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The almost weak specification property for ergodic group automorphisms of abelian groups

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§ 0. Introduction.

The property of specification plays an important role in classifying the class of invariant probability measures preserved under a homeomorphism (see K. Sigmund [**10, 11**] and the author [**4**]). In [**7**] B. Marcus introduced the notion of almost weak specification weaker than that of specification by using toral automorphisms.

The purpose of this paper is to prove that every automorphism of a compact metric abelian group is ergodic under the Haar measure if and only if it satisfies almost weak specification (Corollary of Theorem 1).

Let X be a compact metric space with metric d and σ be a homeomorphism from X onto itself. Then σ satisfies *almost weak specification* if for every $\varepsilon>0$ there is a function M_{ε}: $\boldsymbol{Z}^{+}\rightarrow\boldsymbol{Z}^{+}$ ($\boldsymbol{Z}^{+}$ denotes the set of non-negative integers) with $M_{\varepsilon}(n)/n\rightarrow 0$ as $n\rightarrow\infty$ such that for every $k\geqq 1$ and k points $x_{1}, \cdots, x_{k}\in X$ and for every sequence of integers $a_{1}\leqq b_{1}<a_{2}\leqq b_{2}<\cdots<a_{k}\leqq b_{k}$ with $a_{i}-b_{i-1}\geqq M_{\varepsilon}(b_{i}-a_{i})$ $(2\leqq i\leqq k)$, there is an $x\in X$ with $d(\sigma^{n}x, \sigma^{n}x_{i})\leqq\varepsilon$ $(a_{i}\leqq n\leqq b_{i}, 1\leqq i\leqq k)$. A homeomorphism satisfies *weak specification* if it has the property of almost weak specification with some constant function M_{ε}. It is clear from definition that if (X, σ) satisfies almost weak specification, then it is topologically mixing. Almost weak specification is preserved under direct products and homeomorphic images. A shift of compact metric state space satisfies almost weak specification ((21.2), [**6**]).

Hereafter let X be a compact metric abelian group and σ be an automorphism of X. G denotes the dual group of X. Define the dual automorphism γ of G by $(\gamma g)(x)=g(\sigma x)$ $(g\in G, x\in X)$. Group operations of X and G will be denoted by addition. If X is connected, then G is torsion free (i.e., $ng\neq 0$ for all $0\neq g\in G$ and $0\neq n\in\boldsymbol{Z}$). When X is connected, (X, σ) is said to satisfy *condition* (A) if for every $0\neq g\in G$ there is $0\neq p(x)\in\boldsymbol{Z}[x]$ ($\boldsymbol{Z}[x]$ denotes the ring of polynomials with integral coefficients) such that $p(\gamma)g=0$, and (X, σ) is said to satisfy *condition* (B) if one has $p(\gamma)g\neq 0$ for every $0\neq g\in G$ and $0\neq p(x)$

$\in \boldsymbol{Z}[x]$.

(L.1) ([**1**], Theorem 2). *Let X be a group as above; then X splits into a sum $X=X_1+X_2+X_3$ of σ-invariant subgroups* ($\sigma(X_i)=X_i$, $i=1, 2, 3$) *such that* (i) *X_1 is totally disconnected,* (ii) *X_2 is connected and (X_2, σ) satisfies condition* (A), *and* (iii) *X_3 is connected and (X_3, σ) satisfies condition* (B). *If in particular (X, σ) is ergodic under the Haar measure, then X_i $(i=1, 2, 3)$ can be chosen such that (X_i, σ) is ergodic under the Haar measure.*

(L.2) *Let $\{X_n\}_{n\geqq 0}$ be a sequence of σ-invariant subgroups such that $X=X_0\supset X_1\supset \cdots \supset \bigcap_{n\geqq 0} X_n=\{0\}$ and assume that for $n\geqq 1$, $(X/X_n, \sigma)$ satisfies almost weak specification. Then (X, σ) satisfies almost weak specification.*

