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Canonical stratification of non-degenerate complete intersection varieties

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§1. Introduction.

This paper is a continuation of [19] and [21]. Unless otherwise stated, we use the same notation as in [21] and we assume the results in [19] and [21]. (This paper is a revised version of [20]. §7 and Appendix B in §10 are added.) Let V be a germ of complete intersection variety at the origin of $\boldsymbol{C}^n$. The singularity of V is not necessarily isolated. The purpose of this paper is to describe the canonical toroidal resolution of V, the limits of the tangent spaces and to construct a canonical Whitney b-regular stratification on V under a certain condition (IND-condition). It is very important to get a regular stratification to study non-isolated singularities. In [4], J. Damon considered the topological stability problem of a family of complex hypersurfaces $V_t=\{f_t(\boldsymbol{z})=0\}$ with non-isolated singularities using the vector field argument. He showed that the topological types of V_t do not change if the Newton boundary is strongly non-degenerate and $\Gamma(f_t)=\Gamma(f_0)$. One motivation of this research is to understand this property from the stratification point of view. In [19], we have showed the existence of a canonical stratification for a *good* hypersurface. However, in the process of the stratification of a hypersurface with non-isolated singularities, it turns out that the stratification of a hypersurface which is not good involves the stratification of the complete intersection varieties. See Example (9.3). Thus we consider the following situation. Let $V=\{z\in\boldsymbol{C}^n;\ f_1(\boldsymbol{z})=\cdots=f_\alpha(\boldsymbol{z})=0\}$ and let $V^*=V\cap\boldsymbol{C}^{*n}$ where $f_1,\cdots,f_\alpha$ are analytic functions defined in a neighborhood of the origin. We assume that V is a complete intersection variety with the inductive non-degeneracy condition. (See §6 for the definition.) Let I be a subset of $\{1,\cdots,n\}$ and let $V^{*I}=V\cap\boldsymbol{C}^{*I}$ where $\boldsymbol{C}^{*I}=\{\boldsymbol{z};\ z_i\neq 0\Leftrightarrow i\in I\}$. Let V_{pr} be the closure of V^* in $\boldsymbol{C}^n$ and let $V_{pr}^{*I}=V_{pr}\cap\boldsymbol{C}^{*I}$. Note that $V_{pr}^{*I}\subset V^{*I}$.

In §3, we construct a canonical toroidal resolution of V_{pr}. In §4, we study the geometry of V_{pr}^{*I}. We introduce the concept of the *I-primary boundary components* which play an important role for the stratification of V. Its rough description is as follows. Let $P={}^t(p_1,\cdots,p_n)$ be a positive rational dual vector and let $I=\{i;\ p_i=0\}$. Let $f_{1P},\cdots,f_{\alpha P}$ be the face functions with respect to P

and let $e(P)$ be the set of ν's such that $f_{\nu P}(z)$ is essentially of z_I-variables, i.e. $f_{\nu P}(z)$ is a product $z^{L_\nu} f^e_{\nu P}(z_I)$ where $f^e_{\nu P}(z_I)$ is a function of $\{z_i; i \in I\}$ and z^{L_ν} is a monomial. We consider the varieties $V^*(P)$ and $\partial V^*(P)$ which are defined by

$$V^*(P) = \{z \in \boldsymbol{C}^{*n};\ f_{1P}(z) = \cdots = f_{\alpha P}(z) = 0\},$$

$$\partial V^*(P) = \{z_I \in \boldsymbol{C}^{*I};\ f^e_{\nu P}(z_I) = 0,\ \nu \in e(P)\}.$$

We call $\partial V^*(P)$ an *I-primary boundary component* (with respect to P) if $V^*(P)$ is non-empty. A criterion for the non-emptyness of $V^*(P)$ is given in Appendix B of §10. Then we will show that V^{*I}_{pr} is a union of I-primary boundary components (Lemma (4.1)). In §5, we prove a key lemma (Theorem (5.1)) which states the b-regularity of the pair $(V^*, \partial V^*(P))$. Using this, we will construct in §6 a canonical Whitney b-regular stratification $\mathcal{S}$ of V which depends only on the Newton boundaries $\{\partial\Gamma(f_\nu); \nu = 1, \cdots, \alpha\}$. $\mathcal{S}$ is the simplest regular stratification under the assumption that each V^{*I} is a union of strata in $\mathcal{S}$. By the non-degeneracy assumption, the singular locus of V is a union of certain V^{*I}'s. However the b-regularity for the pair (V^*, V^{*I}) does not hold in general even when V^{*I} is smooth. Thus we have to know the locus where the regularity fails. This is why the notion of the primary boundary component is inevitable.

The idea of the stratification of V is as follows. First we start from the biggest stratum V^*. Suppose that we have obtained a regular stratification $\mathcal{S}(I)$ of V^{*I} for $|I| \geqq n-k$ so that $\bigcup_{|I| \geqq n-k} \mathcal{S}(I)$ is a regular stratification of $V - \bigcup_{|J| < n-k} V^{*J}$. Let J be a subset of $\{1, \cdots, n\}$ with $|J| = n-k-1$. On the subvariety V^{*J}, we consider all the J-primary boundary components of the strata of $\mathcal{S}(I)$ with $|I| \geqq n-k$. Under the IND-condition, they generate a regular stratification $\mathcal{S}(J)$ of V^{*J} and each stratum is a non-degenerate complete intersection variety in $\boldsymbol{C}^{*J}$. By the inductive argument, we obtain a regular stratification $\mathcal{S}$ of V which only depends on the respective Newton boundaries $\partial\Gamma(f_\nu)$ (Theorem (6.1)). In §7, we generalize the result about the principal zeta-function in [**21**] for non-degenerate complete intersection variety with non-isolated singularity. We also show that the IND-condition is stable under a generic hyperplane (or hypersurface) section. In §8, we consider the topological stability problem from the stratification point of view. In §9, we give several examples of the stratifications.

§2. Stratifications.

Let V be an analytic variety in an open set D of $\boldsymbol{C}^n$. We recall the necessary notions of the stratification which is induced by Whitney and Thom. For further details and recent developments, see [**27**], [**24**], [**15**], [**13**] and [**6**]. Let

$\mathcal{S}$ be a family of subsets of V such that V is covered disjointly by elements of $\mathcal{S}$. $\mathcal{S}$ is called *a Whitney stratification* if the following conditions are satisfied.

(i) (*D-strictness*) Each element M of $\mathcal{S}$ (which is called a *stratum*) is a connected smooth analytic variety such that $\overline{M}$ and $\overline{M}-M$ are closed analytic varieties in D. Here $\overline{M}$ is the closure of M in D.

(ii) (*Frontier property*) Let M and N be strata of $\mathcal{S}$ and assume that $M \neq N$ and $M \cap \overline{N} \neq \emptyset$. Then $M \subset \overline{N}-N$.

We recall the Whitney b-condition for a Whitney stratification $\mathcal{S}$. Let (N, M) be a pair of strata of $\mathcal{S}$ with $\overline{N} \supset M$ and let p be a point of M. Let p_i and q_i be sequences on N and on M respectively. We assume that

$$(2.1) \qquad p_i \longrightarrow p, \quad q_i \longrightarrow p, \quad T_{p_i}N \longrightarrow \tau \quad \text{and} \quad [p_i - q_i] \longrightarrow l.$$

Here the arrows imply the convergence in the respective spaces and $[v]$ is the complex line generated by v. Thus $\tau \in G(r, n)\,(r=\dim N)$ and $l \in G(1, n)=\boldsymbol{P}^{n-1}$ where $G(r, n)$ is the Grassmannian manifold of r-planes in $\boldsymbol{C}^n$. We say that (N, M) *satisfies the Whitney b-condition* (respectively *a-condition*) at p if $l \in \tau$ (resp. $\tau \supset T_pM$) for any such sequences. When each pair (N, M) with $M \subset \overline{N}$ satisfies the Whitney b-condition (respectively a-condition) at any point p of M, we call $\mathcal{S}$ *a b-regular* (*resp. a-regular*) *Whitney stratification*. The following proposition is a direct consequence of the Curve Selection Lemma (§3 of [**16**] or [**5**]) and Theorem 17.5 of [**27**].

PROPOSITION (2.2). *Let p_i and q_i be as in* (2.1). *Then there are analytic curves $p(t)$ and $q(t)$ defined on the interval* $[0, 1]$ *such that*

(i) $p(0)=q(0)=p$ *and* $p(t) \in N$ *for* $t \neq 0$ *and* $q(t) \in M$,

(ii) $T_{p(t)}N \rightarrow \tau$ *and* $[p(t)-q(t)] \rightarrow l$ *as* $t \rightarrow 0$.

It is known that the b-condition for analytic varieties follows from the ratio condition (R) by Kuo [**11**]. See also [**26**]. It is known that the Whitney a-condition follows from the b-condition ([**15**]).

REMARK (2.3). Let $\mathcal{S}$ be a stratification and assume that a pair of strata (N, M) satisfies the Whitney b-regular condition. Let $\mathcal{S}'$ be any stratification which is finer than $\mathcal{S}$. Let (N', M') be a pair of strata of $\mathcal{S}'$ with $\overline{N}' \supset M'$ where N' is open dense in N and $M' \subset M$. Then this pair satisfies the Whitney b-regular condition. Though the varieties which we consider in this paper are complex analytic varieties, every argument which follows in later sections can be easily translated into real analytic varieties with few modifications.

§3. Complete intersection variety and its resolution.

Let $f(z)=\sum_\nu a_\nu z^\nu$ be an analytic function of n-variables which is defined in a neighborhood of the origin. The Newton polyhedron $\Gamma_+(f)$ is the convex hull of the union of $\{\nu+\boldsymbol{R}_+^n\}$ for ν such that $a_\nu\neq 0$. The Newton boundary $\Gamma(f)$ is the union of the compact faces of the Newton polyhedron. As we are mainly interested in non-isolated singularities, we also use the notation $\partial\Gamma_+(f)$ which is the union of the boundaries of $\Gamma_+(f)$ which are not necessarily compact. The inclusion $\Gamma(f)\subset\partial\Gamma_+(f)$ is obvious by the definition. We use the same notations as those in § 4 of [21] unless otherwise stated. Let $f_1(\boldsymbol{z}), \cdots, f_\alpha(\boldsymbol{z})$ be analytic functions which are defined in a neighborhood U of the origin $\vec{0}$. We say that the variety $V^*=\{\boldsymbol{z}\in U\,;\, f_1(\boldsymbol{z})=\cdots=f_\alpha(\boldsymbol{z})=0\}$ is a germ of a *smooth complete intersection variety* at the origin if $df_1\wedge\cdots\wedge df_\alpha(\boldsymbol{z})\neq 0$ for any $\boldsymbol{z}\in V^*\cap B_\varepsilon$ for some $\varepsilon>0$ where $B_\varepsilon=\{\boldsymbol{z}\in\boldsymbol{C}^n\,;\,\|\boldsymbol{z}\|<\varepsilon\}$. We say that V^* is a *non-degenerate complete intersection variety* (with respect to the Newton boundary) if for any strictly positive integral dual vector $P={}^t(p_1, \cdots, p_n)$, the α-form $df_{1P}\wedge\cdots\wedge df_{\alpha P}$ does not vanish on $V^*(P)=\{\boldsymbol{z}\in\boldsymbol{C}^{*n}\,;\, f_{1P}(\boldsymbol{z})=\cdots=f_{\alpha P}(\boldsymbol{z})=0\}$ ([8], [17], [21]). Here $f_{\nu P}$ is the face function of f_ν with respect to P. We will see that a non-degenerate complete intersection variety is a smooth intersection variety in an ε-neighborhood of the origin (Lemma (3.7)).

Let Σ^* be a fixed unimodular simplicial subdivision which is compatible with the dual Newton diagrams $\Gamma^*(f_1, \cdots, f_\alpha)$ and let $\hat{\pi}: X\to\boldsymbol{C}^n$ be the associated modification map. See [21] for the definition. One thing which is crucially different comparing with the toroidal modification map for an isolated non-degenerate complete intersection case is that the $|I|$-simplex σ with vertices $\{R_i\,;\, i\in I\}$ is not necessarily an equivalent class in $\Gamma^*(f_1, \cdots, f_\alpha)$. Recall that positive dual vectors P and Q are called equivalent if and only if $\Delta(P;\, f_\nu)=\Delta(Q\,;\, f_\nu)$ for $\nu=1, \cdots, \alpha$ ([21]). Here $R_i={}^t(0, \cdots, \overset{i}{\check{1}}, \cdots, 0)$. Thus Σ^* may have many vertices which is not strictly positive other than $R_1, \cdots, R_n$. X is covered by affine spaces $\boldsymbol{C}_\sigma^n$ with coordinate $\boldsymbol{y}_\sigma=(y_{\sigma 1}, \cdots, y_{\sigma n})$ where σ moves in n-simplices of Σ^*. Let (p_{ij}) be the unimodular matrix corresponding to σ. Then $\hat{\pi}\,|\,\boldsymbol{C}_\sigma^n$ is defined by $\hat{\pi}(\boldsymbol{y}_\sigma)=\boldsymbol{z}=(z_1, \cdots, z_n)$ where $z_i=\prod_{j=1}^n y_{\sigma j}^{p_{ij}}$. Let P be a vertex of Σ^*. Then P defines a divisor $\hat{E}(P)$ of X as follows. Let $\sigma=(P_1, \cdots, P_n)$ be an n-simplex of Σ^* such that $P=P_1$. Then $\hat{E}(P)\cap\boldsymbol{C}_\sigma^n$ is defined by $y_{\sigma 1}=0$. For an n-simplex τ, $\hat{E}(P)\cap\boldsymbol{C}_\tau^n\neq\emptyset$ iff P is a vertex of τ. Let $\hat{E}(P)^*=\hat{E}(P)-\bigcup_{Q\neq P}\hat{E}(Q)$. This is isomorphic to the affine torus $\boldsymbol{C}^{*(n-1)}$. Let I be a subset of $\{1, \cdots, n\}$. Recall that the coordinate subspace $\boldsymbol{C}^I$ and $\boldsymbol{C}^{*I}$ are defined by $\boldsymbol{C}^I=\{\boldsymbol{z}=(z_1, \cdots, z_n)\,;\, z_j=0,\ j\notin I\}$ and $\boldsymbol{C}^{*I}=\{\boldsymbol{z}\in\boldsymbol{C}^n\,;\, z_j=0\Leftrightarrow j\notin I\}$ respectively. If $I=\{1, \cdots, n\}$, we usually write $\boldsymbol{C}^{*n}$ instead of $\boldsymbol{C}^{*I}$. Let $P={}^t(p_1, \cdots, p_n)$ be a vertex of Σ^* and let $I(P)=\{i\,;\, p_i=0\}$. We call $I(P)$ *the kernel of* P. Then $\hat{\pi}(\hat{E}(P))=\boldsymbol{C}^{I(P)}$ (respectively $\hat{\pi}(\hat{E}(P)^*)=\boldsymbol{C}^{*I(P)}$). Thus the fiber dimension of $\hat{\pi}: \hat{E}(P)\to\boldsymbol{C}^{I(P)}$ is $n-1-|I(P)|$.

