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The quasi KO-homology types of the stunted real projective spaces

Dedicated to Professor Akio Hattori on his sixtieth birthday

By Zen-ichi YOSIMURA

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0. Introduction.

Let E be an associative ring spectrum with unit, and X, Y be CW-spectra. We say that X is *quasi E_*-equivalent to* Y if there exists a map $h: Y \to E \wedge X$ such that the composite $(\mu \wedge 1)(1 \wedge h): E \wedge Y \to E \wedge X$ is an equivalence where $\mu: E \wedge E \to E$ stands for the multiplication of E. In this case we write $X \underset{E}{\sim} Y$, and we call such a map $h: Y \to E \wedge X$ a quasi E_*-equivalence. We shall be concerned with the quasi KO_*-equivalence where KO is the real K-spectrum. In [**Y2**] we have determined the quasi KO_*-types of the real projective n-spaces RP^n. The purpose of this note is to determine the quasi KO_*-types of the stunted real projective spaces RP^n/RP^m as a continuation of [**Y2**].

In order to describe our main result precisely we have to introduce some elementary suspension spectra with three or four cells (see [**Y3, Y4**]). The Moore spectrum SZ/n of type Z/n is constructed by the cofiber sequence $\Sigma^0 \xrightarrow{n} \Sigma^0 \xrightarrow{i} SZ/n \xrightarrow{j} \Sigma^1$. Let M_{2m} and V_{2m} denote the cofibers of the maps $i\eta: \Sigma^1 \to SZ/2m$ and $i\bar{\eta}: \Sigma^1 SZ/2 \to SZ/m$ respectively. Here $\eta: \Sigma^1 \to \Sigma^0$ stands for the stable Hopf map of order 2 and $\bar{\eta}: \Sigma^1 SZ/2 \to \Sigma^0$ its extension satisfying $\bar{\eta} i = \eta$. The complex K-spectrum KU possesses the conjugation $t: KU \to KU$ which gives rise to an involution t_* on KU_*X for any CW-spectrum X. By comparing KU_*RP^n with KU_*M_{2m} or KU_*V_{2m} as an abelian group with involution, and then by characterizing a CW-spectrum X which admits the same quasi KO_*-type as M_{2m} or V_{2m}, we have established the following determination [**Y2**, Theorem 5] (cf. [**F**]).

THEOREM 1. *$\Sigma^1 RP^n$ is quasi KO_*-equivalent to $SZ/2^{4r}$, $M_{2^{4r}}$, $V_{2^{4r+1}}$, $\Sigma^4 \vee V_{2^{4r+1}}$, $V_{2^{4r+2}}$, $M_{2^{4r+2}}$, $SZ/2^{4r+3}$, $\Sigma^0 \vee SZ/2^{4r+3}$ according as $n=8r$, $8r+1$, $\cdots$, $8r+7$.*

Let M'_{2m} and MP_{2m} denote the cofibers of the maps $\eta j: SZ/2m \to \Sigma^0$ and $i\eta \vee \tilde{\eta}: \Sigma^1 \vee \Sigma^2 \to SZ/2m$ respectively. Here $\tilde{\eta}: \Sigma^2 \to SZ/2m$ stands for a coextension of η satisfying $j\tilde{\eta}=\eta$. By applying the same method as in the proof of Theorem 1 established in [**Y2**] we will show the following main result (cf. [**FY**]).

THEOREM 2. i) *$\Sigma^{4m}(RP^{4m+n}/RP^{4m})$ is quasi KO_*-equivalent to RP^n.*

ii) *$\Sigma^{4m}(RP^{4m+n}/RP^{4m-1})$ is quasi KO_*-equivalent to the wedge $\Sigma^0\vee RP^n$.*

iii) *$\Sigma^{4m}(RP^{4m+n-2}/RP^{4m-2})$ is quasi KO_*-equivalent to RP^n_σ where $\Sigma^1RP^n_\sigma=SZ/2^{4r}$, $\Sigma^0\vee SZ/2^{4r}$, $SZ/2^{4r+1}$, $M_{2^{4r+1}}$, $V_{2^{4r+2}}$, $\Sigma^4\vee V_{2^{4r+2}}$, $V_{2^{4r+3}}$, $M_{2^{4r+3}}$ according as $n=8r$, $8r+1, \cdots, 8r+7$.*

iv) *$\Sigma^{4m+2}(RP^{4m+n-2}/RP^{4m-3})$ is quasi KO_*-equivalent to $M'_{2^{4r}}$, $\Sigma^1\vee M'_{2^{4r}}$, $M'_{2^{4r+1}}$, $\Sigma^1MP_{2^{4r+2}}$, $\Sigma^4M'_{2^{4r+2}}$, $\Sigma^5\vee\Sigma^4M'_{2^{4r+2}}$, $\Sigma^4M'_{2^{4r+3}}$, $\Sigma^1MP_{2^{4r+4}}$ according as $n=8r$, $8r+1, \cdots, 8r+7$.*

In §1 and §2 we will characterize a CW-spectrum X admitting the same quasi KO_*-type as $SA\vee\Sigma^1SD\vee M'_{2m}$ or $\Sigma^2SB\vee\Sigma^3SE\vee MP_{4m}$ under some restrictions on A, D, B and E (Theorems 1.6 and 2.6), where SG denotes the Moore spectrum of type G. In particular, Theorem 2.6 shows that Σ^4MP_{4m} is quasi KO_*-equivalent to MP_{4m} (Corollary 2.7). In §3 we will first investigate the KU- and KO-homologies of the stunted real projective spaces RP^n/RP^m (cf. [**Ad1**], [**FY**]), and then prove our main result (Theorem 2) by means of results obtained in §1, §2 and [**Y2**]. In fact, Theorem 2 i) and iii) are shown by applying [**Y2**, Theorem 2.5] as Theorem 1 was done in [**Y2**]. Moreover, Theorem 2 iv) is established by applying Theorem 1.6 and Corollary 2.7 (or Theorem 2.6). On the other hand, Theorem 2 ii) is obtained by making use of the Thom isomorphism in KO-theory as was done in [**FY**].

In this note we will work in the stable homotopy category of CW-spectra.

1. The cofiber M'_{2m} of the map $\eta j: SZ/2m\to\Sigma^0$.

1.1. Let KO, KU and KC denote the real, complex and self-conjugate K-spectrum respectively. These K-spectra are closely related each other. Thus we have nice relations among them given by the cofiber sequences as follows ([**An**] or [**B**]):

$$\Sigma^1KO\xrightarrow{\eta\wedge 1}KO\xrightarrow{\varepsilon_u}KU\xrightarrow{\varepsilon_o\pi_u^{-1}}\Sigma^2KO \tag{1.1}$$

$$\Sigma^2KO\xrightarrow{\eta^2\wedge 1}KO\xrightarrow{\varepsilon_c}KC\xrightarrow{\tau\pi_c^{-1}}\Sigma^3KO \tag{1.2}$$

$$KC\xrightarrow{\zeta}KU\xrightarrow{\pi_u^{-1}(1-t)}\Sigma^2KU\xrightarrow{\gamma\pi_u}\Sigma^1KC \tag{1.3}$$

which are related by the commutative diagram below

$$\begin{array}{ccccccc} & & \Sigma^1 KU & = & \Sigma^1 KU & & \\ & & \downarrow & & \downarrow & & \\ KO & \xrightarrow{\varepsilon_c} & KC & \xrightarrow{\tau\pi_c^{-1}} & \Sigma^3 KO & \xrightarrow{\eta^2\wedge 1} & \Sigma^1 KO \\ \| & & \downarrow{\scriptstyle\zeta} & & \downarrow{\scriptstyle \eta\wedge 1} & & \| \\ KO & \xrightarrow{\varepsilon_u} & KU & \xrightarrow{\varepsilon_o\pi_u^{-1}} & \Sigma^2 KO & \xrightarrow{\eta\wedge 1} & \Sigma^1 KO \\ & & \downarrow & & \downarrow & & \\ & & \Sigma^2 KU & = & \Sigma^2 KU. & & \end{array} \tag{1.4}$$

In place of (1.3) we sometimes use the cofiber sequence

$$KC \xrightarrow{\zeta} KU \xrightarrow{1-t} KU \xrightarrow{\gamma} \Sigma^1 KC\,. \tag{1.3$'$}$$

We denote by M_{2m} and M'_{2m}, $m\geqq 1$, the suspension spectra with three cells constructed by the cofiber sequences

$$\Sigma^1 \xrightarrow{i\eta} SZ/2m \xrightarrow{i_M} M_{2m} \xrightarrow{j_M} \Sigma^2 \tag{1.5}$$

$$SZ/2m \xrightarrow{\eta j} \Sigma^0 \xrightarrow{i'_M} M'_{2m} \xrightarrow{j'_M} \Sigma^1 SZ/2m\,. \tag{1.6}$$

Note that M'_{2m} is the Spanier-Whitehead dual of M_{2m}, thus $M'_{2m}=\Sigma^2 DM_{2m}$. The KU- and KO-homologies of these elementary suspension spectra M_{2m} and M'_{2m} are easily calculated in [**Y3**, Propositions 4.1 and 4.2].

PROPOSITION 1.1. i) *$KU_0M_{2m}\cong Z\oplus Z/2m$ on which $t_*=\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$, and $KU_1M_{2m}=0$.*

ii) *$KU_0M'_{2m}\cong Z$, $KU_1M'_{2m}\cong Z/2m$ on both of which $t_*=1$.*

iii) *$KO_iM_{2m}\cong Z/2m \quad 0 \quad Z\oplus Z/2 \quad Z/2 \quad Z/4m \quad 0 \quad Z \quad 0$*

$KO_iM'_{2m}\cong Z \quad Z/4m \quad Z/2 \quad Z/2 \quad Z \quad Z/2m \quad 0 \quad 0$

according as $i=0, 1, \cdots, 7$.