Indeed, define an invariant metric d_n on X/X_n $(n\geqq 1)$ by

$$d_n(x+X_n,\ y+X_n)=\min_{z\in X_n} d(x,\ y+z) \qquad (x,\ y\in X).$$

Take and fix $\varepsilon>0$. Choose $n\geqq 1$ with $\operatorname{diam}(X_n)<\varepsilon/2$. Since $(X/X_n, \sigma)$ satisfies almost weak specification, there is a function $M_{\varepsilon/2}^{(n)}: \boldsymbol{Z}^+\to\boldsymbol{Z}^+$ with $M_{\varepsilon/2}^{(n)}(m)/m \to 0$ as $m\to\infty$ such that for every $k\geqq 1$, k points $x_1, \cdots, x_k\in X$, and a sequence of integers $a_1\leqq b_1<\cdots<a_k\leqq b_k$ with $a_i-b_{i-1}\geqq M_{\varepsilon/2}^{(n)}(b_i-a_i)$ $(2\leqq i\leqq k)$, there is an $x\in X$ with $d_n(\sigma^j x+X_n, \sigma^j x_i+X_n)\leqq\varepsilon/2$ $(a_i\leqq j\leqq b_i,\ 1\leqq i\leqq k)$. Obviously $d(\sigma^j x, \sigma^j x_i)\leqq d_n(\sigma^j x+X_n, \sigma^j x_i+X_n)+\operatorname{diam}(X_n)\leqq\varepsilon$ $(a_i\leqq j\leqq b_i,\ 1\leqq i\leqq k)$. Letting $M_\varepsilon=M_{\varepsilon/2}^{(n)}$ for simplicity, we can easily check that (X, σ) satisfies almost weak specification.

(L.3) *Let X_3 be as in* (L.1). *If (X_3, σ) is ergodic under the Haar measure, then it satisfies almost weak specification.*

This follows from the proof of ([**1**], Lemma 9) together with (L.2).

(L.4) ([**2**], Theorem 3). *Let X_1 be as in* (L.1). *If (X_1, σ) is ergodic under the Haar measure, then it satisfies almost weak specification.*

Let X be as above. Then X is said to be *solenoidal* if X is connected and finite dimensional. Clearly every finite-dimensional torus is solenoidal.

(L.5) ([**1**], Lemma 8). *Let X_2 be as in* (L.1). *Then there is a sequence*

$$X_2=X_{2,0}\supset X_{2,1}\supset\cdots\supset\bigcap_{n\geqq 0} X_{2,n}=\{0\}$$

of σ-invariant subgroups such that each $X_2/X_{2,n}$ is solenoidal.

This follows from the proof of Lemma 8 in [**1**].

For the following statements (L.6)$\sim$(L.14), let X be r-dimensional solenoidal. Since $\operatorname{rank}(G)=r<\infty$ and G is torsion free, there exists an into isomorphism $\varphi: G\to\boldsymbol{Q}^r$ ($\boldsymbol{Q}^r$ denotes the vector space over $\boldsymbol{Q}$), so that $\bar{\gamma}=\varphi\circ\gamma\circ\varphi^{-1}$ is extended to $\boldsymbol{Q}^r$ and further to $\boldsymbol{R}^r$. We denote again by γ the extension to $\boldsymbol{R}^r$.

(L.6) ([**3**], p. 83, (P.2)). *Under the above notations there are a homomorphism* ϕ: $\boldsymbol{R}^r \to X$ *and a totally disconnected subgroup* F *such that* (i) $\phi\circ\gamma=\sigma\circ\phi$, (ii) $X=\phi(\boldsymbol{R}^r)+F$, *and further* (iii) *there is a small closed neighborhood* U *of* 0 *in* $\boldsymbol{R}^r$ *such that* $\phi(U)\cap F=\{0\}$ *and* $U\times F$ *is homeomorphic to* $\phi(U)+F$ *and* $\phi(U)+F$ *is a closed neighborhood of* 0 *in* X (*we write* $\phi(U)\oplus F$ *for such a neighborhood* $\phi(U)+F$).

(L.7) ([**3**], p. 87, (P.8)). *Let* F *be as in* (L.6). *Then* F *contains subgroups* F^{-1}, F^{+}, *and* H *such that* (i) $\sigma(H)=H$, (ii) $F^{-}\supset\sigma^{-1}F^{-}\supset\cdots\supset\bigcap_0^\infty\sigma^{-n}F^{-}=\{0\}$, (iii) $F^{+}\supset\sigma F^{+}\supset\cdots\supset\bigcap_0^\infty\sigma^{n}F^{+}=\{0\}$, *and* (iv) $F=F^{-}\oplus F^{+}\oplus H$.

