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The exterior non-stationary problem for the Navier-Stokes equations in regions with moving boundaries

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1. Introduction.

We consider the motion of a viscous incompressible fluid in an exterior domain with moving boundaries, in other words we have to deal not with a space-time cylinder but with a noncylindrical domain in $R^3\times[0, T]$. To be more precise, we consider a domain

$$\Omega_T=\bigcup_{0\leq t\leq T}\Omega(t)\times\{t\}$$

where each $\Omega(t)$ is the exterior of a bounded connected domain $\Omega^c(t)$ in R^3, and $T>0$ is a finite number.

The exterior problem for the Navier-Stokes equations consists of finding in the region Ω_T exterior to a closed bounded surface, the velocity u and the pressure p which together solve the system (1.1) given below, and are such that the velocity assumes a given value on the surface, for $|x|\rightarrow\infty$, and in $t=0$.

The motion of the fluid in Ω_T is governed by the following equations

$$\partial_t u-\mu\Delta u+u\cdot\nabla u=f-\nabla p\,,\qquad \nabla\cdot u=0\qquad \text{in } \Omega_T \tag{1.1}$$

where $\partial_t=\partial/\partial t$, $u=u(t)=(u_1(x, t), u_2(x, t), u_3(x, t))$ is the velocity, $p=p(t)=p(x, t)$ the pressure, $f=f(t)=(f_1(x, t), f_2(x, t), f_3(x, t))$ the external force, and μ the viscosity. We take the motion of the fluid at $t=0$ to be known, hence $u(x, 0)=u_0$ is a prescribed vector field in $\Omega(0)$. Let

$$\Gamma_T=\bigcup_{0\leq t\leq T}\Gamma(t)\times\{t\}$$

where $\Gamma(t)$ is the boundary of $\Omega^c(t)$. Throughout the paper we suppose that Γ_T is smooth enough and $\Omega(t)$ does not change its topological type as t increases over $[0, T]$. The classical formulation of the problem is the velocity u and the pressure p to satisfy (1.1) and the initial boundary conditions

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$$u(x,t)=b(x,t) \quad \text{on } \Gamma_T$$
(1.2) $$u(x,0)=u_0 \quad \text{in } \Omega(0)$$
$$u(x,t)-b_\infty(t)\to 0 \quad \text{as } |x|\to\infty.$$

In the cylindrical case i.e. $\Omega(t)=\Omega(0)=\Omega$ for all $t>0$ there exists a very extensive literature (see [2] for a bibliography). For the non cylindrical case the theory is much less developped. When $\Omega(t)$ is a bounded domain results are given in [1], [7], [13], [14], [15], [16], [17], [18], [19], [20], [23], [24].

This paper concerns global existence and local-in-time regularity of weak solutions (Hopf's solutions). To prove this we employ the method of the elliptic regularization used in [17] and improved in [20]. For this method also see [10]. Furthermore we prove that weak solutions satisfy the energy inequality as in the case of bounded domains. Note that a proof of this inequality is given in [2], and [21] with additional conditions on the data, and in [12] with no additional assumptions on the data, in cylindrical domains.

Section 2 is devoted to the preliminaries. In Section 3 the initial boundary value problem is posed. Section 4 contains the proof of the existence of weak solutions and of the energy inequality. Section 5 contains the main results on the regularity.

2. Preliminaries.

Throughout this paper $\Omega(t)$ represents a spatial region filled with the fluid and is taken to be an open set of R^3 with bounded connected complement $\Omega^c(t)$ (dependent on t) and $\Gamma(t)$ is the boundary of $\Omega^c(t)$. All functions in this paper are R or R^3-valued. The letter c denotes a constant depending on Ω_T. We employ the usual notations of vector analysis; in particular the jth components of $u\nabla u$ and Δu are $\sum_{i=1}^3 u_i\partial_{x_i}u_j$ and $\sum_{i=1}^3 \partial^2_{x_ix_i}u_j$ respectively. Some additional notation is needed. We let

$$(u,v)_{\Omega(t)}=\sum_{i=1}^3\int_{\Omega(t)}u_iv_i\,dx\,; \qquad |u|^2_{\Omega(t)}=(u,u)_{\Omega(t)}\,;$$

$$((u,v))_{\Omega(t)}=\sum_{i=1}^3\int_{\Omega(t)}\nabla u_i\nabla v_i\,dx\,; \qquad \|u\|^2_{\Omega(t)}=((u,u))_{\Omega(t)}\,;$$

$$(u\cdot\nabla v,w)_{\Omega(t)}=\sum_{i,j=1}^3\int_{\Omega(t)}u_j\partial_{x_j}v_iw_i\,dx\,;$$

$$|u|^2_{\Omega_T}=\int_0^T|u|^2_{\Omega(t)}dt\,; \qquad \|u\|^2_{\Omega_T}=\int_0^T\|u\|^2_{\Omega(t)}dt\,;$$

$$D(\Omega(t))=\{\varphi\mid\varphi\in(C_0^\infty(\Omega(t)))^3,\ \nabla\cdot\varphi=0\}\,;$$

$$D(\Omega_T)=\{\varphi\mid\varphi\in(C^\infty(\Omega_T))^3,\ \operatorname{supp}\varphi\subset\Omega_T,\ \nabla\cdot\varphi=0\}\,;$$

$H(\Omega(t))=$ completion of $D(\Omega(t))$ in the norm $|u|_{\Omega(t)}$;

$$V(\Omega(t)) = \text{completion of } D(\Omega(t)) \text{ in the norm } \|u\|_{\Omega(t)};$$

$$H(\Omega_T) = \text{completion of } D(\Omega_T) \text{ in the norm } |u|_{\Omega_T};$$

$$V(\Omega_T) = \text{completion of } D(\Omega_T) \text{ in the norm } \|u\|_{\Omega_T}.$$

The following lemmas are well known.

LEMMA 1. *For any domain $\Omega \subset R^3$, functions in $H_0^1(\Omega)$ satisfy the Sobolev inequality*

$$\|u\|_{L^4(\Omega)}^4 \leqq 3^{-3/2} |u|_\Omega \|u\|_\Omega^3$$

(in the following $H^s(\Omega)$ denotes the usual Sobolev space of order s on $L^2(\Omega)$).

LEMMA 2. *For any domain Ω and y in R^3, functions in $V(\Omega)$ satisfy*

$$\int_0^T |u(x)/(x-y)|^2 dx \leqq 4\|u\|^2.$$

We assume in the present paper that near $(x_0, t_0) \in \Gamma_T$ the boundary Γ_T is expressed as

$$x_3 = \psi(x_1, x_2, t), \qquad 0 \leqq t \leqq T$$

by translation and rotation of coordinates if necessary, and

(2.1) $\partial_t^h \nabla^k \psi(x_1, x_2, t)$, $(h+k \leqq 3)$ is continuous near (x_0, t_0).

Since Γ_T is compact we have uniformity of bound on $|\partial_t^h \nabla^k \psi(x_1, x_2, t)|$ near (x_0, t_0) ($h \geqq 0$, $k \geqq 0$ are integer).

