta\wedge 1))(\zeta-1)^+\beta_\mu^2dx=\int(\gamma z-\alpha z\log z-\gamma)(\zeta-1)^+\beta_\mu^2dx\leqq 0.$$

Thus we obtain

$$\frac{1}{2}\int|\nabla((\zeta-1)^+\beta_\mu)|^2dx+\int\Big(\gamma-\frac{\mu^2}{2}\Big)(\zeta-1)^{+2}\beta_\mu^2dx\leqq 0,$$

which implies $(\zeta-1)^+=0$. Hence $\zeta\leqq 1$. Thus the desired result is proved. □

2.3. Existence and uniqueness. The set of solutions of (2.2) is equivalent to the set of fixed points of the map $T_{\gamma,\alpha}$. We prove

THEOREM 2.1. *Assume* (1.1), *then the set of solutions of* (2.2) *in* K_α *is not empty and has a minimum and a maximum element.*

PROOF. We know that $T_{\gamma,\alpha}$ maps K_α into itself. Moreover $T_{\gamma,\alpha}$ is monotone increasing on K_α. We follow an argument due to L. Tartar, stated in A. Bensoussan-J. L. Lions [4] (cf. p. 348, Remark 1.5). Let

$$S=\{z\in K_\alpha\mid T_{\gamma,\alpha}z\leqq z\} \tag{2.25}$$

which is not empty. In fact $\zeta_\alpha \in S$ because

$$T_{\gamma,\alpha}(\chi_\alpha \wedge 1) \leqq T_{\gamma,\alpha}\chi_\alpha \leqq \chi_\alpha$$

and $T_{\gamma,\alpha}(\chi_\alpha \wedge 1) \leqq 1$ implies $T_{\gamma,\alpha}\zeta_\alpha \leqq \zeta_\alpha$. Let us next set

$$\Sigma = \{z \in K_\alpha \mid T_{\gamma,\alpha}z \geqq z,\ z \leqq u,\ \forall u \in S\} \tag{2.26}$$

which is not empty since $\phi_\alpha \in \Sigma$. Now $T_{\gamma,\alpha}$ maps Σ into itself. Indeed, let $z \in \Sigma$, then $T_{\gamma,\alpha}z \in K_\alpha$ and

$$T_{\gamma,\alpha}(T_{\gamma,\alpha}z) \geqq T_{\gamma,\alpha}z\,.$$

Moreover, if $u \in S$, $z \leqq u$ implies

$$T_{\gamma,\alpha}z \leqq T_{\gamma,\alpha}u \leqq u\,.$$

We next show that Σ has a maximal element. It is a consequence of Zorn's lemma.

We must prove that every totally ordered subset $\{z_k\}$ of Σ has an upper bound. Let $\{z_k\}$ be such a subset, since $z_k \in L^2_{V,\mu}$ and $\phi_\alpha \leqq z_k \leqq \zeta_\alpha$, z_k converges in $L^2_{V,\mu}$. In fact, being fixed k_0, $\int z_k(\zeta_\alpha - z_{k_0})(1+V)\beta_\mu^2 dx$ are increasing real numbers bounded above and converge. Moreover

$$\int |z_k - z_{k'}|^2(1+V)\beta_\mu^2 dx \leqq \int (z_k - z_{k'})(\zeta_\alpha - z_{k_0})(1+V)\beta_\mu^2 dx$$

for $k_0 \leqq k' \leqq k$. Let $\underline{z}$ be its limit, then $\phi_\alpha \leqq \underline{z} \leqq \zeta_\alpha$ and $\underline{z} \leqq u\ \forall u \in S$. Also from

$$z_k \leqq T_{\gamma,\alpha}z_k \leqq T_{\gamma,\alpha}\underline{z}$$

we deduce $\underline{z} \leqq T_{\gamma,\alpha}\underline{z}$. Therefore $\underline{z} \in \Sigma$ and is the upper bound of the set z_k since, if $\zeta \in \Sigma$ satisfies $z_k \leqq \zeta$, for all k, necessarily $\underline{z} \leqq \zeta$.