We denote by MP_{2m}, $m\geqq 1$, the suspension spectrum with four cells constructed by the cofiber sequence

$$\Sigma^1\vee\Sigma^2 \xrightarrow{i\eta\vee\tilde{\eta}} SZ/2m \xrightarrow{i_{MP}} MP_{2m} \xrightarrow{j_{MP}} \Sigma^2\vee\Sigma^3 \tag{1.7}$$

where $\tilde{\eta}: \Sigma^2\to SZ/2m$ stands for a coextension of η satisfying $j\tilde{\eta}=\eta$. Then there exists a cofiber sequence

$$\Sigma^2 \xrightarrow{i_M\tilde{\eta}} M_{2m} \xrightarrow{k_{MP}} MP_{2m} \xrightarrow{l_{MP}} \Sigma^3 \tag{1.8}$$

making the diagram below commutative

$$
(1.9)\qquad
\begin{array}{ccccccc}
 & & & & \Sigma^2 & = & \Sigma^2 \\
 & & & & \downarrow{\scriptstyle i_M\tilde{\eta}} & & \downarrow{\scriptstyle 0} \\
\Sigma^1 & \xrightarrow{i\eta} & SZ/2m & \longrightarrow & M_{2m} & \longrightarrow & \Sigma^2 \\
\downarrow & & \| & & \downarrow & & \downarrow \\
\Sigma^1\vee\Sigma^2 & \xrightarrow[i\eta\vee\tilde{\eta}]{} & SZ/2m & \longrightarrow & MP_{2m} & \longrightarrow & \Sigma^2\vee\Sigma^3 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \Sigma^3 & = & \Sigma^3 .
\end{array}
$$

PROPOSITION 1.2. i) *$KU_0MP_{2m}\cong Z\oplus Z/m$ on which $t_*=\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$, and $KU_1MP_{2m}\cong Z$ on which $t_*=-1$.*

ii) *$KO_iMP_{2m}\cong Z/2m$, 0, Z, Z according as $i\equiv 0, 1, 2, 3 \bmod 4$.*

PROOF. i) Use the two exact sequences

$$0\longrightarrow KU_1MP_{2m}\longrightarrow KU_0\Sigma^2\xrightarrow{\tilde{\eta}_*} KU_0SZ/2m\longrightarrow KU_0MP_{2m}\longrightarrow KU_{-1}\Sigma^1\longrightarrow 0$$

$$0\longrightarrow KU_1MP_{2m}\longrightarrow KU_0\Sigma^2\xrightarrow{(i_M\tilde{\eta})_*} KU_0M_{2m}\longrightarrow KU_0MP_{2m}\longrightarrow 0$$

induced by the cofiber sequences (1.7), (1.8). Here $\tilde{\eta}_*: KU_0\Sigma^2\to KU_0SZ/2m$ is expressed to be $\tilde{\eta}_*=m: Z\to Z/2m$, as is shown in the proof of [**Y3**, Proposition 4.1]. Hence we obtain that $KU_1MP_{2m}\cong Z$ and $KU_0MP_{2m}\cong Z\oplus Z/m$. Moreover, it follows immediately that $t_*=-1$ on $KU_1MP_{2m}\cong Z$ and $t_*=\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$ on $KU_0MP_{2m}\cong Z\oplus Z/m$ because $t_*=\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$ on $KU_0M_{2m}\cong Z\oplus Z/2m$.

ii) Use the long exact sequence of KO-homology induced by the cofiber sequence (1.7). Then KO_iMP_{2m} is easily calculated except $i=4$. On the other hand, the cofiber sequence (1.8) gives rise to a short exact sequence $0\to KO_4\Sigma^2\to KO_4M_{2m}\to KO_4MP_{2m}\to 0$ in the $i=4$ case. So the result is immediately obtained.

1.2. The short exact sequences

$$(1.10)\quad 0\longrightarrow[\Sigma^2, KU\wedge X]\xrightarrow{j_M^*}[M_{2m}, KU\wedge X]\xrightarrow{i_M^*}[SZ/2m, KU\wedge X]\longrightarrow 0$$

$$(1.11)\quad 0\longrightarrow[\Sigma^1SZ/2m, KU\wedge X]\xrightarrow{j_M'^*}[M'_{2m}, KU\wedge X]\xrightarrow{i_M'^*}[\Sigma^0, KU\wedge X]\longrightarrow 0$$

induced by the cofiber sequences (1.5), (1.6) are split for any CW-spectrum X. Moreover the universal coefficient sequence

$$(1.12)\quad \begin{array}{l} 0\longrightarrow \mathrm{Ext}(KU_0SZ/2m, KU_{i+1}X)\longrightarrow[\Sigma^iSZ/2m, KU\wedge X] \\ \qquad\qquad\xrightarrow{\kappa_i}\mathrm{Hom}(KU_0SZ/2m, KU_iX)\longrightarrow 0\end{array}$$

is also a split exact sequence for each i (cf. [**ArT**]), where the arrow κ_i assigns to any map f its induced homomorphism f_* of KU-homology in dimension i.

Let A, D be a 2-torsion free abelian groups and $m=2^k$, $k\geqq 0$. We now deal with a CW-spectrum X such that

(1.13) $KU_0X\cong A\oplus Z$ and $KU_1X\cong D\oplus Z/2m$ on both of which $t_*=1$, and in addition $KO_1X\cong(A\otimes Z/2)\oplus D\oplus Z/4m$ and $KO_6X=0=KO_7X$.

By means of Proposition 1.1 we note that the wedge sum $SA\vee\Sigma^1SD\vee M'_{2m}$ satisfies the above condition (1.13). In this section we will conversely prove that a CW-spectrum X satisfying (1.13) is quasi KO_*-equivalent to $SA\vee\Sigma^1SD\vee M'_{2m}$. In order to investigate the behaviour of the conjugation t_* on $[M'_{2m}, KU\wedge X]$ for such a CW-spectrum X, we will first show

LEMMA 1.3. *There exists a direct sum decomposition*

$$[\Sigma^1SZ/2m, KU\wedge X]\cong \mathrm{Hom}(KU_0SZ/2m, KU_1X)\oplus\mathrm{Ext}(KU_0SZ/2m, KU_2X)$$
$$\cong Z/2m\oplus(A\oplus Z)\otimes Z/2m$$

on which $t_*=\begin{pmatrix}1 & 0\\ i_2 & -1\end{pmatrix}$ *where* $i_2: Z/2m\to(A\otimes Z/2m)\oplus Z/2m$ *denotes the injection into the last factor.*

PROOF. Denote by t_{2m} the conjugation t_* on $[\Sigma^1SZ/2m, KU\wedge X]$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0\to & \mathrm{Ext}(KU_0SZ/2m, KU_2X) & \to & [\Sigma^1SZ/2m, KU\wedge X] & \xrightarrow{\kappa_1} & \mathrm{Hom}(KU_0SZ/2m, KU_1X) & \to 0\\
 & \downarrow & & \downarrow{\scriptstyle 1-t_{2m}} & & \downarrow & \\
0\to & \mathrm{Ext}(KU_0SZ/2m, KU_2X) & \to & [\Sigma^1SZ/2m, KU\wedge X] & \xrightarrow[\kappa_1]{} & \mathrm{Hom}(KU_0SZ/2m, KU_1X) & \to 0
\end{array}$$

with split exact rows. Note that the left vertical arrow is just multiplication by 2 and the right one is trivial.

In order to give a matrix representation of the central arrow $1-t_{2m}$, we here observe the connecting homomorphism $\delta: \mathrm{Hom}(Z/2m, KU_1X)\to \mathrm{Ext}(Z/2m, KU_2X\otimes Z/2)$ associated with the short exact sequence $0\to KU_2X\otimes Z/2\to KC_1X\to KU_1X\to 0$ induced by the cofiber sequence (1.3)$'$. This short exact sequence is obtained as the canonical exact sequence $0\to(A\otimes Z/2)\oplus Z/2\to(A\otimes Z/2)\oplus D\oplus Z/4m\to D\oplus Z/2m\to 0$ because $\varepsilon_{c*}: KO_1X\to KC_1X$ is an isomorphism. So it is easily seen that the connecting homomorphism $\delta: Z/2m\to(A\otimes Z/2)\oplus Z/2$ is given by $\delta(1)=(0, 1)$. Hence we can express as $1-t_{2m}=\begin{pmatrix}0 & 0\\ -i_2 & 2\end{pmatrix}$ on $[\Sigma^1SZ/2m, KU\wedge X]\cong\mathrm{Hom}(KU_0SZ/2m, KU_1X)\oplus\mathrm{Ext}(KU_0SZ/2m, KU_2X)$ by choosing suitably a splitting of κ_1 if necessary. Thus $[\Sigma^1SZ/2m, KU\wedge X]$ has a direct sum decomposition so that $t_{2m}=\begin{pmatrix}1 & 0\\ i_2 & -1\end{pmatrix}$ on it as desired.

Let P denote the cofiber of the stable Hopf map $\eta:\Sigma^1\to\Sigma^0$. The cofiber sequence $\Sigma^1\xrightarrow{\eta}\Sigma^0\xrightarrow{i_P}P\xrightarrow{j_P}\Sigma^2$ gives rise to a split exact sequence $0\to[\Sigma^2, KU\wedge X]\to[P, KU\wedge X]\to[\Sigma^0, KU\wedge X]\to 0$. As is well known (cf. [**Y3**, (2.3)]), $[P, KU\wedge X]$ has a direct sum decomposition

$$(1.14)\qquad [P, KU\wedge X]\cong KU_0X\oplus KU_2X \quad \text{on which } t_*=\begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}.$$

LEMMA 1.4. *There exists a direct sum decomposition*

$$\begin{aligned}[M'_{2m}, KU\wedge X] &\cong KU_0X\oplus \mathrm{Hom}(KU_0SZ/2m, KU_1X)\oplus \mathrm{Ext}(KU_0SZ/2m, KU_2X)\\ &\cong (A\oplus Z)\oplus Z/2m\oplus(A\oplus Z)\otimes Z/2m\end{aligned}$$

on which $t_*=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -\rho & i_2 & -1\end{pmatrix}$ *where* $\rho: A\oplus Z\to(A\oplus Z)\otimes Z/2m$ *denotes the canonical projection.*

