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Geometric 4-manifolds in the sense of Thurston and Seifert 4-manifolds I

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The notion of geometric manifolds introduced by Thurston plays a fundamental role in the 3-dimensional topology ([**14**], [**15**]). There are just eight 3-dimensional geometries and the six among these correspond to the Seifert 3-manifolds ([**13**]). On the other hand the geometries in dimension 4 were classified by Filipkiewicz [**6**] and the geometric structures of 4-manifolds, in particular those of complex surfaces were studied by Wall ([**18**], [**19**]). For example there is a correspondence between the elliptic surfaces without singular fibers and the certain geometries analogous to that for Seifert 3-manifolds ([**18**], Theorem 7-4). The purpose of this paper (together with Part II [**17**]) is to give the correspondence between Seifert 4-manifolds which are not necessarily complex surfaces and the following eight geometries; E^4, $Nil^3\times E$, Nil^4, $Sol^3\times E$, $S^3\times E$, $S^2\times E^2$, $H^2\times E^2$ and $\widetilde{SL}_2\times E$. Here a Seifert 4-manifold is a 4-manifold which has a structure of a fibered orbifold over a 2-orbifold with general fiber a 2-torus (see § 1). We will characterize the closed orientable geometric 4-manifolds of the above eight types in terms of the Seifert 4-manifolds (Theorems A and B) and will also give the topological classification of such 4-manifolds (cf. Part II [**17**]). The notion of the Seifert 4-manifolds of the above types coincides with that in the usual sense studied in [**21**], [**22**] for example if the base orbifolds have no reflectors. We need to take account of the cases when the bases have (corner) reflectors to prove the converse direction (Theorem B) of the above correspondence. We only consider the closed orientable ones and then a fiber over a (corner) reflector point is a Klein bottle multiply covered by the general fiber. The topology of Seifert 4-manifolds of this type can be described by a series of invariants (which we call the Seifert invariants) analogous to those for Seifert 3-orbifolds ([**3**], [**5**], [**14**]) and will be explained in § 1. In the present paper we will restrict our attention to the cases with euclidean base orbifolds and the corresponding four types of geometries. The other cases will be treated in Part II ([**17**]). But for convenience we will give the results in full generality in the following section.

§ 0. Statements of the results and notation.

THEOREM A. (1) *Every closed orientable Seifert 4-manifold S over a 2-orbifold B admits a geometric structure if B is not hyperbolic. The possible types of the geometries are $S^3\times E$, $S^2\times E^2$ if B is spherical or bad, E^4, $Nil^3\times E$, Nil^4, $Sol^3\times E$ if B is euclidean.*

(2) *If B is hyperbolic then either S has a geometric structure of type $H^2\times E^2$, $\widetilde{SL}_2\times E$ or S is not geometric in the sense of Thurston.*

THEOREM B. *Every closed orientable geometric 4-manifold of one of the above eight types is a Seifert 4-manifold except for just one example in the E^4 case. The unique exception is S^1-fibered but not T^2-fibered.*

Thus we have the following list.

The type of the bases	The corresponding geometries
spherical or bad	$S^2\times E^2$ $S^3\times E$
euclidean	$E^{4(*)}$ $Nil^3\times E$ Nil^4 $Sol^3\times E$
hyperbolic	$H^2\times E^2$ $\widetilde{SL}_2\times E$ non-geometric

(*) There is just one closed orientable euclidean 4-manifold which is not a Seifert 4-manifold in our sense.

We will give the more detailed lists of the types of the Seifert 4-manifolds and the corresponding geometries (or non-geometric cases). See Lists I~IV in §§ 4, 6 and Claims 9, 10 in § 5, Claim 8 in § 7 for the cases with euclidean base orbifolds. The other cases will be described in Part II ([**17**]).

THEOREM C. *Let S and S' be closed orientable Seifert 4-manifolds with $\pi_1S=\pi_1S'$ whose base orbifolds are either euclidean or hyperbolic. Then S is diffeomorphic to S'. Moreover if the bases are hyperbolic or the geometric type of S or S' is Nil^4 or $Sol^3\times E$, then there is a fiber-preserving diffeomorphism between S and S'.*

REMARK. (1) The proof of Theorem C for the cases with hyperbolic base orbifolds is essentially due to Zieschang [**21**].

(2) It is proved in [18] Theorem 10.1 that two closed 4-manifolds with geometric structures of distinct types are not homotopy equivalent. Hence the above list has no overlaps. Moreover to prove Theorem C we may assume that the type of the geometries of S and S' are the same.

(3) The fiberings of the manifolds of type E^4 and $Nil^3\times E$ are not unique in general. In [**16**] we gave the list of such examples when the base orbifolds have no reflectors. The fiberings of the cases with spherical or bad base orbifolds are also far from unique (see Part II).

The classification of the closed euclidean 4-manifolds is classical ([**4**], [**20**]). But in § 8 we reformulate this from the viewpoint of the Seifert 4-manifolds to clarify the relations between them and the other geometries. The proofs of Theorems B and C for the cases with euclidean base orbifolds will be given in § 3. We note that the class of the closed orientable geometric 4-manifolds of type E^4, $Nil^3 \times E$, Nil^4 coincides with that of the closed orientable flat or almost flat Riemannian 4-manifolds by Filipkiewicz's classification. We also note that the class of Seifert 4 manifolds contains all the compact complex surfaces diffeomorphic to the elliptic surfaces with $c_2=0$ (cf. [**18**]) and also contains more examples which have no complex structures. Let us fix some notation used in this paper. We denote the fundamental group of a manifold S by $\pi_1 S$ and the i-th betti number of S by $b_i S$. The fundamental group of an orbifold

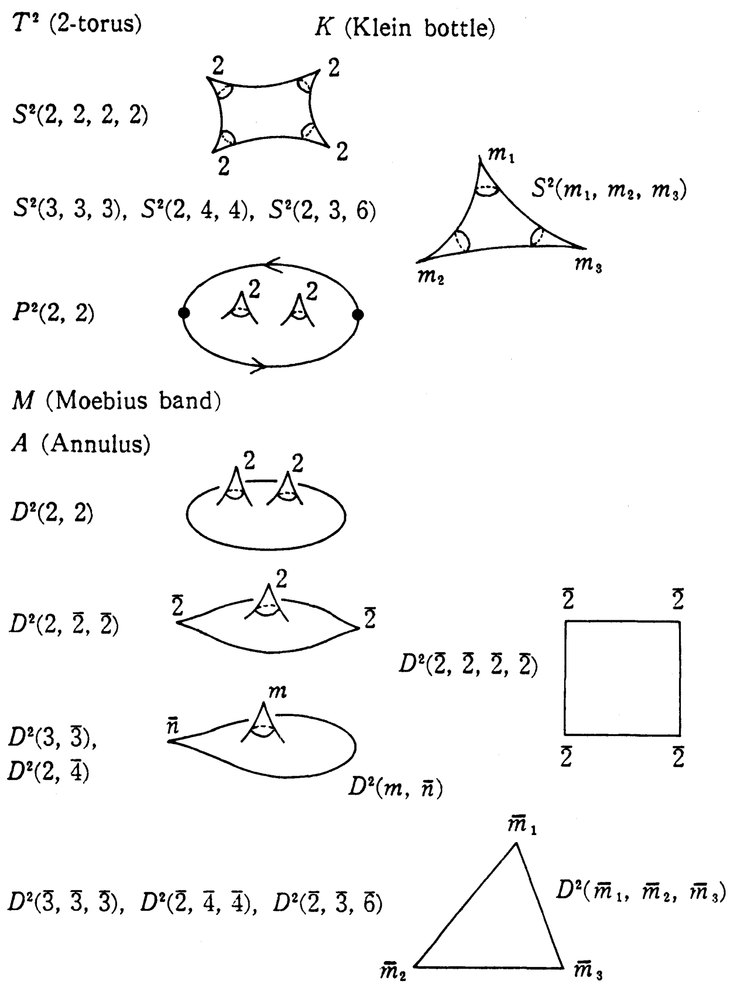

Figure 1. The closed euclidean 2-orbifolds.

B and its underlying space are denoted by $\pi_1^{orb}B$ and $|B|$ respectively. We define $[x, y]=xyx^{-1}y^{-1}$ for the elements $x, y\in\pi_1 S$. We use the symbols for the closed euclidean 2-orbifolds as indicated in Figure 1. In Figure 1 the cone point of angle $2\pi/m$ and the corner reflector of angle π/m are represented by m and $\bar{m}$ respectively. Throughout this paper all the subjects will be considered in the smooth category.

§1. Seifert 4-manifolds over 2-orbifolds with reflectors.

A closed orientable 4-manifold S is called a Seifert 4-manifold if (1) S has a structure of a fibered orbifold $\pi: S\to B$ over a 2-orbifold B with general fiber a 2-torus T^2 and (2) S is non-singular as an orbifold. S is represented by some invariants analogous to those for Seifert 3-orbifolds with general fiber S^1 ([**3**], [**5**]). First we will describe the local pictures of this fibration and then give its global description.

LOCAL PICTURES. A point p of B has a neighborhood of type $D=D^2/G$ where D^2 is a 2-disc centered at $0\in\boldsymbol{R}^2$ corresponding to p and G is a finite subgroup of $O(2)$ corresponding to the stabilizer of p. Then $\pi^{-1}(D)$ is identified with $T^2\times D^2/G$ where the action of G on $T^2\times D^2$ is free and is some lift of that on D^2 so that $\pi|\pi^{-1}(D)$ is the map $T^2\times D^2/G\to D^2/G$ induced from the natural projection from $T^2\times D^2$ to D^2. Here T^2 is identified with $\boldsymbol{R}^2/\boldsymbol{Z}^2$ and the point of $T^2\times D^2$ is represented as (x, y, z) with $(x, y)\in\boldsymbol{R}^2$ (mod $\boldsymbol{Z}^2$) and $z\in\boldsymbol{C}$, $|z|\leqq 1$. Let l and h be the curves represented by $\boldsymbol{R}/\boldsymbol{Z}\times\{0\}$ and $\{0\}\times\boldsymbol{R}/\boldsymbol{Z}$ respectively.

Case 0. $G=\mathrm{id}$. In this case p is a nonsingular point and the fiber over p is called a general fiber.

Case 1. $G=\boldsymbol{Z}_m$ where the generator ρ of $\boldsymbol{Z}_m$ acts on $T^2\times D^2$ by $\rho(x, y, z)=(x-a/m, y-b/m, \exp(2\pi i/m)z)$ with g.c.d. $(m, a, b)=1$. In this case p is a cone point of angle $2\pi/m$ and the fiber over p is called a multiple torus of type (m, a, b).

Case 2. $G=\boldsymbol{Z}_2$ where the generator ι of $\boldsymbol{Z}_2$ acts on $T^2\times D^2$ by $\iota(x, y, z)=(x+1/2, -y, \bar{z})$. In this case p is on the reflector and the fiber over p is a Klein bottle K and $\pi^{-1}(D)$ is a twisted D^2-bundle over K.

Case 3. $G=D_{2m}=\{\iota, \rho\,|\,\iota^2=\rho^m=1, \iota\rho\iota^{-1}=\rho^{-1}\}$ whose action on $T^2\times D^2$ is defined by $\rho(x, y, z)=(x, y-b/m, \exp(2\pi i/m)z)$, $\iota(x, y, z)=(x+1/2, -y, \bar{z})$ with g.c.d. $(m, b)=1$. In this case p is a corner reflector of angle π/m and the fiber over p is a Klein bottle whose fundamental domain is $1/m$-times that of the fiber of the reflector point near p. We call this fiber a multiple Klein bottle of type $(m, 0, b)$. Here we note that the fiber of this type cannot be twisted along l.

PICTURES ALONG THE REFLECTOR CIRCLES. The boundary of $|B|$ consists of a disjoint union of circles $C_1 \cup C_2 \cup \cdots \cup C_s$ each of which we call a reflector circle. Let C be one of them. Let N be a suborbifold of B such that $|N|$ is an annulus bounded by C and a curve $\bar{\gamma}$ parallel to C. We will describe $\pi^{-1}(N)$. Let $p_1, \cdots, p_s$ be the corner reflectors on C of type $(m_1, 0, b_1), \cdots, (m_s, 0, b_s)$ respectively with respect to the framing (l, h) of the general fiber over some base point of N defined above. We consider the double cover $\tilde{B}$ of B with the projection $p: \tilde{B} \to B$ obtained by patching 2 copies of B along the reflector circles and let $\tilde{N}$ be the suborbifold of $\tilde{B}$ covering N. Let $\tilde{\pi}: \tilde{S} \to \tilde{B}$ be the fibration induced from $\pi: S \to B$. Then S is the quotient of $\tilde{S}$ by a free involution ι which is a lift of the standard reflection $\bar{\iota}$ of $\tilde{B}$. The action of ι on the reflection point near the base point is identical to that of ι in Case 3. In the presentation of $\pi_1 S$, ι satisfies

$$(*) \qquad \iota^2 = l, \qquad \iota h \iota^{-1} = h^{-1}.$$

Then the corner reflector p_i is covered by a cone point $\tilde{p}_i$ and the fiber over $\tilde{p}_i$ is a multiple torus of type $(m_i, 0, b_i)$. If we take the oriented meridional circle $\bar{q}_i$ centered at $\tilde{p}_i$ as in Figure 3 then the lifts $q_1, \cdots, q_r$ of $\bar{q}_1, \cdots, \bar{q}_r$ can be taken so that they satisfy the following relations in $\pi_1 S$:

$$(**) \qquad q_i^{m_i} h^{b_i} = 1 \ (i=1, \cdots, r), \ \iota q_r \iota^{-1} = q_r^{-1},$$
$$\iota q_{r-1} \iota^{-1} = q_r^{-1} q_{r-1}^{-1} q_r, \cdots, \iota q_1 \iota^{-1} = q_r^{-1} q_{r-1}^{-1} \cdots q_1^{-1} q_2 \cdots q_r.$$

Next we define two further invariants.