It is thus proved that Σ has a maximal element $\underline{z}$. Necessarily $\underline{z}$ is a fixed point of $T_{\gamma,\alpha}$. Indeed $\underline{z} \in \Sigma$ implies $T_{\gamma,\alpha}\underline{z} \in \Sigma$, $T_{\gamma,\alpha}\underline{z} \geqq \underline{z}$ and by the maximality of $\underline{z}$ in K_α, necessarily one has

$$T_{\gamma,\alpha}\underline{z} = \underline{z}\,.$$

Since $\underline{z} \leqq u\ \forall u$ in S, in particular $\underline{z} \leqq u$ for all u such that $T_{\gamma,\alpha}u = u$. Therefore $\underline{z}$ is the minimum solution in K_α. The existence of maximum solution is proved in a similar way. The proof has been completed. □

THEOREM 2.2. *Assume* (1.1) *and*

$$e^{-\delta\sqrt{V}} \in L^2(\boldsymbol{R}^n), \qquad \exists\delta > 0\,, \tag{2.27}$$

Then the positive solution of (2.2) *in* $H^1_{V,\mu}$ *is unique and belongs to* H^1_V *for* $0 < \alpha < 1/\delta$.

PROOF. *Step* 1. We first show that any solution of (2.2) in $H^1_{V,\mu}$ belongs to H^1_V. Indeed, let u_α be a solution of (2.2) in $H^1_{V,\mu}$, then we have

$$-\frac{1}{2}\Delta u_\alpha+Vu_\alpha+\gamma u_\alpha=\gamma u_\alpha-\alpha u_\alpha\log u_\alpha\leqq\alpha\exp\Big(\frac{\gamma-\alpha}{\alpha}-\frac{V}{2\alpha}\Big)+\frac{V}{2}u_\alpha,$$

therefore

$$-\frac{1}{2}\Delta u_\alpha+\frac{1}{2}Vu_\alpha+\gamma u_\alpha\leqq\alpha\exp\Big(\frac{\gamma-\alpha}{\alpha}-\frac{V}{2\alpha}\Big),\qquad u_\alpha\in H^1_{V,\mu}.$$

Let z be a solution of

$$-\frac{1}{2}\Delta z+\frac{1}{2}Vz+\gamma z=\alpha\exp\Big(\frac{\gamma-\alpha}{\alpha}-\frac{V}{2\alpha}\Big)$$

in $H^1_{V,\mu}$, then z belongs to H^1_V since we deduce $\alpha\exp((\gamma-\alpha)/\alpha-V/(2\alpha))\in L^2$ from the assumptions (2.27) and (1.1). Thus we see that $u_\alpha\in L^2_V$ since $0<u_\alpha\leqq z$. Consequently we deduce,

$$\gamma u_\alpha-\alpha u_\alpha\log u_\alpha\in L^2$$

from $0\leqq\gamma u_\alpha-\alpha u_\alpha\log u_\alpha\leqq\alpha\exp((\gamma-\alpha)/\alpha-\sqrt{V}/\alpha)+\sqrt{V}u$. Hence we conclude that $u_\alpha\in H^1_V$.

Step 2. We next show that if u and u' are two positive solutions of (2.2) in H^1_V such that $u\leqq u'$, then necessarily $u=u'$. In fact, from the equation (2.2) we deduce

$$\int uu'\log u'dx=\int uu'\log u\,dx,$$

which implies $u=u'$.

Step 3. By Step 2 and Theorem 2.1 (2.2) has a unique solution u_α in K_α. Let $\tilde{u}_\alpha$ be another positive solution in $H^1_{V,\mu}$. We set

$$\tilde{K}_\alpha=\{z\in L^2_{V,\mu}\mid\tilde{u}_\alpha\vee\phi_\alpha\leqq z\leqq\zeta_\alpha\}.$$

Note that $\tilde{K}_\alpha$ is not empty. Indeed we have

$$-\frac{1}{2}\Delta(\chi_\alpha-\tilde{u}_\alpha)+(V-\lambda_1)(\chi_\alpha-\tilde{u}_\alpha)+\alpha(\chi_\alpha-\tilde{u}_\alpha)=\alpha e^{-\lambda_1/\alpha}+(\lambda_1-\alpha)\tilde{u}_\alpha+\alpha\tilde{u}_\alpha\log\tilde{u}_\alpha$$

$$\geqq-\alpha\tilde{u}_\alpha\log\tilde{u}_\alpha+\alpha\tilde{u}_\alpha\log\tilde{u}_\alpha=0.$$