PROOF. Use the commutative diagram

$$\begin{array}{ccccccc} SZ/2m & \xrightarrow{\eta j} & \Sigma^0 & \xrightarrow{i'_M} & M'_{2m} & \xrightarrow{j'_M} & \Sigma^1SZ/2m\\ \downarrow{\scriptstyle j} & & \| & & \downarrow{\scriptstyle k'} & & \downarrow{\scriptstyle j}\\ \Sigma^1 & \xrightarrow[\eta]{} & \Sigma^0 & \xrightarrow[i_P]{} & P & \xrightarrow[j_P]{} & \Sigma^2 \end{array}$$

which gives rise to the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [\Sigma^2, KU\wedge X] & \xrightarrow{j_P^*} & [P, KU\wedge X] & \xrightarrow{i_P^*} & [\Sigma^0, KU\wedge X] & \longrightarrow & 0\\ & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle k'^*} & & \| & & \\ 0 & \longrightarrow & [\Sigma^1SZ/2m, KU\wedge X] & \xrightarrow[j'^*_M]{} & [M'_{2m}, KU\wedge X] & \xrightarrow[i'^*_M]{} & [\Sigma^0, KU\wedge X] & \longrightarrow & 0\\ & & \downarrow{\scriptstyle \kappa_1} & & \downarrow{\scriptstyle \kappa_1} & & & & \\ & & \mathrm{Hom}(KU_0SZ/2m, KU_1X) & \xrightarrow{\cong} & \mathrm{Hom}(KU_1M'_{2m}, KU_1X) & & & & \end{array}$$

with two split exact rows. The central composite $\kappa_1k'^*:[P, KU\wedge X]\to \mathrm{Hom}(KU_1M'_{2m}, KU_1X)$ is evidently trivial, and the left vertical arrow $j^*:[\Sigma^2, KU\wedge X]\to[\Sigma^1SZ/2m, KU\wedge X]$ is expressed as the column $\begin{pmatrix}0\\ \rho\end{pmatrix}$ where $[\Sigma^1SZ/2m, KU\wedge X]$ is decomposed as in Lemma 1.3 and $\rho:\mathrm{Hom}(Z, KU_2X)\to \mathrm{Ext}(Z/2m, KU_2X)$ denotes the canonical projection. Hence $k'^*:[P, KU\wedge X]\to$

$[M'_{2m}, KU\wedge X]$ is written into the matrix $\begin{pmatrix}1 & 0\\ 0 & 0\\ a & \rho\end{pmatrix}$ for some homomorphism $a: A\oplus Z\to(A\oplus Z)\otimes Z/2m$, where $[P, KU\wedge X]$ is decomposed as in (1.14) and $[M'_{2m}, KU\wedge X]$ is decomposed by making use of the splitting exact sequence (1.11) and Lemma 1.3.

Denote by t_P and $t_{M'}$ the conjugations t_* on $[P, KU\wedge X]$ and $[M'_{2m}, KU\wedge X]$ respectively. Then (1.14) says that $t_P=\begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}$, and Lemma 1.3 asserts that $t_{M'}$ is written into the matrix $\begin{pmatrix}1 & 0 & 0\\ b & 1 & 0\\ c & i_2 & -1\end{pmatrix}$ for some homomorphisms $b: A\oplus Z\to Z/2m$, $c: A\oplus Z\to(A\oplus Z)\otimes Z/2m$. However $i_2b=0: A\oplus Z\to(A\oplus Z)\otimes Z/2m$ which implies $b=0$, because $t_{M'}^2=1$. Moreover the equality $t_{M'}k'^*=k'^*t_P$ shows that $c=2a-\rho: A\oplus Z\to(A\oplus Z)\otimes Z/2m$. So we may take to be $c=-\rho$ by replacing suitably the splitting of $i_M'^*$ if necessary. Thus $[M'_{2m}, KU\wedge X]$ has a direct sum decomposition so that $t_{M'}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -\rho & i_2 & -1\end{pmatrix}$ on it as desired.

1.3. For any CW-spectrum X satisfying (1.13) we consider the commutative diagram below

$$\begin{array}{ccccccc}
 & 0 & & & & & \\
 & \downarrow & & & & & \\
 & \mathrm{Ext}(KU_0SZ/2m, KU_2X) & & \mathrm{Hom}(KU_0M'_{2m}, KU_0X) & \xrightarrow{\cong} & \mathrm{Hom}(KU_0\Sigma^0, KU_0X) & \\
 & \downarrow & & \uparrow{\scriptstyle \kappa_0} & & \cong\uparrow{\scriptstyle \kappa_0} & \\
0\longrightarrow & [\Sigma^1SZ/2m, KU\wedge X] & \longrightarrow & [M'_{2m}, KU\wedge X] & \longrightarrow & [\Sigma^0, KU\wedge X] & \longrightarrow 0\\
 & \downarrow{\scriptstyle \kappa_1} & & \downarrow{\scriptstyle \kappa_1} & & & \\
 & \mathrm{Hom}(KU_0SZ/2m, KU_1X) & \xrightarrow{\cong} & \mathrm{Hom}(KU_1M'_{2m}, KU_1X) & & & \\
 & \downarrow & & & & & \\
 & 0 & & & & &
\end{array}$$

where the arrows κ_i $(i=0, 1)$ assign to any map f its induced homomorphism of KU-homology in dimension i. Then we can rewrite the direct sum decomposition on $[M'_{2m}, KU\wedge X]$ obtained in Lemma 1.4 as follows:

(1.15) $[M'_{2m}, KU\wedge X]$

$$\cong \mathrm{Hom}(KU_0M'_{2m}, KU_0X)\oplus\mathrm{Hom}(KU_1M'_{2m}, KU_1X)\oplus\mathrm{Ext}(KU_0SZ/2m, KU_2X).$$

PROPOSITION 1.5. *Let A, D be 2-torsion free abelian groups, $m=2^k$, $k\geqq 0$, and X be a CW-spectrum satisfying the condition* (1.13). *Then there exists a map*

$f_{M'}: M'_{2m} \to KU \wedge X$ with $(t \wedge 1) f_{M'} = f_{M'}$, whose induced homomorphisms of KU-homologies in dimensions 0, 1 are respectively the canonical inclusions $i_0: Z \to A \oplus Z$ and $i_1: Z/2m \to D \oplus Z/2m$.

PROOF. Under the direct sum decomposition on $[M'_{2m}, KU \wedge X]$ given in (1.15), we can choose a map $f_{M'}: M'_{2m} \to KU \wedge X$ corresponding to the element $w = (i_0, i_1, 0)$. Then it is immediate that $(t \wedge 1) f_{M'} = f_{M'}$ because $t_{M'}(w) = w$ as is easily calculated by means of the matrix representation of $t_{M'}$ obtained in Lemma 1.4.

We will now prove a main result in this section, which characterize a CW-spectrum X admitting the same quasi KO_*-type as $SA \vee \Sigma^1 SD \vee M'_{2m}$ where SG denotes the Moore spectrum of type G for $G = A$ or D.

THEOREM 1.6. *Let A, D be 2-torsion free abelian groups such that $\mathrm{Ext}(D, A \oplus Z)$ is uniquely 2-divisible, and $m = 2^k$, $k \geqq 0$. Then a CW-spectrum X is quasi KO_*-equivalent to the wedge sum $SA \vee \Sigma^1 SD \vee M'_{2m}$ if and only if $KU_0 X \cong A \oplus Z$ and $KU_1 X \cong D \oplus Z/2m$ on both of which $t_* = 1$ and in addition $KO_1 X \cong (A \otimes Z/2) \oplus D \oplus Z/4m$ and $KO_6 X = 0 = KO_7 X$.*

PROOF. The "only if" part is evident from Proposition 1.1.

The "if" part: By use of Proposition 1.5 we can choose a map $f_{M'}: M'_{2m} \to KU \wedge X$ with $(t \wedge 1) f_{M'} = f_{M'}$ inducing the canonical inclusions $i_0: Z \to A \oplus Z$, $i_1: Z/2m \to D \oplus Z/2m$ in KU-homologies. By virtue of [**Y2**, Lemma 1.1] there exist maps $g_{M'}: M'_{2m} \to KC \wedge X$, $h_0: \Sigma^0 \to KO \wedge X$ and $h_1: SZ/2m \to \Sigma^2 KO \wedge X$ making the diagram below commutative

$$\begin{array}{ccccc}
\Sigma^0 & \xrightarrow{i'_M} & M'_{2m} & \xrightarrow{j'_M} & \Sigma^1 SZ/2m \\
\downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle g_{M'}} & & \downarrow{\scriptstyle h_1} \\
KO \wedge X & \longrightarrow & KC \wedge X & \xrightarrow[\tau \pi_C^{-1} \wedge 1]{} & \Sigma^3 KO \wedge X \\
\| & & \downarrow{\scriptstyle \zeta \wedge 1} & & \downarrow{\scriptstyle \eta \wedge 1} \\
KO \wedge X & \xrightarrow[\varepsilon_u \wedge 1]{} & KU \wedge X & \xrightarrow[\varepsilon_o \pi_u^{-1} \wedge 1]{} & \Sigma^2 KO \wedge X
\end{array}$$

with $(\zeta \wedge 1) g_{M'} = f_{M'}$. However the map $h_1: SZ/2m \to \Sigma^2 KO \wedge X$ becomes trivial because $KO_6 X = 0 = KO_7 X$. Hence we get a map $h_{M'}: M'_{2m} \to KO \wedge X$ with $(\varepsilon_u \wedge 1) h_{M'} = f_{M'}$.

Choose next maps $f_A: SA \to KU \wedge X$ and $f_D: \Sigma^1 SD \to KU \wedge X$ whose induced homomorphisms are respectively the canonical inclusions $i_A: A \to A \oplus Z$ and $i_D: D \to D \oplus Z/2m$ in KU-homologies. By use of [**Y2**, Lemma 1.2] there exists a map $g_D: \Sigma^1 SD \to KC \wedge X$ with $(\zeta \wedge 1) g_D = f_D$ because $\mathrm{Ext}(D, KU_2 X)$ is uniquely

2-divisible. Then the composite maps $(\varepsilon_0\pi_u^{-1}\wedge 1)f_A: SA\to\Sigma^2 KO\wedge X$ and $(\tau\pi_c^{-1}\wedge 1)g_D: SD\to\Sigma^2 KO\wedge X$ are both trivial because $KO_6X=0=KO_7X$. Hence we get maps $h_A: SA\to KO\wedge X$ and $h_D: \Sigma^1 SD\to KO\wedge X$ with $(\varepsilon_u\wedge 1)h_A=f_A$ and $(\varepsilon_u\wedge 1)h_D=f_D$.

We finally apply [**Y3**, Proposition 1.1] to show that the map $h=h_A\vee h_D\vee h_{M'}$: $SA\vee\Sigma^1 SD\vee M'_{2m}\to KO\wedge X$ is a quasi KO_*-equivalence.

2. The cofiber MP_{4m} of the map $i\eta\vee\bar{\eta}: \Sigma^1\vee\Sigma^2\to SZ/4m$.

2.1. Let B, E be 2-torsion free abelian groups and $m=2^k$, $k\geqq 0$. We here deal with a CW-spectrum X such that

(2.1) $KU_0X\cong B\oplus Z\oplus Z/2m$ on which $t_*=\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 1 & 1\end{pmatrix}$, and $KU_1X\cong E\oplus Z$ on which $t_*=-1$, and in addition $KO_iX\cong Z/4m$, 0, $B\oplus Z$ or $E\oplus Z$ according as $i=0$, 1, 2 or 7 (cf. Proposition 1.2).