Let V_{pr} be the closure of V^* and let $\tilde{V}$ be the proper transform of V_{pr} by $\hat{\pi}$. Let $\pi: \tilde{V}\to V_{pr}$ be the restriction of $\hat{\pi}$ to $\tilde{V}$. For finite vertices $Q_1, \cdots, Q_s$

of Σ^*, we define a subvariety $E(Q_1, \cdots, Q_s)$ of $\tilde{V}$ by $\hat{E}(Q_1)\cap\cdots\cap\hat{E}(Q_s)\cap\tilde{V}$ and let $E(Q_1, \cdots, Q_s)^*=E(Q_1, \cdots, Q_s)-\bigcup_{P\neq Q_i}E(P)$. For the non-emptyness condition of $E(Q_1, \cdots, Q_s)$, see Appendix (B) in § 10. Let $\sigma=(P_1, \cdots, P_n)$. Then we have

$$\tilde{V}\cap C_\sigma^n=\{y_\sigma\in C_\sigma^n\ ;\ f_{1\sigma}(y_\sigma)=\cdots=f_{\alpha\sigma}(y_\sigma)=0\} \tag{3.1}$$

where $f_{\nu\sigma}(y_\sigma)=f_\nu(\hat{\pi}(y_\sigma))/\prod_{j=1}^n y_{\sigma j}^{d(P_j;\, f_\nu)}$. We claim

THEOREM (3.2). *There is an $\varepsilon>0$ such that $\tilde{V}$ is a smooth complex manifold and $\pi: \tilde{V}\to V_{pr}$ is a proper modification of V_{pr} over B_ε.*

The assertion is well known if the origin is an isolated singular point of V_{pr}. For the general case, we need several lemmas.

LEMMA (3.3). *Let $P={}^t(p_1, \cdots, p_n)$ be a positive rational dual vector and let $V^*(P)=\{z\in C^{*n}\ ;\ f_{\nu P}(z)=0,\ \nu=1, \cdots, \alpha\}$. Then there exists a positive number ε such that $V^*(P)$ is a non-singular complete intersection over the ε-ball $B_\varepsilon^I=\{z_I\in C^I\ ;\ \sum_{i\in I}|z_i|^2<\varepsilon^2\}$. Here $I=I(P)$ and z_I is the C^I-projection of z. We can take a uniform ε for all P.*

PROOF. The assertion is non-trivial only for P which is not strictly positive. Note that $f_{\nu P}(z)\,(\nu=1, \cdots, \alpha)$ and their partial derivatives are weighted homogeneous with the weight P. Thus $df_{1P}\wedge\cdots\wedge df_{\alpha P}(z)=0$ if and only if $df_{1P}\wedge\cdots\wedge df_{\alpha P}(u\cdot z)=0\ (u\neq 0)$ where the C^*-action $C^*\times C^n\to C^n$ is defined by $u\cdot z=(z_1u^{p_1}, \cdots, z_nu^{p_n})$. Note that $u\cdot z$ converges to z_I as $u\to 0$. Assume that the assertion is false. Using the above observation and the Curve Selection Lemma ([**16**], [**5**]) we can find a real analytic curve $z(t)\,(0\leqq t\leqq 1)$ with the Taylor expansion $z_i(t)=a_it^{b_i}+(\text{higher terms})\ (i=1, \cdots, n)$ such that (i) $f_{\nu P}(z(t))\equiv 0$ $(\nu=1, \cdots, \alpha)$ and (ii) $df_{1P}\wedge\cdots\wedge df_{\alpha P}(z(t))\equiv 0$ where a_i is a non-zero complex number and b_i is a positive integer for $i=1, \cdots, n$. Let $B={}^t(b_1, \cdots, b_n)$ and $a=(a_1, \cdots, a_n)$. We put $R=P+cB$ for a sufficiently small rational number $c>0$. Then it is easy to see that $(f_{\nu P})_B=f_{\nu R}$. Looking at the leading terms of (i), we have that $f_{\nu R}(a)=0$ for $\nu=1, \cdots, \alpha$. As R is strictly positive, the non-degeneracy assumption guarantees the existence of a subset $J=\{j_1, \cdots, j_\alpha\}$ of $\{1, \cdots, n\}$ such that the coefficient $c_J(a)$ of $dz_J=dz_{j_1}\wedge\cdots\wedge dz_{j_\alpha}$ in $df_{1R}\wedge\cdots\wedge df_{\alpha R}(a)$ is non-zero. Here $c_J(a)$ is the determinant of $\alpha\times\alpha$-matrix $(\partial f_\nu/\partial z_{i_\mu}(a))_{\nu,\mu=1,\cdots,\alpha}$. Combining this with the assumption (ii), we get the following contradiction

$$0\equiv \text{the coefficient of } dz_J \text{ in } df_{1P}\wedge\cdots\wedge df_{\alpha P}(z(t))\equiv c_J(a)t^\beta+(\text{higher terms})\neq 0$$

for sufficiently small t where $\beta=\sum_{\nu=1}^\alpha(d(B\,;\,f_{\nu P})-b_{j_\nu})$. Thus we can find a positive number ε which only depends on $\underline{\Delta}=(\Delta(P;\,f_1), \cdots, \Delta(P;\,f_\alpha))$. As there are only finitely many such $\underline{\Delta}$, we can take a uniform ε. This completes the proof.

LEMMA (3.4). *There is a positive number* ε *such that* $E(P_1, \cdots, P_s)=\hat{E}(P_1)\cap\cdots\cap\hat{E}(P_s)\cap\tilde{V}$ *is a smooth complete intersection in* $\hat{\pi}^{-1}(B_\varepsilon)$. *In particular,* $E(P_i)$ *is non-singular over* B_ε.

PROOF. As we may assume that $E(P_1, \cdots, P_s)^*$ is non-empty, we can find an n-simplex $\sigma=(P_1, \cdots, P_n)$. Then $E(P_1, \cdots, P_s)^*\subset C_\sigma^{*n}$. In C_σ^{*n}, $\tilde{V}$ is defined by $\{\boldsymbol{y}_\sigma\in C_\sigma^{*n};\ f_{1\sigma}(\boldsymbol{y}_\sigma)=\cdots=f_{\alpha\sigma}(\boldsymbol{y}_\sigma)=0\}$ where $f_{\nu\sigma}(\boldsymbol{y}_\sigma)$ is defined by the equality: $f_{\nu\sigma}(\boldsymbol{y}_\sigma)\prod_{j=1}^n y_{\sigma j}^{d(P_j;f_\nu)}=f_\nu(\hat{\pi}(\boldsymbol{y}_\sigma))$. Let $\Delta_\nu=\bigcap_{i=1}^s \Delta(P_i; f_\nu)$, $\nu=1, \cdots, \alpha$ and let $P=(P_1+\cdots+P_s)$. (Note that P is not necessarily a vertex of Σ^*.) As $P_i\,(i=1, \cdots, s)$ are positive integral vectors, it is easy to see that $\Delta_\nu=\Delta(P; f_\nu)$. Define $h_\nu(\boldsymbol{y}_\sigma)$ by

$$h_\nu(\boldsymbol{y}_\sigma)\prod_{j=1}^n y_{\sigma j}^{d(P_j;f_\nu)} = f_{\nu P}(\hat{\pi}(\boldsymbol{y}_\sigma)). \tag{3.5}$$

By the definition of $f_{\nu\sigma}$ and h_ν, we have

$$f_{\nu\sigma}(\boldsymbol{y}_\sigma)\equiv h_\nu(\boldsymbol{y}_\sigma) \quad \text{modulo } (y_{\sigma 1}, \cdots, y_{\sigma s}) \tag{3.6}$$

where $(y_{\sigma 1}, \cdots, y_{\sigma s})$ is the ideal generated by $\{y_{\sigma 1}, \cdots, y_{\sigma s}\}$. Recall that $E(P_1, \cdots, P_s)^*$ is defined by

$$y_{\sigma 1}=\cdots=y_{\sigma s}=f_{1\sigma}(\boldsymbol{y}_\sigma)=\cdots=f_{\alpha\sigma}(\boldsymbol{y}_\sigma)=0.$$

This is equivalent to

$$y_{\sigma 1}=\cdots=y_{\sigma s}=h_1(\boldsymbol{y}_\sigma)=\cdots=h_\alpha(\boldsymbol{y}_\sigma)=0.$$

Note that $h_\nu(\boldsymbol{y}_\sigma)$ is independent of the first s-variables $\{y_{\sigma 1}, \cdots, y_{\sigma s}\}$. Let $\boldsymbol{u}=(0, \cdots, 0, \boldsymbol{u}')\in E(P_1, \cdots, P_s)^*$ in this coordinate and let $\boldsymbol{u}_\theta=(\theta, \cdots, \theta, u')$. As $\boldsymbol{u}\in E(P_1, \cdots, P_s)^*$, we have that $h_\nu(\boldsymbol{u}_\theta)=0$ for any θ and $\nu=1, \cdots, \alpha$. Thus $\boldsymbol{z}=\pi(\boldsymbol{u}_\theta)\in V^*(P)$ by (3.5). By Lemma (3.3), there exists a positive number ε such that $df_{1P}\wedge\cdots\wedge df_{sP}(\hat{\pi}(\boldsymbol{u}_\theta))\neq 0$ if $\boldsymbol{z}_I\in B_\varepsilon^I$. As $\hat{\pi}$ is biholomorphic at $\boldsymbol{u}_\theta$, this and (3.5) imply that $dh_1\wedge\cdots\wedge dh_\alpha(\boldsymbol{u}_\theta)\neq 0$. As $\{h_\nu\}$ $(\nu=1, \cdots, \alpha)$ do not contain the variables $y_{\sigma 1}, \cdots, y_{\sigma s}$, there is a subset $I=\{i_1, \cdots, i_\alpha\}$ of $\{s+1, \cdots, n\}$ such that the coefficient of $d\boldsymbol{y}_{\sigma I}=dy_{\sigma i_1}\wedge\cdots\wedge dy_{\sigma i_\alpha}$ in $dh_1\wedge\cdots\wedge dh_\alpha(\boldsymbol{u}_\theta)$ is non-zero. Let $df_{1\sigma}\wedge\cdots\wedge df_{\alpha\sigma}(\boldsymbol{y})=\sum_J c_J(\boldsymbol{y})d\boldsymbol{y}_{\sigma J}$. Then by (3.6), we have $c_I(\boldsymbol{u})=\det(\partial f_{\nu\sigma}/\partial y_{\sigma i_\mu})(\boldsymbol{u})=\det(\partial h_\nu/\partial y_{\sigma i_\mu})(\boldsymbol{u})$. The last determinant is equal to $\det(\partial h_\nu/\partial y_{\sigma i_\mu})(\boldsymbol{u}_\theta)$ which is non-zero by the above assumption. This says that the divisors $\hat{E}(P_1), \cdots, \hat{E}(P_s)$ and $\tilde{V}$ are transverse at $\boldsymbol{u}$. In particular, $\tilde{V}$ is smooth at $\boldsymbol{u}$. As the non-empty strata $\{E(P_1, \cdots, P_s)^*\}$ are finite, we can also take a common ε. This completes the proof of Lemma (3.4).

LEMMA (3.7). (i) V^* *is a smooth complete intersection over* B_ε *for some* $\varepsilon>0$.

(ii) $\tilde{V}$ *is non-singular over* B_ε. *Thus* $\pi: \tilde{V}\to V_{pr}$ *is a resolution.*

PROOF. By Lemma (3.4), $\tilde{V}$ is smooth on $\bigcup_P E(P)$. As $\pi: \tilde{V}-\bigcup_P E(P)\to$

V^* is biholomorphic, it suffices to show (i). Assume that (i) is false. Then we can use the Curve Selection Lemma to find a real analytic curve $\boldsymbol{z}(t)$ with the Taylor expansion $z_i(t)=a_i t^{b_i}+$(higher terms) $(a_i\neq 0,\ b_i>0,\ i=1, \cdots, n)$ such that $f_\nu(\boldsymbol{z}(t))\equiv 0$ and $df_1\wedge\cdots\wedge df_\alpha(\boldsymbol{z}(t))\equiv 0$. Let $B={}^t(b_1, \cdots, b_n)$ and $\boldsymbol{a}=(a_1, \cdots, a_n)$. Looking at the leading terms, we obtain that $f_{\nu B}(\boldsymbol{a})=0$, $\{\nu=1, \cdots, \alpha\}$ and $df_{1B}\wedge\cdots\wedge df_{\alpha B}(\boldsymbol{a})=0$. As B is strictly positive, this is a contradiction to the non-degeneracy assumption. This completes the proof.