Now we define the Stokes operator A. A is the Friedrichs extension of the symmetric operator $-P\Delta$ in $H(\Omega)$ for every $\varphi \in H^2(\Omega) \cap V(\Omega)$ and P is the projection operator from $L^2(\Omega)$ into $H(\Omega)$. $D(A)$ denotes the domain of A. We can see the operator A more explicitly from the following proposition (see Lemma 1 in [**6**], and Proposition 1 in [**11**]).

PROPOSITION 1. *Let Ω be an open set in R^3 the boundary Γ of which is smooth (at least uniformly of class C^3) and Ω^c is bounded. Suppose $u \in V(\Omega) \cap H^1(\Omega)$ is a solution of the Stokes equations*

$$-\Delta u + \nabla p = f \qquad i.e. \qquad ((u, \varphi))_\Omega = (f, \varphi)_\Omega$$

holds for all $\varphi \in D(\Omega)$.

Then u possesses second derivatives in $L^2(\Omega)$ and the inequalities

$$|D^2 u|_\Omega \leqq m_{1\Gamma}(|Pf|_\Omega + \|u\|_\Omega),$$

$$\|\nabla u\|_{L^3(\Omega)} \leqq m_{2\Gamma}(|Pf|_\Omega^{1/2} \|u\|_\Omega^{1/2} + \|u\|_\Omega),$$

$$\|u\|_{L^\infty(\Omega)} \leqq m_{3\Gamma}(|Pf|_\Omega^{1/2} \|u\|_\Omega^{1/2} + \|u\|_\Omega^{1/2} |u|_\Omega^{1/2});$$

hold with constants $m_{i\Gamma}$ dependent only on the regularity of Γ (not on the size of Γ).

We remark if Γ satisfies our assumptions with respect to (x, t) we can consider $m_{i\Gamma}$ independent of t.

3. The initial boundary value problem.

We shall pose the initial boundary value problem for the Navier-Stokes equations in a general form to permit the study of the flow exterior to a non rigid body which may undergo acceleration. In other words the region occupied by the fluid may be time dependent, not only, but the equations cannot be written in a coordinate frame attached to the body without being completely modified.

We shall consider system (1.1), (1.2) assuming that the data $(\bar{b}, b_\infty)$ can be extended continuously into Ω_T as a solenoidal function b which satisfies

i) $b \in L^\infty(\Omega_T) \cap L^2(\Omega(t))$;

ii) $\partial_t b + b\cdot\nabla b - \mu\Delta b \in L^2(\Omega_T)$.

Throughout the paper we set $\mu=1$, and $g=\partial_t b+b\cdot\nabla b-\Delta b-f$. We assume $f\in L^2(\Omega_T)\cap L^{5/3}(\Omega_T)$. Notice, in the following, we need $f\in L^2(\Omega_T)\cap L^{2-\varepsilon}(\Omega_T)$ with ε any positive number, and for simplicity we put $2-\varepsilon=5/3$.

Now we are in the position to give the definition of weak solutions.

u is a weak solution of (1.1), (1.2) if $u=v+b$ and b satisfies the following conditions

i) $v \in L^2(0, T; V(\Omega(t))) \cap L^\infty(0, T; H(\Omega(t)))$;

ii) $$\int_0^T \{(v, \partial_t\varphi)_{\Omega(t)}+(v\cdot\nabla\varphi, v)_{\Omega(t)}+(b, v\cdot\nabla\varphi)_{\Omega(t)}+(v, b\cdot\nabla\varphi)_{\Omega(t)}-((v, \varphi))_{\Omega(t)}$$
$$-(g, \varphi)_{\Omega(t)}\}\,dt = -(v_0, \varphi(0))_{\Omega(0)}, \qquad \varphi\in D(\Omega_T) \quad \text{with } \varphi(T)=0;$$

iii) $$|v(t)|^2_{\Omega(t)}+2\int_s^t \|v\|^2_{\Omega(\sigma)}d\sigma \leqq |v|^2_{\Omega(s)}-2\int_s^t (v\cdot\nabla b+g, v)_{\Omega(\sigma)}d\sigma;$$

holds for almost all $s>0$ including $s=0$ and all $t>s$.

Our results are now given by

THEOREM 1. *Let $v_0\in H(\Omega(0))$ and $g\in L^2(\Omega_T)$. Furthermore* (2.1) *holds with $h=0$ and $k=1$. Then there exists a weak solution of* (1.1), (1.2).

THEOREM 2. *Let $v_0\in H^1(\Omega(0))$, $g\in L^2(\Omega_T)$, $\nabla b\in L^\infty(\Omega_T)$, u a weak solution of* (1.1), (1.2), *and* (2.1) *holds. Then there exists a $\bar{T}>0$ ($\bar{T}\leqq T$) such that*

i) $v \in L^2(0, \bar{T}; H^2(\Omega(t))\cap V(\Omega(t)))\cap L^\infty(0, \bar{T}; V(\Omega(t))\cap H^1(\Omega(t)))$;

$P\partial_t v \in L^2(\Omega_{\bar{T}})$;

ii) *v satisfies the equations*

$$P(\partial_t v-\Delta v+v\cdot\nabla v+b\cdot\nabla v+v\cdot\nabla b+g)=0 \qquad a.e.\ in\ \Omega_{\bar T};$$

$$|v|^2_{\Omega(t)}+2\int_s^t\|v\|^2_{\Omega(\sigma)}=|v(s)|^2_{\Omega(s)}-2\int_s^t(v\cdot\nabla b+g,\ v)_{\Omega(\sigma)}d\sigma$$

for all $t>s$ *with* $t,\ s<\bar T$;

iii) v *is unique.*

4. Proof of Theorem 1.

We consider the following auxiliary problem. We look for $v^{m,\varepsilon}$ such that $\forall\varphi\in H^1(\Omega_T)\cap V(\Omega_T)$

$$\int_0^T\{(1/m)(\partial_t v^{m,\varepsilon},\partial_t\varphi)_{\Omega(t)}+((v^{m,\varepsilon},\varphi))_{\Omega(t)}+(\exp kt)(\tilde v^{m,\varepsilon}\cdot\nabla v^{m,\varepsilon},\ \varphi)_{\Omega(t)}+(v^{m,\varepsilon}\cdot\nabla b,\ \varphi)_{\Omega(t)}$$
$$+(b\cdot\nabla v^{m,\varepsilon},\ \varphi)_{\Omega(t)}+k(v^{m,\varepsilon},\ \varphi)_{\Omega(t)}-(v^{m,\varepsilon},\ \partial_t\varphi)_{\Omega(t)}\}\,dt+(v^{m,\varepsilon}(T),\ \varphi(T))_{\Omega(T)}$$
$$=-\int_0^T\exp(-kt)(g,\ \varphi)_{\Omega(t)}dt+(v_0,\ \varphi(0))_{\Omega(0)}$$

holds where $\tilde v^{m,\varepsilon}$ is a regularization of $v^{m,\varepsilon}$ by using a space-mollifier depending on ε. We set

$$a_{\Omega_T}(v^{m,\varepsilon},\ \varphi)=\int_0^T\{(1/m)(\partial_t v^{m,\varepsilon},\ \partial_t\varphi)_{\Omega(t)}+((v^{m,\varepsilon},\ \varphi))_{\Omega(t)}+k(v^{m,\varepsilon},\ \varphi)_{\Omega(t)}$$
$$+(\exp kt)(\tilde v^{m,\varepsilon}\cdot\nabla v^{m,\varepsilon},\ \varphi)_{\Omega(t)}+(b\cdot\nabla v^{m,\varepsilon},\ \varphi)_{\Omega(t)}$$
$$+(v^{m,\varepsilon}\cdot\nabla b,\ \varphi)_{\Omega(t)}-(v^{m,\varepsilon},\ \partial_t\varphi)_{\Omega(t)}\}\,dt+(v^{m,\varepsilon}(T),\ \varphi(T))_{\Omega(T)};$$

$$L_{\Omega_T}(\varphi)=-\int_0^T(\exp(-kt)(g,\ \varphi))_{\Omega(t)}dt+(v_0,\ \varphi(0))_{\Omega(0)}.$$