For such a CW-spectrum X it is verified that $KO_6X\cong B\oplus Z$ because $(\tau\pi_c^{-1})_*$: $KC_1X\to KO_6X$ is an isomorphism. By means of Proposition 1.2 we note that the wedge sum $\Sigma^2SB\vee\Sigma^3SE\vee MP_{4m}$ satisfies the above condition (2.1). In this section we will conversely prove that a CW-spectrum X satisfying (2.1) is quasi KO_*-equivalent to $\Sigma^2SB\vee\Sigma^3SE\vee MP_{4m}$. For this purpose we will first investigate the behaviour of the conjugations t_* on $[SZ/4m, KU\wedge X]$ and $[M_{4m}, KU\wedge X]$ as in Lemmas 1.3 and 1.4 because we can use the cofiber sequences (1.5), (1.8).

Consider the map $\lambda=\lambda_{4m,2m}: SZ/4m\to SZ/2m$ associated with the canonical epimorphism $\rho_{4m,2m}: Z/4m\to Z/2m$. This map λ gives rise to the following commutative diagram

$$\begin{array}{ccccccccc} 0\to & \mathrm{Ext}(KU_0SZ/2m, KU_1X) & \to & [SZ/2m, KU\wedge X] & \xrightarrow{\kappa_0} & \mathrm{Hom}(KU_0SZ/2m, KU_0X) & \to 0 \\ (2.2) & \downarrow & & \downarrow \lambda^* & & \downarrow & \\ 0\to & \mathrm{Ext}(KU_0SZ/4m, KU_1X) & \to & [SZ/4m, KU\wedge X] & \xrightarrow[\kappa_0]{} & \mathrm{Hom}(KU_0SZ/4m, KU_0X) & \to 0 \end{array}$$

with split exact rows. Hence $\lambda^*: [SZ/2m, KU\wedge X]\to[SZ/4m, KU\wedge X]$ is represented by the matrix

(2.3) $\lambda^*=\begin{pmatrix}1 & 0\\ l & 2\end{pmatrix}$ for some homomorphism $l: Z/2m\to(E\oplus Z)\otimes Z/4m$

where $[SZ/2n, KU\wedge X]\cong\mathrm{Hom}(KU_0SZ/2n, KU_0X)\oplus\mathrm{Ext}(KU_0SZ/2n, KU_1X)\cong Z/2m\oplus(E\oplus Z)\otimes Z/2n$ for $n=m$ or $2m$. In fact, we may take to be $2l=0$ as is shown in the proof of the following lemma.

LEMMA 2.1. *There exists a direct sum decomposition*

$$[SZ/4m,\ KU\wedge X]\cong \mathrm{Hom}(KU_0SZ/4m,\ KU_0X)\oplus \mathrm{Ext}(KU_0SZ/4m,\ KU_1X)$$
$$\cong Z/2m\oplus(E\oplus Z)\otimes Z/4m$$

on which $t_*=\begin{pmatrix}1 & 0\\ 2i_2 & -1\end{pmatrix}$ *where* $2i_2: Z/2m\to(E\otimes Z/4m)\oplus Z/4m$ *denotes the canonical injection into the last factor.*

PROOF. Denote by t_{2n} the conjugation t_* on $[SZ/2n,\ KU\wedge X]$, $n=m$ or $2m$. Obviously we may express as $t_{2n}=\begin{pmatrix}1 & 0\\ a_{2n} & -1\end{pmatrix}$ for some homomorphism $a_{2n}: Z/2m\to(E\oplus Z)\otimes Z/2n$ where $[SZ/2n,\ KU\wedge X]$ is decomposed as in (2.3). In order to represent t_{2n} precisely we first observe the connecting homomorphism $\delta: \mathrm{Hom}(Z/2m,\ KU_0X)\overset{\cong}{\longleftarrow}\mathrm{Hom}(Z/2m,\ Z/2m)\to\mathrm{Ext}(Z/2m,\ KU_1X\otimes Z/2)$ associated with the short exact sequence $0\to KU_1X\otimes Z/2\to KC_0X\to Z/2m\to 0$ induced by the cofiber sequence (1.3)$'$, as in the proof of Lemma 1.3. This short exact sequence is obtained as the canonical exact sequence $0\to(E\otimes Z/2)\oplus Z/2\to(E\otimes Z/2)\oplus Z/4m\to Z/2m\to 0$ because the cofiber sequence (1.2) gives rise to an exact sequence $0\to KO_0X\to KC_0X\to KO_5X\to 0$. So it is easily seen that the connecting homomorphism $\delta: Z/2m\to(E\otimes Z/2)\oplus Z/2$ is given by $\delta(1)=(0,\ 1)$.

Hence the homomorphism $a_{2m}: Z/2m\to(E\otimes Z/2m)\oplus Z/2m$ may be taken to be the injection i_2 into the last factor, by replacing the splitting of the upper κ_0 in (2.2) suitably if necessary. Thus $t_{2m}=\begin{pmatrix}1 & 0\\ i_2 & -1\end{pmatrix}$. Then the equality $\lambda^*t_{2m}=t_{4m}\lambda^*$ shows that $a_{4m}=2i_2+2l: Z/2m\to(E\otimes Z/4m)\oplus Z/4m$. By replacing suitably the splitting of the lower κ_0 in (2.2) if necessary, we may take to be $a_{4m}=2i_2$, and hence $2l=0$. Thus $[SZ/4m,\ KU\wedge X]$ has a direct sum decomposition so that $t_{4m}=\begin{pmatrix}1 & 0\\ 2i_2 & -1\end{pmatrix}$ on it as desired.

LEMMA 2.2. *There exists a direct sum decomposition*

$$[M_{4m},\ KU\wedge X]\cong \mathrm{Hom}(KU_0SZ/4m,\ KU_0X)\oplus \mathrm{Ext}(KU_0SZ/4m,\ KU_1X)\oplus KU_2X$$
$$\cong Z/2m\oplus(E\oplus Z)\otimes Z/4m\oplus(B\oplus Z\oplus Z/2m)$$

on which $t_*=\begin{pmatrix}1 & 0 & 0\\ 2i_2 & -1 & 0\\ i_3 & 0 & -t_0\end{pmatrix}$ *where* $t_0=\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 1 & 1\end{pmatrix}$ *on* $B\oplus Z\oplus Z/2m$ *and* $2i_2: Z/2m\to(E\oplus Z)\otimes Z/4m$ *and* $i_3: Z/2m\to B\oplus Z\oplus Z/2m$ *denote the canonical injections into the last factor respectively.*

PROOF. Use the commutative diagram

$$\begin{array}{ccccccc} \Sigma^1 & \xrightarrow{\eta} & \Sigma^0 & \xrightarrow{i_P} & P & \xrightarrow{j_P} & \Sigma^2 \\ \| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle k} & & \| \\ \Sigma^1 & \xrightarrow[i\eta]{} & SZ/4m & \xrightarrow[i_M]{} & M_{4m} & \xrightarrow[j_M]{} & \Sigma^2 \end{array}$$

which gives rise to the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [\Sigma^2, KU\wedge X] & \xrightarrow{j_M^*} & [M_{4m}, KU\wedge X] & \xrightarrow{i_M^*} & [SZ/4m, KU\wedge X] & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle k^*} & & \downarrow{\scriptstyle i^*} & & \\ 0 & \longrightarrow & [\Sigma^2, KU\wedge X] & \xrightarrow[j_P^*]{} & [P, KU\wedge X] & \xrightarrow[i_P^*]{} & [\Sigma^0, KU\wedge X] & \longrightarrow & 0 \end{array}$$

with split exact rows. Denote by t_P and t_M the conjugations t_* on $[P, KU\wedge X]$ and $[M_{4m}, KU\wedge X]$ respectively. As is easily verified, t_P may be represented by the matrix $\begin{pmatrix} t_0 & 0 \\ t_0 & -t_0 \end{pmatrix}$ on $[P, KU\wedge X]\cong KU_0X\oplus KU_2X$ where $t_0=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ on $B\oplus Z\oplus Z/2m$. Moreover, Lemma 2.1 asserts that t_M is written into the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 2i_2 & -1 & 0 \\ b & c & -t_0 \end{pmatrix}$ for some homomorphisms $b: Z/2m\to B\oplus Z\oplus Z/2m$ and $c: (E\oplus Z)\otimes Z/4m\to B\oplus Z\oplus Z/2m$, where $[M_{4m}, KU\wedge X]$ is decomposed by using the splitting exact sequence (1.10) and Lemma 2.1.

On the other hand, we may express $k^*: [M_{4m}, KU\wedge X]\to[P, KU\wedge X]$ as the matrix $\begin{pmatrix} i_3 & 0 & 0 \\ d & e & 1 \end{pmatrix}$ for some homomorphisms $d: Z/2m\to B\oplus Z\oplus Z/2m$ and $e: (E\oplus Z)\otimes Z/4m\to B\oplus Z\oplus Z/2m$. Then the equality $t_Pk^*=k^*t_M$ shows that $b=-i_3-2d-e(2i_2)$ and $c=0$. So we may take to be $b=i_3$ and $c=0$ by replacing suitably the splitting of i_M^* if necessary. Thus $[M_{4m}, KU\wedge X]$ has a direct sum decomposition so that $t_M=\begin{pmatrix} 1 & 0 & 0 \\ 2i_2 & -1 & 0 \\ i_3 & 0 & -t_0 \end{pmatrix}$ on it as desired.

2.2. The realification map $\varepsilon_0\pi_u^{-1}: KU\to\Sigma^2KO$ gives rise to the following commutative diagram

$$(2.4)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & [\Sigma^2, KU\wedge X] & \xrightarrow{j_M^*} & [M_{4m}, KU\wedge X] & \xrightarrow{i_M^*} & [SZ/4m, KU\wedge X] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle e_0} & & \downarrow{\scriptstyle e_M} & & \downarrow{\scriptstyle e_{4m}} & & \\ 0 & \longrightarrow & [\Sigma^2, \Sigma^2KO\wedge X] & \xrightarrow[j_M^*]{} & [M_{4m}, \Sigma^2KO\wedge X] & \xrightarrow[i_M^*]{} & [SZ/4m, \Sigma^2KO\wedge X] & \longrightarrow & 0 \end{array}$$

with exact rows, for any CW-spectrum X satisfying (2.1). The top exact sequence is evidently split, and the bottom one is also split because

$j^*: [\Sigma^1, \Sigma^2 KO\wedge X]\otimes Z/4m \to [SZ/4m, \Sigma^2 KO\wedge X]$ is an isomorphism. We will explicitly give a matrix representation of the induced homomorphism $e_M: [M_{4m}, KU\wedge X]\to[M_{4m}, \Sigma^2 KO\wedge X]$.