THE MONODROMY ALONG THE REFLECTOR CIRCLE. If we take $\bar{\gamma}$ (and $\bar{\iota}\bar{\gamma}^{-1}\bar{\iota}^{-1}$) on N as in Figure 3 then the curve represented by $\bar{\gamma}^{-1}\bar{q}_1\bar{q}_2 \cdots \bar{q}_r \bar{\iota}\bar{\gamma}\bar{\iota}^{-1}$ is null-homologous in $\tilde{N} - \bigcup$(the disk neighborhood of $\tilde{p}_i$). Hence the monodromy matrix A along $\bar{\gamma}$ with respect to (l, h) must satisfy $JAJ^{-1} = A$ where $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a monodromy matrix for $\bar{\iota}$. Then we must have $A = \pm I$ (I is the identity matrix).

THE EULER CLASS OF C. Take a lift γ of $\bar{\gamma}$ which also determines the lift $\iota\gamma\iota^{-1}$ of $\bar{\iota}\bar{\gamma}\bar{\iota}^{-1}$. Then we have the relation $\gamma^{-1} q_1 \cdots q_r \iota\gamma\iota^{-1} = l^a h^b$ in $\pi_1 S$. Here by taking the conjugate by ι on the both sides and using the relations (*) and (**) above we have $\iota\gamma^{-1}\iota^{-1} q_r^{-1} \cdots q_1^{-1} \gamma l^{\varepsilon} = l^a h^{-b}$ where $\varepsilon = 0$ if $A = I$ and $\varepsilon = -2$ if $A = -I$. Hence we have $a = -1$ if the monodromy A along $\bar{\gamma}$ is $-I$ and $a = 0$ if $A = I$. We call (a, b) the euler class of C which is the obstruction to extending $\gamma \cup \iota\gamma\iota^{-1} \cup q_1 \cup \cdots \cup q_r$ to the cross section on $\tilde{\pi}^{-1}(\tilde{N} - \bigcup$the neighborhood of $\tilde{p}_i)$. The value of b depends on the choices of the lifts γ, q_i of $\bar{\gamma}$, $\bar{q}_i$.

TRANSFORMATIONS OF THE INVARIANTS. Let (a, b), $(m_1, 0, b_1), \cdots, (m_r, 0, b_r)$ be the euler class, the types of the fibers on the corner reflectors on C. We can choose another lift of $\bar{q}_i$ of the form $q_i h^s$ satisfying (**). Then (a, b), $(m_i, 0, b_i)$

are replaced by $(a, b+s)$, $(m_i, 0, b_i-sm_i)$ respectively and the others remain unchanged. If we take $\gamma l^p h^q$ as another lift of $\bar{\gamma}$ then (a, b) is replaced by $(a, b-2q)$ and the others remain unchanged.

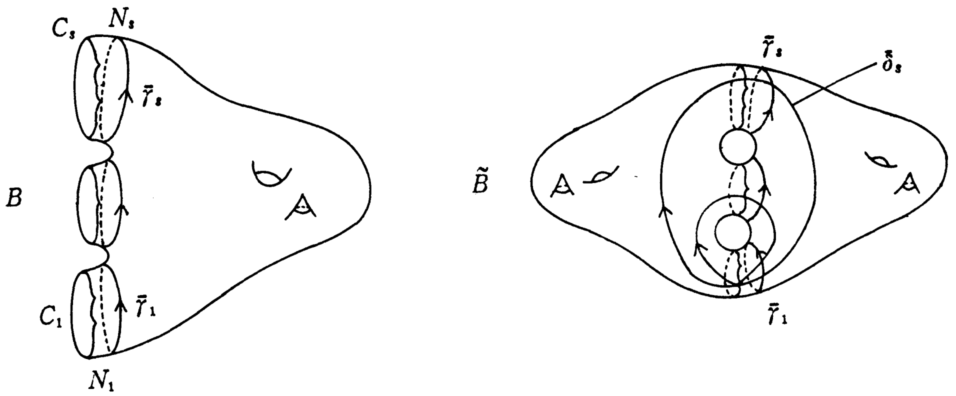

Figure 2.

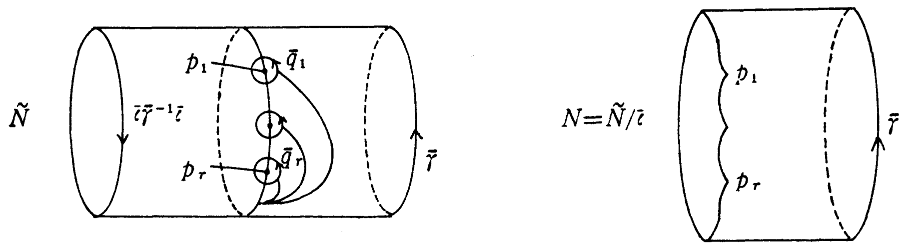

Figure 3.

GLOBAL DESCRIPTION. Take a neighborhood N_i of each reflector circle C_i with boundaries C_i and $\bar{\gamma}_i$ for each i. Choose the base point near C_1 and fix the framing (l, h) of the general fiber satisfying $(*)$. Then if we fix the lift γ_i of $\bar{\gamma}_i$ the fibration over $B_0=B-\cup N_i$ is described by the following data;

(1) the monodromy matrices $A_i, B_i \in SL_2\boldsymbol{Z}$ along the set of standard generators s_i, t_i $(i=1, \cdots, g)$ of $\pi_1|B_0|$ if $|B_0|$ is orientable,

(1′) the monodromy matrices $A'_i \in GL_2\boldsymbol{Z}$ with $\det A'_i=-1$ along the set of standard generators v_i $(i=1, \cdots, g)$ of $\pi_1|B_0|$ if $|B_0|$ is non-orientable,

(2) the type (m_i, a_i, b_i) of the multiple torus over the cone point p'_i $(i=1, \cdots, t)$,

(3) the obstruction (a', b') to extending $(\cup\gamma_i)\cup(\cup q'_i)$ to the cross section in $\pi^{-1}(B_0-\cup$the disk neighborhood of $p'_i)$ where q'_i is the lift of the meridional circle centered at p'_i. This is called an euler class.

The fibration on N_i is described as before with respect to the framing (l_i, h_i) of the general fiber on N_i and the lift ι_i of the reflection along C_i (where $(l_1, h_1)=(l, h)$) satisfying $\iota_i^2=l_i$, $\iota_i h_i \iota_i^{-1}=h_i^{-1}$. Then $\pi^{-1}(N_i)$ is attached to $\pi^{-1}(B_0)$ so that $(l_i, h_i)=(l, h)P_i$ for some $P_i \in SL_2\boldsymbol{Z}$ $(P_1=I)$. This implies that if we take the lift δ_i of the curve $\bar{\delta}_i$ on $\tilde{B}$ as in Figure 2, then the monodromy along δ_i is $B_i=P_iJP_i^{-1}J$ with respect to (l, h). We can take δ_i so that $\iota_i=\delta_i\iota$. For, $\bar{\iota}\bar{\delta}_i\bar{\iota}^{-1}=\bar{\delta}_i^{-1}$ on $\pi_1\tilde{B}$ and hence $\delta_i\iota\delta_i\iota=l^{s+1}h^t$ for some $s, t\in\boldsymbol{Z}$ (note that $\iota^2=l$). Then considering $\iota^2\delta_i\iota^{-2}$ we can see that $B_i\begin{pmatrix}s+1\\ -t\end{pmatrix}=\begin{pmatrix}s+1\\ t\end{pmatrix}$ and hence $JP_i^{-1}\begin{pmatrix}s+1\\ t\end{pmatrix}=P_i^{-1}\begin{pmatrix}s+1\\ t\end{pmatrix}$. It follows that $(\delta_i\iota)^2=l_i^{s'}$ for some $s'\in\boldsymbol{Z}$. We also note that $\delta_i\iota(l_i, h_i)\iota^{-1}\delta_i^{-1}=(l_i, h_i)J$. On the other hand the existence of the curve $\bar{\delta}_i$ implies that $\tilde{B}$ (and also B) is euclidean or hyperbolic and in this case S is aspherical. It follows that π_1S has no nontrivial torsion and hence s' must be odd (we consider δ_i, ι as elements in π_1S). Then replacing δ_i by $l_i^k\delta_i$ for some k if necessary we have $(\delta_i\iota)^2=l_i$. Here we note that the Seifert 4-manifold S with $P_i=-P_0$ for the i-th reflector circle is the same (up to fiber-preserving diffeomorphisms) as some Seifert 4-manifold with $P_i=P_0$ for the same i-th reflector circle (replace (l_i, h_i) by (l_i^{-1}, h_i^{-1}), δ_i by $l_i^{-1}\delta_i$ and hence ι_i by ι_i^{-1}). Hence it suffices to consider one of these types; the ones with $P_i=P_0$ or the ones with $P_i=-P_0$. Finally we have the relation among the monodromies. Let $I_i=\pm I$ be the monodromy along C_i (with respect to any framing) and A_i, B_i (or A_i' if B_0 is non-orientable) be the monodromies along the standard curves on B_0 as before. Then $\prod[A_i, B_i]\prod I_i=I$ (or $\prod A_i'^2\prod I_i=I$). These informations determine the total fibration $\pi: S\to B$.

If B has no reflectors then the fibration is determined by the genus g of B, the monodromy matrices A_i, B_i (or A_i') for $i=1, \cdots, g$ with respect to the set of the standard generators of $\pi_1|B|$, the euler class (a, b), and the types (m_i, a_i, b_i) for $i=1, \cdots, t$ of the multiple tori (in [**16**] we used $(-a, -b)$ as the definition of the euler class instead of (a, b)). We have further transformations of the Seifert invariants by replacing the lift q_i' around the cone point p_i. In particular if S has a multiple fiber (a, b) can be taken to be $(0, 0)$. If B has reflectors we can take (a, b) to be $(0, 0)$ without changing the invariants of the multiple tori by replacing the lift of $\bar{\gamma}_1$ for example (in this case the euler class of C_1 changes). We do not normalize these invariants in this paper since the further normalizations are unnecessary for the proof of our theorems. But if every monodromy is trivial (and $P_i=I$ for any i when B has reflectors) we can define the rational euler class of S as follows:

Case 1. B has no reflectors. Let (a, b), (m_i, a_i, b_i) for $i=1, \cdots, t$ be the euler class and the types of the multiple tori of S. Then we put $e=(a+\sum a_i/m_i, b+\sum b_i/m_i)\in\boldsymbol{Q}^2$ (mod the action of $GL_2\boldsymbol{Z}$). We can normalize e so that one of the factors is 0 by the change of the framing of the fiber.

Case 2. B has reflectors. In this case we have $(l_k, h_k)=(l, h)$ for any k by the assumption. Let $(0, b_i)$, $(m_{ij}, 0, b_{ij})$ $j=1, \cdots, r_i$ be the euler class and the types of the multiple Klein bottles on the reflector circle C_i ($i=1, \cdots, s$) and (a, b), (m_j, a_j, b_j) for $j=1, \cdots, t$ be the euler class and the types of the multiple tori of S. Then we put $e=b+\sum_{j=1}^{t} b_j/m_j+\sum_{i=1}^{s}(\sum_{j=1}^{r_i} b_{ij}/m_{ij}+b_i)/2\in \boldsymbol{Q}$.

In either case the rational euler class $(\bmod GL_2\boldsymbol{Z})$ depends only on the fibration $S\to B$ (not depending on the choices of the cross sections etc.).

§ 2. Four geometries in dimension 4.

Let X be a complete 1-connected Riemannian manifold which is a geometry in the sense of Thurston (cf. [**13**]). For such X put G_X=the group of all the isometries of X, G_X^+=the group of orientation-preserving isometries of X, G_X^0= the identity component of G_X. Four dimensional geometries and G_X, G_X^0 for every such X were completely determined by [**6**], [**18**], [**19**]. A closed orientable geometric manifold of type X is of the form Γ/X where Γ is a discrete subgroup of G_X^+ which acts freely on X. Since in this paper we will be concerned with the following four kinds of geometries; E^4, $Nil^3\times E$, Nil^4, $Sol^3\times E$, let us recall their definitions ([**18**]).

$X=E^4$. In this case $G_X^+=\boldsymbol{R}^4\cdot SO_4$, a semi-direct product of $\boldsymbol{R}^4$ (translations) and SO_4 (rotations).

$X=Nil^3\times E$. Nil^3 is a nilpotent Lie group consisting of all the matrices of the form $\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}$, $x, y, z\in\boldsymbol{R}$ ([**13**]). Nil^3 is a bundle over $\boldsymbol{R}^2$ spanned by x, y with fiber $\boldsymbol{R}$ spanned by z. $G_{X'}$ has just two components for $X'=Nil^3$ and $G_{X'}^0$ is a semi-direct product of Nil^3 which acts as left multiplication and SO_2 which is a maximal compact subgroup of Aut Nil^3. The action of $\theta\in SO_2$ $(0\leqq\theta<2\pi)$ on Nil^3 is given by $(x, y, z)\to(cx+sy, -sx+cy, z+s(cy^2-cx^2-2sxy)/2)$ where $s=\sin\theta$, $c=\cos\theta$. $G_{X'}/G_{X'}^0$ is represented, for example, by the automorphism of the form $(x, y, z)\to(x, -y, -z)$. For $X=Nil^3\times E$ we have $G_X^+=(\mathrm{Isom}\, Nil^3)\times\boldsymbol{R}$. $Nil^3\times E$ is identified with $\boldsymbol{C}^2$ where the multiplication on $\boldsymbol{C}^2$ is defined by $(w, z)(w', z')=(w+w'-i\bar{z}z', z+z')$. SO_2 acts on $Nil^3\times E$ by $t(w, z)=(w, tz)$ $(t\in\boldsymbol{C}, |t|=1)$ ([**18**]).

$X=Nil^4$. Nil^4 is a semi-direct product of $\boldsymbol{R}^3$ and $\boldsymbol{R}$ where $t\in\boldsymbol{R}$ acts on $\boldsymbol{R}^3$ by the matrix $C(t)=\exp t\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$. In this case $G_X^0=Nil^4$ (left multiplication) and G_X/G_X^0 is represented by the following automorphisms: $(x, y, z, t)\to(\varepsilon x, \varepsilon\eta y, \varepsilon z, \eta t)$ where $(x, y, z)\in\boldsymbol{R}^3$, $t\in\boldsymbol{R}$, ε, $\eta=\pm1$ ([**18**], § 3).