Now Theorem (3.2) is an immediate consequence of Lemmas (3.4) and (3.7).

§ 4. Primary boundary components.

Let V_{pr} be the closure of V^* as in § 3. Let I be a subset of $\{1, \cdots, n\}$. We define the I-proper boundary V_{pr}^{*I} of V in $\boldsymbol{C}^{*I}$ by $V_{pr}^{*I}=V_{pr}\cap\boldsymbol{C}^{*I}$. We will describe the structure of the proper boundaries. We say that an analytic function $g(\boldsymbol{z})$ is essentially of $\boldsymbol{z}_I$-variables if $g(\boldsymbol{z})$ is a product of a monomial $\boldsymbol{z}^L$ of variables $\{z_j;\ j\notin I\}$ and an analytic function $g^e(\boldsymbol{z}_I)$ which only contains the variables $\{z_i;\ i\in I\}$. Then we call $g^e(\boldsymbol{z}_I)$ the essential part of g. Let $Q_+(I)$ be the set of the prositive rational dual vectors $P={}^t(p_1, \cdots, p_n)$ such that $I(P)=I$. For a given $P\in Q_+(I)$, we consider the face functions $f_{1P}(\boldsymbol{z}), \cdots, f_{\alpha P}(\boldsymbol{z})$. Though $f_{\nu P}(\boldsymbol{z})$ is not necessarily a polynomial in the variables $\{z_i;\ i\in I\}$, it is a polynomial in the variables $\{z_j;\ j\notin I\}$ for $\nu\notin e(P)$. Let $e(P)$ be the set of ν's such that $f_{\nu P}(\boldsymbol{z})$ is essentially of $\boldsymbol{z}_I$-variables. We define the varieties $\partial V^*(P)$ and $V^*(P)$ by

$$\partial V^*(P)=\{\boldsymbol{z}_I\in\boldsymbol{C}^{*I}\cap B_\varepsilon^I;\ f_{\nu P}^e(\boldsymbol{z}_I)=0,\ \nu\in e(P)\}\quad\text{and}$$

$$V^*(P)=\{\boldsymbol{z}\in\boldsymbol{C}^{*n}\cap p_I^{-1}(B_\varepsilon^I);\ f_{\nu P}(\boldsymbol{z})=0,\ \nu=1, \cdots, \alpha\}$$

where ε is a small enough positive number and $p_I\colon \boldsymbol{C}^n\to\boldsymbol{C}^I$ is the canonical projection. If $V^*(P)$ is not empty, we call $\partial V^*(P)$ the I-primary boundary component of V^* with respect to P. The necessary and sufficient condition for non-emptyness of $V^*(P)$ is given in Appendix B in § 10. Note that $\partial V^*(P)=\boldsymbol{C}^{*I}$ by the definition if $e(P)$ is empty. Let $q_I\colon V^*(P)\to\partial V^*(P)$ be the restriction of p_I to $V^*(P)$. We say that V_{pr} satisfies the primary non-degeneracy condition (or simply the PND-condition) if the following conditions are satisfied for any primary boundary component $\partial V^*(P)$. Let $\hat{f}_\nu=f_\nu-f_{\nu P}$ and let $\hat{f}_{\nu P}$ be the face function of $\hat{f}_\nu$ with respect to P. We call $\hat{f}_{\nu P}$ the secondary face function of f_ν with respect to P. Let $p_{\max}=\max\{p_1, \cdots, p_n\}$ and $p_{\min}=\min\{p_j;\ j\notin I\}$. The $p_{\max}\geqq p_{\min}>0$.

(PND1) (a) For each $\nu\in e(P)$, either (i) $d(P;\ f_\nu)=0$ or (ii) $d(P;\ f_\nu)>0$ and $d(P;\ \hat{f}_\nu)\geqq d(P;\ f_\nu)+p_{\max}$.

(b) $\partial V^*(P)$ is a non-degenerate complete intersection in $\boldsymbol{C}^{*I}$ in an ε-ball B_ε^I for some ε.

(PND2) For each fixed $\boldsymbol{z}_I \in \partial V^*(P) \cap B_\varepsilon^I$, the fiber $q_I^{-1}(\boldsymbol{z}_I)=\{f_{\nu P}(\boldsymbol{z})=0,\ \nu \notin e(I)\}$ is a smooth complete intersection variety in $\boldsymbol{C}^{*I^c} \times \{\boldsymbol{z}_I\}$. Here I^c is the complement of I in $\{1, \cdots, n\}$.

If $\alpha=1$, (PND1)-(a) can be replaced by the following weaker condition. Let $f=f_1$ and assume that $f_P(\boldsymbol{z})$ is essentially of $\boldsymbol{z}_I$-variables. Write $f_P(\boldsymbol{z})=\boldsymbol{z}^K f_P^e(\boldsymbol{z}_I)$ where $K=(k_1, \cdots, k_n)$. (a)$'$ $(\alpha=1)$ (i) $d(P;\ f)=0$ or (ii)$'$ $d(P;\ f)>0$ and $d(P;\ \hat{f}) \geqq d(P;\ f)+p_{\min}$ or (iii) $\{\boldsymbol{z} \in \boldsymbol{C}^{*n};\ f_P(\boldsymbol{z})=0,\ z_j(\partial \hat{f}_P/\partial z_j)(\boldsymbol{z})-k_j \hat{f}_P(\boldsymbol{z})=0\ \ j \notin I\}=\emptyset$. (See [**19**].)

The projection $q_I: \partial V^*(P) \to V^{*I}(P)$ is a submersion over an ε-ball B_ε^I by the PND2-condition. Note that $Q_+(I)$ is infinite but the primary boundary components are finite. Note also that $\partial V^*(P) \subset V_{pr}^{*I}$ but $V^*(P)$ has no inclusion relation with V_{pr}. Usually the PND2-condition is more difficult to be checked and we will give sufficient conditions for the PND-condition in § 10. We assume that V satisfies the PND-condition hereafter. The main result of this section is:

LEMMA (4.1). *The I-proper boundary V_{pr}^{*I} of V is the union of the I-primary boundary components.*

PROOF. Let $\pi: \tilde{V} \to V_{pr}$ be the resolution of V_{pr} constructed in § 3. Let $\tilde{V}^{*I}$ be the union of the strata $E(P_1, \cdots, P_s)^*$ of the stratification $\tilde{S}$ of $\tilde{V}$ such that $\pi(E(P_1, \cdots, P_s)^*) \subset \boldsymbol{C}^{*I}$. As π is a proper surjective mapping, it is clear that $\pi(\tilde{V}^{*I})=V_{pr}^{*I}$. Let $E(P_1, \cdots, P_s)^*$ be such a stratum and let $\sigma=(P_1, \cdots, P_n)$ be an n-simplex of Σ^*. The assumption $\pi(E(P_1, \cdots, P_s)^*) \subset \boldsymbol{C}^{*I}$ implies that $\bigcap_{i=1}^s I(P_i)=I$. Let $P=P_1+\cdots+P_s$. Then P is a positive dual vector in $Q_+(I)$. We may assume that $I=\{m+1, \cdots, n\}$ $(m \geqq s)$ for simplicity and let $\sigma=(p_{ij})$.

SUBLEMMA (4.2). *The restriction of $\hat{\pi}$ to $E(P_1, \cdots, P_s)^*$ is a submersion onto $\partial V^*(P)$.*

PROOF. Recall that $E(P_1, \cdots, P_s)^*$ is defined by

$$\{y_{\sigma 1}=\cdots=y_{\sigma s}=h_1(\boldsymbol{y}_\sigma)=\cdots=h_\alpha(\boldsymbol{y}_\sigma)=0\}$$

where h_ν is characterized by

$$h_\nu(\boldsymbol{y}_\sigma) \prod_{i=1}^n y_{\sigma i}^{d(f_\nu;\, P_i)}=f_{\nu P}(\hat{\pi}(\boldsymbol{y}_\sigma)). \tag{4.3}$$

Note that $\Delta(f_\nu;\ P)=\bigcap_{i=1}^s \Delta(P_i;\ f_\nu)$. Thus $h_\nu(\boldsymbol{y}_\sigma)$ does not contain the variables $y_{\sigma 1}, \cdots, y_{\sigma s}$. Let E^* be the subvariety of $\boldsymbol{C}_\sigma^{*n}$ defined by $\{\boldsymbol{y}_\sigma \in \boldsymbol{C}_\sigma^{*n};\ h_1(\boldsymbol{y}_\sigma)=\cdots=h_\alpha(\boldsymbol{y}_\sigma)=0\}$. E^* is nothing but the product of $\boldsymbol{C}^{*s} \times E(P_1, \cdots, P_s)^*$. Recall that $V^*(P)=\{\boldsymbol{z} \in \boldsymbol{C}^{*n};\ f_{1P}(\boldsymbol{z})=\cdots=f_{\alpha P}(\boldsymbol{z})=0\}$. It is clear that $\hat{\pi}: E^* \to$

$V^*(P)$ is an isomorphism by (4.3). Let $q_I: V^*(P) \to \partial V^*(P)$ and $p: E^* \to E(P_1, \cdots, P_s)^*$ be the canonical projections. We have the commutative diagram:

$$\begin{array}{ccc} E^* & \xrightarrow{\hat{\pi}} & V^*(P) \\ \big\downarrow p & & q_I \big\downarrow \\ E(P_1, \cdots, P_s)^* & \xrightarrow{\pi} & \partial V^*(P). \end{array}$$

Let ϕ be the composition $q_I \circ \hat{\pi}: E^* \to \partial V^*(P)$. As q_I is a submersion by PND2-condition, ϕ is a submersion. As p is obviously a submersion, this implies that $\pi: E(P_1, \cdots, P_s)^* \to \partial V^*(P)$ is a submersion. This completes the proofs of Sublemma (4.2).

Let $P \in Q_+(I)$ and assume that P gives a primary boundary component $\partial V^*(P)$. Taking a subdivision of Σ^* if necessary, we may assume that P is a vertex of Σ^*. Then the above sublemma says that $E(P)^* \xrightarrow{\hat{\pi}} \partial V^*(P)$ is a submersion. In particular, $\partial V^*(P) \subset V^{*I}_{pr}$. This completes the proof of Lemma (4.1).

REMARK (4.4). The proof of Lemma (4.1) shows that $\partial V^*(P)$ is the image of the intersection $\tilde{V} \cap \hat{E}(P)^* = E(P)^*$. Thus it is necessary to add $\partial V^*(P)$ as a stratum in the stratification of V_{pr}. Assume that $f^I_\nu(z_I)$ is not identically zero for each $\nu = 1, \cdots, \alpha$. Then V^{*I} is defined by $f^I_1(z_I) = \cdots = f^I_\alpha(z_I) = 0$. In this case, $f_{\nu P}(z) = f^I_\nu(z_I)$ and $e(P) = \{1, \cdots, \alpha\}$ for any $P \in Q_+(I)$. Thus V^{*I} itself is the unique I-primary boundary component. In this case, V is non-singular on V^{*I}.

REMARK (4.5). Consider a primary boundary component $\partial V^*(P)$. Then Lemma (4.1) implies that $\dim \partial V^*(P) \leqq \dim E(P) = \dim V^* - 1$.

§ 5. Whitney conditions for the primary boundary components.

In this section, we prove the following theorem which is essentially equivalent to the Whitney b-condition for the pair $(V^*, \partial V^*(P))$. Let $p(t) = (p_1(t), \cdots, p_n(t))$ be an analytic curve defined in the interval $[0, 1]$ with the Taylor expansion $p_i(t) = a_i t^{b_i} + (\text{higher terms})$. We assume that (i) $f_\nu(p(t)) \equiv 0$, $\nu = 1, \cdots, \alpha$, (ii) $a_j \neq 0$, $j = 1, \cdots, n$ and (iii) $b_i = 0$ if and only if $i \in I$.

Let $B = {}^t(b_1, \cdots, b_n)$, $\boldsymbol{a} = (a_1, \cdots, a_n)$. The condition (iii) implies that $I(B) = I$. Let $b_{\min} = \min\{b_j;\ j \notin I\}$, $b_{\max} = \max\{b_1, \cdots, b_n\}$ and let $J_{\min} = \{j;\ b_j = b_{\min}\}$. Let $q(t)$ be an analytic curve in $\partial V^*(B)$ with $q(0) = p(0)$. Note that $p(t) \in V^*$ for $0 < t < 1$ by (i) and (ii). We assume that (iv) $T_{p(t)} V^* \to \tau$ and $[p(t) - q(t)] \to l$ as $t \to 0$. Then we assert

THEOREM (5.1). *l is contained in τ.*

This is the heart of the stratification theory of V which will be discussed in the next section. The proof of Theorem (5.1) occupies the rest of this section.