By the following well known theorem (see [3], page 106) one obtains the existence of a solution in $H^1(\Omega_T)\cap V(\Omega_T)$ of the equation

$$a_{\Omega_T}(v^{m,\varepsilon},\ \varphi)=L_{\Omega_T}(\varphi). \tag{4.1}$$

THEOREM 3. *If* i) *there exists a constant* $c>0$ *such that*

$$a_{\Omega_T}(v^{m,\varepsilon},\ v^{m,\varepsilon})\geqq c\|v^{m,\varepsilon}\|^2_{H^1(\Omega_T)};$$

ii) *the form* $v^{m,\varepsilon}\to a_{\Omega_T}(v^{m,\varepsilon},\ \varphi)$ *is weakly continuous in* $H^1(\Omega_T)\cap V(\Omega_T)$ *i.e.* $v_n^{m,\varepsilon}\to v^{m,\varepsilon}$ *weakly in* $H^1(\Omega_T)\cap V(\Omega_T)$ *implies*

$$\lim_{n\to\infty}a_{\Omega_T}(v_n^{m,\varepsilon},\ \varphi)=a_{\Omega_T}(v^{m,\varepsilon},\ \varphi).$$

Then (4.1) *has a solution in* $H^1(\Omega_T)\cap V(\Omega_T)$.

The condition i) can be easily proved; in fact

$$a_{\Omega_T}(v^{m,\varepsilon},\ v^{m,\varepsilon})\geqq\int_0^T\{(1/m)|\partial_t v^{m,\varepsilon}|^2_{\Omega(t)}-(1/2)\|b\|^2_{L^\infty(\Omega_T)}|v^{m,\varepsilon}|^2_{\Omega(t)}+(1/2)\|v^{m,\varepsilon}\|^2_{\Omega(t)}$$

$$+k|v^{m,\varepsilon}|^2_{\Omega(t)}\}dt+(1/2)|v^{m,\varepsilon}(T)|^2_{\Omega(T)}$$
$$+(1/2)|v^{m,\varepsilon}(0)|^2_{\Omega(0)}\geqq c\|v^{m,\varepsilon}\|^2{}_{H^1(\Omega_T)\cap V(\Omega_T)}$$

(for suitable k); hence i) holds.

For ii) we consider $v_n^{m,\varepsilon}\to v^{m,\varepsilon}$ weakly in $H^1(\Omega_T)\cap V(\Omega_T)$. We need to prove the convergence of the non linear term. We note that $v_n^{m,\varepsilon}\to v^{m,\varepsilon}$ strongly in $L^2_{\text{loc}}(\Omega_T)$, then

$$\tilde{v}_n^{m,\varepsilon}\cdot\nabla v_n^{m,\varepsilon}\longrightarrow\tilde{v}^{m,\varepsilon}\cdot\nabla v^{m,\varepsilon}\qquad\text{in the distributions sense.}$$

Now

$$\tilde{v}_n^{m,\varepsilon}\cdot\nabla v_n^{m,\varepsilon}\longrightarrow\beta^{m,\varepsilon}\qquad\text{weakly in } L^{4/3}(\Omega_T),$$

consequently

$$\beta^{m,\varepsilon}=\tilde{v}^{m,\varepsilon}\cdot\nabla v^{m,\varepsilon}.$$

So

$$\lim_{n\to\infty}\int_0^T(\tilde{v}_n^{m,\varepsilon}\cdot\nabla v_n^{m,\varepsilon},\varphi)_{\Omega(t)}dt=\int_0^T(\tilde{v}^{m,\varepsilon}\cdot\nabla v^{m,\varepsilon},\varphi)_{\Omega(t)}dt$$

$\forall\varphi\in H^1(\Omega_T)\cap V(\Omega_T)$, hence

$$a_{\Omega_T}(v_n^{m,\varepsilon},\varphi)\longrightarrow a_{\Omega_T}(v^{m,\varepsilon},\varphi)\qquad\forall\varphi\in H^1(\Omega_T)\cap V(\Omega_T).$$

Then there exists a solution of (4.1).

To passing to the limit in (4.1) we will need a priori estimates of the approximations $v^{m,\varepsilon}$. To do this, we replace in (4.1) φ by $v^{m,\varepsilon}$, it comes

$$\int_0^T\{(1/m)|\partial_t v^{m,\varepsilon}|^2_{\Omega(t)}+(1/2)\|v^{m,\varepsilon}\|^2_{\Omega(t)}+k|v^{m,\varepsilon}|^2_{\Omega(t)}-(1/2)\|b\|^2_{L^\infty(\Omega_T)}|v^{m,\varepsilon}|^2_{\Omega(t)}$$
$$-(v^{m,\varepsilon},\partial_t v^{m,\varepsilon})_{\Omega(t)}+(\exp(-kt))(g,v^{m,\varepsilon})_{\Omega(t)}\}dt\leqq(v_0^m,v^{m,\varepsilon}(0))_{\Omega(0)}-|v^{m,\varepsilon}(T)|^2{}_{\Omega(T)}.$$

After some calculations, one has

$$(4.2)\qquad\begin{aligned}&(1/m)\int_0^T|\partial_t v^{m,\varepsilon}|^2_{\Omega(t)}dt\leqq c;\\&\int_0^T\|v^{m,\varepsilon}\|^2_{\Omega(t)}dt\leqq c;\qquad\int_0^T|v^{m,\varepsilon}|^2_{\Omega(t)}dt\leqq c;\\&|v^{m,\varepsilon}(T)|^2_{\Omega(T)}\leqq c;\qquad|v^{m,\varepsilon}(0)|^2_{\Omega(0)}\leqq c\end{aligned}$$

(the constant in (4.2) is independent of m and ε).

It follows

$$(4.3)\qquad v^{m,\varepsilon}\longrightarrow v^{\varepsilon}\qquad\text{weakly in } V(\Omega_T)\cap H^1(\Omega_T).$$

To passing to the limit with respect to m (and after with respect to ε) in (4.1) we need the convergence of $\{v^{m,\varepsilon}\}$ in a suitable topology e.g. in $L^2(0,T;L^2_{\text{loc}}(\Omega(t)))$. For this we shall prove a time difference quotients estimates.