Therefore we see that $\chi_\alpha-\tilde{u}_\alpha\geqq0$ since $\chi_\alpha-\tilde{u}_\alpha\in H^1_{V,\mu}$. Thus we have $\tilde{u}_\alpha\leqq\zeta_\alpha$. Now we shall see that $\tilde{u}_\alpha\vee\phi_\alpha$ is a subsolution of (2.2). In fact, noting that $T_{\gamma,\alpha}$ is monotone on K^0_α for sufficiently large γ, we have

$$T_{\gamma,\alpha}(\tilde{u}_\alpha\vee\phi_\alpha)\geqq T_{\gamma,\alpha}\tilde{u}_\alpha=\tilde{u}_\alpha$$

and

$$T_{\gamma,\alpha}(\tilde{u}_\alpha\vee\phi_\alpha)\geqq T_{\gamma,\alpha}\phi_\alpha\geqq\phi_\alpha,$$

from which we deduce $T_{\gamma,\alpha}(\tilde{u}_\alpha\vee\phi_\alpha)\geqq\tilde{u}_\alpha\vee\phi_\alpha$.

Therefore in the same way as the proof of Theorem 2.1, we see the existence of the solution of (2.2) in $\tilde{K}_\alpha$, which is moreover unique because of Step

1 and Step 2. Let u_α^* be the solution in $\tilde{K}_\alpha$, then we have $u_\alpha^*=u_\alpha$ since $\tilde{K}_\alpha \subset K_\alpha$. Moreover $u_\alpha^* \geqq \tilde{u}_\alpha$ by definition, which implies $u_\alpha^*=\tilde{u}_\alpha$ and $\tilde{u}_\alpha=u_\alpha$. The proof has been completed. □

3. Study of the limit as $\alpha \to 0$.

3.1. Limit of $-\alpha \log u_\alpha$.

THEOREM 3.1. *Assume* (1.1), *then for any solution of* (2.2) *in* $K_\alpha \cap H^1_{V,\mu}$ *one has*

$$\lim_{\alpha \to 0}(-\alpha \log u_\alpha(x)) = \lambda_1 . \tag{3.1}$$

PROOF. We have

$$-\alpha \log \phi_\alpha(x) \geqq -\alpha \log u_\alpha(x) \geqq -\alpha \log \chi_\alpha(x) . \tag{3.2}$$

Therefore

$$\varliminf_{\alpha \to 0}(-\alpha \log u_\alpha(x)) \geqq \varliminf_{\alpha \to 0}(-\alpha \log \chi_\alpha(x)) \geqq \lambda_1$$

by (2.20). On the other hand

$$\varlimsup_{\alpha \to 0}(-\alpha \log u_\alpha(x)) \leqq \varlimsup_{\alpha \to 0}(-\alpha \log \phi_\alpha(x)) = \lambda_1 + \varlimsup_{\alpha \to 0}(-\alpha \log c\phi(x)) = \lambda_1 .$$

Hence we obtain (3.1). □

3.2. Limit of v_α. Let u_α be a solution of (2.2) in $K_\alpha \cap H^1_{V,\mu}$, then it is locally smooth by regularity properties of elliptic equations and $0<u_\alpha \leqq 1$. Therefore we deduce from (2.2) that the function $v_\alpha=-\log u_\alpha$ satisfies

$$0 \leqq v_\alpha < \frac{\lambda_1}{\alpha} - \log c\phi(x) \tag{3.3}$$

and is a solution of (1.7). Moreover

$$\lambda_\alpha = \alpha \int v_\alpha \phi^2 dx = -\int (\alpha \log u_\alpha)\phi^2 dx < +\infty .$$

From Theorem 3.1 and (3.3) we can assert that

$$\lim_{\alpha \to 0} \lambda_\alpha = \lambda_1 . \tag{3.4}$$

We then prove

THEOREM 3.2. *Assume* (1.1) *and* (2.27). *Let u_α be the unique solution of* (2.2) *in H^1_V with $0<u_\alpha \leqq 1$ and $v_\alpha=-\log u_\alpha$, then $v_\alpha - \int v_\alpha \phi^2 dx$ converges to $w= -\log \phi + \int \phi^2 \log \phi \, dx$ in H^1_ϕ.*

PROOF. Since $u_\alpha \in H^1_V$, $\phi/u_\alpha \in L^\infty$ and $\phi^2/u_\alpha \in H^1_V$, from (2.2) it follows that

$$(3.5)\qquad \frac{1}{2}\int\nabla u_\alpha\cdot\nabla\Big(\frac{\phi^2}{u_\alpha}\Big)dx+\int Vu_\alpha\Big(\frac{\phi^2}{u_\alpha}\Big)dx=-\alpha\int(u_\alpha\log u_\alpha)\frac{\phi^2}{u_\alpha}dx.$$