(I) Case of hypersurfaces ($\alpha=1$). This case coincides with the Key Lemma (4.1) of [**19**]. As the proof is also important for the case $\alpha>1$, we repeat the proof. We put $f=f_1$. It is well-known that the tangent space $T_{\boldsymbol{z}}V^*$ is characterized by $df(\boldsymbol{z})^{\perp}=\{\boldsymbol{v}\in T_{\boldsymbol{z}}\boldsymbol{C}^n\,;\ df(\boldsymbol{z})(\boldsymbol{v})=0\}$. Let us consider the limit of $df(p(t))$. For a real analytic function $k(t)$, we define an integer $\operatorname{ord}(k(t))$ by the order of the zeros $k(t)=0$ at $t=0$. Similarly we define the order of a vector-valued analytic function by the minimum of the order of the coordinate functions. Thus $\operatorname{ord}(df(p(t)))$ is the minimum of $\operatorname{ord}(\partial f/\partial z_i(p(t)))$ for $i=1,\cdots,n$. Let $m=\operatorname{ord}(df(p(t)))$ and let $\vec{\gamma}=df(p(t))/t^m|_{t}=0$. Let $\vec{\gamma}=\sum_{i=1}^n\gamma_i dz_i$. Then we have an obvious equality $\tau=\vec{\gamma}^{\perp}$. Considering the leading term of (i), we obtain the equality $f_B(\boldsymbol{a})=0$.

Case (I-a). Assume first that $f_B(\boldsymbol{z})$ is not essentially of $\boldsymbol{z}_I$-variables. Then $\partial V^*(B)=\boldsymbol{C}^{*I}$ by the definition. Then by PND2, there exists an index j with $j\notin I$ such that $\partial f_B/\partial z_j(\boldsymbol{a})\neq 0$ if $\sum_{i\in I}|a_i|^2$ is small enough. Thus we have $m\leqq d(B\,;\ f)-b_{\min}$. As $d(B\,;\ \partial f/\partial z_j)\geqq d(B\,;\ f)-b_j$, we must have

$$\text{(5.2)}\qquad \text{if } m=d(B\,;\ f)-b_{\min},\ \frac{\partial f_B}{\partial z_j}(\boldsymbol{a})=\begin{cases}0, & j\notin J_{\min}\cup I\\ \gamma_j, & j\in J_{\min},\end{cases}$$

$$\text{(5.3)}\qquad \text{if } m<d(B\,;\ f)-b_{\min},\quad \gamma_j=0,\quad j\in J_{\min}\cup I.$$

Note that $\gamma_i=0$ for $i\in I$ in both cases. This implies that $\vec{\gamma}|\boldsymbol{C}^I=0$ and the Whitney a-condition follows immediately.

Now we consider the line $[p(t)-q(t)]$. Let $k=\operatorname{ord}(p(t)-q(t))$. As $q(t)\in\boldsymbol{C}^{*I}$, it is easy to see that $1\leqq k\leqq b_{\min}$. Let $\vec{l}=(p(t)-q(t))/t^k|_{t=0}$. By the definition of l, we have that $[\vec{l}]=l\in\boldsymbol{P}^{n-1}$. If $k<b_{\min}$, $\vec{l}$ is a vector in $\boldsymbol{C}^I$. In this case, it is clear that $\vec{\gamma}(\vec{l})=0$. Assume that $k=b_{\min}$. Then $l_j=a_j$ for $j\in J_{\min}$ and $l_j=0$ for $j\notin J_{\min}\cup I$. We consider the equality

$$0\equiv\sum_{j=1}^n\frac{\partial f}{\partial z_j}(p(t))\frac{dp_j(t)}{dt}\equiv\Big(\sum_{j\notin I}\frac{\partial f_B}{\partial z_j}(\boldsymbol{a})b_ja_j\Big)t^{d(B;f)-1}+(\text{higher terms}).$$

In particular, this implies the following.

$$\text{(5.4)}\qquad \sum_{j\notin I}\frac{\partial f_B}{\partial z_j}(\boldsymbol{a})b_ja_j=0.$$

If $m<d(B\,;\ f)-b_{\min}$, the equality $\vec{\gamma}(\vec{l})=0$ is immediate from (5.3). Assume that $m=d(B\,;\ f)-b_{\min}$. By (5.2) and (5.4), we can see easily that $\vec{\gamma}(\vec{l})=0$. Here $\vec{l}$ is identified with the tangent vector $\sum_{j=1}^n l_j\partial/\partial z_j$ at $p(0)$.

Case (I-b). Assume that $f_B(\boldsymbol{z})$ is essentially of $\boldsymbol{z}_I$-variables. Let $f_B(\boldsymbol{z})=\boldsymbol{z}^L f_B^e(\boldsymbol{z})$ where $\boldsymbol{z}^L$ is a monomial in the variables $\{z_j\,;\ j\notin I\}$. Then $V^{*I}(B)=\{\boldsymbol{z}_I\in\boldsymbol{C}^{*I}\,;\ f_B^e(\boldsymbol{z}_I)=0\}$ and $\operatorname{ord}(f_B(p(t)))=\operatorname{ord}(p(t)^L)+\operatorname{ord}(f_B^e(p(t)))$. We have two equalities:

$$\sum_{j=1}^{n}\frac{\partial f}{\partial z_j}(p(t))\frac{dp_j(t)}{dt}\equiv 0,\qquad \sum_{i\in I}\frac{\partial f_B^e}{\partial z_i}(q(t))\frac{dq_i(t)}{dt}\equiv 0. \tag{5.5}$$

Let $\beta=\mathrm{ord}(f_B^e(p(t)))$ and $\delta=\mathrm{ord}(\hat{f}(p(t)))$. We assume the PND1-(a)-(ii)-condition. As $f(p(t))=f_B(p(t))+\hat{f}(p(t))\equiv 0$, we have

$$\beta+d(B;f)=\delta\geqq d(B;\hat{f}) \tag{5.6}$$

where $\hat{f}_B(z)$ is the secondary face function of f with respect to the weight B. The equality holds if and only if $\hat{f}_B(\boldsymbol{a})\neq 0$. We consider the equality which follows immediately from (5.5).

$$\sum_{j=1}^{n}\frac{\partial f}{\partial z_j}(p(t))\frac{d}{dt}(p_j(t)-q_j(t))+\sum_{i\in I}\Big(\frac{\partial f}{\partial z_i}(p(t))-\frac{\partial f_B}{\partial z_i}(p(t))\Big)\frac{dq_i(t)}{dt} \tag{5.7}$$
$$+\sum_{i\in I}p(t)^L\Big(\frac{\partial f_B^e}{\partial z_i}(p(t))-\frac{\partial f_B^e}{\partial z_i}(q(t))\Big)\frac{dq_i(t)}{dt}\equiv 0.$$

By the assumption, $p_j(t)\equiv q_j(t)$, modulo (t^k) for any j. This implies that $\mathrm{ord}(\partial f_B^e/\partial z_i(p(t))-\partial f_B^e/\partial z_i(q(t)))\geqq k$. Thus the order of the last sum is at least $d(B;f)+k$. On the other hand, we have $\mathrm{ord}(\partial f/\partial z_i(p(t))-\partial f_B/\partial z_i(p(t)))\geqq d(B;\hat{f})\geqq d(B;f)+b_{\max}$ $(i\in I)$ by PND1-(a)-(ii). As $k\leqq b_{\min}$, the order of the second sum in (5.7) is also at least $d(B;f)+k$. The order of the first sum in (5.7) is (at least) $m+k-1$. As $m\leqq d(B;f)$ by PND1-(b) and $k\leqq b_{\min}$, the coefficient of t^{m+k-1} of (5.7) is equal to $k\vec{\gamma}(\vec{l})$. Thus we conclude that $\vec{\gamma}(\vec{l})=0$. We assert that $m=d(B;f)$. In fact, as $\partial f/\partial z_j(p(t))=\partial\hat{f}/\partial z_j(p(t))+\partial \boldsymbol{z}^L/\partial z_j\cdot f_B^e(p(t))$, we have $\mathrm{ord}\,\partial f/\partial z_j(p(t))\geqq\min\{d(B;\hat{f})-b_j,\ \beta+d(B;f)-b_j\}\geqq d(B;f)$ by (5.6) and PND1-(a)-(ii).

Assume (a)-(i): $d(B;f)=0$. We consider the following equality instead of (5.7).

$$\sum_{j=1}^{n}\frac{\partial f}{\partial z_j}(p(t))\frac{d}{dt}(p_j(t)-q_j(t))+\sum_{i\in I}\Big(\frac{\partial f}{\partial z_i}(p(t))-\frac{\partial f}{\partial z_i}(q(t))\Big)\frac{dq_i(t)}{dt}\equiv 0.$$

Here we have used the equality $\partial f/\partial z_i(q(t))=\partial f_B/\partial z_i(q(t))$. By PND1-(b), $m=0$. Thus by a similar argument, we have $\vec{\gamma}(\vec{l})=0$. Note that $\gamma_i=\boldsymbol{a}^L\partial f_B^e(\boldsymbol{a})/\partial z_i$ for $i\in I$ in the both cases of PND1-(a).

(II) General case $(\alpha>1)$. The case of $\alpha>1$ requires more delicate care than the case of $\alpha=1$. The main difficulty comes from the fact that the limits of the 1-forms $df_1(p(t)),\cdots,df_\alpha(p(t))$ as $t\to 0$ are not linearly independent in general. We first prepare several lemmas. A non-singular holomorphic p-form ω is called *decomposable* at $\boldsymbol{z}$ if there exist linearly independent holomorphic 1-forms $\omega_1,\cdots,\omega_p$ in a neighborhood of $\boldsymbol{z}$ such that $\omega(\boldsymbol{z})=\omega_1\wedge\cdots\wedge\omega_p(\boldsymbol{z})$. To such an ω, we associate an $(n-p)$-dimensional subspace $\omega(\boldsymbol{z})^\perp$ of $T_{\boldsymbol{z}}\boldsymbol{C}^n$ by

$$\omega(\boldsymbol{z})^\perp=\{v\in T_{\boldsymbol{z}}\boldsymbol{C}^n;\ v\dashv\omega(\boldsymbol{z})=0\}.$$

Here $v\dashv$ is the inner derivative by v. We will use the following formula later.

$$v\dashv(\omega_1\wedge\cdots\wedge\omega_p)=\sum_{j=1}^{p}(-1)^{j-1}\omega_j(v)\omega_1\wedge\overset{j}{\check{\cdots}}\wedge\omega_p\,. \tag{5.8}$$

Here $\omega_1\wedge\overset{i}{\check{\cdots}}\wedge\omega_p$ is the exterior product of ω_k's with ω_j being omitted. This can be proved by an elementary argument. (Use for example (4.10) of [22].) Now we consider the tangent space $T_{p(t)}V$. As V is a complete intersection variety at $z=p(t)\,(t>0)$ by Lemma (3.7), $T_{p(t)}V$ is characterized by

$$T_{p(t)}V=\{v\in T_{\boldsymbol{z}}\boldsymbol{C}^n\,;\ v\dashv df_1\wedge\cdots\wedge df_\alpha(p(t))=0\}\,. \tag{5.9}$$

By (5.8), the right side of (5.9) is equal to the intersection $\bigcap_{\nu=1}^{\alpha}df_\nu(p(t))^\perp$. Let $m=\mathrm{ord}(df_1\wedge\cdots\wedge df_\alpha(p(t)))$ and let $\omega(t)=df_1\wedge\cdots\wedge df_\alpha(p(t))/t^m$. We define ω by $\omega(0)$. As $\omega(t)^\perp=T_{p(t)}V$, ω is an α-covector (i.e., $\omega\in\wedge^\alpha T^*_{p(0)}\boldsymbol{C}^n$) such that $\tau=\omega^\perp$. In fact, we will see later that ω is a decomposable covector. We may assume that $f_{\nu B}(z)$ is essentially of z_I-variables if and only if $s<\nu\leqq\alpha$. By multiplying suitable monomials if necessary, we may also assume that

$$d(B\,;\ f_1)=\cdots=d(B\,;\ f_s). \tag{5.10}$$

This makes our calculation much easier. Let $d=d(B\,;\ f_1)$.

LEMMA (5.11). *After renumbering $f_1,\cdots,f_s$ if necessary, we can find polynomials $c_{\nu\mu}(t)$ for $1\leqq\mu<\nu\leqq s$ such that the following conditions are satisfied. Let*

$$d'f_\nu(p(t))=df_\nu(p(t))-\sum_{\mu=1}^{\nu-1}c_{\nu\mu}(t)df_\mu(p(t)) \tag{5.11.1}$$

and let $d'f_\nu(p(t))=\omega_\nu t^{m_\nu}+$(higher terms) with $\omega_\nu\neq0$ for $\nu=1,\cdots,s$.

(i) *$\omega_1,\cdots,\omega_s$ are linearly independent cotangent vectors which are linear combinations of $\{dz_j\,;\ j\notin I\}$.* (ii) *$m_1\leqq m_2\leqq\cdots\leqq m_s\leqq d-b_{\min}$.*

PROOF. We first define *the relative order* $\mathrm{ord}(df_\nu\,;\ df_1,\cdots,df_{\nu-1})$ by the maximum of the order of $(df_\nu-\sum_{j=1}^{\nu-1}g_j(t)df_j)(p(t))$ where $g_1(t),\cdots,g_{\nu-1}(t)$ move in the arbitrary polynomials. By renumbering $f_1,\cdots,f_s$ if necessary, we may assume that $\mathrm{ord}(df_1(p(t)))\leqq\mathrm{ord}(df_\nu(p(t)))$ for $\nu=2,\cdots,s$. Let $df_1(t)=\omega_1t^{m_1}+$(higher terms). By the PND2-condition, we have the inequality $m_1\leqq d-b_{\min}$ and ω_1 is a linear combination of $\{dz_j\,;\ j\notin I\}$. By the assumption, we have the inequality $\mathrm{ord}(df_\nu\,;\ df_1)\geqq m_1$ $(\nu=2,\cdots,s)$. We prove the assertion by the induction. We assume that we have chosen polynomials $c_{\nu\mu}(t)$ $(1\leqq\mu<\nu\leqq k)$ and that we have renumbered $f_1,\cdots,f_s$ so that

$(0)_k$ $d'f_\nu(p(t))=\omega_\nu t^{m_\nu}+$(higher terms), $\nu=1,\cdots,k$. $(\mathrm{i})_k$ $\omega_1,\cdots,\omega_k$ are linearly independent cotangent vectors which are linear combinations of $\{dz_j\,;\ j\notin I\}$. $(\mathrm{ii})_k$ $m_1\leqq\cdots\leqq m_k\leqq d-b_{\min}$ and $\mathrm{ord}(df_\nu\,;\ df_1,\cdots,df_k)\geqq m_k$ for $\nu=k+1,\cdots,s$.