We denote by $\bar{v}^{m,\varepsilon}$ the natural extension, by zero, to R^3 of $v^{m,\varepsilon}(x,t)$ for every $t\in[0,T]$; moreover we put $v^{m,\varepsilon}=0$ for $t<0$ and for $t>T$. We let

$$v_h^{m,\varepsilon}=(1/h)\int_{t-h}^{t}\bar{v}^{m,\varepsilon}(x,s)ds \qquad (h>0).$$

Let $w_h^{m,\varepsilon}$ be the solution of the system

$$(4.4)\qquad \begin{aligned} &-\Delta w_h^{m,\varepsilon}+\lambda w_h^{m,\varepsilon}+\nabla q=0 && \\ &\nabla\cdot w_h^{m,\varepsilon}=0 && \text{in } \Omega(t)\\ &w_h^{m,\varepsilon}=v_h^{m,\varepsilon} && \text{on } \Gamma(t)\\ &w_h^{m,\varepsilon}\longrightarrow 0 && \text{as } |x|\to\infty. \end{aligned}$$

Here λ is an arbitrary positive number.

For the estimates of $\partial_t w_h^{m,\varepsilon}$, we formally differentiate (4.4) with respect to t, and we consider $\partial_t w_h^{m,\varepsilon}$ as a generalized solution of the problem

$$\begin{aligned} &-\Delta\partial_t w_h^{m,\varepsilon}+\lambda\partial_t w_h^{m,\varepsilon}+\nabla\partial_t q=0 && \\ &\nabla\cdot\partial_t w_h^{m,\varepsilon}=0 && \text{in } \Omega(t)\\ &\partial_t w_h^{m,\varepsilon}=-\bar{v}^{m,\varepsilon}(t-h)/h && \text{on } \Gamma(t)\\ &\partial_t w_h^{m,\varepsilon}\longrightarrow 0 && \text{as } |x|\to\infty. \end{aligned}$$

We need $w_h^{m,\varepsilon}\in H^1(\Omega(t))$ and $\partial_t w_h^{m,\varepsilon}\in L^2(\Omega(t))$.

From standard results (see [8] or [22]), and bearing in mind $\bar{v}^{m,\varepsilon}(t-h)=0$ on $\Gamma(t-h)$, we have

$$\|w_h^{m,\varepsilon}\|_{H^1(\Omega(t))}\leqq c\left\|(1/h)\int_{t-h}^{t}v^{m,\varepsilon}ds\right\|_{H^{1/2}(\Gamma(t))}\leqq c(1/\sqrt{h})\left\|\int_{t-h}^{t}v^{m,\varepsilon}ds\right\|_{H^1(\Omega^c(t))},$$

and

$$\begin{aligned}|\partial_t w_h^{m,\varepsilon}|^2_{\Omega(t)}&\leqq(c/h^2)\|\bar{v}^{m,\varepsilon}(t-h)\|^2_{H^{1/2}(\Gamma(t))}\\ &\leqq(c/h^2)(\text{measure}(\Omega^c(t)-\Omega^c(t-h))\|\bar{v}^{m,\varepsilon}(t-h)\|^2_{R^3}\leqq(c/h)\|\bar{v}^{m,\varepsilon}\|^2_{R^3}.\end{aligned}$$

Now we can replace in (4.1) φ by $v_h^{m,\varepsilon}-w_h^{m,\varepsilon}$ and we get

$$\begin{aligned}&\int_0^T\{(1/m)(\partial_t v^{m,\varepsilon},(\bar{v}^{m,\varepsilon}(t)-\bar{v}^{m,\varepsilon}(t-h))/h-\partial_t w_h^{m,\varepsilon})_{\Omega(t)}-(1/h)(v^{m,\varepsilon}(t),\bar{v}^{m,\varepsilon}(t)\\ &\quad-\bar{v}^{m,\varepsilon}(t-h))_{\Omega(t)}+(v^{m,\varepsilon},\partial_t w_h^{m,\varepsilon})_{\Omega(t)}+(\exp kt)(\tilde{v}^{m,\varepsilon}\cdot\nabla v_h^{m,\varepsilon},v_h^{m,\varepsilon}-w_h^{m,\varepsilon})_{\Omega(t)}\\ &\quad+(\exp(-kt))(g,v_h^{m,\varepsilon}-w_h^{m,\varepsilon})_{\Omega(t)}+(b\cdot\nabla v^{m,\varepsilon}+v^{m,\varepsilon}\cdot\nabla b,v_h^{m,\varepsilon}-w_h^{m,\varepsilon})_{\Omega(t)}\}dt\\ &=-(v^{m,\varepsilon}(T),v_h^{m,\varepsilon}(T)-w_h^{m,\varepsilon}(T))_{\Omega(T)}+(v_0^m,v_h^{m,\varepsilon}(0)-w_h^{m,\varepsilon}(0))_{\Omega(0)}.\end{aligned}$$

As in [17], page 218 we obtain

$$(4.5)\qquad \int_0^T|\bar{v}^{m,\varepsilon}(t)-\bar{v}^{m,\varepsilon}(t-h)|^2_{\Omega(t)}dt\leqq c\sqrt{h}$$

(c is independent of m and ε).

By the classical characterization of M. Riesz and A. Kolmogorov of compact sets, we can prove that the set $\{v^{m,\varepsilon}\}$ of $v^{m,\varepsilon}$ satisfying (4.2), (4.5) is relatively compact in $L^2(0, T; L^2_{\text{loc}}(\Omega(t)))$. From (4.3) and the relatively compactness of $\{v^{m,\varepsilon}\}$ in $L^2(0, T; L^2_{\text{loc}}(\Omega(t)))$ we can choose a subsequence again denoted by $\{v^{m,\varepsilon}\}$ such that $\forall\varphi\in D(\Omega_T)$

$$\lim_{m\to\infty}\int_0^T(\tilde{v}^{m,\varepsilon}\cdot\nabla v^{m,\varepsilon},\ \varphi)_{\Omega(t)}dt = \int_0^T(\tilde{v}^{\varepsilon}\cdot\nabla v^{\varepsilon},\ \varphi)_{\Omega(t)}dt\,.$$

Now passing to the limit $m\to\infty$ in (4.1) we obtain $\forall\varphi\in D(\Omega_T)$ with $\varphi(T)=0$

$$\begin{aligned}\int_0^T\{&-(v^{\varepsilon},\ \partial_t\varphi)_{\Omega(t)}+(\exp kt)(\tilde{v}^{\varepsilon}\cdot\nabla v^{\varepsilon},\ \varphi)_{\Omega(t)}+k(v^{\varepsilon},\ \varphi)_{\Omega(t)}+(b\cdot\nabla v^{\varepsilon},\ \varphi)_{\Omega(t)}\\ &+(v^{\varepsilon}\cdot\nabla b,\ \varphi)_{\Omega(t)}+(\exp(-kt))(g,\ \varphi)_{\Omega(t)}+((v^{\varepsilon},\ \varphi))_{\Omega(t)}\}dt = (v_0,\ \varphi(0))_{\Omega(0)}\,.\end{aligned}\tag{4.6}$$

If we denote again $\varphi=(\exp kt)\varphi$, and $v^{\varepsilon}=v^{\varepsilon}\exp kt$, we have proved the existence of a solution of

$$\begin{aligned}&\int_0^T\{-(v^{\varepsilon},\ \partial_t\varphi)_{\Omega(t)}+((v^{\varepsilon},\ \varphi))_{\Omega(t)}+(\tilde{v}^{\varepsilon}\cdot\nabla v^{\varepsilon}+b\cdot\nabla v^{\varepsilon}+v^{\varepsilon}\cdot\nabla b+g,\ \varphi)_{\Omega(t)}\}dt\\ &= (v_0^{\varepsilon},\ \varphi(0))_{\Omega(0)}\,.\end{aligned}\tag{4.7}$$