Let $(\boldsymbol{R}^r, \gamma)$ be a lifting system of (X, σ) by ϕ. Then $\boldsymbol{R}^r$ splits into a direct sum $\boldsymbol{R}^r=E^u\oplus E^s\oplus E^c$ of γ-invariant subspaces E^u, E^s, and E^c such that the eigenvalues of $\gamma_{|E^u}$ have modulus>1, the eigenvalues of $\gamma_{|E^s}$ modulus<1 and the eigenvalues of $\gamma_{|E^c}$ modulus one. We call $(\boldsymbol{R}^r, \gamma)$ *hyperbolic* if $E^c=\{0\}$; i.e., $\boldsymbol{R}^r=E^u\oplus E^s$. If $E^c\neq\{0\}$, by using Jordan's normal form in the real field for (E^c, γ) the subspace E^c splits into a finite direct sum $E^c=E^{c_0}\oplus E^{c_1}\oplus\cdots\oplus E^{c_k}$ of subspaces of E^c satisfying the following three conditions; (a) for $0\leqq i\leqq k$, the dimension of E^{c_i} is 1 or 2,

(b) $$\gamma_{|E^c}=\begin{pmatrix} \gamma_0 & I_0 & & 0 \\ & \gamma_1 & \ddots & \\ & & \ddots & I_k \\ 0 & & & \gamma_k \end{pmatrix}$$

where γ_i: $E^{c_i}\to E^{c_i}$ is an isometry under some norm $\|\cdot\|_{c_i}$ of E^{c_i} and each I_i: $E^{c_i}\to E^{c_{i-1}}$ is either a zero-map or a map corresponding to the identity matrix. We call that $(\boldsymbol{R}^r, \gamma)$ has *central spin* if $E^c\neq\{0\}$ and each I_i: $E^{c_i}\to E^{c_{i-1}}$ is a zero-map. If $(\boldsymbol{R}^r, \gamma)$ has central spin, then each E^{c_i} is γ-invariant. Let I denote the identity map of $\boldsymbol{R}^r$. For every $m>0$, the eigenvalues of $I-\gamma^m$ on E^{c_i} are $1-\lambda_i^m$ where λ_i's are eigenvalues of $\gamma_{|E^{c_i}}$. It is easily proved that there is a constant $c_{(i)}$ such that $\|(I-\gamma^m)x\|_{c_i}<c_{(i)}|1-\lambda_i^m|\,\|x\|_{c_i}$ $(x\in E^{c_i}, m>0)$. We define a norm $\|\cdot\|_c$ of E^c by $\|x\|_c=\max_{0\leq i\leq k}\{\|x^i\|_{c_i}\}$ $(x=x^0+\cdots+x^k\in E^{c_0}\oplus\cdots\oplus E^{c_k})$.

There are $0<\lambda_0<1$ and norms $\|\ \|_u$ and $\|\ \|_s$ on E^u and E^s respectively such that $\|\gamma^n x\|_u\leqq\lambda_0^n\|x\|_u$ $(n\leqq 0, x\in E^u)$ and $\|\gamma^n x\|_s\leqq\lambda_0^n\|x\|_u$ $(n\geqq 0, x\in E^s)$. Define a norm $\|\ \|$ on $\boldsymbol{R}^r$ by

$$\|x\|=\max\{\|x_u\|_u, \|x_s\|_s, \|x_c\|_c\} \qquad (x=x_u+x_s+x_c\in E^u\oplus E^s\oplus E^c)$$

and define a metric d_0 on $\boldsymbol{R}^r$ by

$$d_0(x, y)=\|x-y\| \qquad (x, y\in\boldsymbol{R}^r).$$

(L.8) ([**3**], p. 89, (P.10)). *There is* $\alpha_1>0$ *such that* (i) *for* $\varepsilon\in(0, \alpha_1]$, $B(\varepsilon)=\{x\in\boldsymbol{R}^r;\ d_0(x, 0)\leqq\varepsilon\}$ *splits into a direct sum* $B(\varepsilon)=B^u(\varepsilon)\oplus B^s(\varepsilon)\oplus B^c(\varepsilon)$ *where*

$B^u(\varepsilon)=B(\varepsilon)\cap E^s$, $B^s(\varepsilon)=B(\varepsilon)\cap E^s$ and $B^c(\varepsilon)=B(\varepsilon)\cap E^c$, (ii) *$B(\alpha_1)\oplus F^-\oplus F^+\oplus H$ is a closed neighborhood of* 0 *in* X.