Note that $\mathrm{ord}(df_\nu\,;\ df_1,\cdots,df_k)=\mathrm{ord}(df_\nu\,;\ d'f_1,\cdots,d'f_k)$. We renumber $f_{k+1},\cdots,f_s$ if necessary so that

(5.12) $\operatorname{ord}(df_{k+1}; df_1, \cdots, df_k) \leqq \operatorname{ord}(df_\nu; df_1, \cdots, df_k), \qquad \nu=k+2, \cdots, s.$

Let $m_{k+1}=\operatorname{ord}(df_{k+1}; df_1, \cdots, df_k)$. As $df_1\wedge\cdots\wedge df_{k+1}(p(t))\neq 0$, the analyticity implies that m_{k+1} is finite. We can find polynomials $c_{k+1,1}(t), \cdots, c_{k+1,k}(t)$ such that $d'f_{k+1}(p(t))=df_{k+1}(p(t))-\sum_{j=1}^{k} c_{k+1,j}df_j(p(t))$ has order m_{k+1}. Let $d'f_{k+1}(p(t))=\omega_{k+1}t^{m_{k+1}}+(\text{higher terms})$. Then we have $m_{k+1}\geqq m_k$ and $\omega_1\wedge\cdots\wedge\omega_{k+1}\neq 0$. (If not, the relative order of df_{k+1} is strictly larger than m_{k+1}.) Assume that

$$(5.13) \qquad m_{k+1} > d-b_{\min}.$$

We will show that this gives a contradiction. Let $df_1\wedge\cdots\wedge df_k(p(t))=\sum_{|J|=k} d_J(t)dz_J$ where $dz_J=dz_{j_1}\wedge\cdots\wedge dz_{j_k}$ for $J=\{j_1, \cdots, j_k\}$. Recall that $d_J(t)$ is the determinant of $k\times k$ matrix $(\partial f_\nu/\partial z_{j_\mu}(p(t)))$, $(\nu, \mu=1, \cdots, k)$. Thus we have

$$(5.14) \qquad \operatorname{ord}(d_J(t)) \geqq kd-(b_{j_1}+\cdots+b_{j_k})$$

and the equality holds if and only if

$$(5.15) \qquad \det\left(\frac{\partial f_{\nu B}}{\partial z_{j_\mu}}(\boldsymbol{a})\right)_{\nu,\mu=1,\cdots,k} \neq 0.$$

Let $df_1\wedge\cdots\wedge df_{k+1}(p(t))=\sum_{|K|=k+1} e_K(t)dz_K$. As $df_1\wedge\cdots\wedge df_{k+1}=d'f_1\wedge\cdots\wedge d'f_{k+1}$, (5.13) and (5.14) imply that

$$(5.16) \qquad \operatorname{ord}(e_K(t)) > (k+1)d-(b_{j_1}+\cdots+b_{j_{k+1}})$$

where $K=\{j_1, \cdots, j_{k+1}\}$. On the other hand, the PND2-condition guarantees the existence of K such that $K\cap I=\emptyset$ and $\operatorname{ord}(e_K(t))=(k+1)d-(b_{j_1}+\cdots+b_{j_{k+1}})$. This is a contradiction to (5.16). Thus we have proved that $m_{k+1}\leqq d-b_{\min}$. As the order of the coefficient of $dz_i\,(i\in I)$ in $df_\nu(p(t))\,(\nu\leqq s)$ is greater than or equal to d, ω_{k+1} does not contain any non-zero $dz_i\,(i\in I)$ terms. This proves $(0)_{k+1}$, $(\mathrm{i})_{k+1}$ and $(\mathrm{ii})_{k+1}$ and completes the proof of Lemma (5.11).

COROLLARY (5.17). *Let $m_1, \cdots, m_s$ be as in Lemma* (5.11) *and let $df_\nu(p(t))=\omega_\nu t^{m_\nu}+(\text{higher terms})$ for $\nu=s+1, \cdots, \alpha$. Then $m=\operatorname{ord}(df_1\wedge\cdots\wedge df_\alpha(p(t)))$ is equal to $m_1+\cdots+m_\alpha$ and $\omega=\omega_1\wedge\cdots\wedge\omega_\alpha$. In particular, ω is a decomposable α-covector and $T_{p(t)}V$ converges to $\omega^\perp$.*

PROOF. In the proof of Theorem (5.1) in Case (I-b), we have shown that $m_\nu=d(B; f_\nu)$ for $\nu=s+1, \cdots, \alpha$. There exists a subset $L=\{l_1, \cdots, l_{\alpha-s}\}$ of I such that $\omega_{s+1}\wedge\cdots\wedge\omega_\alpha$ has a non-zero coefficient in dz_L by the PND1-(b)-condition. Thus $\omega_1\wedge\cdots\wedge\omega_\alpha\neq 0$ by (i) of Lemma (5.11). As $df_1\wedge\cdots\wedge df_\alpha=(\wedge_{\nu=1}^{s} d'f_\nu)\wedge df_{s+1}\wedge\cdots\wedge df_\alpha$, the assertion is now immediate.

Now we are ready to prove Theorem (5.1) for the case $\alpha>1$. Let $k=\operatorname{ord}(p(t)-q(t))$ and let $\bar{l}=(l_1, \cdots, l_n)=(p(t)-q(t))/t^k|_{t=0}$. By the argument in Case (I), we have that

$$\omega_\nu(\tilde{l}) = 0, \qquad \text{for} \quad \nu=s+1, \cdots, \alpha. \tag{5.18}$$

On the other hand, by differentiating the equality $f_\nu(p(t))-\sum_{\mu=1}^{\nu-1} c_{\nu\mu}(t)f_\mu(p(t))\equiv 0$, we get

$$\Big(df_\nu(p(t))-\sum_\mu c_{\nu\mu}(t)df_\mu(p(t))\Big)\Big(\frac{dp(t)}{dt}\Big) \equiv 0. \tag{5.19}$$

Here we have used the equality $f_\mu(p(t))\equiv 0$ and $dp(t)/dt$ is identified with the tangent vector $\sum_{j=1}^n (dp_j(p(t))/dt)(\partial/\partial z_j)$. Looking at the coefficient of t^{d-1} in (5.19), we obtain the equality

$$\sum_{j\notin I}\Big(\frac{\partial f_{\nu B}}{\partial z_j}(\boldsymbol{a})-\sum_\mu c_{\nu\mu}(0)\frac{\partial f_{\mu B}}{\partial z_j}(\boldsymbol{a})\Big)b_j a_j = 0. \tag{5.20}$$

Case (II-a). Assume that $\tilde{l}\in \boldsymbol{C}^I$. In this case, by (ii) of Lemma (5.11) we have that

$$\omega_\nu(\tilde{l}) = 0, \qquad \nu=1, \cdots, s. \tag{5.21}$$

Thus by (5.18), (5.21) and the formula (5.8), we obtain $\tilde{l}\dashv\omega=\tilde{l}\dashv(\omega_1\wedge\cdots\wedge\omega_\alpha)=0$ i.e., $\tilde{l}\in\omega^\perp$.

Case (II-b). Assume that $l_j\neq 0$ for some $j\notin I$. This implies that

$$k = b_{\min}, \qquad l_j = \begin{cases} a_j & \text{if } j\in J_{\min} \\ 0 & \text{if } j\notin J_{\min}\cup I. \end{cases} \tag{5.22}$$

Here $J_{\min}=\{j\,;\, b_j=b_{\min}\}$. Fix a ν with $\nu\leqq s$ and let $\omega_\nu=\sum_{j\notin I}\omega_{\nu j}dz_j$. There are two possible cases.

(II-b-1) $\omega_{\nu j}\neq 0$ for some j such that $j\in J_{\min}$ or (II-b-2) $\omega_{\nu j}=0$ for any j with $j\in J_{\min}$.

In Case (II-b-2), $\omega_\nu(\tilde{l})=0$ is immediate from (5.22) and (i) of Lemma (5.11). Assume (II-b-1). Then we have $m_\nu=d-b_{\min}$. In general, we have the canonical inequality

$$\operatorname{ord}\Big(\frac{\partial f_\nu}{\partial z_j}(p(t))-\sum_\mu c_{\nu\mu}(t)\frac{\partial f_\mu}{\partial z_j}(p(t))\Big) \geqq d-b_j. \tag{5.23}$$

Combining this inequality and the assumption that $m_\nu=d-b_{\min}$, we must have

$$\frac{\partial f_{\nu B}}{\partial z_j}(\boldsymbol{a})-\sum_\mu c_{\nu\mu}(0)\frac{\partial f_{\mu B}}{\partial z_j}(\boldsymbol{a}) = 0, \qquad \text{for} \quad j\notin J_{\min}\cup I \quad \text{and} \tag{5.24}$$

$$\omega_{\nu j} = \frac{\partial f_{\nu B}}{\partial z_j}(\boldsymbol{a})-\sum_\mu c_{\nu\mu}(0)\frac{\partial f_{\mu B}}{\partial z_j}(\boldsymbol{a}), \qquad \text{for} \quad j\in J_{\min}. \tag{5.25}$$

Thus by (i) of Lemma (5.11), (5.20), (5.22), (5.24) and (5.25), we obtain $\omega_\nu(\tilde{l})=0$. Therefore we have the equality $\omega_\nu(\tilde{l})=0$ in any cases for $\nu=1, \cdots, \alpha$. By the formula (5.8), this implies that $\tilde{l}\in\omega^\perp$. This completes the proof of Theorem (5.1).

§ 6. Canonical stratification of V.

Let $V=\{z\in \boldsymbol{C}^n;\ f_1(z)=\cdots=f_\alpha(z)=0\}$, $V^*=V\cap \boldsymbol{C}^{*n}$ and V_{pr} be the closure of V^* as before. Note that $V\supset V_{pr}$ but V may have other irreducible components in general. We will construct a b-regular stratification $\mathcal{S}$ of V in a canonical way using the results of § 5. For a subset I of $\{1, \cdots, n\}$, we define $V^{*I}=V\cap \boldsymbol{C}^{*I}$. Recall that $V_{pr}^{*I}=V_{pr}\cap \boldsymbol{C}^{*I}$. If $V^{*I}\neq V_{pr}^{*I}$, V is not irreducible.

Assume first the following general situation. Let W be a smooth analytic variety in an open set D of $\boldsymbol{C}^n$ and let $W_i\,(i\in A)$ be a finite family of the smooth analytic subvarieties of W. We define the 'stratification' $\mathcal{S}$ generated by W_i $(i\in A)$ by the collection of strata W_I^* where I is a subset of A and $W_I^*=\bigcap_{i\in I}W_i-\bigcup_{j\notin I}W_j$. Strictly speaking, a stratum is a connected component of W_I^*. For $I=\emptyset$, $W_\emptyset^*=W-\bigcup_{i\in A}W_i$ by definition. If $\{W_I^*\}$ is a smooth complete intersection variety for each I, $\mathcal{S}$ gives a regular stratification of W.

Our construction of the stratification of V is inductive. Namely we construct a stratification $\mathcal{S}(I)$ of V^{*I} by the induction on $n-|I|$. Then we take the union $\mathcal{S}=\bigcup_I \mathcal{S}(I)$. We will show that $\mathcal{S}$ is a regular stratification of V under a suitable condition. We start from the biggest stratum V^*. Let $I(i)=\{1, \cdots, n\}-\{i\}$. V has the unique $I(i)$-primary boundary component $V_{pr}^{*I(i)}$ which is defined by $V_{pr}^{*I(i)}=\{z_{I(i)}\in \boldsymbol{C}^{*I(i)};\ f_{\nu R_i}^e(z_{I(i)})=0,\ \nu=1, \cdots, \alpha\}$ where $R_i={}^t(0, \cdots, \overset{i}{\check{1}}, \cdots, 0)$. Note that $f_\nu(z)\equiv z_i^{d_{\nu i}}f_{\nu R_i}^e(z_{I(i)})$ modulo $z_i^{d_{\nu i}+1}$ where $d_{\nu i}=d(R_i;\ f_\nu)$ for $\nu=1, \cdots, \alpha$. Thus $e(R_i)=\{1, \cdots, \alpha\}$. As the stratification $\mathcal{S}(I(i))$ of $V^{*I(i)}$, we simply take the stratification generated by $V_{pr}^{*I(i)}$. Namely $\mathcal{S}(I(i))=\{V^{*I(i)}-V_{pr}^{*I(i)}, V_{pr}^{*I(i)}\}$. Of course, $V^{*I(i)}-V_{pr}^{*I(i)}$ can be empty. This simple description is no more valid in general for higher codimensional cases. Assume that we have obtained stratifications $\mathcal{S}(I)$ of V^{*I} for $|I|\geqq n-k$ which satisfies the following conditions.

(i) For $I_0=\{1, \cdots, n\}$, $\mathcal{S}(I_0)$ is V^*.