$X=Sol^3\times E$. $X_0=Sol^3$ is a semi-direct product of $\boldsymbol{R}^2$ and $\boldsymbol{R}$ where $t\in\boldsymbol{R}$ acts on $\boldsymbol{R}^2$ by $\phi(t)=\begin{pmatrix}\exp t & 0\\ 0 & \exp(-t)\end{pmatrix}$ ([**13**]). $G_{X_0}^0=Sol^3$ and $G_{X_0}/G_{X_0}^0=D_8$ generated by the automorphisms $(x, y, z)\to(\pm x, \pm y, z)$, $(x, y, z)\to(\pm y, \pm x, -z)$. For X we have $G_X=\mathrm{Isom}\,X_0\times\mathrm{Isom}\,E$ and G_X^+ has 8 components.

§ 3. Proofs of Theorems B and C when the bases are euclidean.

First we will prove Theorem B for the geometric 4-manifold $\Gamma\backslash X$ with $X=Nil^4$, $Nil^3\times E$ or $Sol^3\times E$. Strategy of the proof is suggested in [**18**], § 2. In any case we put $\Gamma_0=\Gamma\cap G_X^0$.

Case 1. $X=Nil^4$. In this case $G_X^0=Nil^4$ and its commutator subgroup $(G_X^0)'$ is isomorphic to $\boldsymbol{R}^2$ spanned by $x, y\in\boldsymbol{R}$ (see § 2) with $G_X^0/(G_X^0)'=\boldsymbol{R}^2$. Since Γ_0 is a lattice of the nilpotent Lie group, $\Gamma_0\cap(G_X^0)'$ is also a lattice of $(G_X^0)'$ ([**18**] § 3, [**11**]). This implies that the image of Γ in this quotient which we denote by $\bar{\Gamma}$ is also a lattice in the quotient$=\boldsymbol{R}^2$. Then from the bundle structure $\boldsymbol{R}^2\to Nil^4\to\boldsymbol{R}^2$ we derive the T^2-bundle over T^2 of the form $\boldsymbol{R}^2\cap\Gamma_0\backslash\boldsymbol{R}^2\to\Gamma_0\backslash X\to\bar{\Gamma}_0\backslash\boldsymbol{R}^2$. On the other hand the action of G_X^+/G_X^0 preserves the bundle structure of Nil^4 (see the representatives of G_X^+/G_X^0 in § 2) and hence the action of the finite group Γ/Γ_0 on $\Gamma_0\backslash X$ induces a desired Seifert fibration of $\Gamma\backslash X$ from that of $\Gamma_0\backslash X$ (note that the action of Γ on X is free and orientation preserving). Γ/Γ_0 acts on the base $\bar{\Gamma}_0\backslash\boldsymbol{R}^2=T^2$ so that the quotient is either T^2, K, A, or M (Figure 1) and we will see later that any of them actually occurs.

Case 2. $X=Nil^3\times E$. This case was treated in [**19**] by the geometric argument. In fact the action of Γ_0 yields the Seifert fibration of the form $\Gamma_0\cap\boldsymbol{R}\times E\backslash\boldsymbol{R}\times E\to\Gamma_0\backslash Nil^3\times E\to\bar{\Gamma}_0\backslash\boldsymbol{R}^2$ where $\bar{\Gamma}_0$ is the image of Γ_0 into $\mathrm{Isom}\,\boldsymbol{R}^2$ derived from the projection $\mathrm{Isom}\,Nil^3\to\mathrm{Isom}\,\boldsymbol{R}^2$. Since the action of G_X^+/G_X^0 preserves the bundle structure $\boldsymbol{R}^2\times E\to Nil^3\times E\to\boldsymbol{R}^2$ the action of Γ/Γ_0 again yields a Seifert fibration of $\Gamma\backslash X$ as desired.

Case 3. $X=Sol^3\times E$. Since $G_X^0=Sol^3\times\boldsymbol{R}$ and its nilradical is $\boldsymbol{R}^2\times\boldsymbol{R}$ where the $\boldsymbol{R}^2$-factor is the $\boldsymbol{R}^2$-fiber of Sol^3 and the $\boldsymbol{R}$-factor corresponds to E, we can see that $\Gamma_0'=\Gamma\cap\boldsymbol{R}^2\times\boldsymbol{R}$ is a lattice of $\boldsymbol{R}^2\times\boldsymbol{R}$ and the image $\bar{\Gamma}_0$ of Γ_0 by the projection of the fibration $\boldsymbol{R}^2\times\boldsymbol{R}\to Sol^3\times\boldsymbol{R}\to\boldsymbol{R}$ induced from the fibration of Sol^3 is again a lattice of $\boldsymbol{R}$ ([18], § 3). We have another fibration of the form $\boldsymbol{R}^2\to Sol^3\times\boldsymbol{R}\to\boldsymbol{R}\times\boldsymbol{R}$ where the fiber and the first factor of the base are the same as those of the fibration of Sol^3 and the second factor of the base corresponds to the E-factor. We claim that the restriction of Γ_0' to $\boldsymbol{R}^2\times\{0\}$ is also a lattice in $\boldsymbol{R}^2\times\{0\}$ which corresponds to the fiber of the second fibration. To prove this fix $\gamma\in\Gamma_0$ such that the image $\bar{\gamma}\in\bar{\Gamma}_0$ $(=\boldsymbol{Z})$ generates $\bar{\Gamma}_0$. Then conjugation by γ defines an automorphism of the lattice Γ_0' which acts by a multiplication of an Anosov matrix (of trace $\geqq 3$) on the first $\boldsymbol{R}^2$-factor and acts trivially on the third factor. Hence if we choose an element $\gamma_0\in\Gamma_0'$ whose first two coordinates $\neq(0, 0)$ then $\gamma_1=\gamma\gamma_0\gamma^{-1}-\gamma_0$ is a non-trivial element of Γ_0' contained in $\boldsymbol{R}^2\times\{0\}$ since there is no nonzero vector on $\boldsymbol{R}^2$ invariant by an Anosov map. On the other hand conjugation by γ is represented by an integral matrix A with respect to Γ_0' which maps $\boldsymbol{R}^2\times\{0\}$ to itself by an Anosov map. Then γ_1 and $\gamma\gamma_1\gamma^{-1}$ are the elements in $\Gamma_0'\cap\boldsymbol{R}^2\times\{0\}$ which are linearly independent in $\boldsymbol{R}^2\times\{0\}$ for otherwise A must have a rational eigenvalue other than 1 which is a contradiction. This proves the claim and we can see that $\Gamma_0\backslash Sol^3\times\boldsymbol{R}$ is a T^2-bundle over T^2 as in the previous cases. Observing the representatives of G_X^+/G_X^0 we can also see that the action of Γ/Γ_0 preserves this fibration and hence we have the desired fibration of $\Gamma\backslash X$.

Next we go on to the proof of Theorem C. If $X=E^4$, $Nil^3\times E$ or Nil^4 then the geometric manifold of type X is either flat or almost flat. In these cases there are rigidity theorems due to Bieberbach, Lee and Raymond ([**20**], [7], [8]) from which the first part of the claim immediately follows. To prove the second part we need the following (well-known) lemma whose proof is omitted.

LEMMA 1. *Let B be a 2-orbifold which is either K (Klein bottle), A (annulus), or M (Moebius band). Then any normal abelian subgroup of $\pi_1^{\mathrm{orb}}B$ is contained in a free abelian normal subgroup of rank 2 in $\pi_1^{\mathrm{orb}}B$ which is the fundamental group of T^2 covering B.*

Suppose that $\pi: S\to B$ and $\pi': S'\to B'$ are geometric 4-manifolds of type Nil^4 or $Sol^3\times E$ (which are Seifert 4-manifolds over some euclidean 2-orbifolds by Theorem B) and there is an isomorphism ϕ from $\pi'=\pi_1S'$ to $\pi=\pi_1S$. Let H and H' be the subgroups of π and π' respectively which are generated by the canonical curves on the general fiber. Then H and H' are isomorphic to $\boldsymbol{Z}^2$ since B and B' are euclidean ([**16**], Proposition 0.1). We will see later (§§5, 7) that B and B' are either T^2, K, A, or M under the above condition. Then H (resp. H') is maximal among the normal free abelian subgroups of rank 2 in π (resp. π'), i.e., H (resp. H') cannot be contained properly in the subgroup of π (resp. π') with the same property. For otherwise there is a subgroup $\hat{H}$ of π containing H such that $\hat{H}/H$ is a non-trivial normal abelian torsion subgroup of $\pi_1^{\mathrm{orb}}B$. But this contradicts Lemma 1. Then the proof of Theorem C is reduced to the following claim.

CLAIM. *ϕ induces the isomorphism from H onto H'.*

Assuming this we have the isomorphism $\bar{\phi}: \pi_1^{\mathrm{orb}}B\to\pi_1^{\mathrm{orb}}B'$ induced from ϕ. By Bieberbach's theorem $\bar{\phi}$ is induced by some isomorphism between B and B'. Taking the fibration induced by this isomorphism we may assume that $\bar{\phi}=\mathrm{id}$. Then replacing the choice of the lifts of the elements of $\pi_1^{\mathrm{orb}}B$ or $\pi_1^{\mathrm{orb}}B'$ and performing some coordinate change of the fiber we can make all the invariants of S coincide with those of S' (cf. §6 for the representation of π when the base is A or M). This proves Theorem C.

PROOF OF CLAIM. Consider the isomorphism $\psi: \pi' \to \pi$. Then by the above remark H' is a maximal free abelian normal subgroup of rank 2 in π' and hence $\psi(H')$ has the same property in π. Here we note that the type of the geometric structure on S is the same as that of S' by [18], Theorem 10.1. Considering $\pi_{\#}(\psi(H'))$ for $\pi_{\#}: \pi \to \pi_1^{\mathrm{orb}} B$ we can see by Lemma 1 that $\pi_{\#}(\psi(H'))$ is contained in $\pi_1 B_0$ $(=\boldsymbol{Z}^2)$ where $B_0 = T^2$ is some covering of B. Consider the induced fibration $\tilde{S} \to B_0$ where $\tilde{S}$ is the unbranched covering of S and let $\tilde{\pi} = \pi_1 \tilde{S}$. Then $\psi(H')$ is contained in $\tilde{\pi}$. On the other hand $\tilde{\pi}/\psi(H')$ is a subgroup of $\pi/\psi(H')$ which is isomorphic to $\pi'/H' = \pi_1^{\mathrm{orb}} B'$. Hence $\tilde{\pi}/\psi(H')$ is isomorphic to $\pi_1^{\mathrm{orb}} \hat{B}$ for some orbifold covering $\hat{B}$ of B' and taking a further covering we have a subgroup $\hat{\pi}$ of $\tilde{\pi}$ containing $\psi(H')$ such that $\hat{\pi}/\psi(H') = \pi_1 \hat{B}_0$ where $\hat{B}_0 = T^2$ which covers $\hat{B}$. Note that the above isomorphism is induced by ψ, i.e., the exact sequence $1 \to \psi(H') \to \hat{\pi} \to \hat{\pi}/\psi(H') \to 1$ is isomorphic via ψ to $1 \to H' \to \psi^{-1}(\hat{\pi}) \to \pi_1 \hat{B}_0 = \boldsymbol{Z}^2 \to 1$ which is the exact sequence of the fundamental group for some T^2-bundle over T^2 which covers the fibration of S'. Hence this bundle inherits the structure of type Nil^4 or $Sol^3 \times E$. We shall see in §5 that in this case the first betti number of this T^2-bundle is 2 and so the above sequence coincides with $1 \to \hat{K} \to \hat{\pi} \to H_1(\hat{\pi})/\mathrm{Torsion} \to 1$ where $\hat{K}$ is the kernel of the natural projection. On the other hand the sequence $1 \to H \to \tilde{\pi} \to \pi_1^{\mathrm{orb}} B_0 \to 1$ is also induced from the fibering of the T^2-bundle $\tilde{S}$ over T^2 of type Nil^4 or $Sol^3 \times E$ (induced from that of S). Therefore this sequence is also identified with $1 \to K \to \tilde{\pi} \to H_1(\tilde{\pi})/\mathrm{Torsion} \to 1$ where K is the kernel of the natural projection. Then the inclusion $\hat{\pi} \to \tilde{\pi}$ induces the map from $H_1(\hat{\pi})/\mathrm{Torsion}$ to $H_1(\tilde{\pi})/\mathrm{Torsion}$ and hence $\hat{K}$ is contained in K. It follows that $\psi(H') \subset H$. By the maximality of $\psi(H')$ and H as was remarked above we have $\psi(H') = H$ which proves Claim.

§4. Classification of the T^2-bundles over T^2 and K.

In this section we give a complete list of T^2-bundles $\pi: S \to B$ with $B = T^2$ or K (Lists I and II). The statements for the geometric structures will be proved in §5.

Case 1. $B = T^2$.

In this case S is represented as $\{C, D, (a, b)\}$ where C and D are the monodromy matrices along the curves $\bar{\gamma}$ and $\bar{\delta}$ on B generating $\pi_1 B$ with $C, D \in SL_2\boldsymbol{Z}$, $CDC^{-1}D^{-1} = I$ and (a, b) is the euler class. For some lifts γ, δ of $\bar{\gamma}, \bar{\delta}$ on S we have the representation $\pi_1 S = \{\gamma, \delta, l, h \mid [l, h] = 1, \gamma(l, h)\gamma^{-1} = (l, h)C, \delta(l, h)\delta^{-1} = (l, h)D, [\gamma, \delta] = l^a h^b\}$. Then S is classified according to the value of $b_1 = b_1 S$ and is diffeomorphic to one of those in List I below (see [12], [16] for the details).

List I. $B=T^2$.