The short exact sequence $0\to KO_2X\to KU_2X\xrightarrow{e_0} KO_0X\to 0$ is obtained as the exact sequence $0\to B\oplus Z\xrightarrow{\varphi} B\oplus Z\oplus Z/2m\xrightarrow{\psi} Z/4m\to 0$, where φ and ψ are represented by the matrices $\begin{pmatrix}1 & 0\\ 0 & 2\\ 0 & -1\end{pmatrix}$ and $(0\ \ 1\ \ 2)$. Thus

(2.5) $\quad e_0: KU_2X\to KO_0X$ is expressed as the row $(0\ \ 1\ \ 2)$.

We will next investigate the right arrow e_{4m} in (2.4) by making use of the commutative diagram

$$\begin{array}{ccccccc}
0\to \mathrm{Ext}(KU_0SZ/2n, KU_1X) & \to & [SZ/2n, KU\wedge X] & \xrightarrow{\kappa_0} & \mathrm{Hom}(KU_0SZ/2n, KU_0X)\to 0\\
\downarrow & & \downarrow{\scriptstyle e_{2n}} & & \\
\mathrm{Ext}(KO_0SZ/2n, KO_7X) & \xrightarrow{\cong} & [SZ/2n, \Sigma^2 KO\wedge X] & &
\end{array} \tag{2.6}$$

with a split exact row, where $n=m$ or $2m$. The short exact sequence $0\to KU_1X\to KO_7X\to Z/2\to 0$ induced by the cofiber sequence (1.1) is obtained as the canonical exact sequence $0\to E\oplus Z\to E\oplus Z\to Z/2\to 0$. Hence the left vertical arrow is expressed as the matrix $\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$ on $(E\otimes Z/2n)\oplus Z/2n$. Therefore $e_{2n}: [SZ/2n, KU\wedge X]\to[SZ/2n, \Sigma^2 KO\wedge X]$ is written into the matrix $\begin{pmatrix}u_{2n} & 1 & 0\\ v_{2n} & 0 & 2\end{pmatrix}$ for some homomorphisms $u_{2n}: Z/2m\to E\otimes Z/2n$ and $v_{2n}: Z/2m\to Z/2n$ where $[SZ/2n, KU\wedge X]\cong Z/2m\oplus(E\oplus Z)\otimes Z/2n$ is decomposed as in (2.3) and $[SZ/2n, \Sigma^2 KO\wedge X]\cong(E\oplus Z)\otimes Z/2n$.

In order to express e_{2n} $(n=m, 2m)$ precisely we here use the commutative diagram

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \to & KU_1X & \to & KO_7X & \to & KO_0X & \to & KU_0X & \to & KO_6X \to 0\\
 & & \| & & \downarrow & & \downarrow & & \| & & \downarrow\cong \\
0 & \to & KU_1X & \to & KU_7X & \to & KC_0X & \to & KU_0X & \to & B\oplus Z \to 0\\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & KO_5X & = & KO_5X & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}$$

where the two long exact sequences are induced by the cofiber sequences (1.1) and (1.3). Since the left column is obtained as the canonical exact sequence $0\to E\oplus Z\to E\oplus Z\to E\otimes Z/2\to 0$, the discussion given in the proof of Lemma 2.1 shows that $2u_{2m}=0$ and $v_{2m}=-1$. So we may take to be $u_{2m}=0$ and $v_{2m}=-1$, thus $e_{2m}=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$ where $[SZ/2m, KU\wedge X]$ is decomposed as in the proof of Lemma 2.1 and $[SZ/2m, \Sigma^2 KO\wedge X]$ might be changed by a suitable direct sum decomposition if necessary.

On the other hand, the induced homomorphism $\lambda^*: [SZ/2m, KU\wedge X]\to [SZ/4m, KU\wedge X]$ is represented by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ l_1 & 2 & 0 \\ l_2 & 0 & 2 \end{pmatrix}$ for some homomorphism $l=\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}: Z/2m\to(E\oplus Z)\otimes Z/4m$ with $2l=0$, because of (2.3). Moreover the induced homomorphism $\lambda^*: [SZ/2m, \Sigma^2 KO\wedge X]\to[SZ/4m, \Sigma^2 KO\wedge X]$ is given by the canonical inclusion $i_{2m,4m}: (E\oplus Z)\otimes Z/2m\to(E\oplus Z)\otimes Z/4m$. Therefore the equality $\lambda^* e_{2m}=e_{4m}\lambda^*$ shows that $u_{4m}=-l_1$ and $v_{4m}=-2$. So we may take to be $u_{4m}=0$, $v_{4m}=-2$ by replacing suitably the splitting of κ_0 in (2.6) if necessary. Thus we see that

(2.7) $e_{4m}: [SZ/4m, KU\wedge X]\to[SZ/4m, \Sigma^2 KO\wedge X]$ is expressed as the matrix $\begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & 2 \end{pmatrix}$.

Remark that the conjugation t_{4m} on $[SZ/4m, KU\wedge X]$ remains to be expressed by the same matrix as given in Lemma 2.1 because $2l_1=0$, in spite of changing the direct sum decomposition on $[SZ/4m, KU\wedge X]$ slightly in the above discussion.

LEMMA 2.3. *There exist direct sum decompositions*

$$[M_{4m}, KU\wedge X]\cong \mathrm{Hom}(KU_0SZ/4m, KU_0X)\oplus\mathrm{Ext}(KU_0SZ/4m, KU_1X)\oplus KU_2X$$
$$\cong Z/2m\oplus(E\oplus Z)\otimes Z/4m\oplus(B\oplus Z\oplus Z/2m),$$
$$[M_{4m}, \Sigma^2 KO\wedge X]\cong \mathrm{Ext}(KO_0SZ/4m, KO_7X)\oplus KO_0X\cong(E\oplus Z)\otimes Z/4m\oplus Z/4m$$

so that $(\varepsilon_0\pi_u^{-1})_*: [M_{4m}, KU\wedge X]\to[M_{4m}, \Sigma^2 KO\wedge X]$ *is represented by the matrix*
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 2 \end{pmatrix}.$$

PROOF. From (2.5) and (2.7) it follows that $e_M: [M_{4m}, KU\wedge X]\to [M_{4m}, \Sigma^2 KO\wedge X]$ is written into the matrix $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ r & s & t & 0 & 1 & 2 \end{pmatrix}$ for some homomorphisms $r: Z/2m\to Z/4m$, $s: E\otimes Z/4m\to Z/4m$ and $t: Z/4m\to Z/4m$. Since the conjugation t_M on $[M_{4m}, KU\wedge X]$ is explicitly given in Lemma 2.2, the equality $e_M t_M = -e_M$ then implies that $2t=-2r-2: Z/2m \to Z/4m$. So we may take to be $r=0$, $s=0$ and $t=-1$ by replacing suitably splittings of i_M^*'s in (2.4) if necessary. Thus we have direct sum decompositions on $[M_{4m}, KU \wedge X]$ and $[M_{4m}, \Sigma^2 KO \wedge X]$ as desired.

We again remark that the conjugation t_M on $[M_{4m}, KU \wedge X]$ remains to be expressed by the same matrix as given in Lemma 2.2, in spite of changing slightly the direct sum decomposition on $[M_{4m}, KU \wedge X]$ in the above lemma.

2.3. For any CW-spectrum X satisfying (2.1) we use the commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathrm{Ext}(KU_0M_{4m}, KU_1X) & \xrightarrow{\cong} & \mathrm{Ext}(KU_0SZ/4m, KU_1X) \\
 & & & & \downarrow & & \downarrow \\
0 \longrightarrow & [\Sigma^2, KU \wedge X] & \longrightarrow & & [M_{4m}, KU \wedge X] & \longrightarrow & [SZ/4m, KU \wedge X] \\
 & \cong \downarrow \kappa_0 & & & \downarrow \kappa_0 & & \downarrow \kappa_0 \\
0 \to & \mathrm{Hom}(KU_0\Sigma^2, KU_0X) & \longrightarrow & & \mathrm{Hom}(KU_0M_{4m}, KU_0X) & \longrightarrow & \mathrm{Hom}(KU_0SZ/4m, KU_0X) \to 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
\end{array}$$

in order to rewrite the direct sum decomposition on $[M_{4m}, KU \wedge X]$ given in Lemma 2.3 as follows:

$$\begin{aligned}
(2.8) \qquad [M_{4m}, KU \wedge X] &\cong \mathrm{Hom}(KU_0M_{4m}, KU_0X) \oplus \mathrm{Ext}(KU_0M_{4m}, KU_1X) \\
&\cong \mathrm{Hom}(KU_0SZ/4m, KU_0X) \oplus \mathrm{Hom}(KU_0\Sigma^2, KU_0X) \oplus \mathrm{Ext}(KU_0M_{4m}, KU_1X) \\
&\cong Z/2m \oplus (B \oplus Z \oplus Z/2m) \oplus (E \oplus Z) \otimes Z/4m\,.
\end{aligned}$$

The cofiber sequence (1.8) gives rise to a short exact sequence

$$0 \longrightarrow [\Sigma^3, KU \wedge X] \xrightarrow{i_{MP}^*} [MP_{4m}, KU \wedge X] \xrightarrow{k_{MP}^*} [M_{4m}, KU \wedge X] \longrightarrow 0\,.$$

Notice that the universal coefficient sequence

$$0 \to \mathrm{Ext}(KU_0MP_{4m}, KU_1X) \to [MP_{4m}, KU \wedge X] \to \bigoplus_{i=0,1} \mathrm{Hom}(KU_iMP_{4m}, KU_iX) \to 0$$

is a pure exact sequence (use [**Y1**, Theorem 5]). Then we see by means of [**HM**, Lemma 3.6] that its pure exact sequence is split because $\mathrm{Pext}((B \oplus E) * Q/Z, (E \oplus Z) \otimes Z/2m) = 0$ for $m=2^k$, $k \geqq 0$. We will here give a matrix representation of the induced homomorphism k_{MP}^* explicitly.