Therefore we have

$$(3.6)\qquad -\frac{1}{2}\int\Big|\frac{\nabla u_\alpha}{u_\alpha}\Big|^2\phi^2dx+\int\frac{\nabla u_\alpha}{u_\alpha}\frac{\nabla\phi}{\phi}\phi^2dx+\int V\phi^2dx=-\alpha\int\phi^2\log u_\alpha dx.$$

Since $v_\alpha=-\log u_\alpha$ and $\nabla u_\alpha=-\nabla u_\alpha/u_\alpha$ we see that $v_\alpha\in H^1_\phi$ and

$$(3.7)\qquad \frac{1}{2}\int\Big|\nabla v_\alpha+\frac{\nabla\phi}{\phi}\Big|^2\phi^2dx+\alpha\int v_\alpha\phi^2dx=\int V\phi^2dx+\frac{1}{2}\int|\nabla\phi|^2dx=\lambda_1.$$

As noted above $\alpha\int v_\alpha\phi^2dx\to\lambda_1$ as $\alpha\to0$. Therefore

$$(3.8)\qquad \frac{1}{2}\int\Big|\nabla v_\alpha+\frac{\nabla\phi}{\phi}\Big|^2\phi^2dx\longrightarrow0,\qquad\text{as}\quad\alpha\to0,$$

Let us set $\tilde{v}_\alpha=v_\alpha-\int v_\alpha\phi^2dx$, then from Lemma 1.3 we deduce

$$\int|\tilde{v}_\alpha-w|^2\phi^2dx\leqq\frac{1}{2(\lambda_2-\lambda_1)}\int|\nabla\tilde{v}_\alpha-\nabla w|^2\phi^2dx=\frac{1}{2(\lambda_2-\lambda_1)}\int\Big|\nabla v_\alpha+\frac{\nabla\phi}{\phi}\Big|^2\phi^2dx.$$

Hence by (3.8) we obtain the desired result. □

4. Example

We illustrate a simple example. Let

$$V(x)=\frac{1}{2}|x|^2=\frac{1}{2}(x_1^2+x_2^2+\cdots+x_n^2),$$

then the principal eigenvalue of $-(1/2)\Delta+V$ in $L^2(\boldsymbol{R}^n)$ is $n/2$ and the corresponding normalized eigenfunction is $\phi(x)=Ae^{-(1/2)|x|^2}$, where A is a normalized constant. Let us take C such that $\sup_x c\phi(x)=1$ and set

$$\phi_\alpha(x)=e^{-\lambda_1/\alpha}c\phi(x)=\exp\Big\{-\frac{1}{2}|x|^2-\frac{n}{2\alpha}\Big\}.$$

We can find the solution u_α of (2.2)

$$u_\alpha(x)=\exp\Big\{\frac{-|x|^2}{\alpha+\sqrt{\alpha^2+4}}-\frac{n}{\alpha(\alpha+\sqrt{\alpha^2+4})}\Big\},$$

and the solution v_α of (1.7)

$$v_\alpha(x)=\frac{|x|^2}{\alpha+\sqrt{\alpha^2+4}}+\frac{n}{\alpha(\alpha+\sqrt{\alpha^2+4})},$$

We see that as $\alpha\to0$

$$\alpha v_\alpha(x)\longrightarrow\frac{n}{2}$$

and

$$v_\alpha(x)-\int v_\alpha(x)\phi^2(x)dx \longrightarrow \frac{1}{2}|x|^2-\frac{n}{2}=-\log\phi(x)+\int\phi^2\log\phi(x)dx$$

REMARK. We can develop our all arguments without using more regularities than 1^{st} order differentiabilities on $u_\alpha(x)$. Then the assumption that $V(x)$ is smooth can be weakened, for example, as $V(x)\in L^2_{loc}$ if $n\leqq 3$ and $V(x)\in L^p_{loc}$, $p>n/2$ if $n>3$ since, under these assumptions besides the conditions that $V(x)\geqq 0$, $V(x)\to\infty$, as $|x|\to\infty$, the principal eigenvalue of $-(1/2)\Delta+V$ is simple and a corresponding eigenfunction satisfies (1.3) (cf. [5]).

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