Now to prove the strong convergence of $\{v^{\varepsilon}\}$ in $L^2(\Omega_T)$ we need some estimates on $\partial_t v^{\varepsilon}$. We shall prove that

$$\partial_t v^{\varepsilon}\in L^2(0,\ T;\ V^{-2}(\Omega(t)))\,;$$

uniformly with respect to ε (V^{-2} is the dual of $H_0^2(\Omega(t))\cap V(\Omega(t))$). First we shall prove that

$$\partial_t v^{m,\varepsilon}\cdot\nu = 0 \qquad \text{on } \Gamma(t)\,,\tag{4.8}$$

where ν is the unit exterior normal vector to $\Gamma(t)$. It is well known (see [22]) that it exists a linear continuous operator $\gamma_\nu: E(\Omega(t))\to H^{-1/2}(\Gamma(t))$ with $E=\{\varphi\,|\,\varphi\in L^2(\Omega(t)),\ \nabla\varphi\in L^2(\Omega(t))$ with the natural norm$\}$ (we denote $\gamma_\nu\varphi=\varphi\cdot\nu$ on $\Gamma(t)$). We consider time difference quotient for $\bar{v}^{m,\varepsilon}$ ($\bar{v}^{m,\varepsilon}$ is as above) on $\Gamma(t)$

$$\begin{aligned}&\|((\bar{v}^{m,\varepsilon}(t+h)-\bar{v}^{m,\varepsilon}(t))/h)\cdot\nu\|_{H^{-1/2}(\Gamma(t))} = \|(\bar{v}^{m,\varepsilon}(t+h)/h)\cdot\nu\|_{H^{-1/2}(\Gamma(t))}\\ &\leqq c\,|\bar{v}^{m,\varepsilon}(t+h)/h\,|_{\Omega^c(t)} \leqq (c/h)\,\text{measure}\,(\Omega^c(t)-\Omega^c(t+h))\|\bar{v}^{m,\varepsilon}(t+h)\|_{\Omega^c(t)}\\ &\leqq c\|\bar{v}^{m,\varepsilon}(t)\|_{\Omega^c(t)-\Omega^c(t+h)}+c\|\bar{v}^{m,\varepsilon}(t+h)-\bar{v}^{m,\varepsilon}(t)\|_R\,.\end{aligned}$$

Bearing in mind the $L^2(\Omega_T)$-continuity of a square summable function, we have

$$\|\partial_t v^{m,\varepsilon}\cdot\nu\|_{H^{-1/2}(\Gamma(t))} = 0 \qquad \text{a.e. in } (0,\ T)\,.$$

This last relation implies that $\partial_t v^{m,\varepsilon}\in L^2(0, T; H(\Omega(t)))\subset L^2(0, T; V^{-2}(\Omega(t)))$. Furthermore $\{\partial_t v^{m,\varepsilon}\}$ is a bounded set in $D'(\Omega_T)$ the dual of $D(\Omega_T)$ uniformly with respect to m; so $\partial_t v^{\varepsilon}\in D'(\Omega_T)$. Thank to (4.7) $\{\partial_t v^{\varepsilon}\}$ is a bounded set in $L^2(0, T; V^{-2}(\Omega(t)))$.

From standard arguments, we have that exists a distribution p^{ε} such that $\forall\varphi\in C^\infty(0, T; C_0^\infty(\Omega(t)))$ with $\varphi(T)=0$

(4.11)
$$\int_0^T \{-(v^\varepsilon, \partial_t \varphi)_{\Omega(t)} + ((v^\varepsilon, \varphi))_{\Omega(t)} + (\tilde{v}^\varepsilon \cdot \nabla v^\varepsilon, \varphi)_{\Omega(t)} + (p^\varepsilon, \nabla \cdot \varphi)_{\Omega(t)}$$
$$+ (b \cdot \nabla v^\varepsilon + v^\varepsilon \cdot \nabla b, \varphi)_{\Omega(t)} + (g, \varphi)_{\Omega(t)}\} dt = (v_0, \varphi(0))_{\Omega(0)}.$$

From (4.11) follows that p^ε satisfies, in the sense of distributions

(4.12) $$\Delta p^\varepsilon = -\nabla \cdot (\tilde{v}^\varepsilon \cdot \nabla v^\varepsilon + b \cdot \nabla v^\varepsilon + v^\varepsilon \cdot \nabla b + g) \quad \text{in } \Omega(t) \text{ (a.e. in } (0, T)).$$

We note that (4.2) and (4.6), for suitable k, imply

$\{v^\varepsilon\}$ is a bounded set in $L^\infty(0, T; L^2(\Omega(t)))$,

hence

(4.13)
$\{v^\varepsilon \cdot v^\varepsilon\}$ is a bounded set in $L^{5/3}(\Omega_T) \cap L^2(0, T; L^{3/2}(\Omega(t)))$,
$\{\tilde{v}^\varepsilon \cdot \nabla v^\varepsilon\}$ is a bouded set in $L^{5/4}(\Omega_T)$.

Now we localize (4.12). Let $\Omega_1 = \{x \in R^3 \mid |x| > \rho\}$, ρ is a positive number chosen such that $\Omega_1^c \supset \Omega^c(t)$ for every t. And let $\gamma \in C^\infty(R^3)$ with $\gamma = 0$ in a neighborhood of Ω_1^c and $=1$ for $|x| > 2\rho$. Then in any time in R^3

(4.14) $$\Delta(\gamma p^\varepsilon) = p^\varepsilon \Delta\gamma + 2\nabla\gamma\nabla p^\varepsilon + \gamma\Delta p^\varepsilon.$$

Let now $\alpha \in \mathcal{D}(R^3)$ with $\alpha = 1$ in a neighborhood of the origin. Then

(4.15) $$\Delta(-\alpha/3r) = -(1/3)((\Delta\alpha)/r + 2\nabla\alpha\nabla 1/r) + \delta = \zeta + \delta$$

where $\zeta \in \mathcal{D}(R^3)$, δ is the Dirac measure, and $r = x_1^2 + x_2^2 + x_3^2$. From (4.14), (4.15) we have, in any time, for $|x| > 3\rho$

(4.16)
$$\gamma p^\varepsilon = (\Delta(-\alpha/3r)) * \gamma p^\varepsilon - \zeta * \gamma p^\varepsilon$$
$$= \sum_{i,j=1}^3 (-\partial^2_{y_i y_j}(\alpha/3r)) * \gamma \cdot (v_i^\varepsilon v_j^\varepsilon + b_i v_j^\varepsilon + b_j v_i^\varepsilon + g) - \zeta * \gamma p^\varepsilon.$$

In (4.16) $f * g = \int_{R^3} f(y) g(y-x) dy$.