LEMMA 2.4. *There exists a direct sum decomposition*

$$\begin{aligned}
&[MP_{4m},\ KU\wedge X]\\
&\cong \mathrm{Hom}(KU_0MP_{4m},\ KU_0X)\oplus\mathrm{Hom}(KU_1MP_{4m},\ KU_1X)\oplus\mathrm{Ext}(KU_0MP_{4m},\ KU_1X)\\
&\cong (Z/2m\oplus B\oplus Z\oplus Z/2m)\oplus(E\oplus Z)\oplus(E\oplus Z)\otimes Z/2m
\end{aligned}$$

so that $k_{MP}^*: [MP_{4m},\ KU\wedge X]\to[M_{4m},\ KU\wedge X]$ *is represented by the matrix* $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 2\end{pmatrix}$ *where* $[M_{4m},\ KU\wedge X]\cong\mathrm{Hom}(KU_0M_{4m},\ KU_0X)\oplus\mathrm{Ext}(KU_1M_{4m},\ KU_1X)\cong$ $(Z/2m\oplus B\oplus Z\oplus Z/2m)\oplus(E\oplus Z)\otimes Z/4m$.

PROOF. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{Ext}(KU_0MP_{4m},\ KU_1X) & \longrightarrow & \mathrm{Ext}(KU_0M_{4m},\ KU_1X) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & [\Sigma^3,\ KU\wedge X] & \xrightarrow{\iota_{MP}^*} & [MP_{4m},\ KU\wedge X] & \xrightarrow{k_{MP}^*} & [M_{4m},\ KU\wedge X] & \longrightarrow & 0\\
 & & & & \downarrow\binom{\kappa_0}{\kappa_1} & & \downarrow\kappa_0 & & \\
 & & \cong\downarrow\kappa_1 & & \mathrm{Hom}(KU_0MP_{4m},\ KU_0X) & \xrightarrow{\cong} & \mathrm{Hom}(KU_0M_{4m},\ KU_0X) & & \\
 & & & & \oplus & & \downarrow & & \\
 & & \mathrm{Hom}(KU_1\Sigma^3,\ KU_1X) & \longrightarrow & \mathrm{Hom}(KU_1MP_{4m},\ KU_1X) & & 0 & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

with exact row and columns. The top arrow is the canonical monomorphism $i_{2m,4m}: (E\oplus Z)\otimes Z/2m\to(E\oplus Z)\otimes Z/4m$ and the bottom one is just multiplication by 2 on $E\oplus Z$. Observe the connecting homomorphism $\delta: \mathrm{Hom}(KU_1MP_{4m}, KU_1X)$ $\to\mathrm{Ext}(Z/2, KU_1X)$ associated with the short exact sequence $0\to KU_1MP_{4m}\to KU_1\Sigma^3\to Z/2\to 0$ induced by the cofiber sequence (1.8). Since the connecting homomorphism $\delta: E\oplus Z\to(E\oplus Z)\otimes Z/2$ is evidently the canonical epimorphism, $k_{MP}^*: [MP_{4m},\ KU\wedge X]\to[M_{4m},\ KU\wedge X]$ may be expressed as the matrix $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 2\end{pmatrix}$. Here $[M_{4m},\ KU\wedge X]$ is decomposed as in (2.6) and $[MP_{4m},\ KU\wedge X]$ is done by choosing splittings of κ_0 and κ_1 suitably.

PROPOSITION 2.5. *Let* B, E *be 2-torsion free abelian groups,* $m=2^k$, $k\geqq 0$, *and* X *be a CW-spectrum satisfying the condition* (2.1). *Then there exists a map* $f_{MP}: MP_{4m}\to KU\wedge X$ *such that the composite* $(\varepsilon_0\pi_u^{-1}\wedge 1)f_{MP}k_{MP}: M_{4m}\to MP_{4m}\to KU\wedge X\to\Sigma^2KO\wedge X$ *is trivial, whose induced homomorphisms of KU-homologies in dimensions* 0, 1 *are respectively the canonical inclusions* $i_0: Z\oplus Z/2m\to B\oplus Z\oplus Z/2m$

and $i_1: Z\to E\oplus Z$.

PROOF. Among maps $f: MP_{4m}\to KU\wedge X$ inducing the canonical inclusions i_0, i_1 in KU-homologies we pick up the map $f_{MP}: MP_{4m}\to KU\wedge X$ corresponding to the element $w=(1, 0, 1, 0, 0, 1, 0, 0)$ under the direct sum decomposition on $[MP_{4m}, KU\wedge X]$ given in Lemma 2.4. By means of Lemmas 2.3 and 2.4 we can easily compute that $e_M k_{MP}^*(w)=0$. Thus the composite $(\varepsilon_0\pi_u^{-1}\wedge 1)f_{MP}k_{MP}: M_{4m}\to\Sigma^2 KO\wedge X$ becomes trivial.

2.4. We will now prove a main result in this section, which characterize a CW-spectrum X admitting the same quasi KO_*-type as $\Sigma^2SB\vee\Sigma^3SE\vee MP_{4m}$.

THEOREM 2.6. *Let B, E be 2-torsion free abelian groups such that* $\mathrm{Ext}(E, B\oplus Z)$ *is uniquely 2-divisible, and $m=2^k$, $k\geqq 0$. A CW-spectrum X is quasi KO_*-equivalent to the wedge sum $\Sigma^2SB\vee\Sigma^3SE\vee MP_{4m}$ if and only if $KU_0X\cong B\oplus Z\oplus Z/2m$ on which $t_*=\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 1 & 1\end{pmatrix}$, and $KU_1X\cong E\oplus Z$ on which $t_*=-1$, and in addition $KO_iX\cong Z/4m$, 0, $B\oplus Z$ or $E\oplus Z$ according as $i=0$, 1, 2 or 7.*

PROOF. The "only if" part is evident from Proposition 1.2.

The "if" part: By means of Proposition 2.5 we can choose a map $f_{MP}: MP_{4m}\to KU\wedge X$ inducing the canonical inclusions $i_0: Z\oplus Z/2m\to B\oplus Z\oplus Z/2m$, $i_1: Z\to E\oplus Z$ in KU-homologies such that the composite $(\varepsilon_0\pi_u^{-1}\wedge 1)f_{MP}k_{MP}: M_{4m}\to\Sigma^2KO\wedge X$ is trivial. So there exist maps $h_0: M_{4m}\to KO\wedge X$, $h_1: \Sigma^1\to KO\wedge X$ making the diagram below commutative

$$\begin{array}{ccccc} M_{4m} & \xrightarrow{k_{MP}} & MP_{4m} & \xrightarrow{l_{MP}} & \Sigma^3 \\ \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle f_{MP}} & & \downarrow{\scriptstyle h_1} \\ KO\wedge X & \xrightarrow[\varepsilon_u\wedge 1]{} & KU\wedge X & \xrightarrow[\varepsilon_0\pi_u^{-1}\wedge 1]{} & \Sigma^2KO\wedge X. \end{array}$$

Since the map h_1 is trivial, we get a map $h_{MP}: MP_{4m}\to KO\wedge X$ with $(\varepsilon_u\wedge 1)h_{MP}=f_{MP}$.

Choose next maps $f_B: \Sigma^2SB\to KU\wedge X$ and $f_E: \Sigma^3SE\to KU\wedge X$ whose induced homomorphisms are respectively the canonical inclusions $i_B: B\to B\oplus Z\oplus Z/2m$ and $i_E: E\to E\oplus Z$ in KU-homologies. The composite $(\varepsilon_0\pi_u^{-1}\wedge 1)f_B: SB\to KO\wedge X$ becomes trivial because the realification $(\varepsilon_0\pi_u^{-1})_*: KU_2X\to KO_0X$ restricted to B is trivial by (2.5) and $KO_1X=0$. On the other hand, there exists a map $g_E: \Sigma^3SE\to KC\wedge X$ with $(\zeta\wedge 1)g_E=f_E$ by means of [**Y2**, Lemma 1.2] because $\mathrm{Ext}(E, KU_4X)$ is uniquely 2-divisible. Making use of this map g_E we will show that the composite $(\varepsilon_0\pi_u^{-1}\wedge 1)f_E: \Sigma^1SE\to KO\wedge X$ is trivial, too.

Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}(E, KO_1X) & \longrightarrow & [SE, KO\wedge X] & \longrightarrow & \mathrm{Hom}(E, KO_0X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow {\scriptstyle (\eta\wedge 1)_*} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ext}(E, KO_2X) & \longrightarrow & [SE, \Sigma^{-1}KO\wedge X] & \longrightarrow & \mathrm{Hom}(E, KO_1X) & \longrightarrow & 0
\end{array}$$

with $KO_1X=0$. In order to show that the central arrow $(\eta\wedge 1)_*$ is trivial, we observe the connecting homomorphism $\delta: \mathrm{Hom}(E, KO_0X)\to \mathrm{Ext}(E, KO_2X)$ associated with the short exact sequence $0\to KO_2X\to KU_2X\to KO_0X\to 0$ induced by the cofiber sequence (1.1). Since $KO_0X\cong Z/4m$ and $\mathrm{Ext}(E, KO_2X)\cong \mathrm{Ext}(E, B\oplus Z)$ is uniquely 2-divisible, the connecting homomorphism δ is trivial. Thus $(\eta\wedge 1)_*: [SE, KO\wedge X]\to[SE, \Sigma^{-1}KO\wedge X]$ is trivial, and hence $(\varepsilon_0\pi_u^{-1}\wedge 1)f_E: \Sigma^1 SE\to KO\wedge X$ becomes trivial.

Consequently we get maps $h_B: \Sigma^2SB\to KO\wedge X$ and $h_E: \Sigma^3SE\to KO\wedge X$ as well as $h_{MP}: MP_{4m}\to KO\wedge X$ with $(\varepsilon_u\wedge 1)h_H=f_H$ for $H=B$, E and MP. We can now apply [**Y3**, Proposition 1.1] to show that the map $h=h_B\vee h_E\vee h_{MP}: \Sigma^2SB\vee\Sigma^3SE\vee MP_{4m}\to KO\wedge X$ is a quasi KO_*-equivalence.

Combining Theorem 2.6 with Proposition 1.2 we obtain the following result immediately.