(ii)$_k$ Let I be a subset of $\{1, \cdots, n\}$ with $|I|\geqq n-k$. Then $\mathcal{S}(I)$ is the stratification of V^{*I} generated by I-primary boundary components of the strata Y of $\mathcal{S}(J)$ with $J\supset I$ and $J\neq I$.

(iii)$_k$ Let $X\in \mathcal{S}(I)\,(|I|\geqq n-k)$. Then $\bar{X}$ is a non-degenerate complete intersection variety in $\boldsymbol{C}^I$ which satisfies the PND-condition. If $X, Y\in \mathcal{S}(I)$ and $\bar{X}\supset Y$, Y is a smooth submanifold of $\bar{X}$.

Assume $|I|=n-k-1$. We define $\mathcal{S}(I)$ by be the stratification generated by the primary boundary components of the strata Y of $\mathcal{S}(K)$ for K such that $I\subset K$, $I\neq K$. We say that V satisfies the inductive non-degeneracy condition (or IND-*condition*) if (iii)$_k$ is satisfied for every k. We assume that V satisfies the IND-condition hereafter. We admit that the IND-condition is a hysteric condition in general but this is a necessary condition and it is usually satisfied for a complete intersection variety which is not too bad. Then we can complete the stratification of V^{*I} for any I by the inductive argument.

THEOREM (6.1). *Assume that V satisfies the* IND-*condition and let $\mathcal{S}$ be the stratification of V which is defined by the union of $\mathcal{S}(I)$. Then $\mathcal{S}$ is a b-regular stratification of V.*

PROOF. Let Y and Z be a pair of strata of $\mathcal{S}$ such that $\bar{Y}\cap Z\neq\emptyset$. We assume that $Y\in\mathcal{S}(J)$ and $Z\in\mathcal{S}(K)$. Then we must have $J\supset K$. If $J=K$, the b-regularity follows from the transversality assumption (iii) in the IND-condition. Thus we may assume that $J\neq K$. If Y is an open dense stratum in $\boldsymbol{C}^{*J}$, the b-regularity for (Y, Z) is obvious. Thus we assume that $\bar{Y}\neq\boldsymbol{C}^{J}$. Let $p(t)$ and $q(t)$ be real analytic curves defined on $[0, 1]$ such that (i) $p(0)=q(0)\in Z$. (ii) $p(t)\in Y$ for $t>0$. (iii) $q(t)\in Z$ for $t\geqq 0$. Assume that the tangent space $T_{p(t)}Y$ converges to τ and the line $[p(t)-q(t)]$ converges to l. Let $h_1(\boldsymbol{z}_J)=\cdots=h_\delta(\boldsymbol{z}_J)=0$ be the defining equations of Y. Y is a non-degenerate complete intersection variety by the IND-condition. Assume that $p_j(t)=a_j t^{b_j}+$(higher terms) for $j\in J$. For brevity's sake, we assume that $J=\{1, \cdots, m\}$. Let $I=\{i\in J;\ b_i=0\}$. Let $B={}^t(b_1, \cdots, b_m)$ and $\boldsymbol{a}=(a_1, \cdots, a_m)$. As $p(0)=q(0)=\boldsymbol{a}_I\in Z$ and $I=K$. By looking at the leading terms of the equality $h_\nu(p(t))\equiv 0$ for $\nu=1, \cdots, \delta$, we can see that $\boldsymbol{a}_I$ belongs to the I-primary boundary component $\partial Y^*(B)$. As $\partial Y^*(B)$ is a member of the subvarieties which generate $\mathcal{S}(K)$, our construction of $\mathcal{S}(K)$ implies that $Z\subset\partial Y^*(B)$. Thus Theorem (6.1) follows immediately from Theorem (5.1) and Remark (2.3).

In the rest of this section, we give a few remarks about the IND-condition. Let X be a strata in $\mathcal{S}(I)$. By the construction of the stratification $\mathcal{S}$ of V, there are three possibilities: (S1) X is open dense in V^{*I}. (S2) There are a sequence of strata $X_0, \cdots, X_r$ of $\mathcal{S}$ such that $X_0=V^*$ and $X=X_r$ and X_i is an open dense subvariety of a proper primary boundary component of X_{i-1}. X is called a *primary boundary component of V^* of order r*. (S3) The other case.

We first study a primary boundary component of order two. Let $Y=\partial V^*(P)$ be a proper primary boundary component and we assume that $e(P)=\{1, \cdots, s\}$. Then we have $\partial V^*(P)=\{\boldsymbol{z}_I\in\boldsymbol{C}^{*I};\ f_{1P}^e(\boldsymbol{z}_I)=\cdots=f_{sP}^e(\boldsymbol{z}_I)=0\}$. Let $Q={}^t(q_1, \cdots, q_n)$ be a rational weight vector and let $J=\{j;\ q_j=0\}$. We assume that $J\subset I$ and that $\boldsymbol{Q}$ gives a primary boundary component $\partial Y^*(Q)$ of Y. Let $R=P+rQ$ for a sufficiently small $r>0$. Then it is an easy linear algebra to see the following. (i) $(f_{\nu P})_Q=f_{\nu R}$ for $\nu=1, \cdots, s$. (ii) The secondary face function $\hat{f}_{\nu R}$ of f_ν with respect to R is equal to the secondary face function of $f_{\nu P}$ with respect to Q for $\nu=1, \cdots, s$. In general, it is possible that $e(R)\cap\{s+1, \cdots, \alpha\}\neq\emptyset$. If we have the inclusion $e(R)\subset e(P)$, the $\boldsymbol{Q}$-primary boundary component of Y is equal to the R-primary boundary component of V. Namely we have the transitivity: $\partial Y^*(Q)=\partial X^*(R)$. In general, we have $\partial Y^*(Q)\supset\partial V^*(R)$. For a subset Ξ of

$\{1, \cdots, \alpha\}$, we define the variety V_Ξ by $V_\Xi=\{z\in C^{*n};\ f_\nu(z)=0,\ \nu\in\Xi\}$. Then we have shown that $\partial Y^*(Q)=\partial(V_{e(P)})^*(R)$. In particular, the PND-condition for Y with respect to Q is equivalent to the PND-condition for $V_{e(P)}$ with respect to R. By an inductive argument, we can show the following. Let X be a stratum of $\mathcal{S}(I)$ of type (S2). We can find a weight vector $P\in Q_+(I)$ and subset Ξ of $\{1, \cdots, \alpha\}$ such that X is open dense in $\partial(V_\Xi)^*(P)=\{z_I\in C^{*I};\ f_{\nu P}^e(z_I)=0,\ \nu\in\Xi\cap e(P)\}$. The description of a stratum of type (S3) is more complicated and unpleasant.

We say that V is a *good* non-degenerate complete intersection if the stratification $\mathcal{S}$ does not have any strata of type (S3). In this case, the IND-condition follows from the PND-condition for V_Ξ's. In the case of a hypersurface, V is a *good* hypersurface if V has at most one proper I-primary boundary component for each I with $|I|>2$. For a good hypersurface, the IND-condition is equivalent to the PND-condition and the stratification is much simpler than the general case ([**19**]).

We assume that $V=V_{pr}$. Let s be the dimension of the singular locus V_{sg} of V. Let $\mathcal{I}=\{I\subset\{1, \cdots, n\};\ \exists\nu,\ f_\nu^I\equiv 0\}$. Then $V_{sg}=\bigcup_{I\in\mathcal{I}}V^{*I}$. The IND-condition for small s is quite simple. In fact, the case that $s=1$, the PND-condition is enough for the IND-condition. Assume that $s=2$. Take any primary boundary component $\partial V^*(P)$. If $\dim\partial V^*(P)\geqq 3$, we have that $e(P)=\{1, \cdots, \alpha\}$, $\partial V^*(P)\subset V-V_{sg}$ and any proper primary boundary component of $\partial V^*(P)$ is a primary boundary component of V^*. If the dimension of $\partial V^*(P)$ is two, its possible primary components have dimension one or zero. Let $\mathcal{S}_i=\{\partial V^*(P);\ \partial V^*(P)\subset V_{sg},\ \dim\partial V^*(P)=i\}$ $(i=1, 2)$ and let $\mathcal{S}_1'$ be the set of primary boundary components of order 2 which are not contained in $\mathcal{S}_1$. Each strata $X\in\mathcal{S}_1'$ has dimension one. In this case, the IND-condition is equivalent with the PND-condition for $\mathcal{S}_2$, $\mathcal{S}_1$ and $\mathcal{S}_1'$.

§ 7. Generic hyperplane section and its zeta-function.

In this section, we consider generic hyperplane sections of a non-degenerate hypersurface which has non-isolated singularity at the origin. Let $H=\{z\in C^n;\ f(z)=0\}$ be a given non-degenerate hypersurface with non-isolated singularity of dimension s at the origin. Let $f_1(z), \cdots, f_{k-1}(z)$ be given analytic function defined on a neighbourhood of the origin. Let $V_{k-1}=\{z\in C^n;\ f_1(z)=\cdots=f_{k-1}(z)=0\}$ and $V_k=\{z\in C^n;\ f_1(z)=\cdots=f_k(z)=0\}$. Here $f_k(z)=f(z)$. We assume that (i) the hypersurface $H_i=f_i^{-1}(0)$ is non-degenerate with isolated singularity at the origin and $f_i(z)$ is convenient for $i=1, \cdots, k-1$ and (ii) V_{k-1} and $V=V_k$ are non-degenerate complete intersection varieties. Then we can consider the restriction of the Milnor fibration of the mapping $\boldsymbol{f}=(f_1, \cdots, f_k)\colon (C^n, \vec{0})\to(C^k, \vec{0})$ to $\{f_1=\cdots=f_{k-1}=0,\ |f_k|\neq 0\}$. We use the same notation as in § 6 of

[**21**]. Let B_ε be a small disc of radius ε and let $U_\delta=\{\eta\in \boldsymbol{C}\,;\ |\eta|<\delta\}$ and let $U_\delta^*=U_\delta-\{0\}$ where δ is sufficiently small comparing with ε. Let $X_{k-1}=V_{k-1}\cap B_\varepsilon\cap f_k^{-1}(U_\delta)$ and $X_{k-1}^*=V_{k-1}\cap B_\varepsilon\cap f_k^{-1}(U_\delta^*)$. By the assumption, V_{k-1}^* is non-singular and the restriction of f_k to V_{k-1}^* is a fibration: $f_k: X_{k-1}^*\to U_\delta^*-\{0\}$. Let $\tilde{X}_{k-1}=\pi_{k-1}^{-1}(X_{k-1})$ and $\tilde{X}_{k-1}^*=\pi_{k-1}^{-1}(X_{k-1}^*)$. We may assume that the unimodular simplicial subdivision Σ^* is chosen so that

$$(\#)\quad P\in\Sigma^*,\ I(P)\neq\emptyset \quad\text{and}\quad P\neq R_1,\cdots,R_n \Longrightarrow f^{I(P)}\equiv 0\,.$$

As $\pi_{k-1}: \tilde{X}_{k-1}^*\to X_{k-1}^*$ is biholomorphic, the above fibration is equivalent to $f_k': \tilde{X}_{k-1}^*\to U_\delta^*$ where $f_k'=\pi_{k-1}\circ f_k$. Now $\pi_{k-1}: \tilde{X}_{k-1}\to X_{k-1}$ is already a good resolution of X_{k-1} which satisfies the conditions of Theorem (3.2) of §3, [**21**]. Therefore by the same argument as [**21**], Theorem (6.8) of [**21**] can be generalized in this situation as follows. Let $\mathcal{S}_I$ be the set of primitive strictly positive weight vectors Q in N_I such that (a) (Non-emptiness) $\{\Delta(Q\,;f_i^I)\,;\, i=1,\cdots,k-1\}$ satisfies the (A_0)-condition, (b) $f_k^I\neq 0$ and (c) (Maximal dimension) $\dim(\Delta(Q\,;f_1^I)+\cdots+\Delta(Q\,;f_k^I))=|I|-1$. $\mathcal{S}_I$ is called *the I-data set* of V_k. (If $f_k^I\equiv 0$, $f_k'^{-1}(0)\supset \hat{E}(Q)\cap\tilde{X}_{k-1}^*$.) Then we have

THEOREM (7.1). *We have*

$$\zeta_k(t)=\prod_{|I|\geq k}\prod_{Q\in\mathcal{S}_I}(1-t^{d(Q;f_k^I)})^{-\chi(Q)}\,.$$

Here the integer $\chi(Q)$ is as in §6 of [**21**]. Compare the definition of $\mathcal{S}_I$ with that in the case of an isolated non-degenerate complete intersection variety. Assume that $f_1,\cdots,f_{k-1}$ are generic linear forms so that $V=V_k$ is nothing but $(k-1)$-times iterated generic hyperplane sections. Then Theorem (7.1) describes the zeta-function of f^L where $L=V_{k-1}$. If $k\geqq s+1$, V_k has an isolated singularity at the origin and the Milnor fiber is homotopically a bouquet of $(n-k)$-spheres. Thus the Milnor number μ is well-defined. Thus we have

COROLLARY (7.2). *Assume that* $k\geqq s+1$. *Then*

$$1+(-1)^{n-k}\mu=\sum_{|I|\geq k}\sum_{Q\in\mathcal{S}_I} d(Q\,;f_k^I)\chi(Q)\,.$$

Now we consider the stratifications of H and V_k. Assume that the hypersurface H satisfies the IND-condition and let $\mathcal{S}=\bigcup_I\mathcal{S}(I)$ be the stratification of H as in §6.