By standard arguments, we have that the first term in the right side of (4.16) belongs to $L^2(0, T; L^{3/2}(\bar{\Omega}))$, where $\bar{\Omega} = \{x \in R^3 \mid |x| > 3\rho\}$. Now we note that

(4.17) $$\Delta(\zeta * (\gamma p^\varepsilon)) = \zeta * \Delta(\gamma p^\varepsilon) = \zeta * ((\Delta\gamma) p^\varepsilon) + 2\zeta * (\nabla\gamma\nabla p^\varepsilon) + \zeta * (\gamma\Delta p^\varepsilon).$$

Since $\{p^\varepsilon\}$, $\{\nabla p^\varepsilon\}$ are bounded sets in $L^2(0, T; \mathcal{D}'(\Omega(t)))$, the first two terms in the right side of (4.17) are continuous functions in x with support compact, and square summable in t, uniformly with respect to ε, and from (4.12) it follows $\zeta * \gamma\Delta p^\varepsilon$ can be considered as a sum of continuous functions with support compact in x, square summable in t, of second derivatives of functions belonging to $L^2(0, T; L^{3/2}(R^3))$, and of derivatives of function belonging to $L^2(\Omega_T) \cap L^{5/3}(\Omega_T)$ for example. Now thank to (4.16), (4.17) we have $p^\varepsilon = p_1^\varepsilon + p_2^\varepsilon$ such that

(4.18)
$\{p_1^\varepsilon\}$ is a bounded set in $L^2(0, T; L^{3/2}(\bar{\Omega}))$,
$\{p_2^\varepsilon\}$ is a bounded set in $L^2(0, T; L^5(\bar{\Omega}))$.

Now following Leray [9], we introduce the cut-off function $\vartheta\in C^{\infty}(R^3)$, $\vartheta=0$ for $|x|<d$ and $\vartheta=1$ for $|x|>2d$ (d is a number big enough). Replacing in (4.11) φ by $\vartheta v^{\varepsilon}$, we obtain

$$\begin{aligned}(1/2)\partial_t|\vartheta^{1/2}v^{\varepsilon}|_{R^3}^2 \leqq -|\vartheta^{1/2}\nabla v^{\varepsilon}|_{R^3}^2+(1/2)|\Delta\vartheta|^{1/2}|v^{\varepsilon}|_{R^3}^2+(\nabla\vartheta,\ \tilde{v}^{\varepsilon}|v^{\varepsilon}|^2)_{R^3}\\ +((\nabla\vartheta)p^{\varepsilon},\ v^{\varepsilon})_{R^3}-(g+v^{\varepsilon}\cdot\nabla b+b\cdot\nabla v^{\varepsilon},\ \vartheta v^{\varepsilon})_{R^3}.\end{aligned} \tag{4.19}$$

Bearing in mind that $|\nabla\vartheta|\leqq c/d$; $|\Delta\vartheta|\leqq c/d^2$; $H^1(\Omega)\subset L^3(\Omega)$, the following inequalities hold

$$\begin{aligned}
&\int_0^T|(\nabla\vartheta,\ \tilde{v}^{\varepsilon}|v^{\varepsilon}|^2)_{R^3}dt \leqq (c/d)\int_0^T\|v^{\varepsilon}\|_{L^3(\Omega(t))}^3dt \leqq c/d\,;\\
&\int_0^T\|\Delta\vartheta\|^{1/2}|v^{\varepsilon}|_{\Omega(t)}^2dt \leqq c/d^2\,;\\
&\left|\int_0^T((\nabla\vartheta)p^{\varepsilon},\ v^{\varepsilon})_{R^3}dt\right| \leqq c/(d+d^{1/10})\,;\\
&\left|\int_0^T(b,\ \nabla\vartheta|v^{\varepsilon}|^2)_{\Omega(t)}dt\right| \leqq c/d\,;\\
&\left|\int_0^T(v^{\varepsilon}\cdot\nabla b,\ v^{\varepsilon}\vartheta)_{\Omega(t)}dt\right| \leqq c\int_0^T\|b\|_{L^{\infty}(R^3)}^2|\vartheta^{1/2}v^{\varepsilon}|_{R^3}^2+(1/2)\int_0^T|\vartheta^{1/2}\nabla v^{\varepsilon}|_{R^3}^2+c/d\,;\\
&\left|\int_0^T(g,\ \vartheta v^{\varepsilon})_{\Omega(t)}dt\right| \leqq \int_0^T(|\vartheta^{1/2}g|_{R^3}^2+|\vartheta^{1/2}v^{\varepsilon}|_{R^3}^2)dt.
\end{aligned} \tag{4.20}$$

Integrating (4.19) with respect to t, and using (4.20), and Gronwall's lemma, we deduce

$$\begin{aligned}\int_0^T|v^{\varepsilon}|_{L^2(\{x\in R^3||x|>d+1\})}^2dt \leqq c|v_0|_{\{x\in R^3||x|>d+1\}}^2+(c/d)^{1/10}\\ +c\int_0^T|\vartheta^{1/2}g|_{\{x\in R^3||x|>d+1\}}^2.\end{aligned} \tag{4.21}$$

Thanks to the estimates (4.2), (4.5), (4.7), (4.21) and to the characterization of compact sets in $L^2(\Omega_T)$ of M. Riesz and A. Kolmogorov, it is now routine to show that from the set $\{v^{\varepsilon}\}$ it is possible to select a subsequence again denoted $\{v^{\varepsilon}\}$ such that

$$v^{\varepsilon} \longrightarrow v \qquad \text{weakly in } L^2(0,\ T;\ V(\Omega(t))) \text{ and strongly in } L^2(\Omega_T). \tag{4.22}$$

Now replacing in (4.7) φ by $v^{\varepsilon}\eta$ where η is the characteristic function of the interval $(s,\ t)$, we have

$$|v^{\varepsilon}(t)|_{\Omega(t)}^2+2\int_s^t\|v^{\varepsilon}\|_{\Omega(\sigma)}^2d\sigma \leqq |v^{\varepsilon}(s)|_{\Omega(s)}^2-2\int_s^t(g+v^{\varepsilon}\cdot\nabla b,\ v^{\varepsilon})_{\Omega(\sigma)}d\sigma. \tag{4.23}$$

Passing to the limit $\varepsilon\to0$ in (4.23), and bearing in mind (4.22) we have

$$|v(t)|_{\Omega(t)}^2+2\int_s^t\|v\|_{\Omega(\sigma)}^2d\sigma \leqq |v(s)|_{\Omega(s)}^2-2\int_s^t(g+v\cdot\nabla b,\ v)_{\Omega(\sigma)}d\sigma.$$

Finally passing to the limit $\varepsilon \rightarrow 0$ in (4.7), we have that v satisfies ii) of Theorem 1. Theorem 1 is now completely proved.

5. Proof of Theorem 2.

Now we prove the regularity of weak solutions using the method developed in [**18**]. For this reason we shall give only a sketch of the proof. First we need the following uniqueness theorem proved in [**15**].

PROPOSITION 2. *Let u, v be weak solutions of* (1.1), (1.2) *in the interval* $[0, T]$. *Suppose*

$$\int_0^T \|v\|_{L^s(\Omega(t))}^{s'} ds < \infty$$

for some pair (s, s') *with* $3s^{-1}+2(s')=1$ *and with* $s>3$. *Suppose that*

$$|v(t)|_{\Omega(t)}^2 + 2\int_0^t \|v\|_{\Omega(\sigma)}^2 d\sigma + 2\int_0^t (v\cdot\nabla b+g,\, v)_{\Omega(\sigma)} d\sigma = |v(0)|_{\Omega(0)}^2$$

holds for $0 \leqq t \leqq T$. *Then we have*

$$|u(t)-v(t)|_{\Omega(t)} \leqq |u(0)-v(0)|_{\Omega(0)} \exp\left(c\int_0^t \|v(\sigma)\|_{L^s(\Omega(\sigma))}^{s'} d\sigma\right).$$

In particular, if $u(0)=v(0)$, *then* $u=v$ *in* $[0, T]$.