	S	type of geometry
1-1 $b_1=4$	$\{I, I, (0,0)\}=T^4$	E^4
1-2 $b_1=3$	$\{I, I, (a,b)\}$ with $(a,b)\neq(0,0)$	$Nil^3\times E$
$b_1=2$	$\left\{\begin{pmatrix}0&-1\\1&-1\end{pmatrix}, I, (0,0)\right\}$	
	$\left\{\begin{pmatrix}0&-1\\1&-1\end{pmatrix}, I, (-1,0)\right\}$	
1-3(a)	$\left\{\begin{pmatrix}0&-1\\1&0\end{pmatrix}, I, (0,0)\right\}$	
	$\left\{\begin{pmatrix}0&-1\\1&0\end{pmatrix}, I, (-1,0)\right\}$	E^4
	$\left\{\begin{pmatrix}1&-1\\1&0\end{pmatrix}, I, (0,0)\right\}$	
	$\{-I, I, (0,0)\}$	
	$\{-I, I, (-1,0)\}$	
1-3(b)	$\left\{\begin{pmatrix}1&\lambda\\0&1\end{pmatrix}, I, (a,b)\right\}$ with $\lambda\neq0, b\neq0$	Nil^4
1-3(c)	$\left\{\begin{pmatrix}-1&\lambda\\0&-1\end{pmatrix}, I, (a,b)\right\}$ with $\lambda\neq0$	$Nil^3\times E$
1-3(d)	$\{C, I, (a,b)\}$ with $\lvert\mathrm{tr}\, C\rvert\geqq3$	$Sol^3\times E$
1-4(a)	$\left\{\begin{pmatrix}1&\lambda\\0&1\end{pmatrix}, -I, (a,b)\right\}$ with $\lambda\neq0$	$Nil^3\times E$
1-4(b)	$\{C, -I, (a,b)\}$ with $\mathrm{tr}\, C\geqq3$	$Sol^3\times E$

Case 2. $B=K$.

In this case S is represented as $\{C, D, (a, b)\}$. Here $C, D\in GL_2\boldsymbol{Z}$ are the monodromy matrices along $\bar{\gamma}$, $\bar{\delta}$ with $\det C=-1$, $\det D=1$ respectively where $\bar{\gamma}$ (resp. $\bar{\delta}$) is the orientation preserving (resp. reversing) curve in K such that $\bar{\gamma}\bar{\delta}\bar{\gamma}^{-1}\bar{\delta}=1$ in π_1K. We have $CDC^{-1}D=I$. (a, b) is the euler class and π_1S has the representation of the form $\{\gamma, \delta, l, h\mid[l, h]=1, \gamma(l, h)\gamma^{-1}=(l, h)C, \delta(l, h)\delta^{-1}=(l, h)D, \gamma\delta\gamma^{-1}\delta=l^ah^b\}$ where γ and δ are the lifts of $\bar{\gamma}$ and $\bar{\delta}$ respectively and l, h are the generators of π_1(fiber). Let us give the classification of S which was partially proved in [**16**], §2. If $b_1=2$ then $\mathrm{rank}(C-I, D-I)=1$ and we may assume that $C=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ or $\begin{pmatrix}-1&1\\0&1\end{pmatrix}$, $D=\begin{pmatrix}1&\lambda\\0&1\end{pmatrix}$, $b=0$ or 1 and S is diffeomorphic to a T^2-bundle over T^2 ([**16**], §2). If $b_1=1$ there is a T^2-bundle $\tilde{S}$ over T^2 which is a double covering of S associated to the canonical map $\pi_1S\to H_1S/\mathrm{Torsion}=\boldsymbol{Z}\to\boldsymbol{Z}_2$. $\tilde{S}$ has the representation of the form $\left\{C^2, D, (C-D)\begin{pmatrix}a\\b\end{pmatrix}\right\}$ and $\tilde{b}_1=b_1\tilde{S}$ is either 3 or 2 ([**16**], §2). If $b_1=1$ and $\tilde{b}_1=3$ then we may assume that $C=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ or $\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$, $D=\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}$, $b=0$ and S is diffeomorphic to a Seifert 4-manifold over $S^2(2, 2, 2, 2)$ ([**16**], Proposition 2.5). If $b_1=1$ and $\tilde{b}_1=2$ or equivalently

$$(*)\qquad \operatorname{rank}(C-I,\ D-I)=\operatorname{rank}\left(C^2-I,\ D-I,\ (C-D)\begin{pmatrix}a\\ b\end{pmatrix}\right)=2\,,$$

then the structure of S as a T^2-bundle over K is unique ([**16**]). This case is divided into 6 subclasses (List II) according to the trace of D.

LEMMA 2. *Let $C\in GL_2\boldsymbol{Z}$ with $\det C=-1$. Then $\operatorname{tr} C=0$ if and only if $C^2=I$ and C is conjugate in $GL_2\boldsymbol{Z}$ to $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ or $\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$ and $\operatorname{tr} C\neq 0$; if and only if $\operatorname{tr} C^2\geqq 3$. In this case C is conjugate in $GL_2\boldsymbol{R}$ to $\begin{pmatrix}\exp t & 0\\ 0 & -\exp(-t)\end{pmatrix}$ with $t\in\boldsymbol{R}$, $t\neq 0$. Furthermore $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ is not conjugate in $GL_2\boldsymbol{Z}$ to $\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$.*

The proof is straightforward. Let us recall that the representation $\{C, D, (a, b)\}$ of S is transformed into one of the followings by fiber-preserving diffeomorphisms ([**16**], §2).

(I) $\left\{P^{-1}CP,\ P^{-1}DP,\ P^{-1}\begin{pmatrix}a\\ b\end{pmatrix}\right\}$ (base change of the fiber),

(II-1) $\left\{C^{-1},\ D,\ D^{-1}C^{-1}\begin{pmatrix}a\\ b\end{pmatrix}\right\}$, (II-2) $\left\{C,\ D^{-1},\ -D\begin{pmatrix}a\\ b\end{pmatrix}\right\}$,

(II-3) $\left\{DCD^{-1},\ D,\ D\begin{pmatrix}a\\ b\end{pmatrix}\right\}$, (II-4) $\left\{C^{-1}D,\ D^{-1},\ C^{-2}\begin{pmatrix}a\\ b\end{pmatrix}\right\}$,

(III) $\left\{C,\ D,\ \begin{pmatrix}a\\ b\end{pmatrix}+D^{-1}C(D^{-1}-I)\begin{pmatrix}p\\ q\end{pmatrix}+(D^{-1}C+I)\begin{pmatrix}s\\ t\end{pmatrix}\right\}$.

(II-1~4) are induced by some automorphisms of the base K and (III) is realized by the replacement of the lifts (γ, δ) by $(\gamma l^p h^q, \delta l^s h^t)$. Now we give the complete list and explain the subclasses in (2-3).

NOTE. For the cases (2-1) and (2-2) in List II below the bundle structures of S over K are not unique and there are some overlaps (up to diffeomorphisms). See [**16**] for the details (which we do not need for the proof of the main theorems in this paper). But there are no overlaps in case (2-3) (up to fiber-preserving diffeomorphisms). Any two classes which belong to the different blocks in case (2-3) cannot give the same manifold since any transformation of type (I)~(III) preserves the conjugacy classes of $D^{\pm 1}$ and the bundle structure on K is unique in case (2-3) (see below for the details).

List II. $B=K$.

	S	type of geometry
2-1 $b_1=2$.		
2-1-1	$\left\{\begin{pmatrix}-1 & (1)\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}, (a,b)\right\}$ with $\lambda\neq 0$	$Nil^3\times E$
2-1-2	$\left\{\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, I, (a,b)\right\}$ with $a\neq 0$	$Nil^3\times E$
2-1-3	$\left\{\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, I, (0,b)\right\}$	E^4
2-1-4	$\left\{\begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix}, I, (a,b)\right\}$ with $2a\neq b$	$Nil^3\times E$
2-1-5	$\left\{\begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix}, I, (a,2a)\right\}$	E^4
2-2 $b_1=1, \tilde{b}_1=3$.		
	$\left\{\begin{pmatrix}1 & (1)\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}, (a,0)\right\}$ with $\lambda\neq 0$	$Nil^3\times E$
2-3 $b_1=1, \tilde{b}_1=2$.		
2-3-1(a)	$\left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}, (0,0)\right\}$ $\left\{\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}, (0,0)\right\}$ $\left\{\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}, (0,1)\right\}$	E^4
2-3-1(b)	$\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, (0,0)\right\}$ $\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, (1,0)\right\}$	E^4
2-3-1(c)	$\left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}, (0,0)\right\}$	E^4
2-3-2	$\{C, I, (a,b)\}$ with $\mathrm{tr}C\neq 0$	$Sol^3\times E$
2-3-3(a)	$\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, -I, (0,0)\right\}$ $\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, -I, (1,0)\right\}$ $\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, -I, (1,1)\right\}$	E^4

2-3-3(a′)	$\left\{\begin{pmatrix}0&1\\1&0\end{pmatrix}, -I, (0,0)\right\}$ $\left\{\begin{pmatrix}0&1\\1&0\end{pmatrix}, -I, (0,1)\right\}$	E^4
2-3-3(b)	$\{C, -I, (a,b)\}$ with $\mathrm{tr}C\neq 0$	$Sol^3\times E$
2-3-4	$\left\{\begin{pmatrix}1&(1)\\0&-1\end{pmatrix}, \begin{pmatrix}1&\lambda\\0&1\end{pmatrix}, (a,b)\right\}$ with $\lambda\neq 0, b\neq 0$	Nil^4
2-3-5	$\left\{\begin{pmatrix}1&(1)\\0&-1\end{pmatrix}, \begin{pmatrix}-1&\lambda\\0&-1\end{pmatrix}, (a,b)\right\}$ with $\lambda\neq 0$	$Nil^3\times E$
2-3-6	$\{C, D, (a,b)\}$ with $\lvert\mathrm{tr}D\rvert\geqq 3$	$Sol^3\times E$

Here $\begin{pmatrix}1&(1)\\0&-1\end{pmatrix}$ means $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ or $\begin{pmatrix}1&1\\0&-1\end{pmatrix}$.

2-3-1. $|\mathrm{tr}\,D|\leqq 1$. In this case D is conjugate in $SL_2\boldsymbol{Z}$ to either $\begin{pmatrix}0&-1\\1&-1\end{pmatrix}$, $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, $\begin{pmatrix}1&-1\\1&0\end{pmatrix}$ or one of their inverses. By the transformation of type (II-2) it suffices to consider the first 3 cases.

2-3-1(a). $D=\begin{pmatrix}0&-1\\1&-1\end{pmatrix}$. By the condition $CD=D^{-1}C$ we can see that $C=\pm\begin{pmatrix}1&-1\\0&-1\end{pmatrix}$, $\pm\begin{pmatrix}0&1\\1&0\end{pmatrix}$ or $\pm\begin{pmatrix}1&0\\1&-1\end{pmatrix}$. But the pair (C, D) can be transformed as $(C, D)\to(C^{-1}D, D^{-1})\to(C^{-1}D, D)$ (use (II) above). Hence it suffices to consider the following cases; $C=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $(a, b)=(0, 0)$ or $C=-\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $(a, b)=(0,0)$, $(0,1)$ (use (II-1) and then (III)). Hence S is one of the 3 cases in the above list. These three are mutually distinct. For $H_1S=\boldsymbol{Z}+\boldsymbol{Z}_2+\boldsymbol{Z}_3$ for the first one and $H_1S=\boldsymbol{Z}+\boldsymbol{Z}_2$ for the last two. On the other hand consider the double cover $\tilde{S}$ of the form $\left\{C^2, D, (C-D)\begin{pmatrix}a\\b\end{pmatrix}\right\}$ associated to the projection $\pi_1\to H_1(\pi_1)/\mathrm{Torsion}\to\boldsymbol{Z}_2$ to see that $H_1\tilde{S}=\boldsymbol{Z}^2+\boldsymbol{Z}_3$ for the second one while $H_1\tilde{S}=\boldsymbol{Z}^2$ for the third one.

2-3-1(b). $D=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. Considering as before we may assume that $C=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and $(a, b)=(0,0)$ or $(1,0)$ (use the transformations of type $(C, D)\to(C^{-1}D, D)$, $(C, D)\to(DCD^{-1}, D)$ and (III)). Hence we get the two classes in 2-3-1(b) which are distinguished by the first homology group.

2-3-1(c). $D=\begin{pmatrix}1&-1\\1&0\end{pmatrix}$. In this case we may assume that $C=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $(a,b)=(0,0)$ by the transformations as in the previous case.

2-3-2. $D=I$. In this case by (*) and Lemma 2 $\mathrm{tr}\,C\neq 0$ and C is conjugate in $GL_2\boldsymbol{Z}$ to $\begin{pmatrix}\exp t & 0\\ 0 & -\exp(-t)\end{pmatrix}$ with $t\neq 0$.

2-3-3. $D=-I$. We consider two subclasses (a), (a′) ($\operatorname{tr} C=0$) and (b) ($\operatorname{tr} C\neq 0$). If $\operatorname{tr} C=0$ then we may assume that $C=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ or $\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$ $\Big($which is conjugate to $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ in $GL_2\boldsymbol{Z}\Big)$. In the first case we may assume that $(a, b)=(0,0)$, $(1,1)$, $(0,1)$ or $(1,0)$ (use (III)). But the last two cases can be identified to get the 3 classes in the list 2-3-3(a) $\Big($apply (II-4) and then conjugation by $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Big)$. Note that $H_1S=\boldsymbol{Z}+\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2$ for the first one and $=\boldsymbol{Z}+\boldsymbol{Z}_4+\boldsymbol{Z}_2$ for the last two. We can see that the second one is not diffeomorphic to the third one. For if they are diffeomorphic there is a fiber-preserving diffeomorphism between them which is a finite number of products of the transformations of type (I)~(III) or their inverses. By any transformation of type (II) $(C,-I)$ is changed to $(\pm C, -I)$. On the other hand if $PCP^{-1}=\pm C$ $\Big(C=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Big)$ for some $P\in GL_2\boldsymbol{Z}$ then $P=\begin{pmatrix}\pm 1 & 0\\ 0 & \pm 1\end{pmatrix}$ or $\begin{pmatrix}0 & \pm 1\\ \pm 1 & 0\end{pmatrix}$. Then we can observe that the second one cannot be transformed to the third one by any kind of transformation. If $C=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ then (a, b) can be reduced to $(0,0)$ or $(0,1)$ by (III). Then we get the two classes in 2-3-3(a′) which are distinguished by the first homology group. The class 2-3-3(a) is also distinguished from the one in 2-3-3(a′) by the first homology group.