THEOREM (7.3). *Assume that the coefficients of* $f_1,\cdots,f_{k-1}$ *are sufficiently generic. Then* V_k *satisfies the* IND-*condition and the corresponding stratification* $\mathcal{S}'$ *of* V_k *is simply* $\bigcup_I\mathcal{S}'(I)$ *where*

$$\mathcal{S}'(I)=\bigcup_{X\in\mathcal{S}(I)}X',\qquad X'=\{z_I\in \boldsymbol{C}^{*I}\,;\ z_I\in X,\ f_1^I(z_I)=\cdots=f_{k-1}^I(z_I)=0\}\,.$$

Namely we have $X'=X\cap V_{k-1}^{I}$. *If* H *is a good hypersurface,* $V=V_k$ *is also a good complete intersection variety.*

PROOF. By the definition of $\mathcal{S}(I)$, $X\in\mathcal{S}(I)$ is defined by an open dense subset of $\partial X_1^*(P_1)\cap\cdots\cap\partial X_t^*(P_t)$. Here $X_i\in\mathcal{S}(J_i)$ with $J_i\supset I$ and $J_i\neq I$. We prove the assertion by the descending induction on $|I|$. Let $X_i=\{z_{J_i}\in C^{*J_i};\ h_{i1}(z_{J_i})=\cdots=h_{i\nu_i}(z_{J_i})=0\}$. By the induction's assumption, we have $X_i'=X_i\cap V_{k-1}^{J_i}\in\mathcal{S}'(J_i)$. It is easy to see that

$$X_i'^*(P_i)=\{z_{J_i}\in C^{*J_i}\ ;\ h_{i1P_i}(z_{J_i})=\cdots=h_{i\nu_iP_i}(z_{J_i})=f_1^I(z_I)=\cdots=f_{k-1}^I(z_I)=0\}$$

$$\partial X_i'^*(P_i)=\{z_I\in C^{*I}\ ;\ h_{i1P_i}^e(z_I)=\cdots=h_{i\nu_iP_i}^e(z_I)=f_1^I(z_I)=\cdots=f_{k-1}^I(z_I)=0\}$$

where $h_{ijP_i}^e\equiv 0$ if h_{ijP_i} is not essentially of z_I-variables. Thus the PND-condition for $(X_i', \partial X_i'^*(P_i))$ follows immediately from that of $(X_i, \partial X_i^*(P_i))$. Now the non-degeneracy of the intersection variety $\bigcap_{i=1}^t\partial X_i'^*(P_i)$ also follows from the non-degeneracy assumption of X as $\bigcap_{i=1}^t\partial X_i'^*(P_i)=X\cap V_{k-1}^I$. Note that if H is a good hypersurface, V_k is also a good complete intersection variety. This completes the proof.

REMARK (7.4). Theorem (7.3) can be obviously extended to the case that H is a non-degenerate complete intersection variety with the IND-condition: $H=\{z\in C^n;\ f_k(z)=\cdots=f_{k+s}(z)=0\}$.

§ 8. Topological stability.

In this section, we consider a family of the complete intersection varieties and we study the topological stability. Let $f_1(z, u),\cdots, f_\alpha(z, u)$ be analytic functions defined on $W\times U$ where W is a neighborhood of the origin of C^n and U is an open connected subset of C^m. Let

$$\mathcal{V}=\{(z, u)\in W\times U\ ;\ f_\nu(z, u)=0,\ \nu=1,\cdots,\alpha\}.$$

Let $\pi:\mathcal{V}\to U$ be the projection map and let $V_u=\pi^{-1}(u)$. We assume that V_u satisfies the simultaneous IND-condition for each $u\in U$ in the following sense.

(i) $\partial\Gamma(f_\nu)$ is independent of $u\in U$.

(ii) For each weight vector P, the degrees of $f_{\nu P}(z, u)$ and $\hat{f}_{\nu P}(z, u)$ are independent of $u\in U$.

(iii) For each $u\in U$, V_u satisfies IND-condition.

Here u is considered as a fixed parameter when we say something about the Newton boundary, face functions and so on. Under this assumption, the argument in §§ 5, 6 can be extended easily with parameter u to obtain a regular stratification $\mathcal{S}$ of $\mathcal{V}$. Only thing we have to do is to extend Theorem (5.1) with a parameter u: Let $p(t)=(z(t), u(t))$ be an real analytic function defined on the interval $[0, 1]$. Let $z_i(t)=a_i t^{b_i}+$(higher terms) as in § 5. Let $I=\{i\,;\, b_i=0\}$, $B={}^t(b_1, \cdots, b_n)$ and $\boldsymbol{a}=(a_1, \cdots, a_n)$ as before. Let $q(t)$ be a real analytic curve in $\mathcal{V}^{*I}(B)$ such that $p(0)=q(0)$. Let τ be the limit of the tangent space $T_{p(t)}\mathcal{V}$ of $\mathcal{V}$ at $p(t)$ and let l be the limit line of $[p(t)-q(t)]$ when t goes to zero. In this case, τ is a subspace of codimension α of $\boldsymbol{C}^{n+m}$ and l is a line of $\boldsymbol{C}^{n+m}$. Then we assert

THEOREM (5.1)$'$. *l is contained in τ.*

The proof is completely parallel to that of Theorem (5.1). We only use the fact that $\mathrm{ord}(\partial f_\nu/\partial u_j(p(t)))\geqq d(B\,;\, f_\nu)$. For instance, assume that $\alpha=1$. For brevity's sake, we use the notation that $z_{n+j}=u_j\,(j=1, \cdots, m)$, $f=f_1$ and $\hat{I}=I\cup\{n+1, \cdots, n+m\}$. We start the canonical identities

$$\sum_{i=1}^{n+m}\frac{\partial f}{\partial z_i}(p(t))\frac{dz_i(p(t))}{dt}\equiv 0 \quad\text{and}\quad \sum_{j\in\hat{I}}\frac{\partial f_B^e}{\partial z_j}(q(t))\frac{dz_j(q(t))}{dt}\equiv 0\,.$$

Let $m=\mathrm{ord}(df(p(t)))$ and $k=\mathrm{ord}(p(t)-q(t))$. We put $\vec{\gamma}=df(p(t))/t^m|_{t=0}$ and $\vec{l}=(p(t)-q(t))/t^k|_{t=0}$. If $f_B(\boldsymbol{z}, \boldsymbol{u})$ is not essentially of $\boldsymbol{z}_I$-variables, $\vec{\gamma}=\sum_{j\notin\hat{I}}\gamma_j dz_j$ and the proof is exactly the same as that of Case (I-a) in the proof of Theorem (5.1). Assume that $f_B(\boldsymbol{z}, \boldsymbol{u})$ is essentially of $\boldsymbol{z}_I$-variables. Let $f_B(\boldsymbol{z}, \boldsymbol{u})=\boldsymbol{z}^L f_B^e(\boldsymbol{z}, \boldsymbol{u})$. We use the following equality instead of (5.6).

$$\sum_{j=1}^{n+m}\frac{\partial f}{\partial z_j}(p(t))\frac{d}{dt}(z_j(p(t))-z_j(q(t)))+\sum_{j\in\hat{I}}\Big(\frac{\partial f}{\partial z_j}(p(t))-\frac{\partial f_B}{\partial z_j}(p(t))\Big)\frac{dz_j(q(t))}{dt}$$
$$+\sum_{j\in\hat{I}}p(t)^L\Big(\frac{\partial f_B^e}{\partial z_j}(p(t))-\frac{\partial f_B^e}{\partial z_j}(q(t))\Big)\frac{dz_j(q(t))}{dt}=0\,.$$

Assume that the PND1-(a)-(ii)-condition holds. Then the order of the first sum is $m+k-1$. The order of second sum is at least $d(B\,;\, f)+b_{\max}\geqq d(B\,;\, f)+k$. The order of the last sum is at least $d(B\,;\, f)+k$. Thus we conclude that $\vec{\gamma}(\vec{l})=0$ as in the proof of Case (I-b) of Theorem (5.1). We have also the equality $m=d(B\,;\, f)$. The case of PND1-(a)-(i) can be treated similarly. The general case $\alpha>1$ can be proved in the exact same way using modified Lemma (5.11) with parameter $\boldsymbol{u}$.

Let U_1 be an arbitrary relatively compact connected subset of U. We use the same argument as in § 6 to construct a regular stratification $\mathcal{S}$ of $\mathcal{V}\cap(B_\varepsilon\times U_1)$ for some $\varepsilon>0$ such that $\mathcal{S}=\bigcup_I\mathcal{S}(I)$ and each stratum Y of $\mathcal{S}(I)$ is described as $Y=\{(\boldsymbol{z}_I, \boldsymbol{u})\in B_\varepsilon^{*I}\times U_1\,;\, h_1(\boldsymbol{z}_I, \boldsymbol{u})=\cdots=h_\delta(\boldsymbol{z}_I, \boldsymbol{u})=0\}$ for some h_ν $(\nu=1, \cdots, \delta)$ where $B_\varepsilon^{*I}=\{\boldsymbol{z}_I\in\boldsymbol{C}^{*I}\,;\, \sum_{i\in I}|z_i|^2<\varepsilon\}$ and Y is a non-degenerate complete intersection for each fixed $\boldsymbol{u}$. This implies that the projection $\pi: Y\to U_1$ is a submersion. Consider the b-regular stratification $\mathcal{T}$ of $B_\varepsilon\times U_1$ which is the union $\bigcup_I\mathcal{T}(I)$ where $\mathcal{T}(I)=\mathcal{S}(I)\cup\{B_\varepsilon^{*I}\times U_1-\mathcal{V}^{*I}\}$ where $\mathcal{V}^{*I}=\mathcal{V}\cap(B_\varepsilon^{*I}\times U_1)$. We apply the Thom's first isotopy theorem ([**24**, **15**]) to obtain the following.

THEOREM (8.2). *Under the simultaneous* IND-*condition for* V_u, *the topological type of* (B_ε, V_u) *is constant for* $u \in U_1$.

§ 9. Examples.

In this section, we study several examples.

(I) Hypersurfaces. Let V be a hypersurface defined by $f(z)=0\ (\alpha=1)$. Let V_{sg} be the singular locus.

EXAMPLE (9.1). Let $f(z_1, \cdots, z_4)=\sum_{i=1}^4(z_i z_{i+1} z_{i+3})^3$, $(z_{i+4}=z_i)$. For brevity's sake, we use the variable x, y, z, w for z_1, z_2, z_3 and z_4 respectively. Then V_{sg} is the union of the 2-dimensional coordinate planes. V^{*I} is empty for each I such that $|I|=3$. Let us consider the case of $I=\{1, 2\}$. We consider I-primary boundary components. Let $P={}^t(0, 0, a, b)$. If $a \neq b$, $f_P(z)$ is a monomial. Assume $a=b$. Then $f_P(x, y, z, w)=(xy)^3(z^3+w^3)$. This satisfies PND2-condition and the corresponding primary boundary component is $C^{*\{1,2\}}$. Thus $\mathcal{S}(\{1, 2\})$ consist of a stratum $C^{*\{1,2\}}$. The same is true for any I with $|I|=2$. Namely $\mathcal{S}(\{i, j\})=\{C^{*\{i,j\}}\}$. Thus the stratification $\mathcal{S}$ of V is given by 12 strata $\mathcal{S}=V^*, C^{*\{i,j\}}, C^{*\{i\}}, \{0\}$. V is irreducible as $V^{*I}=V_{pr}^{*I}$ for any I. This is a good hypersurface.

EXAMPLE (9.2) (Damon [4]). Let $f(z)=xyz^3(x^3+y^3)+x^2y^2w^4+yz^2w^5+xzw^6$. It is easy to see that V^{*I} is empty for $I=\{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 4\}$. For $I=\{1, 2, 3\}$, V^{*I} is defined by $x^3+y^3=0$ and V is non-singular on V^{*I}. It has three connected components $L_i=\{x+\omega^i y=0\}\ (i=0, 1, 2)$ where $\omega=\exp(2\pi\sqrt{-1}/3)$. Now we consider the case $|I|=2$. First let $I=\{1, 2\}$. We consider a dual vector $P={}^t(0, 0, a, b)$. If $3a=4b$, P gives a non-trivial face function $f_P(z)=xyz^3(x^3+y^3)+x^2y^2w^4$. This is the face where f is not strongly non-degenerate in the terminology of Damon ([4]). However f_P satisfies PND2-condition as $\partial f_P/\partial z=\partial f_P/\partial w=0$ has no solution in $C^{*\{1,2,3\}}$. The corresponding primary boundary component is $C^{*\{1,2\}}$. If $3a<4b$, P gives the face function $f_P(z)=xyz^3(x^3+y^3)$. This satisfies the PND1-condition. As the secondary face function $\hat{f}_B(z)$ is a monomial, PND1-(a)$'$-(iii) is satisfied. Thus P gives the primary boundary component $x^3+y^3=0$ which gives three strata C_i: $\{x+\omega^i y=0\}$.