Now we consider the following auxiliary problem. Let

$$\mathcal{F} = \{\varphi \mid \varphi \in L^2(0, T; H^2(\Omega(t)) \cap V(\Omega(t))) \text{ with the natural norm}\};$$

$$\mathcal{P} = \{\varphi \mid \varphi \in L^2(0, T; H^2(\Omega(t)) \cap V(\Omega(t))),$$
$$\partial_t \varphi \in L^2(0, T; H^2(\Omega(t)) \cap V(\Omega(t))),\ \varphi(T)=0\}.$$

We consider on $\mathcal{P}$ the norm

$$\|\varphi\|_{\mathcal{P}} = \|\varphi\|_{\mathcal{F}} + \|\varphi(0)\|_{H^1(\Omega(0))}.$$

We note that, for $\varphi \in \mathcal{P}$, has sense $\varphi(x, T)$ in $\Omega(T)$ and $\varphi(x, 0)$ in $\Omega(0)$; furthermore, $\|\varphi\|_{H^1(\Omega(t))}$ is continuous in $[0, T]$ hence $\varphi(x, 0) \in H^1(\Omega(0))$.

We consider the following problem.

Find a $v \in \mathcal{F}$ such that for all $\varphi \in \mathcal{P}$

$$\int_0^T \{-(v, (I+A)\partial_t \varphi)_{\Omega(t)} + (Av, (I+A)\varphi)_{\Omega(t)} + k(v, (I+A)\varphi)_{\Omega(t)} + (b\cdot\nabla v, (I+A)\varphi)_{\Omega(t)}$$

$$(5.1) \qquad + (v\cdot\nabla b, (I+A)\varphi)_{\Omega(t)}\} dt = \int_0^T (\exp(-kt))(-u\cdot\nabla u + g, (I+A)\varphi)_{\Omega(t)}\} dt$$
$$+ (v_0, (I+A)\varphi(0))_{\Omega(0)}$$

holds. Here I is the unit operator. In (5.1) k is a suitable constant and $u \in L^\infty(0, T; H^1(\Omega(t))) \cap \mathcal{F}$, $v_0 \in H^1(\Omega(0))$ are given functions. We let

$$E(v, \varphi)=\int_0^T\{-(v, (I+A)\partial_t\varphi)_{\Omega(t)}+(Av, (I+A)\varphi)_{\Omega(t)}+k(v, (I+A)\varphi)_{\Omega(t)}$$
$$+(v\cdot\nabla b, (I+A)\varphi)_{\Omega(t)}+(b\cdot\nabla v, (I+A))_{\Omega(t)}\}\,dt;$$
$$L(\varphi)=-\int_0^T(\exp(-kt))(u\cdot\nabla u+g, (I+A))_{\Omega(t)}dt+(v_0, (I+A)\varphi(0))_{\Omega(0)}.$$

First $L(\varphi)$ is a linear continuous form on $\mathcal{P}$ with respect to the norm $\|\varphi\|_{\mathcal{P}}$. Moreover, bearing in mind that

$$\|\nabla\varphi\|_{L^2(\Gamma(t))}\leqq c(|D^2\varphi|_{\Omega(t)}^{1/2}\|\varphi\|_{\Omega(t)}^{1/2}+\|\varphi\|_{\Omega(t)})$$
$$\leqq cm_{1\Gamma}(|A\varphi|_{\Omega(t)}^{1/2}\|\varphi\|_{\Omega(t)}^{1/2}+\|\varphi\|_{\Omega(t)})+\|\varphi\|_{\Omega(t)},$$

if $c_T=\sup_t c(m_{1\Gamma}+1)$, $t\in[0, T]$, one has

$$\int_0^T-(\varphi, A\partial_t\varphi)_{\Omega(t)}dt=\int_0^T-(\nabla\varphi, \nabla\partial_t\varphi)_{\Omega(t)}dt$$
$$\geqq-(1/2)\int_0^T\partial_t\|\varphi\|_{\Omega(t)}^2dt-\|\partial_t\psi\|_{L^\infty(\bar\Omega_T)}\int_0^T\|\nabla\varphi\|_{\Gamma(t)}^2dt>-(1/8)\int_0^T|A\varphi|_{\Omega(t)}^2dt$$
$$-2(c_T^4\|\partial_t\psi\|_{L^\infty(\bar\Omega_T)}^2+c_T^2\|\partial_t\psi\|_{L^\infty(\bar\Omega_T)})\int_0^T\|\varphi\|_{\Omega(t)}^2dt+(1/2)\|\varphi\|_{\Omega(0)}^2$$

($\bar\Omega_T$=domain where is defined ψ).

Consequently

$$E(\varphi, \varphi)=\int_0^T\{-(\varphi, (I+A)\partial_t\varphi)_{\Omega(t)}+\|\varphi\|_{\Omega(t)}^2+|A\varphi|_{\Omega(t)}^2+(b\cdot\nabla\varphi, (I+A)\varphi)_{\Omega(t)}$$
$$+(\varphi\cdot\nabla b, (I+A)\varphi)_{\Omega(t)}+k\|\varphi\|_{\Omega(t)}^2+k|\varphi|_{\Omega(t)}^2\}\,dt\geqq(1/2)\int_0^T|A\varphi|_{\Omega(t)}^2dt$$
$$+k\int_0^T(|\varphi|_{\Omega(t)}^2+\|\varphi\|_{\Omega(t)}^2)dt-2(c_T^4\|\partial_t\psi\|_{L^\infty(\bar\Omega_T)}^2+c_T^2\|\partial_t\psi\|_{L^\infty(\bar\Omega_T)})$$
$$\times\int_0^T(\|\varphi\|_{\Omega(t)}^2+c(|\varphi|_{\Omega(t)}^2+\|\varphi\|_{\Omega(t)}^2))dt+(1/2)\|\varphi\|_{H^1(\Omega(0))}^2$$
$$\geqq c\|\varphi\|_{\mathcal{P}}^2;\quad\text{for suitable } k.$$

Then there exists a $v\in\mathcal{F}$ such that (5.1) is satisfied for every $\varphi\in\mathcal{P}$ (see [**23**], page 208). Now $(I+A)$ is one to one and onto from $H^2(\Omega(t))\cap V(\Omega(t))$ to $H(\Omega(t))$, so if $h(t)\in H(\Omega(t))$ for all $[0, T]$, there exists a $\varphi(t)\in H^2(\Omega(t))\cap V(\Omega(t))$ such that $h(t)=(I+A)\varphi(t)$. Hence if we substitute $h(t)$ in (5.1), by density, we obtain that $v\in\mathcal{F}$ satisfies (5.1) with $\partial_t h\in L^2(0, T; L^2(\Omega(t)))$ and $h(T)=0$.