2-3-4. $\operatorname{tr} D=2$ with $D\neq I$. In this case we may assume that $D=\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}$ with $\lambda\neq 0$ and $C=\pm\begin{pmatrix}1 & \eta\\ 0 & -1\end{pmatrix}$ for some η. Taking the conjugate of C and D by some matrix of the form $\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}$ we may assume that $\eta=0$ or 1. Hence by the condition (*) we get the list 2-3-4 above.

2-3-5. $\operatorname{tr} D=-2$ with $D\neq -I$. Arguing as before we may assume that $D=\begin{pmatrix}-1 & \lambda\\ 0 & -1\end{pmatrix}$, $C=\begin{pmatrix}1 & \eta\\ 0 & -1\end{pmatrix}$ with $\eta=0$ or 1 (use the transformation of type $(C, D)\rightarrow(C^{-1}D, D)$).

2-3-6. $|\operatorname{tr} D|\geqq 3$. In this case there is a matrix $P\in SL_2\boldsymbol{R}$ such that $PDP^{-1}\equiv\pm\begin{pmatrix}\exp t & 0\\ 0 & \exp(-t)\end{pmatrix}$ with $t\neq 0$ and then $PCP^{-1}=\begin{pmatrix}0 & s\\ s^{-1} & 0\end{pmatrix}$ for some $s\neq 0$. Taking the conjugate by $\begin{pmatrix}q & 0\\ 0 & q^{-1}\end{pmatrix}$ for some q and $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ we may assume that $PCP^{-1}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

§5. Proof of Theorem A when the bases are euclidean—Part 1.

In this section we will see that any Seifert 4-manifold S over a euclidean base orbifold B without reflectors admits a geometric structure. The proof is similar to that of the analogous statements for the Seifert 3-manifolds. We try to lift the geometric structure of B to that of S by giving a faithful discrete representation ρ from $\pi_1 S$ to G_X^+ for some geometry X. The standard representation of $\pi_1 S$ indicates the construction of S so that we can see directly that $\rho(\pi_1 S)$ acts freely on X and $\rho(\pi_1 S)\backslash X$ coincides with S in each case given below. Therefore we only indicate how to construct $\rho: \pi_1 S \to G_X^+$. In some cases below we only sketch the proofs since they are quite similar to each other.

Case 1. $B=T^2$. We proceed according to List I in §4. Fix the representation of $\pi_1 S$ as in §4. The proofs for 1-1 and 1-2 are straightforward. Case 1-3(a) is the class of hyperelliptic surfaces and each one has the euclidean structure (cf. [**16**]).

CLAIM 1. $\left\{\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}, I, (a, b)\right\}$ *with* $\lambda\neq 0$, $b\neq 0$ (1-3(b)) *has a* Nil^4 *structure.*

PROOF. Let (x, y, z, t) be the coordinates of Nil^4 (§2) and define the representation ρ by $\rho(l)(x, y, z, t)=(x+l_0, y, z, t)$, $\rho(h)(x, y, z, t)=(x, y+h_0, z, t)$, $\rho(\gamma)(x, y, z, t)=(a_0, b_0, 0, 1)(x, y, z, t)=(x+y+z/2+a_0, y+z+b_0, z, t+1)$, $\rho(\delta)(x, y, z, t)=(x+a_1, y+b_1, z+1, t)$ with $l_0, h_0\neq 0$. Then $\rho(l)$ and $\rho(h)$ form a lattice in $\boldsymbol{R}^2$ and $\rho(\gamma)$, $\rho(\delta)$ give the lift of the standard representation of $\pi_1 T^2$. We deduce the following conditions on the parameters: $\lambda=h_0/l_0$ from $\gamma(l, h)\gamma^{-1}=(l, l^\lambda h)$, $a_1+b_1+1/2=al_0+a_1$, $b_1+1=bh_0+b_1$ from $[\gamma, \delta]=l^a h^b$. Hence if we put $h_0=1/b$, $l_0=1/b\lambda$, $b_1=a/b\lambda-1/2$ (a_1 is arbitrary) we obtain the desired representation.

CLAIM 2. $\left\{\begin{pmatrix}-1 & \lambda\\ 0 & -1\end{pmatrix}, I, (a, b)\right\}$ *with* $\lambda\neq 0$ (1-3(c)) *and* $\left\{\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}, -I, (a, b)\right\}$ *with* $\lambda\neq 0$ (1-4(a)) *admit* $Nil^3\times E$ *structures.*

PROOF. Let (x, y, z, w) be the coordinates of $Nil^3\times E$ with $\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\in Nil^3$. In the first case S is a T^3-bundle over S^1 whose fiber is spanned by l, h, δ. Then define ρ by $\rho(l)(x, y, z, w)=(x, y, z+l_0, w)$, $\rho(h)(x, y, z, w)=(x+h_0, y, z+h_0 y, w)$ with $l_0\neq 0$, $h_0\neq 0$, $\rho(\delta)(x, y, z, w)=(x+s, y, z+sy+t, w+1)$, $\rho(\gamma)(x, y, z, w)=(-x, y+1, -z+q, w)$. Here the image of $\rho(\xi)$ is a left multiplication by $\begin{pmatrix}1 & 0 & l_0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ $(\xi=l)$, $\begin{pmatrix}1 & h_0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ $(\xi=h)$, $\begin{pmatrix}1 & s & t\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ $(\xi=\delta)$. $\rho(l)$, $\rho(h)$, $\rho(\delta)$ form a lattice in $\boldsymbol{R}^3$ with coordinates (x, z, w) if y is fixed. $\rho(\gamma)$ is defined as a composition of the automorphism $(x, y, z, w)\to(-x, y, -z, w)$ and a left multiplication by $\begin{pmatrix}1 & 0 & q\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$.

Then it is easy to find the desired parameters and ρ. In the second case modifv ρ so that $\rho(\gamma)(x, y, z, w)=(x+p, y+1, z+py+q, w)$, $\rho(\delta)(x, y, z, w)=(-x, y, -z, w+1)$ and choose the parameters appropriately.

Claim 3. *$\{C, I, (a, b)\}$ with $|\mathrm{tr}C|\geqq 3$ (1-3(d)) and $\{C, -I, (a, b)\}$ with* $\mathrm{tr}C\geqq 3$ (1-4(b)) *admit $Sol^3\times E$ structures.*

Proof. There is a matrix $P=\begin{pmatrix} p & r \\ q & s\end{pmatrix}\in GL_2\boldsymbol{R}$ such that $PCP^{-1}=\begin{pmatrix} \varepsilon\exp t_0 & 0 \\ 0 & \varepsilon\exp(-t_0)\end{pmatrix}$ with $t_0\neq 0$ where $\varepsilon=1$ if $\mathrm{tr}C\geqq 3$ and $\varepsilon=-1$ if $\mathrm{tr}C\leqq -3$. Let (x, y, t, u) be the coordinates of $Sol^3\times E$ (§2). In the first case define ρ by $\rho(l)(x, y, t, u)=(x+p, y+q, t, u)$, $\rho(h)(x, y, t, u)=(x+r, y+s, t, u)$, $\rho(\delta)(x, y, z, u)=(x+x_1, y+y_1, t, u+1)$, $\rho(\gamma)(x, y, t, u)=(x_0, y_0, t_0, 0)(\varepsilon x, \varepsilon y, t, u)=(\varepsilon\exp t_0\cdot x+x_0, \varepsilon\exp(-t_0)\cdot y+y_0, t+t_0, u)$. Then we deduce from $\gamma\delta\gamma^{-1}=l^a h^b\delta$ that $\begin{pmatrix} x_1 \\ y_1\end{pmatrix}=(PCP^{-1}-I)^{-1}P\begin{pmatrix} a \\ b\end{pmatrix}$ which determines x_1 and y_1 since $t_0\neq 0$. This gives the desired representation. In the second case modify $\rho(\delta)$ so that $\rho(\delta)(x, y, t, u)=(-x+x_1, -y+y_1, t, u+1)$ and the proof goes similarly.

Case 2. $B=K$. Fix the representation of $\pi_1 S$ for $S=\{C, D, (a, b)\}$ as in §4. We proceed according to List II in §4.

Claim 4. $\left\{\begin{pmatrix} -1 & (1) \\ 1 & 1\end{pmatrix}, \begin{pmatrix} 1 & \lambda \\ 0 & 1\end{pmatrix}, (a, b)\right\}$ *with* $\lambda\neq 0$ (2-1-1), $\left\{\begin{pmatrix} 1 & (1) \\ 0 & -1\end{pmatrix}, \begin{pmatrix} 1 & \lambda \\ 0 & 1\end{pmatrix}, (a, 0)\right\}$ *with* $\lambda\neq 0$ (2-2) *and* $\left\{\begin{pmatrix} 1 & (1) \\ 0 & -1\end{pmatrix}, \begin{pmatrix} -1 & \lambda \\ 0 & -1\end{pmatrix}, (a, b)\right\}$ *with* $\lambda\neq 0$ (2-3-5) *admit $Nil^3\times E$ structures.*

Proof. Let (x, y, z, w) be the coordinates of $Nil^3\times E$ as before. In the first case define ρ by $\rho(l)(x, y, z, w)=(x, y, z+l_0, w)$, $\rho(h)(x, y, z, w)=(x, y+h_0, z, w)$ with $l_0\neq 0$, $h_0\neq 0$, $\rho(\gamma)(x, y, z, w)=\left(\begin{pmatrix} 1 & c & u \\ 0 & 1 & v \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & -x & -z \\ 0 & 1 & y \\ 0 & 0 & 1\end{pmatrix}, w+1\right)=(-x+c, y+v, -z+cy+u, w+1)$, $\rho(\delta)(x, y, z, w)=\left(\begin{pmatrix} 1 & 1 & u' \\ 0 & 1 & v' \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1\end{pmatrix}, w\right)=(x+1, y+v', z+y+u', w)$. Note that $\rho(\gamma)$ and $\rho(\delta)$ define the lift of the representation of $\pi_1 K=\{\bar{\gamma}, \bar{\delta}\,|\,\bar{\gamma}\bar{\delta}\bar{\gamma}\bar{\delta}^{-1}=1\}$ of the form $\bar{\gamma}: (x, w)\to(-x+c, w+1)$, $\bar{\delta}: (x, w)\to(x+1, w)$. Then it is easy to find the parameters to get the desired results. The proof for the second one is similar. The third case was proved in [**16**], Proposition 3.17.

Claim 5. $\left\{\begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}, I, (a, b)\right\}$ (2-1-2, 3) *admits a $Nil^3\times E$ structure if $a\neq 0$ and an E^4 structure if $a=0$.* $\left\{\begin{pmatrix} -1 & 1 \\ 0 & 1\end{pmatrix}, I, (a, b)\right\}$ (2-1-4, 5) *admits a $Nil^3\times E$ structure if $2a\neq b$ and an E^4 structure if $2a=b$.*

PROOF. We can define the desired representations for the $Nil^3\times E$ cases as in Claim 4. The proof for E^4 is an easy modification of that for $Nil^3\times E$.

CLAIM 6. $\{C, D, (a, b)\}$ *with* $\mathrm{tr}C=0$, D *periodic* $(\neq I)$ *admits an* E^4*-structure* (2-3-1(a)$\sim$(c), 2-3-3(a), (a')).

PROOF. Let $R(\theta)=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$. Suppose that $D=\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}$. Then there is a matrix $P=\begin{pmatrix}p & r\\ q & s\end{pmatrix}\in GL_2\boldsymbol{R}$ such that $R(2\pi/3)P=PD$ and $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}P=P\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ (for example $P=\begin{pmatrix}\sqrt{3}-2 & 1\\ 1 & \sqrt{3}-2\end{pmatrix}$). Then for the coordinates (x, y, z, w) of E^4 define ρ by $\rho(l)(x, y, z, w)=(x, y, z+p, w+q)$, $\rho(h)(x, y, z, w)=(x, y, z+r, w+s)$, $\rho(\gamma)(x, y, z, w)=(x+1, -y, \varepsilon w+e, \varepsilon z+f)$, $\rho(\delta)(x, y, z, w)=(x, y+1, cz+sw, -sz+cw)$ where $\varepsilon=\pm 1$ according as $C=\begin{pmatrix}0 & \varepsilon\\ \varepsilon & 0\end{pmatrix}$, $c=\cos(2\pi/3)$, $s=\sin(2\pi/3)$. Then we deduce $(I-R(-2\pi/3))\begin{pmatrix}e\\ f\end{pmatrix}=P\begin{pmatrix}a\\ b\end{pmatrix}$ from $\gamma\delta\gamma^{-1}\delta=l^a h^b$ from which e and f are well-determined and we get the desired result. The proofs for the remaining cases go similarly.

CLAIM 7. $\left\{\begin{pmatrix}1 & (1)\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}, (a, b)\right\}$ *with* $\lambda\neq 0$, $b\neq 0$ *admits a* Nil^4 *structure* (2-3-4).

PROOF. Let $(\boldsymbol{x}, t)=(x, y, z, t)$ be the coordinates of Nil^4 and $C_t\boldsymbol{x}=\begin{pmatrix}1 & t & t^2/2\\ 0 & 1 & t\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$. Recall that there is an automorphism $(\in G^+_{Nil^4})$ of the form $(x, y, z, t)\rightarrow(x, -y, z, -t)$ and put $\bar{\boldsymbol{x}}=(x, -y, z)$. Then we can define ρ for appropriate parameters by $\rho(\gamma)(\boldsymbol{x}, t)=(C_s\bar{\boldsymbol{x}}+\boldsymbol{p}, -t+s)$, $\rho(\delta)(\boldsymbol{x}, t)=(C_1\boldsymbol{x}+\tilde{\boldsymbol{p}}, t+1)$, $\rho(l)(\boldsymbol{x}, t)=(x+l_0, y, z, t)$, $\rho(h)(\boldsymbol{x}, t)=(x, y+h_0, z, t)$ with $\boldsymbol{p}=(p, q, 1)$, $\tilde{\boldsymbol{p}}=(\tilde{p}, \tilde{q}, 0)$, $l_0\neq 0$, $h_0\neq 0$. (Note that the representations of γ, δ are the lifts of that of the Klein bottle $(z, t)\rightarrow(z+1, -t+s)$, $(z, t)\rightarrow(z, t+1)$.)