COROLLARY 2.7. *Σ^4MP_{4m} is quasi KO_*-equivalent to MP_{4m} for any m, $m\geqq 1$.*

3. The stunted real projective spaces RP^n/RP^m.

3.1. Let RP^n be the real projective n-space, and X_n be the suspension spectrum $\Sigma^{-n}SP^2S^n$ whose n-th term is the symmetric square SP^2S^n of n-sphere. The spectrum X_{n+1} is exhibited by the following two cofiber sequences [**L**, **JTTW**]:

$$\Sigma^n \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow \Sigma^{n+1} \tag{3.1}$$

$$RP^n \longrightarrow \Sigma^0 \longrightarrow X_{n+1} \longrightarrow \Sigma^1 RP^n \tag{3.2}$$

which are related by the commutative diagram below

$$\begin{array}{ccccccc}
 & & & & \Sigma^n & = & \Sigma^n \\
 & & & & \downarrow & & \downarrow \\
RP^{n-1} & \longrightarrow & \Sigma^0 & \longrightarrow & X_n & \longrightarrow & \Sigma^1RP^{n-1} \\
\downarrow & & \| & & \downarrow & & \downarrow \\
RP^n & \longrightarrow & \Sigma^0 & \longrightarrow & X_{n+1} & \longrightarrow & \Sigma^1RP^n \\
 & & & & \downarrow & & \downarrow \\
 & & & & \Sigma^{n+1} & = & \Sigma^{n+1}.
\end{array} \tag{3.3}$$

Hence we may regard that the stunted real projective space RP^n/RP^m is obtained by the cofiber sequence

(3.4) $$RP^n/RP^m \longrightarrow X_{m+1} \longrightarrow X_{n+1} \longrightarrow \Sigma^1(RP^n/RP^m), \qquad m<n.$$

In [**Y2**, Theorem 2.7] we have determined the quasi KO_*-type of X_{n+1} as follows.

THEOREM 3.1. *X_{n+1} is quasi KO_*-equivalent to Σ^0, P, Σ^4, $\Sigma^4\vee\Sigma^4$, Σ^4, P, Σ^0, $\Sigma^0\vee\Sigma^0$ according as $n=8r, 8r+1, \cdots, 8r+7$.*

As a result we note that

(3.5) $\Sigma^{4m}X_{4m+n}$ is quasi KO_*-equivalent to X_n.

The conjugation t on KU gives rise to an involution t_* on KU_*X for any CW-spectrum X. Thus the KU-homology KU_*X is regarded as a $Z/2$-graded abelian group with involution. In order to investigate the structure of $KU_*(RP^n/RP^m)$ as a $Z/2$-graded abelian group with involution, we recall the following result (see [**Ad1**], [**F**] or [**Y2**, Proposition 2.6]).

PROPOSITION 3.2. i) *$KU_0RP^n=0$, and $KU_{-1}RP^n\cong Z/2^s$ or $Z\oplus Z/2^s$ according as $n=2s$ or $2s+1$.*

ii) *The conjugation t_* on $KU_{-1}RP^n$ behaves as $t_*=1$ if $n\not\equiv 1 \bmod 4$ and $t_*=\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$ if $n\equiv 1 \bmod 4$.*

iii) *$KO_0RP^n=0$ if $n\equiv 2, 3, 4 \bmod 8$, $KO_4RP^n=0$ if $n\equiv 0, 6, 7 \bmod 8$ and $KO_6RP^n=0$ for all n.*

Let RP^n_σ be a fixed CW-spectrum such that $KU_*RP^n_\sigma\cong KU_*RP^n$ and the conjugation t_* on $KU_{-1}RP^n_\sigma$ behaves as

(3.6) $t_*=1$ if $n\not\equiv 3 \bmod 4$ and $t_*=\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$ if $n\equiv 3 \bmod 4$, and in addition $KO_0RP^n_\sigma=0$ if $n\equiv 4, 5, 6 \bmod 8$, $KO_4RP^n_\sigma=0$ if $n\equiv 0, 1, 2 \bmod 8$ and $KO_6RP^n_\sigma=0$ for all n.

As an abelian group with involution $KU_*RP^n_\sigma$ differs from KU_*RP^n when n is odd, although they coincide when n is even. For example, as in Theorem 2 ii) we may set $\Sigma^1RP^n_\sigma$ to be $SZ/2^{4r}$, $\Sigma^0\vee SZ/2^{4r}$, $SZ/2^{4r+1}$, $M_{2^{4r+1}}$, $V_{2^{4r+2}}$, $\Sigma^4\vee V_{2^{4r+2}}$, $V_{2^{4r+3}}$, $M_{2^{4r+3}}$ according as $n=8r, 8r+1, \cdots, 8r+7$. By applying [**Y2**, Theorem 2.5] with (3.6) we notice that RP^n_σ is uniquely determined up to quasi KO_*-equivalence.

3.2. We will first study the KU-homology of RP^n/RP^m with the involution t_* (cf. [**Ad1**]). For simplicity RP^n/RP^m is sometimes abbreviated to be RP^n_{m+1}.

PROPOSITION 3.3. *As abelian groups with involution the KU-homologies of the stunted real projective spaces are isomorphic to the following KU-homologies:*

X	$=RP^{4m+n}/RP^{4m}$	RP^{4m+n}/RP^{4m-1}	RP^{4m+n-2}/RP^{4m-2}	RP^{4m+n-2}/RP^{4m-3}
$KU_0X\cong$	0	$KU_0\Sigma^{4m}$	0	$KU_0\Sigma^{4m-2}$
$KU_{-1}X\cong$	$KU_{-1}RP^n$	$KU_{-1}RP^n$	$KU_{-1}RP^n_\sigma$	$KU_{-1}RP^n_\sigma$.

PROOF. i) The $X=RP^{2t+n}/RP^{2t}$ case: Use the two cofiber sequences $RP^{2t}\to RP^{2t+n}\to RP^{2t+n}_{2t+1}\to\Sigma^1RP^{2t}$ and $X_{2t+1}\to X_{2t+n+1}\to RP^{2t+n}_{2t+1}\to\Sigma^1X_{2t+1}$. Then it follows from Theorem 3.1 and Proposition 3.2 that $KU_0RP^{2t+n}_{2t+1}=0$, and hence the sequence $0\to KU_{-1}RP^{2t}\to KU_{-1}RP^{2t+n}\to KU_{-1}RP^{2t+n}_{2t+1}\to 0$ is exact. The result is now immediate.

ii) The $X=RP^{2t+n}/RP^{2t-1}$ case: Use the two cofiber sequences $\Sigma^{2t}\to RP^{2t+n}_{2t}\to RP^{2t+n}_{2t+1}\to\Sigma^{2t+1}$ and $\Sigma^{2t-1}\to RP^{2t+n}_{2t-1}\to RP^{2t+n}_{2t}\to\Sigma^{2t}$. Assume that $KU_0RP^{2t+n}_{2t}=0$. Then there exist two exact sequences $0\to KU_1RP^{2t+n}_{2t}\to KU_1RP^{2t+n}_{2t+1}\to KU_0\Sigma^{2t}\to 0$ and $0\to KU_0\Sigma^{2t}\to KU_{-1}RP^{2t+n}_{2t-1}\to KU_{-1}RP^{2t+n}_{2t}\to 0$. By use of the former sequence we see that $KU_1RP^{2t+n}_{2t+1}\cong Z\oplus KU_1RP^{2t+n}_{2t}$. When $n=2s$, this is a contradiction because $KU_1RP^{2t+n}_{2t+1}\cong KU_1RP^n\cong Z/2^s$ by the above i) and Proposition 3.2. On the other hand, the latter sequence is obtained in the form of the short exact sequence $0\to Z\to Z\oplus Z/2^{s+1}\to Z/2^s\to 0$ when $n=2s+1$, because $KU_{-1}RP^{2t+n}_{2t-1}\cong KU_{-1}RP^{n+2}\cong Z\oplus Z/2^{s+1}$. This is obviously a contradiction, too. Therefore it is easily verified that $KU_0RP^{2t+n}_{2t}\cong Z$, and hence $KU_0RP^{2t+n}_{2t}\cong KU_0\Sigma^{2t}$ and $KU_1RP^{2t+n}_{2t}\cong KU_1RP^{2t+n}_{2t+1}$. The result is now immediate from the above i).

In order to determine the quasi KO_*-types of RP^{2t+n}/RP^{2t} we will next show that $KO_i(RP^{2t+n}/RP^{2t})=0$ for certain dimensions i as so are KO_iRP^n and $KO_iRP^n_\sigma$.

LEMMA 3.4. i) $KO_{4m}(RP^{4m+n}/RP^{4m})=0=KO_{4m}(RP^{4m+n}/RP^{4m-2})$ *if* $n\equiv 1$, 2, 3, 4, 5 mod 8.

ii) $KO_{4m+4}(RP^{4m+n}/RP^{4m})=0=KO_{4m+4}(RP^{4m+n}/RP^{4m-2})$ *if* $n\equiv 0$, 1, 5, 6, 7 mod 8.

iii) $KO_{4m+6}(RP^{4m+n}/RP^{4m})=0=KO_{4m+6}(RP^{4m+n}/RP^{4m-2})$ *for all* n.

PROOF. Since $\varepsilon_{u*}: KO_jRP^{2t+n}_{2t+1}\otimes Z[1/2]\to KU_jRP^{2t+n}_{2t+1}\otimes Z[1/2]$ is a monomorphism, Proposition 3.3 implies that $KO_jRP^{2t+n}_{2t+1}$ is 2-torsion whenever j is even. Use the cofiber sequence $RP^{2t+n}_{2t+1}\to X_{2t+1}\to X_{2t+n+1}\to\Sigma^1RP^{2t+n}_{2t+1}$, $t=2m$ or $2m-1$. By means of (3.5) we then get epimorphisms $KO_{i+1}X_{n+1}\to KO_{i+4m}RP^{4m+n}_{4m+1}$ and $KO_{i+1}X_{n-1}\to KO_{i+4m}RP^{4m+n-2}_{4m-1}$ for $i=0$, 4 or 6, because $X_1=\Sigma^0$ and $X_7\underset{KO}{\sim}\Sigma^0$. The result is now immediate from Theorem 3.1.

PROOF OF THEOREM 2 i) and iii). Combine Proposition 3.3 with Lemma 3.4 and then apply Theorem 2.5 in [**Y2**], as was previously done in [**Y2**] to prove Theorem 1.

3.3. In order to determine the quasi KO_*-type of RP^{4m+n-2}/RP^{4m-3} we will here calculate the KO-homology of RP^{4m+n-2}/RP^{4m-3} although it has completely done by [**FY**].