In the above result we have used the following relation

$$P\partial_t\Delta\varphi=P\partial_tP\Delta\varphi+P\partial_t(I-P)\Delta\varphi=P\partial_tP\Delta\varphi+P\partial_t\nabla p=-\partial_tA\varphi.$$

Now, if $h(t)\in C_0^\infty(0, T; H(\Omega(t)))$, one obtains

$$\left|\int_0^T(v(t), \partial_th(t))_{\Omega(t)}dt\right|\leqq\left|\int_0^T\{(Av(t), h(t))_{\Omega(t)}+k(v(t), h(t))_{\Omega(t)}+(b\cdot\nabla v(t)\right.$$
$$\left.+v(t)\cdot\nabla b, h(t))_{\Omega(t)}+(\exp(-kt))(u\cdot\nabla u-g, h(t))_{\Omega(t)}\}\,dt\right|\leqq c\Big(\int_0^T|h|^2{}_{\Omega(t)}dt\Big)^{1/2},$$

and by results in [20] we have $P\partial_t v \in L^2(\Omega_T)$, and by standard arguments,

$$P(\partial_t v - \Delta v + (\exp(-kt))(u\cdot\nabla u - g) + kv + b\cdot\nabla v + v\cdot\nabla b) = 0 \tag{5.2}$$

a. e. in Ω_T.

Bearing in mind $P(\exp kt) = (\exp kt)P$, multiplying (5.2) by $\exp kt$, and denoting again $v = (\exp kt)v$, we have proved the auxiliary problem.

We notice that $\partial_t u \in H(\Omega(t))$ a. e. in $(0, T)$. In what follows, we do not make any explicit use of this. In any case, later, we will give, formally, a proof of this.

Now we complete the proof of Theorem 2.

The existence and the uniqueness of the equation

$$P(\partial_t v - \Delta v + b\cdot\nabla v + v\cdot\nabla b + u\cdot\nabla u - g) = 0 \tag{5.3}$$

enables us to define the map $v = \tau u$. The fixed point of τ are just the solutions of (1.1). Consider the set

$$\mathcal{Y} = \{\varphi \mid \|\varphi\|^2_{L^\infty(0,\bar{T};H^1(\Omega(t)))} + \|\varphi\|^2_{L^2(0,\bar{T};H^2(\Omega(t)))} + |\partial_t\varphi|^2_{L^2(0,\bar{T};H(\Omega(t)))} \leqq K_T\|v_0\|^2_{H^1(\Omega(0))}\}$$

(the constant K_T will be defined below, and $\bar{T} \leqq T$).

$\mathcal{Y}$ is a compact set in $L^2_{\text{loc}}(\Omega_{\bar{T}})$. We have to prove that $\tau\mathcal{Y} \subset \mathcal{Y}$ and τ is continuous in $\mathcal{Y}$ with respect to the $L^2_{\text{loc}}(\Omega_{\bar{T}})$ norm. We prove that $\tau\mathcal{Y} \subset \mathcal{Y}$ for suitable $\bar{T}$. In fact, multiplying (5.3) by $P\partial_t v + Av + v$ and integrating over Ω_t we have

$$\begin{aligned}&\int_0^t (|P\partial_t v|^2_{\Omega(s)} + (Av, P\partial_t v)_{\Omega(s)} + |Av|^2_{\Omega(s)} + \|v\|_{\Omega(s)})ds + |v|_{\Omega(t)}\\ &\leqq c\int_0^t (|u\cdot\nabla u|^2_{\Omega(s)} + |b\cdot\nabla v|^2_{\Omega(s)} + |v\cdot\nabla b|^2_{\Omega(s)} + |g|^2_{\Omega(s)})ds + |v(0)|_{\Omega(0)}.\end{aligned} \tag{5.4}$$

Bearing in mind

$$(Av, P\partial_t v)_{\Omega(t)} = (1/2)(\partial_t\|v\|^2_{\Omega(t)}) - \sum_{i=1}^3 \int_{\Gamma(t)} \nabla v_i \nabla v_i \partial_t\psi \cos(\nu, x_3)d\Gamma$$

(see [20]), (5.3), (5.4), and Proposition 1 imply

$$\begin{aligned}&\int_0^t (|P\partial_s v|^2_{\Omega(s)} + \|v\|^2_{\Omega(s)} + |Av|^2_{\Omega(s)})ds + \|v\|^2_{H^1(\Omega(t))}\\ &\leqq c_1\|v_0\|^2_{H^1(\Omega(0))} + c_3\bar{T}(\sup\|u\|^4_{\Omega(t)}) + |v|_{\Omega(0)}\\ &\quad + c_2\sup\|u\|^3_{\Omega(t)}\cdot\left(\int_0^{\bar{T}} |Au|^2_{\Omega(t)}dt\right)^{1/2}\bar{T}^{1/2} + c_4\int_0^{\bar{T}} |g|^2_{\Omega(t)}dt.\end{aligned} \tag{5.5}$$

In (5.5) $c_1, \cdots, c_4$ are constants dependent on Γ_T and on the data. At this point we define K_T. We set

$$K_T = 2(\inf(1, (2m_{1\Gamma})^{-1}))^{-1}(c_1 + 1).$$

Now from (5.5), choosing $\bar{T}$ sufficiently small, it follows that $\tau\mathcal{Y} \subset \mathcal{Y}$. To prove the continuity in $L^2_{\mathrm{loc}}(\Omega_{\bar{T}})$ of τ we observe that if $\{u_n\}\subset \mathcal{Y}$ then $u_n\to u$ strongly in $L^2_{\mathrm{loc}}(\Omega_{\bar{T}})$ and weakly $*$ in $L^\infty(0, \bar{T}; H(\Omega(t)))$ and $\{u_n\cdot\nabla u_n\}$ converges weakly in $L^2(\Omega_{\bar{T}})$ so

$$u_n\cdot\nabla u_n \longrightarrow u\cdot\nabla u \qquad \text{weakly in } L^2(\Omega_{\bar{T}}).$$

It follows from the linear equation (5.3) that $v_n\to v$ strongly in $L^2_{\mathrm{loc}}(\Omega_{\bar{T}})$ where v_n and v are the solutions of (5.3) corresponding to u_n and u respectively. Hence τ is continuous and the existence of a local solution is completely proved. It is routine matter to prove the energy equality. By Proposition 2 we have the uniqueness of the solution.

Now we prove, formally, that $\partial_t u\in H(\Omega(t))$. First, from $\nabla\cdot u=0$ we have $\nabla\cdot\partial_t u=0$. Then, let τ be any unit vector tangent to Γ_T. Bearing in mind $u=0$ on Γ_T, and differentiating in the direction τ, we get $\partial_\tau u=0$ on $\Gamma(t)$. This fact implies

$$\partial_t u+\partial_\nu u\cos(\nu, \tau)/\cos(t, \tau) = 0 \qquad \text{on } \Gamma(t) \tag{5.6}$$

(ν is the unit normal to $\Gamma(t)$).

Thanks to $\nabla\cdot u=\partial_\nu u\cdot\nu$ on $\Gamma(t)$, (5.6) implies $\partial_t u\cdot\nu=0$ on $\Gamma(t)$. Since the vectors in $H(\Omega(t))$ are divergence free and have vanishing normal component on $\Gamma(t)$, we get

$$\partial_t u \in H(\Omega(t)).$$

Now Theorem 2 is completely proved.

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