CLAIM 8. $\{C, D, (a, b)\}$ *with* $\mathrm{tr}C\neq 0$ *or* $|\mathrm{tr}D|\geqq 3$ *admits a* $Sol^3\times E$ *structure* (2-3-2, 2-3-3(b), 2-3-6).

PROOF. First consider 2-3-2, 2-3-3(b) ($\mathrm{tr}C\neq 0$, $D=\pm I$). In either case we have $P=\begin{pmatrix}c & e\\ d & f\end{pmatrix}\in GL_2\boldsymbol{R}$ such that $PC=\begin{pmatrix}\exp s & 0\\ 0 & -\exp(-s)\end{pmatrix}P$ with $s\neq 0$ (§4). Let (x, y, t, w) be the coordinate of $Sol^3\times E$ as before. Note that there are automorphisms of $Sol^3\times E$ of the form $(x, y, t, w)\rightarrow(x, -y, t, -w)$, $(x, y, t, w)\rightarrow(-x, -y, t, w)$ (§2). Then define ρ by $\rho(\gamma)(x, y, t, w)=(\exp s\cdot x+p, -\exp(-s)\cdot y+q, t+s, -w)$, $\rho(\delta)(x, y, t, w)=(\varepsilon x+u, \varepsilon y+v, t, w+1)$, $\rho(l)(x, y, t, w)=(x+c, y+d, t, w)$, $\rho(h)(x, y, t, w)=(x+e, y+f, t, w)$ where $\varepsilon=\pm 1$ according as $D=\pm I$.

Choose u, v so that $\begin{pmatrix}(1+\exp s)u \\ (1-\exp(-s))v\end{pmatrix}=P\begin{pmatrix}a \\ b\end{pmatrix}$ (if $D=I$), $\begin{pmatrix}(\exp s-1)u+2p \\ -(\exp(-s)+1)v+2q\end{pmatrix}=P\begin{pmatrix}a \\ b\end{pmatrix}$ (if $D=-I$) then all other relations are satisfied automatically. If $|\mathrm{tr}D|\geqq 3$ we have observed in § 4 that there is $P=\begin{pmatrix}c & e \\ d & f\end{pmatrix}\in GL_2\boldsymbol{R}$ such that $PCP^{-1}=\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$ and $PDP^{-1}=\varepsilon\begin{pmatrix}\exp s & 0 \\ 0 & \exp(-s)\end{pmatrix}$ with $s\neq 0$ where $\varepsilon=1$ if $\mathrm{tr}D\geqq 3$ and $\varepsilon=-1$ if $\mathrm{tr}D\leqq -3$. Then define ρ by $\rho(\gamma)(x, y, t, w)=(y+u, x+v, -t, w+1)$, $\rho(\delta)(x, y, t, w)=(\varepsilon\exp s\cdot x+p, \varepsilon\exp(-s)\cdot y+q, t+s, w)$, $\rho(l)(x, y, t, w)=(x+c, y+d, y, w)$, $\rho(h)(x, y, t, w)=(x+e, y+f, t, w)$. We have only to choose the parameters so that $\begin{pmatrix}\varepsilon\exp(-s) & 1 \\ 1 & \varepsilon\exp s\end{pmatrix}\begin{pmatrix}p \\ q\end{pmatrix}+\begin{pmatrix}(1-\varepsilon\exp(-s))u \\ (1-\varepsilon\exp s)v\end{pmatrix}=P\begin{pmatrix}a \\ b\end{pmatrix}$.

Case 3. $B=P^2(2, 2)$. In this case $S=\left\{\begin{pmatrix}1 & (1) \\ 0 & -1\end{pmatrix}, (a, b), (2, a_1, b_1), (2, a_2, b_2)\right\}$ and $\pi_1 S=\{\gamma, q_1, q_2, l, h \mid [l, h]=[q_i, l]=[q_i, h]=1$ for $i=1, 2$, $\gamma(l, h)\gamma^{-1}=(l, h)\begin{pmatrix}1 & (1) \\ 0 & -1\end{pmatrix}$, $q_i^2 l^{a_i} h^{b_i}=1$ for $i=1, 2$, $\gamma^2 q_1 q_2=l^a h^b\}$. We have proved in [**16**] Proposition 3.16 the following claim.

CLAIM 9. *$\left\{\begin{pmatrix}1 & (1) \\ 0 & -1\end{pmatrix}, (a, b), (2, a_1, b_1), (2, a_2, b_2)\right\}$ admits a E^4 structure if $b+(b_1+b_2)/2=0$ and a $Nil^3\times E$ structure if $b+(b_1+b_2)/2\neq 0$.*

Case 4. B is of genus 0. In this case S is represented as $\{(a, b), (m_1, a_1, b_1), \cdots, (m_k, a_k, b_k)\}$ where $B=S^2(m_1, \cdots, m_k)$ (§ 1). We have defined the rational euler class $e=(a+\sum a_i/m_i, b+\sum b_i/m_i)$ in § 1. We may assume that $a+\sum a_i/m_i=0$ (see § 1).

CLAIM 10. *S admits an E^4 structure if $e=(0, 0)$ and an $Nil^3\times E$ structure if $e\neq(0, 0)$.*

PROOF. There are just 7 classes with $e=(0, 0)$ each of which has a euclidean structure ([**16**]). Then suppose that $e=(0, b+\sum b_i/m_i)\neq(0, 0)$. Here we use the coordinates (w, z) of $Nil^3\times E$ in § 2. First fix the representation $\bar{\phi}$ of $\pi_1^{\mathrm{orb}}B=\{\bar{q}_1, \cdots, \bar{q}_k \mid \bar{q}_1^{m_1}=\cdots=\bar{q}_k^{m_k}=\bar{q}_1\cdots\bar{q}_k=1\}$ to Isom^+E^2 so that $\bar{\phi}(q_j)(z)=\rho(z-z_j)+z_j$ where $\rho=\exp(2\pi i/m_j)$, $z_j\in\boldsymbol{C}$. Define the lift of $\bar{\phi}(\bar{q}_j)$ by $\phi(q_j)(w, z)=(w+w_j+i\bar{z}_j(z-\rho(z-z_j)), \rho(z-z_j)+z_j)$ (this is defined by the composition of multiplication by $(w_j, -z_j)$, $\rho:(w, z)\to(w, \rho z)$ and multiplication by $(0, z_j)$). Thus we have $\rho(q_j)^{m_j}(w, z)=(w+m_j w_j+im_j|z_j|^2, z)$. Putting $\rho(l)(w, z)=(w+il_0, z)$, $\rho(h)(w, z)=(w+h_0, z)$ with $l_0\neq 0$, $h_0\neq 0$, $l_0, h_0\in\boldsymbol{R}$ we deduce $m_j(w_j+i|z_j|^2)=-b_j h_0-ia_j l_0$ from $q^{m_j}l^{a_j}h^{b_j}=1$. These relations $(i=1, \cdots, k)$ and that derived from $q_1\cdots q_k=l^a h^b$ lead us to the condition of the form $(b+\sum b_i/m_i)h_0+i(a+\sum a_i/m_i)l_0=c$ for some nontrivial real number c. (If $B=S^2(3, 3, 3)$ for example we choose the parameters so that $z_1=0$, $z_2=1$, $z_3=-\exp(-2\pi/3)$ and then $c=-\sqrt{3}$.) Then we have $h_0=$

$c/(b+\sum b_i/m_i)$ and choose l_0 arbitrarily to obtain the desired representation. In this case S is diffeomorphic to an elliptic surface [**17**].

§6. Classification of the Seifert 4-manifolds over the euclidean base orbifolds with reflectors.

In §§6, 7 we will be concerned with the Seifert 4-manifold S over B with reflectors, i.e., $B=A$, M, or $|B|=D^2$. In either case there is a standard double covering $\tilde{B}$ of B without reflectors and the induced Seifert 4-manifold $\tilde{S}$ over $\tilde{B}$ is an unbranched double covering of S with covering translation ι. In this section we give the representation of S with $B=A$ or M as $\tilde{S}/\iota$ (List III, IV) and in §7 we prove the statements for the geometric structures.

Case 1. $B=A$. For the representation $\pi_1^{orb}A=\{\bar{\gamma}, \bar{\delta}, \bar{\iota}\,|\,[\bar{\gamma}, \bar{\delta}]=\bar{\iota}^2=1, \bar{\iota}(\bar{\gamma}, \bar{\delta})\bar{\iota}^{-1}=(\bar{\gamma}, \bar{\delta}^{-1})\}$ S is determined by the following data;

(a) the monodromy $C=\pm I$ along the reflector circle $\bar{\gamma}_i$ ($\bar{\gamma}_1$ and $\bar{\gamma}_2^{-1}$ are homologous to $\bar{\gamma}$ and the monodromies over them are the same),

(b) the euler class of the lift γ_i of $\bar{\gamma}_i$ with respect to the base (l_i, h_i) of the general fiber near γ_i $(i=1, 2)$,

(c) the transformation of the fibers between (l_1, h_1) and (l_2, h_2).

We always assume that $\gamma_1=\gamma_2^{-1}$ (in $\pi_1 S$) so that the obstruction to extending $\gamma_1\cup\gamma_2$ to the cross section is 0. (a)~(c) are described in π_1-level as follows;

(a) $\gamma_i(l_i, h_i)\gamma_i^{-1}=(l_i, h_i)C$,

(b) $\gamma_i^{-1}(\iota_i\gamma_i\iota_i^{-1})=l_i^{a_i}h_i^{b_i}$ where ι_i is the lift of the reflector along $\bar{\gamma}_i$ such that $\iota_i^2=l_i$, $\iota_i h_i\iota_i^{-1}=h_i^{-1}$,

(c) $(l_1, h_1)P=(l_2, h_2)$.

Here we note that $a_1=a_2=0$ if $C=I$, $a_1=a_2=-1$ if $C=-I$. Furthermore we recall that the lift δ of the curve $\bar{\delta}$ (Figure 2) can be chosen so that $\delta\iota_1=\iota_2$ (§1). Put $\gamma=\gamma_1=\gamma_2^{-1}$, $\iota=\iota_1$, $(l, h)=(l_1, h_1)$. Then $\pi_1\tilde{S}$ is generated by γ, δ, l, h and ι is the generator of the covering translation of $\tilde{S}$. Now we can describe the relations of $\pi_1 S$ in terms of γ, δ, l, h, ι:

(0) $\iota^2=l$, $\iota(l, h)\iota^{-1}=(l, h^{-1})$,

(1) $\gamma(l, h)\gamma^{-1}=(l, h)C$ with $C=\pm I$,

(2) $\delta(l, h)\delta^{-1}=(l, h)D$ with $D=PJP^{-1}J\left(J=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)$ which is derived from $\iota_2(l_2, h_2)\iota_2^{-1}=h_2^{-1}$ where $\iota_2=\delta\iota$, $(l_2, h_2)=(l, h)P$,

(3) $[\gamma, \delta]=(l, h)\left(CD\begin{pmatrix}a_1\\ b_1\end{pmatrix}+P\begin{pmatrix}a_2\\ b_2\end{pmatrix}\right)$ which is derived from (b) for $i=2$ which is equivalent to $\delta\iota\gamma^{-1}\iota^{-1}\delta^{-1}=\gamma^{-1}(l, h)P\begin{pmatrix}a_2\\ b_2\end{pmatrix}$ and (4) below,

(4) $\iota\gamma\iota^{-1}=\gamma l^{a_1}h^{b_1}$ which is equivalent to (b) for $i=1$,

(5) $\iota\delta\iota^{-1}=\delta^{-1}(l, h)(P-I)\begin{pmatrix}1\\0\end{pmatrix}$ derived from $(\delta\iota)^2=(l, h)P\begin{pmatrix}1\\0\end{pmatrix}$.

It is easy to see that (a), (b), (c) can be derived from (0)~(5), (1)~(3) define the relations of $\pi_1\tilde{S}$ and (0), (4), (5) indicate the action of ι on $\pi_1\tilde{S}$. Now we describe the classification of S with $B=A$. Put $P=\begin{pmatrix}p & r\\ q & s\end{pmatrix}$ with $ps-qr=1$. Then $D=PJP^{-1}J=\begin{pmatrix}1+2qr & 2pr\\ 2qs & 1+2qr\end{pmatrix}$ and hence $|\mathrm{tr}D|\geqq 2$ and we obtain the following list.

List III. $B=A$.

	C	D	P		type of geometry
3-1-1	I	$\begin{pmatrix}1 & 2\lambda\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}$	with $\lambda\neq 0, b_1+b_2\neq 0$	Nil^4
				with $\lambda\neq 0, b_1+b_2=0$	$Nil^3\times E$
3-1-2	I	$\begin{pmatrix}1 & 0\\ 2\lambda & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ \lambda & 1\end{pmatrix}$	with $\lambda\neq 0$ or $b_1+b_2\neq 0$	$Nil^3\times E$
				with $\lambda=b_1+b_2=0$	E^4
3-1-3	$-I$	$\begin{pmatrix}1 & 2\lambda\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}$	with $\lambda\neq 0$	$Nil^3\times E$
3-1-4	$-I$	$\begin{pmatrix}1 & 0\\ 2\lambda & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ \lambda & 1\end{pmatrix}$	with $\lambda\neq 0$	$Nil^3\times E$
				with $\lambda=0$	E^4
3-2-1	I	$\begin{pmatrix}-1 & 2\lambda\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}\lambda & 1\\ -1 & 0\end{pmatrix}$	with $\lambda\neq 0$	$Nil^3\times E$
				with $\lambda=0$	E^4
3-2-2	I	$\begin{pmatrix}-1 & 0\\ 2\lambda & -1\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ 1 & \lambda\end{pmatrix}$	with $\lambda\neq 0$	$Nil^3\times E$
3-2-3	$-I$	$\begin{pmatrix}-1 & 2\lambda\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}\lambda & 1\\ -1 & 0\end{pmatrix}$	with $\lambda\neq 0$	$Nil^3\times E$
				with $\lambda=0$	E^4
3-2-4	$-I$	$\begin{pmatrix}-1 & 0\\ 2\lambda & -1\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ 1 & \lambda\end{pmatrix}$	with $\lambda\neq 0$	$Nil^3\times E$
3-3	$\pm I$	$\|\mathrm{tr}D\|\geqq 3$	$ps\geqq 2$ or $\leqq -1$		$Sol^3\times E$

Case 2. $B=M$. First fix the representation $\pi_1^{\mathrm{orb}}M=\{\bar{\gamma}, \bar{\delta}, \bar{\iota}\,|\,\bar{\gamma}\bar{\delta}\bar{\gamma}^{-1}\bar{\delta}=\bar{\iota}^2=1, \bar{\iota}\bar{\gamma}\bar{\iota}^{-1}=\bar{\delta}\bar{\gamma}, \bar{\iota}\bar{\delta}\bar{\iota}^{-1}=\bar{\delta}^{-1}\}$ where $\bar{\gamma}$, $\bar{\delta}$ generate the fundamental group of the double covering K of M. Let l, h be the base of the general fiber and ι be the lift of the reflection $\bar{\iota}$ along the reflector circle such that $\iota^2=l$, $\iota(l, h)\iota^{-1}=(l, h^{-1})$. Fix the lift γ of $\bar{\gamma}$ so that the curve parallel to the lift of the reflector circle is represented by γ^2. Then S is determined by;

(a) the monodromy C along γ with $C\in GL_2\boldsymbol{Z}$, $\det C=-1$, $C^2=I$. (Note that $C^2=\pm I$ by the argument in §1 but C^2 cannot be $-I$.)