LEMMA 3.5. *The KO-homology* $KO_{i+4m}(RP^{4m+n-2}/RP^{4m-3})$ *is isomorphic to the following abelian group* $A_{i,n}$ *for each* i *and* n*:*

$$A_{i,n}=KO_4RP^{n+2},\ KO_5RP^{n+2},\ KO_4\Sigma^0,\ KO_7RP^{n+2},\ KO_0RP^{n+2},\ KO_5RP^n_\sigma,$$
$$KO_0\Sigma^0,\ KO_3RP^{n+2}/KO_2\Sigma^0 \qquad \textit{according as } i=0, 1, \cdots, 7.$$

PROOF. Use the three cofiber sequences $\Sigma^{4m-3}\to RP^{4m+n-2}_{4m-3}\to RP^{4m+n-2}_{4m-2}\to\Sigma^{4m-2}$, $\Sigma^{4m-2}\to RP^{4m+n-2}_{4m-2}\to RP^{4m+n-2}_{4m-1}\to\Sigma^{4m-1}$ and $RP^{4m+n-2}_{4m-2}\to X_{4m-2}\to X_{4m+n-1}\to\Sigma^1RP^{4m+n-2}_{4m-2}$. By means of Theorems 2 i), iii) and 3.1 we notice that $RP^{4m+n-2}_{4m-3}\underset{KO}{\sim}\Sigma^{4m-4}RP^{n+2}$, $RP^{4m+n-2}_{4m-1}\underset{KO}{\sim}\Sigma^{4m}RP^n_\sigma$ and $X_{4m-2}\underset{KO}{\sim}P$. Consider the long exact sequences of KO-homologies associated with the three cofiber sequences. By use of the first two exact sequences we see easily that $A_{4,n}\cong KO_0RP^{n+2}$, $A_{3,n}\cong KO_7RP^{n+2}$, $A_{5,n}\cong KO_5RP^n_\sigma$, $A_{2,n}\cong Z$ and $A_{0,n}$ is 2-torsion. By use of the third exact sequence we then get that $A_{6,n}\cong KO_6P$, $A_{0,n}\cong KO_1X_{n-1}$ and hence $A_{6,n}\cong KO_0\Sigma^0$, $A_{0,n}\cong KO_4RP^{n+2}$. So there exist short exact sequences $0\to KO_2\Sigma^0\to KO_3RP^{n+2}\to A_{7,n}\to 0$ and $0\to A_{1,n}\to KO_1RP^n_\sigma\to KO_2\Sigma^0\to 0$. Therefore it follows that $A_{7,n}\cong KO_3RP^{n+2}/KO_2\Sigma^0$, $A_{1,n}\cong KO_5RP^{n+2}$, and hence $A_{2,n}\cong KO_4\Sigma^0$.

In particular, Lemma 3.5 shows that

(3.7) i) $KO_{4+4m}(RP^{4m+n-2}/RP^{4m-3})=0=KO_{5+4m}(RP^{4m+n-2}/RP^{4m-3})$ if $n\equiv 0, 1, 2$ mod 8, and $KO_{7+4m}(RP^{4m+n-2}/RP^{4m-3})\cong Z/2^{4r+1}$, $Z\oplus Z/2^{4r+1}$ or $Z/2^{4r+2}$ according as $n=8r$, $8r+1$ or $8r+2$.

ii) $KO_{4m}(RP^{4m+n-2}/RP^{4m-3})=0=KO_{1+4m}(RP^{4m+n-2}/RP^{4m-3})$ if $n\equiv 4, 5, 6$ mod 8, and $KO_{3+4m}(RP^{4m+n-2}/RP^{4m-3})\cong Z/2^{4r+3}$, $Z\oplus Z/2^{4r+3}$ or $Z/2^{4r+4}$ according as $n=8r+4$, $8r+5$ or $8r+6$.

iii) $KO_{4m}(RP^{4m+n-2}/RP^{4m-3})=0=KO_{4+4m}(RP^{4m+n-2}/RP^{4m-3})$ if $n\equiv 3$ mod 4.

PROOF OF THEOREM 2 iv). The $n\not\equiv 3$ mod 4 case: Combine Proposition 3.3 with (3.7) i) and ii), and then apply Theorem 1.6. The result is easily shown.

The $n\equiv 3$ mod 4 case: Set $n=4s-1$. Putting Theorems 1 and 2 i) together we see that RP^{4m+n-2}_{4m-3} is quasi KO_*-equivalent to $\Sigma^{4m-5}M_{2^{2s}}$. Thus there exists a map $h_M:\Sigma^{4m-5}M_{2^{2s}}\to KO\wedge RP^{4m+n-2}_{4m-3}$ which induces the canonical isomorphism in KU-homology. Using the cofiber sequence (1.8) we consider the following diagram

$$\begin{array}{ccccccc}
\Sigma^{4m-3} & \longrightarrow & \Sigma^{4m-5}M_{2^{2s}} & \longrightarrow & \Sigma^{4m-5}MP_{2^{2s}} & \longrightarrow & \Sigma^{4m-2} \\
\downarrow{\scriptstyle \iota\wedge 1} & & \downarrow{\scriptstyle h_M} & & \downarrow{\scriptstyle h_{MP}} & & \downarrow{\scriptstyle \iota\wedge 1} \\
KO\wedge\Sigma^{4m-3} & \longrightarrow & KO\wedge RP^{4m+n-2}_{4m-3} & \longrightarrow & KO\wedge RP^{4m+n-2}_{4m-2} & \longrightarrow & KO\wedge\Sigma^{4m-2}.
\end{array}$$

The complexification $\varepsilon_{u*}: KO_{4m-3}RP^{4m+n-2}_{4m-3}\to KU_{4m-3}RP^{4m+n-2}_{4m-3}$ is a monomorphism because of (3.7) iii). Therefore the left square in the above diagram becomes commutative by means of Propositions 1.1, 1.2 and 3.3. Hence there exists a map $h_{MP}: \Sigma^{4m-5}MP_{2^{2s}} \to KO \wedge RP^{4m+n-2}_{4m-2}$ making the above diagram commutative. Obviously the map h_{MP} is a quasi KO_*-equivalence. Thus $\Sigma^{4m+2}PR^{4m+n-2}_{4m-2}$ is quasi KO_*-equivalent to $\Sigma^5 MP_{2^{2s}}$, which is also so to $\Sigma^1 MP_{2^{2s}}$ by Corollary 2.7.

REMARK. We may directly apply Theorem 2.6 combining Proposition 3.3 with Lemma 3.5 in the $n \equiv 3 \bmod 4$ case, in place of the above discussion using the cofiber sequence (1.8) and Corollary 2.7.

3.4. Let E be an associative ring spectrum with unit and ξ be an n-dimensional real vector bundle over a CW-complex X. Let $T(\xi)$ denote the Thom complex of ξ, thus $T(\xi)=D(\xi)/S(\xi)$ where $D(\xi)$ and $S(\xi)$ are respectively the associated disc and sphere bundle. We say ξ to be E-orientable if there exists a Thom class $u_\xi \in E^n T(\xi)$ such that the composite $(u_\xi \wedge p^+)\Delta: T(\xi) \to T(\xi) \wedge D(\xi)^+ \to \Sigma^n E \wedge X^+$ gives rise to an isomorphism $E_* T(\xi) \to E_{*-n} X^+$. Here Δ denotes the map induced by the diagonal map and p denotes the projection of the disc bundle $D(\xi)$ over X, and Y^+ stands for the based CW-complex with the additional base point $+$ for any CW-complex Y.

Hence we notice

(3.8) the Thom complex $T(\xi)$ is quasi E_*-equivalent to $\Sigma^n X^+$ whenever its n-dimensional vector bundle ξ over X is E-orientable.

PROOF OF THEOREM 2 ii). Let ξ_n be the canonical line bundle over RP^n and θ be the trivial line bundle over RP^n. As is well known, the $8m$-dimensional vector bundle $4m\xi_n \oplus 4m\theta$ over RP^n is KO-orientable because it has a spin reduction, thus its first and second Stiefel-Whitney classes vanish (see [**ABS**]). On the other hand, the Thom complex $T(4m\xi_n)$ is homeomorphic to the stunted real projective space RP^{4m+n}/RP^{4m-1} (see [**A**]). The result follows immediately from (3.8).

References

[Ad1] J.F. Adams, Vector fields on spheres, Ann. of Math., **75** (1962), 603-622.

[Ad2] J.F. Adams, Stable homotopy and generalized homology, Univ. of Chicago, Chicago, 1974.

[An] D.W. Anderson, A new cohomology theory, Thesis, Univ. of California, Berkeley, 1964.

[ArT] S. Araki and H. Toda, Multiplicative structures in mod q cohomology theories, I, Osaka J. Math., **2** (1965), 71-115.

[At] M.F. Atiyah, Thom complexes, Proc. London Math. Soc., **11** (1961), 291-310.

[ABS] M.F. Atiyah, R. Bott and A. Shapiro, Clifford modules, Topology, **3** (1964), 3-38.

[B] A.K. Bousfield, A classification of K-local spectra, preprint.

[F] M. Fujii, K_O-groups of projective spaces, Osaka J. Math., **4** (1967), 141–149.
[FY] M. Fujii and T. Yasui, K_O-groups of the stunted real projective spaces, Math. J. Okayama Univ., **16** (1973), 47–54.
[HM] M. Huber and W. Meier, Cohomology theories and infinite CW-complexes, Comment. Math. Helv., **53** (1978), 239–257.
[JTTW] I.M. James, E. Thomas, H. Toda and G.W. Whitehead, On the symmetric square of an sphere, J. Math. Mech., **12** (1963), 771–776.
[L] S.D. Liao, On the topology of cyclic products of spheres, Trans. Amer. Math. Soc., **77** (1954), 520–551.
[Y1] Z. Yosimura, Universal coefficient sequences for cohomology theories of CW-spectra, II, Osaka J. Math., **16** (1979), 201–217.
[Y2] Z. Yosimura, The quasi KO-homology types of the real projective spaces, Homotopy theory and related topics, Lecture Notes in Math., **1418**, Springer, 1990, pp. 156–174.
[Y3] Z. Yosimura, Quasi K-homology equivalences, I, Osaka J. Math., **27** (1990), to appear.
[Y4] Z. Yosimura, Quasi K-homology equivalences, II, Osaka J. Math., **27** (1990), to appear.

Zen-ichi YOSIMURA
Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi, Osaka 558
Japan