EXAMPLE (9.3). Let $f(z)=x^5(x^4+z^4+w^4+u^5)+(xy)^2zwu+y^5(ay^4+bz^4+cw^4+du^4)$. $(n=5, u=z_5)$. If I contains 1 or 2, $V_{pr}^{*I}=V^{*I}$ and it is non-singular. Let $I=\{3, 4, 5\}$. Then $V^{*I}=C^{*I}$. Let $P={}^t(s, t, 0, 0, 0)$. If $3s<2t$ or $3t<2s$, P gives face functions $x^4(z^4+w^4+u^4)$ and $y^5(bz^4+cw^4+du^4)$ respectively. Assume that $s \leqq 2t/3$ for example. PND1-(b) is clearly satisfied. $d(P; \hat{f})<d(P; f)+p_{\min}$ if and only if $t/2<s<2t/3$. In this case, $\hat{f}_P(z)=(xy)^2zwu$ and therefore PND1-(a)$'$-(iii) is satisfied. The other case is similar. The corresponding primary boundary components are $W_1=\{z^4+w^4+u^4=0\}$ and $W_2=\{bz^4+cw^4+du^4=0\}$. If

$3t/2 \geqq s \geqq 2t/3$, it is easy to see that the corresponding face functions satisfy PND2-condition. The corresponding primary boundary component is $\boldsymbol{C}^{*I}$. Thus $\mathcal{S}(I)=\{W_1^*, W_2^*, W_{\{1,2\}}^*, W_\emptyset^*\}$ where $W_{\{1,2\}}^*=W_1 \cap W_2$ and $W_\emptyset^*=\boldsymbol{C}^{*I}-W_1 \cup W_2$. Let $I=\{4, 5\}$. Let $P={}^t(a, b, c, 0, 0)$ be a dual vector. As PND1-type face functions, P can give $x^5(w^4+u^4)$ $(a<b,\ 3a<2b+c)$ and $y^5(cw^4+du^4)$ $(b<a,\ 3b<2a+c)$. They satisfy the PND1-condition. The corresponding primary boundary components are $w^4+u^4=0$ and $cw^4+du^4=0$ (8 strata). As PND2-type face functions, f_P can be $x^5(w^4+u^4)+(xy)^2zwu+y^5(cw^4+du^4)$ (if $a=b=c$), $x^5(w^4+u^4)+y^5(cw^4+du^4)$ (if $a=b<c$), $x^5(w^4+u^5)+(xy)^2zwu$ (if $3a=2b+c,\ a<b$) and $(xy)^2zwu+y^5(cw^4+du^4)$ (if $3b=2a+c,\ b<a$). They satisfy the PND2-condition. This example shows that in general cases, the stratification of a hypersurface involves the stratification of a complete intersection variety. V is not a good hypersurface.

(II) General case. Let $V=\{f_1(\boldsymbol{z})=\cdots=f_\alpha(\boldsymbol{z})=0\}$. We give two examples for $\alpha=2$.

EXAMPLE (9.4). Let $f_i(x, y, z, w)=\sum_{j=1}^4 a_{ij}(z_j z_{j+1} z_{j+2})^3$ for $i=1, 2$. Here $z_j=z_{j+4}$. Using the calculation of Example (9.1), we can see easily that V satisfies the IND-condition if $a_{ij}\neq 0$ and $a_{1j}a_{2k}-a_{1k}a_{2j}\neq 0$ for $j\neq k$. The stratification $\mathcal{S}$ is given by 12 strata V^*, $\boldsymbol{C}^{*\{i,j\}}$, $\boldsymbol{C}^{*\{i\}}$, $\{0\}$. For I with $|I|=2$, $V^{*I}=\emptyset$ while we have $V^{*I}=\boldsymbol{C}^{*I}$. This implies that $\boldsymbol{C}^I$ $(|I|=2)$ are irreducible components of V.

EXAMPLE (9.5) (cf. Example (9.2)). Let $f_i(x, y, z, w)=xyz^3(a_i x^3+b_i y^3)+c_i x^2y^2w^4+d_i yz^2w^5+e_i xzw^6$ for $i=1$ and 2. We need the condition that $a_i\neq 0, \cdots, e_i\neq 0$ and any 2×2 minor of

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \end{pmatrix}$$

is non-zero. V^{*I} is empty for $|I|=3$. Let $|I|=2$ and suppose that $I\neq\{1, 2\}$. Then PND2 is satisfied. Let $I=\{1, 2\}$ and let $P={}^t(0, 0, 4a, 3a)$. Then the corresponding non-trivial face functions are $f_{iP}(\boldsymbol{z})=xyz^3(a_i x^3+b_i y^3)+c_i x^2y^2w^4=0$ for $i=1, 2$. This does not satisfy the PND2-condition along

$$C\ :(a_1c_2-a_2c_1)x^3+(b_1c_2-b_2c_1)y^3=0. \tag{9.6}$$

Thus this time, V does not satisfy the IND-condition. To obtain a regular stratification of V, we need to add a stratum defined by (9.6).

§ 10. Appendix.

(A) PND2-condition. We first consider sufficient conditions for the PND2-condition. Let P, $e(P)$ and I be as in the PND-condition in § 4. We will replace the PND2-condition for a given P by the condition on the coefficients of f_ν on

$\Gamma(f_\nu)$. Let Q be a strictly positive rational dual vector and let

$$\partial V^*(P, Q) = \{z_I \in \boldsymbol{C}^{*I}\ ; (f^e_{\nu P})_Q(z_I)=0,\ \nu\in e(P)\} \quad \text{and}$$
$$V^*(P, Q) = \{z \in \boldsymbol{C}^{*n}\ ; (f_{\nu P})_Q(z)=0,\ \nu=1, \cdots, \alpha\}$$

and let $q_I : V^*(P, Q) \to \partial V^*(P, Q)$ be the projection. Here $(f_{\nu P})_Q$ is the face function of $f_{\nu P}$ with respect to Q and it has a compact support on a face of $\Gamma(f_\nu)$. We assert

LEMMA (10.1). *The following is a sufficient condition for the* PND2-*condition for a given P.* (PND2)′ *For any strictly positive rational dual vector Q such that $V^*(P, Q)$ is not empty, the fiber $q^{-1}(z_I)\cap V^*(P, Q)$ is a smooth complete intersection for each fixed $z_I \in \partial V^*(P, Q)$.*

PROOF. Assume that PND2 does not hold for P. We may assume that $e(P)=\{s+1, \cdots, \alpha\}$. Let $df_1\wedge\cdots\wedge df_s(z)=\sum_K c_K(z)dz_K$. Here K is a subset of $\{1, \cdots, n\}$ with $|K|=s$ and $dz_K=dz_{k_1}\wedge\cdots\wedge dz_{k_s}$, if $K=\{k_1, \cdots, k_s\}$. Then we apply the Curve Selection Lemma to find a real analytic curve $p(t)\ (0\leqq t<1)$ such that (i) $p(0)=\vec{0}$, $f_{\nu P}(p(t))\equiv 0$ for $\nu=1, \cdots, \alpha$ and $p(t)\in \boldsymbol{C}^{*n}$ for $t\neq 0$ and (ii) $c_K(p(t))\equiv 0$ for any K with $K\cap I=\emptyset$. Let $p_i(t)=a_i t^{b_i}+$(higher terms) where $a_i\neq 0$ and $b_i>0$ for $i=1, \cdots, n$. Let $Q={}^t(b_1, \cdots, b_n)$ and $\boldsymbol{a}=(a_1, \cdots, a_n)$. Considering the leading terms of the above equalities, we obtain $(f_{\nu P})_Q(\boldsymbol{a})=0$, for $\nu=1, \cdots, \alpha$ and $\det(\partial(f_{\nu P})_Q/\partial z_{k_\mu}(\boldsymbol{a}))_{(1\leqq\nu,\mu\leqq s)}=0$ for any $K=\{k_1, \cdots, k_s\}$ with $K\cap I=\emptyset$. This contradicts to (PND2)′-condition.

In general (PND2)′ is still not so easy to be checked. We have also the following sufficient condition for (PND2)′. For a positive rational dual vector $\boldsymbol{R}={}^t(r_1, \cdots, r_n)$, recall that we have defined $I(\boldsymbol{R})=\{j\ ;\ r_j=0\}$. For brevity's sake, we assume that $e(P)=\{s+1, \cdots, \alpha\}$. Let $h_\nu(z)=f_{\nu P}(z)$ (resp. $(f_{\nu P})_Q$) for $\nu=1, \cdots, s$. We say that Q is *compatible* with $h_1, \cdots, h_s$ if each $h_\nu(z)$ is a weighted homogeneous polynomial with the weight Q.

LEMMA (10.2). *Assume that there exist positive rational dual vectors $R_0(=P)$, $\cdots$, R_{m-1} which are compatible with $h_1, \cdots, h_s$ such that $I(R_0)\supset\cdots\supset I(R_{m-1})$. Here $m=|I|$ and $|I(R_i)|=m-i$. Assume that $W=\{z\in \boldsymbol{C}^{*n}\cap B_\varepsilon\ ;\ h_1(z)=\cdots=h_s(z)=0\}$ is a smooth complete intersection variety. Let $q_I : W\to \boldsymbol{C}^{*I}\cap B_\varepsilon$ be the canonical projection. Then for any fixed $z_I\in \boldsymbol{C}^{*I}\cap B_\varepsilon$, $q^{-1}(z_I)$ is a smooth complete intersection. In particular, the* PND2-*condition for P (resp.* (PND2)′-*condition for P and Q) is true.*

PROOF. We prove the assertion by the induction on m. The assertion is trivial if $m=0$. Assume that the assertion is true for $|I|=m-1$. By changing the ordering of the coordinates if necessary, we may assume that $I=I(R_0)=\{1, \cdots, m\}$ and $I(R_1)=\{1, \cdots, m-1\}$. Let $dh_1\wedge\cdots\wedge dh_s(z)=\sum_{|K|=s} c_K(z)dz_K$.

Assume that the assertion does not hold. Then there exists a point $\boldsymbol{a}$ of W such that $c_K(\boldsymbol{a})=0$ for any $K\subset\{m+1, \cdots, n\}$. We will prove that this implies that $c_K(\boldsymbol{a})=0$ for any $K\subset\{m, \cdots, n\}$. Let $\Lambda=\{l_1, l_2, \cdots, l_{s-1}\}$ where $l_1=m$ and $l_j>m$. By the assumption, we can write $R_1={}^t(r_{11}, \cdots, r_{1n})$ where $r_{1j}=0$ for $j<m$ and $r_{1j}>0$ for $j\geqq m$. Let d_ν be the degree of $h_\nu(z)$ under the weight R_1. Then h_ν satisfies the following equation:

$$(10.3) \qquad d_\nu h_\nu(\boldsymbol{z}) = \sum_{j=m}^{n} r_{1j} z_j \frac{\partial h_\nu}{\partial z_j}(\boldsymbol{z}), \qquad \nu=1, \cdots, s.$$

As $c_\Lambda(\boldsymbol{a})$ is equal to $\det(\partial h_\nu/\partial z_{l_\mu}(\boldsymbol{a}))_{\nu,\mu=1,\cdots,s}$, we use (10.3) and the equality $h_\nu(\boldsymbol{a})=0$ to eliminate the $\partial h_\nu/\partial z_m(\boldsymbol{a})$ of the first column of the above matrix to obtain

$$(10.4) \qquad r_1 m a_m c_\Lambda(\boldsymbol{a}) = -\sum_{j=m+1}^{n} r_{1j} a_j c_{\Lambda_j}(\boldsymbol{a})$$

where $\Lambda_j=\{j, l_2, \cdots, l_s\}$. As $\Lambda_j\subset\{m+1, \cdots, n\}$, the right side of (10.4) is zero by the induction's assumption. As $r_{1m}a_m\neq 0$, we obtain $c_\Lambda(\boldsymbol{a})=0$. This is true for any such Λ which is a contradiction to the induction's assumption.

(B) Non-emptyness for $V^*(P)$. We consider the variety $V^*(P)=\{\boldsymbol{z}\in \boldsymbol{C}^{*n}\cap B_\varepsilon;\ f_{1P}(\boldsymbol{z})=\cdots=f_{\alpha P}(\boldsymbol{z})=0\}$ which appeared in the definition of the primary boundary components. Let $V(P)_{pr}$ be the closure of $V^*(P)$. Let $\Delta_1, \cdots, \Delta_\alpha$ be compact convex polyhedra in $\boldsymbol{R}^n$. Recall that A_0-condition is defined by the following: (A_0) For any subset $K\subset\{1, \cdots, \alpha\}$, $\dim\sum_{\nu\in K}\Delta_\nu\geqq|K|$. We consider the non-emptyness condition of $V^*(P)$ as a germ of variety at the origin. For this purpose, we fix a toric resolution of $V^*(P)$, $\pi: X\to V(P)_{pr}$ which is associated to $\Sigma^*(P)$, a unimodular simplicial subdivision of $\Gamma^*(f_{1P}, \cdots, f_{\alpha P})$. Then it is obvious that $V(P)_{pr}$ is non-empty as a germ of an analytic variety at the origin if and only if there exists a strictly positive vertex $Q\in\Sigma^*(P)$ such that the corresponding exceptional divisor $E(Q)$ is non-empty. However the non-emptyness of $E(P)$ is equivalent to the A_0-condition for $\{\Delta(Q; f_{1P}), \cdots, \Delta(Q; f_{\alpha P})\}$ (Proposition (5.4), [21]). Note also the existence of such a vertex Q does not depend on the choice of $\Sigma^*(P)$. Thus we have proved the following.

LEMMA (10.5). *The germ of $V(P)_{pr}$ at the origin is non-empty if and only if there exists a strictly positive dual vector $Q\in N^+$ such that $\{\Delta(Q; f_{1P}), \cdots, \Delta(Q; f_{\alpha P})\}$ satisfies the A_0-condition.*

REMARK (10.6). Assume that $h_1, \cdots, h_\alpha$ be polynomials and let $Z=\{\boldsymbol{z}\in\boldsymbol{C}^{*n};\ h_1(\boldsymbol{z})=\cdots=h_\alpha(\boldsymbol{z})=0\}$ be a non-degenerate complete intersection variety. Let $\bar{Z}$ be the closure of Z in $\boldsymbol{C}^n$. Then the A_0-condition for $\{\Delta(h_1), \cdots, \Delta(h_\alpha)\}$ is enough for the non-emptyness of Z but it is not enough for the non-emptyness of $\bar{Z}$

as a germ of a variety at the origin.

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