(b) the euler class for γ^2 ($\iota\gamma^2\iota^{-1}=\gamma^2 h^{b_1}$).

Define the lift δ of $\bar{\delta}$ by $\delta=\iota\gamma\iota^{-1}\gamma^{-1}$ then $\pi_1\tilde{S}$ for the induced fibration $\tilde{S}$ over K is generated by γ, δ, l, h and $\pi_1 S$ is represented as follows;

(0) $\iota^2=l$, $\iota(l, h)\iota^{-1}=(l, h^{-1})$,

(1) $\gamma(l, h)\gamma^{-1}=(l, h)C$, $\delta(l, h)\delta^{-1}=(l, h)D$ with $D=JCJC$, $C^2=I$,

(2) $\gamma\delta\gamma^{-1}\delta=(l, h)D^{-1}\begin{pmatrix}0\\ b_1\end{pmatrix}$ which is derived from (b) above,

(3) $\iota\gamma\iota^{-1}=\delta\gamma$, $\iota\delta\iota^{-1}=\delta^{-1}(l, h)D(I-C)\begin{pmatrix}1\\ 0\end{pmatrix}$ which is derived from (0)~(2). If we put $C=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ with $ad-bc=-1$ then $d=-a$ since $C^2=I$. Then $D=\begin{pmatrix}2a^2-1 & 2ab\\ -2ac & 2a^2-1\end{pmatrix}$ and hence $|\mathrm{tr}D|\geqq 2$. Thus we obtain the following list.

List IV. $B=M$.

	C	D		type of geometry
4-1-1	$\begin{pmatrix}1 & \lambda\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 2\lambda\\ 0 & 1\end{pmatrix}$	with $\lambda\neq 0, b_1\neq 0$	Nil^4
			with λ or $b_1=0$, $(\lambda, b_1)\neq(0, 0)$	$Nil^3\times E$
			with $\lambda=b_1=0$	E^4
4-1-2	$\begin{pmatrix}-1 & \lambda\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & -2\lambda\\ 0 & 1\end{pmatrix}$	with $\lambda\neq 0$	$Nil^3\times E$
			with $\lambda=0$	E^4
4-1-3	$\begin{pmatrix}1 & 0\\ \lambda & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ -2\lambda & 1\end{pmatrix}$	with $\lambda\neq 0$	$Nil^3\times E$
4-1-4	$\begin{pmatrix}-1 & 0\\ \lambda & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 2\lambda & 1\end{pmatrix}$	with $\lambda\neq 0$	$Nil^3\times E$
4-2	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$-I$		E^4
4-3	$\lvert a\rvert\geqq 2$	$\mathrm{tr}D\geqq 3$		$Sol^3\times E$

REMARK. The case with $C=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$, $D=-I$ can be reduced to the case with $C=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $D=-I$.

§7. Proof of Theorem A when the bases are euclidean—Part 2.

In this section we will prove the claims for the geometric structures in the lists in §6. In either case we give the faithful discrete representation ρ from $\pi_1 S$ to G_X^+ for some X such that ρ defines a structure on the double cover $\tilde{S}$ on which $\rho(\iota)$ acts as an isometric involution and hence induces the desired structure on S.

Case 1. $B=A$. We proceed according to List III in §6.

CLAIM 1. *S with* $C=I$, $D=\begin{pmatrix}1 & 2\lambda\\ 0 & 1\end{pmatrix}$ *admits a* Nil^4 *structure if* $\lambda\neq 0$, $b_1+b_2\neq 0$, *and a* $Nil^3\times I$ *structure if* $\lambda\neq 0$, $b_1+b_2=0$ (3-1-1). *S with* $C=-I$, $D=\begin{pmatrix}1 & 2\lambda\\ 0 & 1\end{pmatrix}$ $\lambda\neq 0$ *admits a* $Nil^3\times E$ *structure* (3-1-3).

PROOF. Let $(\boldsymbol{x}, t)=(x, y, z, t)$ be the coordinates of Nil^4 as before. Then in the first case define ρ by $\rho(\iota)(x, y, z, t)=(x+l_0/2, -y, z, -t)$ (and hence $\rho(l)$ $(x, y, z, t)=(x+l_0, y, z, t)$), $\rho(h)(x, y, z, t)=(x, y+h_0, z, t)$ with l_0, $h_0\neq 0$, $\rho(\gamma)(x, t)$ $=(x+a_0, y+b_0, z+1, t)$, $\rho(\delta)(x, t)=(C(1)\boldsymbol{x}+\bar{a}, t+1)$ where $C(t)$ is defined as in the proof of Claim 7 in §5 and $\bar{a}=(a_0', b_0', 0)$ and choose the parameters appropriately. If $b_1+b_2=0$ in the first case for the coordinates of $Nil^3\times E$ as before define ρ by $\rho(\iota)(x, y, z, w)=(-x, -y, z+1/2, w)$, $\rho(h)(x, y, z, w)=(x, y+1, z, w)$, $\rho(\gamma)(x, y, z, w)=(x, y+d, z, w+1)$, $\rho(\delta)(x, y, z, w)=(x+b_0, y, z+b_0 y, w)$ with $b_0\neq 0$ and choose the parameters appropriately. The proof for the last case is similar.

CLAIM 2. *S with* $C=I$, $D=\begin{pmatrix}1 & 0\\ 2\lambda & 1\end{pmatrix}$ *admits a* $Nil^3\times E$ *structure if* $\lambda\neq 0$ *or* $b_1+b_2\neq 0$ *and an* E^4 *structure if* $\lambda=b_1+b_2=0$ (3-1-2). *S with* $C=-I$, $D=\begin{pmatrix}1 & 0\\ 2\lambda & 1\end{pmatrix}$ *admits a* $Nil^3\times E$ *structure if* $\lambda\neq 0$ *and an* E^4 *structure if* $\lambda=0$ (3-1-4).

ADDENDUM 2. *S is diffeomorphic to the* T^2*-bundle over* T^2 *of the form* $\left\{\begin{pmatrix}-1 & \lambda\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & b_1+b_2\\ 0 & 1\end{pmatrix}, (b_1, 0)\right\}$ *in the first case and is diffeomorphic to the* T^2*-bundle over K of the form* $\left\{\begin{pmatrix}1 & 0\\ \lambda-b_1+b_2 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ \lambda & -1\end{pmatrix}, (0, b_1)\right\}$ *in the second case.*

PROOF. We can get the desired ρ by modifying that given in the proof of Claim 1. The proof for the E^4 cases is an easy modification of that for $Nil^3\times E$. To prove Addendum it suffices to give the isomorphism between the fundamental groups by the rigidity theorem. If we denote the standard curves of the T^2-bundle over T^2 by γ', δ', ι', h' then the correspondence in the first case is given by $\gamma'=\iota$, $\delta'=\gamma$, $(l', h')=(h, \delta)$. In the second case the correspondence $\gamma'=\gamma$, $\iota=\delta'$, $(\delta, h)=(l', h')$ for the standard curves γ', δ', l', h' of the T^2-bundle over K defines the isomorphism required in Addendum 2.

CLAIM 3. *S with* $C=\pm I$, $D=\begin{pmatrix}-1 & 2\lambda\\ 0 & -1\end{pmatrix}$ *or* $\begin{pmatrix}-1 & 0\\ 2\lambda & -1\end{pmatrix}$ *admits a* $Nil^3\times E$ *structure if* $\lambda\neq 0$ *and* E^4 *structure if* $\lambda=0$ (3-2-1~3-2-4).

PROOF. The proof in any case is similar to that given for Claim 1 or 2 so we omit them.

CLAIM 4. *S with* $C=\pm I$ *and* $|\mathrm{tr}D|\geqq 3$ *admits a* $Sol^3\times E$ *structure.*

PROOF. We note that $D=PJP^{-1}J$ for $P=\begin{pmatrix} p & r \\ q & s\end{pmatrix}\in SL_2\boldsymbol{Z}$, $J=\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$. First we claim that there is a matrix $Q\in GL_2\boldsymbol{R}$ such that $QDQ^{-1}=\begin{pmatrix} \varepsilon\exp t_0 & 0 \\ 0 & \varepsilon\exp(-t_0)\end{pmatrix}$ with $t_0\neq 0$ and $QJQ^{-1}=\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$ where $\varepsilon=1$ if $\mathrm{tr}D\geqq 3$ and $\varepsilon=-1$ if $\mathrm{tr}D\leqq -3$. To see this first put $Q=\begin{pmatrix} \xi & -\eta \\ \xi & \eta\end{pmatrix}$ with $\det Q=2\xi\eta\neq 0$. Then Q satisfies the second condition. Let $P'=QPQ^{-1}=\begin{pmatrix} p' & q' \\ r' & s'\end{pmatrix}$. Then QDQ^{-1} is a diagonal matrix if and only if $p'q'=s'r'$ which is equivalent by simple calculation to $\xi^2/\eta^2=qs/pr$. If $\mathrm{tr}D\geqq 3$ then $ps=qr+1\geqq 2$ and if $\mathrm{tr}D\leqq -3$ then $ps=qr+1\leqq -1$. In either case $qs/pr>0$ and hence there are $\xi, \eta\in\boldsymbol{R}$ satisfying the above conditions which prove the claim. First suppose that $C=I$. Let (x, y, t, u) be the coordinates of $Sol^3\times E$ as before and define ρ by $\rho(\iota)(x, y, t, u)=(y+\xi/2, x+\xi/2, -t, u)$, $\rho(h)(x, y, t, u)=(x-\eta, y+\eta, t, u)$, $\rho(\gamma)(x, y, t, u)=(x+x_1, y+y_1, t, u+1)$, $\rho(\delta)(x, y, t, u)=(\varepsilon\exp t_0\cdot x+x_0, \varepsilon\exp(-t_0)\cdot y+y_0, t+t_0, u)$ where ξ, η, ε, t_0 are defined above. Then from the relations in $\pi_1 S$ we deduce the following conditions.

(a) $\begin{pmatrix}(1-\varepsilon\exp t_0)x_1 \\ (1-\varepsilon\exp(-t_0))y_1\end{pmatrix}=\begin{pmatrix} \xi & -\eta \\ \xi & \eta\end{pmatrix}\left(D\begin{pmatrix} 0 \\ b_1\end{pmatrix}+P\begin{pmatrix} 0 \\ b_2\end{pmatrix}\right)$,

(b) $x_1-y_1=\eta b_1$,

(c) $\begin{pmatrix}\varepsilon\exp(-t_0)x_0+y_0 \\ x_0+(\varepsilon\exp t_0)y_0\end{pmatrix}=\begin{pmatrix}\varepsilon\exp(-t_0)(p'+r'-1)\xi+\xi(\varepsilon\exp(-t_0)-1)/2 \\ \varepsilon\exp t_0(q'+s'-1)\xi+\xi(\varepsilon\exp t_0-1)/2\end{pmatrix}$.

Here (x_1, y_1) is determined uniquely by (a). We claim that then (x_1, y_1) satisfies (b) automatically. Putting $\phi=\varepsilon\exp t_0$ and $P'=QPQ^{-1}=\begin{pmatrix} p' & r' \\ q' & s'\end{pmatrix}$ as before we deduce from (a) that $x_1-y_1=(1-\phi)^{-1}(-b_1\eta\phi+b_2\eta(r'-p'))-(1-\phi^{-1})^{-1}(b_1\eta\phi^{-1}+b_2\eta(s'-q'))$. Then $x_1-y_1=\eta b_1$ which follows from $\det P'=p's'-q'r'=1$, $p'q'=s'r'$ (see the above setting). Next we deduce (the first equality)$\times\varepsilon\exp t_0=$(the second one) in (c) from the same conditions on P'. Hence (x_0, y_0) can be well-determined so that it satisfies (c) and we obtain the desired representation. If $C=-I$ then ρ is defined similarly except for $\rho(\gamma)$ which is modified so that $\rho(\gamma)(x, y, t, u)=(-x+x_1, -y+y_1, t, u+1)$. Then by analogous calculation we obtain the desired result.

Case 2. $D=M$. First note that $\pi_1^{\mathrm{orb}}M$ has the representation to $\mathrm{Isom}\,E^2$ of the form $\bar{\gamma}:(z, t)\to(z+z_0, -t+t_0)$, $\bar{\delta}:(z, t)\to(z, t+t_1)$, $\bar{\iota}:(z, t)\to(z, -t)$ with $t_0+t_1/2=0$, $t_1\neq 0$, $z_0\neq 0$.

CLAIM 5. *S with* $C=\begin{pmatrix} 1 & \lambda \\ 0 & -1\end{pmatrix}$, $D=\begin{pmatrix} 1 & 2\lambda \\ 0 & 1\end{pmatrix}$ *admits a* Nil^4 *structure if* $\lambda\neq 0$, $b_1\neq 0$, *a* $Nil^3\times E$ *structure if* λ *or* $b_1=0$ *but* $(\lambda, b_1)\neq(0, 0)$, *an* E^4 *structure if* $\lambda=$

$b_1=0$ (4-1-1).

ADDENDUM 5. *If $\lambda=0$, S is diffeomorphic to a T^2-bundle over T^2 of the form* $\left\{\begin{pmatrix}-1 & b_1\\ 0 & -1\end{pmatrix}, -I, (0, -1)\right\}$.

PROOF. The proof is similar to those for the previous cases. For example if $\lambda\neq 0$, $b_1=0$ define ρ by $\rho(\iota)(x, y, z, w)=(-x, -y, z+1/2, w)$, $\rho(h)(x, y, z, w)=(x, y+1, z, w)$, $\rho(\gamma)(x, y, z, w)=(-x+x_0, -y+y_0, z-x_0y+z_0, w+1)$, $\rho(\delta)(x, y, z, w)=(x+x_1, y+y_1, z+x_1y+z_1, w)$ for appropriate parameters with $x_0+x_1/2=0$, $x_0\neq 0$. On the other hand if we put $l'=h$, $h'=\delta$, $\gamma'=\gamma$, $\delta'=\iota$ in the case with $\lambda=0$, we can see that $\{l', h', \gamma', \delta'\}$ give the generator of the fundamental group of the T^2-bundle over T^2 of the type required in Addendum (via the rigidity theorem).

CLAIM 6. *S with $C=\begin{pmatrix}-1 & \lambda\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0\\ \lambda & -1\end{pmatrix}$, or $\begin{pmatrix}-1 & 0\\ \lambda & 1\end{pmatrix}$ admits a $Nil^3\times E$ structure if $\lambda\neq 0$ and an E^4 structure if $\lambda=0$ (4-1-2～4-1-4). S admits an E^4 structure if $C=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $D=-I$ (4-2).*

PROOF. The proof goes similarly to those of the previous claims.

CLAIM 7. *S admits a $Sol^3\times E$ structure if* $\mathrm{tr}D\geqq 3$ (4-3).

PROOF. First note that in this case $C=\begin{pmatrix}a & b\\ c & -a\end{pmatrix}$ with $a^2+bc=1$, $|a|\geqq 2$ and $D=JCJC$. Then arguing as in the proof of Claim 4 we have a matrix $Q=\begin{pmatrix}\xi & -\eta\\ \xi & \eta\end{pmatrix}\in GL_2\boldsymbol{R}$ with $\det Q=2\xi\eta\neq 0$ such that $QJQ^{-1}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $C'=QCQ^{-1}=\begin{pmatrix}0 & \varepsilon\exp(-t_0)\\ \varepsilon\exp t_0 & 0\end{pmatrix}$ for some $t_0\neq 0$, $\varepsilon=\pm 1$, and hence $D'=QDQ^{-1}=\begin{pmatrix}\exp 2t_0 & 0\\ 0 & \exp(-2t_0)\end{pmatrix}$ (choose ξ, η satisfying $\xi^2/\eta^2=-c/b>0$). Then we define ρ for the coordinates (x, y, t, z) of $Sol^3\times E$ by $\rho(\iota)(x, y, t, z)=(y+\xi/2, x+\xi/2, -t, z)$ (note that there is an automorphism of the form $(x, y, t, z)\to(y, x, -t, z)$), $\rho(h)(x, y, t, z)=(x-\eta, y+\eta, t, z)$, $\rho(\gamma)(x, y, t, z)=(\varepsilon\exp(-t_0)y+x_0, \varepsilon\exp t_0\cdot x+y_0, -t-t_0, z+1)$, $\rho(\delta)(x, y, t, z)=(\exp(2t_0)x+x_1, \exp(-2t_0)y+y_1, t+2t_0, z)$. Then we deduce the following conditions from the relation of $\pi_1 S$;

(a) $D'^{-1}\begin{pmatrix}x_1-x_0\\ y_1-y_0\end{pmatrix}+\begin{pmatrix}x_0\\ y_0\end{pmatrix}+C'\begin{pmatrix}x_1\\ y_1\end{pmatrix}=D'^{-1}Q\begin{pmatrix}0\\ b_1\end{pmatrix}$,

(b) $y_0+(1-\varepsilon\exp t_0)\xi/2=x_0\exp 2t_0+x_1$, $x_0+(1-\varepsilon\exp(-t_0))\xi/2=y_0\exp(-2t_0)+y_1$,

(c) $D'^{-1}\begin{pmatrix}-\xi/2\\ -\xi/2\end{pmatrix}+\begin{pmatrix}\xi/2\\ \xi/2\end{pmatrix}+\begin{pmatrix}y_1\\ x_1\end{pmatrix}=D'^{-1}\left\{QD(I-C)\begin{pmatrix}1\\ 0\end{pmatrix}-\begin{pmatrix}x_1\\ y_1\end{pmatrix}\right\}$.

Here (c) is equivalent to $x_1+\exp(2t_0)y_1=\xi(1+\exp(2t_0)-2\varepsilon\exp t_0)/2$. Then it is easy to determine the parameters to get the desired result (for example put $x_0=0$ for simplicity).

Case 3. $|B|=D^2$. In this case S is represented as $\{(0, b), (m_i, 0, b_i)(i=1, \cdots, k), (m'_j, a'_j, b'_j)(j=1, \cdots, k')\}$ where $(0, b)$ is the euler class of the reflector circle, $(m_i, 0, b_i)$ is the type of the i-th multiple Klein bottle and (m'_j, a'_j, b'_j) be the type of the j-th multiple torus. (We always assume that the euler class $(a', b')=(0, 0)$ by choosing the cross sections appropriately (§1).) Let $e=(b+\sum b_i/m_i)/2+\sum b'_j/m'_j$ be the rational euler class of S.

CLAIM 8. *S admits a $Nil^3\times E$ structure if $e\neq 0$ and an E^4 structure if $e=0$.*

PROOF. For the case with $e\neq 0$ we seek a representation ρ using the coordinates (w, z) of $Nil^3\times E$. Let $\tilde{B}$ be the standard covering of B without reflectors. Fix the representation of $\pi_1^{\mathrm{orb}}\tilde{B}$ to $\mathrm{Isom}E^2$ so that we obtain the geometric realization of B as the quotient of $\tilde{B}$ by an involution $\bar{\iota}$ of the form $z\to\bar{z}$ for $z\in \boldsymbol{C}$. Then ρ is defined by $\rho(\iota)(w, z)=(-\bar{w}+il_0/2, \bar{z})$, $\rho(l)(w, z)=(w+il_0, z)$, $\rho(h)(w, z)=(w+h_0, z)$ with $l_0, h_0\in\boldsymbol{R}$, $l_0, h_0\neq 0$ where l and h are the base curves for the general fiber and ι is the lift of $\bar{\iota}$ satisfying $\iota^2=l$ as before and further define $\rho(q_j)$ for the lift q_j of the rotation $\bar{q}_j$ (which generate $\pi_1\tilde{B}$) by $\rho(q_j)(w, z)=(w+w_j+i\bar{z}_j(z-\gamma(z-z_j)), \gamma(z-z_j)+z_j)$ where $\bar{q}_j$ is represented by the map $z\to\gamma(z-z_j)+z_j$ with $\gamma=\exp(2\pi i/m)$ for some $m\in\boldsymbol{Z}$. Then we proceed as in the proof of Claim 10 in §5 to obtain the desired result. We can easily modify the representation in each case to give the result for the case with $e=0$.

§8. The euclidean cases—a comparison to Calabi construction.

There is a method called Calabi construction which describes any closed euclidean manifold S with $b_1S\geqq 1$ in terms of the euclidean manifolds of lower dimensions ([4], [**20**]). All the closed orientable euclidean 4-manifolds can be classified by this method since we have always $b_1\geqq 1$ for such manifolds ([**20**]). In this section we reformulate this construction from the viewpoint of Seifert 4-manifolds (the claim for E^4 case in Theorem B).

PROPOSITION 3. *Any closed orientable euclidean 4-manifold is a Seifert 4-manifold over some euclidean 2-orbifold with only one exception. The exceptional case is described as a Seifert manifold over a euclidean 3-orbifold with general fiber S^1.*

PROOF. First consider the following data: a closed orientable flat manifold N of dimension $4-q$ $(q\geqq 1)$, a finite abelian group $\Delta=\boldsymbol{Z}_{r_1}\times\cdots\times\boldsymbol{Z}_{r_q}$ where r_i may be 1, a homomorphism $\delta\colon \Delta\to T^q=S^1\times\cdots\times S^1$ which is the product of the standard inclusion $Z_{r_i}\to S^1$, a Δ-action on T^q defined by $x\to\delta(g)x$ for $g\in\Delta$, $x\in T^q$, and an orientation preserving affine action of Δ on N satisfying

(*) there is no nontrivial Δ-invariant parallel vector field on N.

Then the orbit space $N\underset{\Delta}{\times}T^q$ of the diagonal action of Δ on $N\times T^q$ where the action on each factor is described above is a closed orientable euclidean 4-manifold with $b_1=q$. Conversely any such 4-manifold S can be described in this way ([**20**]). Here we note that the Δ-action on N can be assumed to be isometric since any finite smooth action on a euclidean 3-manifold preserves the geometric structure ([**9**]). If $b_1=4$ then $S=T^4$. Then case with $b_1=3$ cannot occur since there is no Δ-action on S^1 satisfying (*). If $b_1=2$ then $N=T^2$ and $\Delta=Z_{r_1}\times Z_{r_2}$. Then the natural projection $p: T^2\underset{\Delta}{\times}T^2\to T^2/\Delta$ induced by the projection to the first factor defines a Seifert 4-manifold over the euclidean 2-orbifold T^2/Δ. If $b_1=1$ we also consider the projection $p: N\underset{\Delta}{\times}S^1\to N/\Delta$ where $\Delta=Z_r$ for some $r\in Z$. Here N is a closed orientable euclidean 3-manifold which is a Seifert fibered space ([**10**], [**20**]) and N/Δ is a euclidean 3-orbifold ([**2**], [**14**]). If the Δ-action on N preserves some fibration of N then N/Δ has the induced fibration and hence is a Seifert 3-orbifold whose general fiber is S^1. The above condition does not hold exactly when the holonomy group of N is $Z_2\times Z_2$ (in this case N is a Seifert fibration over $P^2(2, 2)$) and $\Delta=Z_3$ whose generator acts on N by rotating the three axes of the nontrivial holonomies. In this case the underlying space of N/Δ is S^3 and the singular set of N/Δ is a figure eight knot whose cone angle is $2\pi/3$ (cf. [**2**], [**14**]). The classification of the euclidean 3-orbifolds ([**2**], [**5**]) shows that the above case is the unique non-fibered orbifold which is obtained from the euclidean 3-manifold divided by the cyclic action. Hence in the remaining cases the composition of p and the projection of the Seifert fibration N/Δ to some euclidean 2-orbifold gives the desired structure. It is easy to see that in these remaining cases the holonomy groups are either cyclic or dihedral whereas the holonomy group of the exceptional case is the tetrahedral group. On the other hand the holonomy group of a euclidean Seifert 4-manifold over some orientable 2-orbifold is cyclic of order 1, 2, 3, 4, or 6. (They are T^4 and the hyperelliptic surfaces.) If a euclidean Seifert 4-manifold S has a non-orientable base orbifold B then B has an orientable double cover $\tilde{B}$ (note that if $B=M$ (the Moebius band) then we choose the representation of $\pi_1^{orb}M$ of the form $\{\alpha', \beta, \iota \mid \iota^2=1, \iota\alpha'\iota^{-1}=\beta\alpha', \iota\beta\iota^{-1}=\beta^{-1}, [\alpha', \beta]=1\}$ and take the double cover $\tilde{M}$ corresponding to the subgroup generated by α' and β). Then the Seifert 4-manifold $\tilde{S}$ over $\tilde{B}$ induced by the projection from $\tilde{B}$ to B is an unbranched double cover of S and is again euclidean. Thus we have an exact sequence $1\to\Gamma_0\to\pi_1\tilde{S}\to Z_r\to 1$ where Γ_0 is a maximal normal free abelian subgroup of rank 4 in $\pi_1\tilde{S}$ and Z_r is the holonomy group of $\tilde{S}$ (a cyclic group of order $\leqq 6$). Then the quotient G of $\pi_1 S$ by the subgroup Γ_0 ($\pi_1\tilde{S}$ is a subgroup of $\pi_1 S$ of index 2) is the extension of $\boldsymbol{Z}_r$ by $\boldsymbol{Z}_2 (1\to\boldsymbol{Z}_r\to G\to\boldsymbol{Z}_2\to 1)$. Since the maximal normal free abelian subgroup of rank 4 in $\pi_1 S$ (the translation parts) is unique it contains Γ_0 and then the holonomy group G' of $\pi_1 S$ is some quotient of G (and hence the order of $G'\leqq 12$). Thus we can see that G' cannot be a tetrahedral group and the unique exceptional case is certainly not diffeomorphic to a Seifert 4-manifold in our sense.

§ 9. A remark concerning complex structures.

The Enriques-Kodaira classification ([**1**], Chapter VI, table 10) shows that there are just 8 complex surfaces with euclidean structures. Any of them is diffeomorphic to a T^2-bundle over T^2 and just 7 classes among them have alternative fibering over the euclidean 2-orbifolds of genus 0 (hyperelliptic surfaces). The complex surfaces of type $Nil^3\times E$ are called Kodaira surfaces ([**1**]). A Seifert 4-manifold S over a euclidean 2-orbifold B is diffeomorphic to a primary Kodaira surface if $B=T^2$ with $b_1(S)=3$ and diffeomorphic to a secondary Kodaira surface if B is of genus 0 and the rational euler class is nonzero. (But some cases have some alternative fibrations over K. cf. [**16**].) We note that in either case there is a representation of $\pi_1 S$ to G_X^0 with $X=Nil^3\times E$ and hence S has a compatible complex structure ([**18**], Theorem 1.1). We can also see that if a Seifert 4-manifold S over a euclidean 2-orbifold is homeomorphic to a complex surface then S is diffeomorphic to one of the above cases from the classification of the T^2-bundles over T^2 by passing to some finite unbranched coverings. For example there are many Seifert 4-manifolds with $b_1=1$, $b_2=0$ in our classes but none of them is homeomorphic to a complex surface other than a secondary Kodaira surface.

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