(L.9) ([**3**], p. 89, (P.11) and p. 90, (P.12)). *There is an invariant metric d on X and a positive number α_0 with $\alpha_0<\alpha_1$ such that* (i) *for* $\varepsilon\in(0, \alpha_0]$, $W(\varepsilon)=\{x\in X;\ d(x, 0)\leqq\varepsilon\}$ *is expressed as* $W(\varepsilon)=W^u(\varepsilon)\oplus W^s(\varepsilon)\oplus W^c(\varepsilon)$ *where* $W^u(\varepsilon)=W(\varepsilon)\cap\{\phi B^u(\varepsilon)\oplus F^-\}$, $W^s(\varepsilon)=W(\varepsilon)\cap\{\phi B^s(\varepsilon)\oplus F^+\}$ *and* $W^c(\varepsilon)=W(\varepsilon)\cap\{\phi B^c(\varepsilon)\oplus H\}$, *and* (ii) *for* $\varepsilon\in(0, \alpha_0]$ *$W(\varepsilon)\cap H$ is a subgroup of H and there is an $n\geqq 0$ such that $W(\varepsilon)\cap F^-=\sigma^{-n}F^-$ and $W(\varepsilon)\cap F^+=\sigma^n F^+$ and moreover that $W(\varepsilon)\cap F^-=\sigma^{-n}F^-$ and $W(\varepsilon)\cap F^+=\sigma^n F^+$.*

(L.10) ([**3**], p. 90, (P.13)). *There is $0<\lambda_0<1$ such that for $\varepsilon\in(0, \alpha_0]$ and $x=x^u+x^s+x^c\in W^u(\varepsilon)\oplus W^s(\varepsilon)\oplus W^c(\varepsilon)$ the following hold* (i) $d(x, 0)=\max\{d(x^u, 0), d(x^s, 0), d(x^c, 0)\}$, (ii) $d(\sigma^n x, 0)\leqq\lambda_0^{-n}d(x, 0)$ $(x\in W^u(\varepsilon),\ n\leqq 0)$, (iii) $d(\sigma^n x, 0)\leqq\lambda_0^n d(x, 0)$ $(x\in W^s(\varepsilon),\ n\geqq 0)$, *and* (iv) $d(\sigma^n x, 0)=d(x, 0)$ $(x\in W^c(\varepsilon)\cap H,\ n\in\boldsymbol{Z})$.

X is said to have *property* (*) if G is finitely generated under γ; i.e., there is a finite set Λ in G such that $G=\mathrm{gp}\bigcup_{-\infty}^{\infty}\gamma^j\Lambda$ (the notation gp$E$ means the subgroup generated by a set E). We say (X, σ) is *hyperbolic* if (X, σ) has property (*) and $(\boldsymbol{R}^r, \gamma)$ is hyperbolic, and we say (X, σ) has *central spin* if either $(\boldsymbol{R}^r, \gamma)$ is hyperbolic and X does not have property (*), or $(\boldsymbol{R}^r, \gamma)$ has central spin.

(L.11) ([**3**], p. 91, (P.14)). *Assume that (X, σ) is ergodic under the Haar measure. Then $W^u(\alpha_0)\neq\{0\}$ and $W^s(\alpha_0)\neq\{0\}$.*

(L.12) *For $\varepsilon\in(0, \alpha_0]$, the following hold:*

(i) $\sigma^{-1}W^u(\varepsilon)\subset W^u(\lambda_0\varepsilon)$,

(ii) $\sigma W^s(\varepsilon)\subset W^s(\lambda_0\varepsilon)$,

(iii) $\sigma W^c(\varepsilon)=W^c(\varepsilon)$ *if (X, σ) has central spin,*

(iv) *$W^u(\varepsilon)$, $W^s(\varepsilon)$, and $W^c(\varepsilon)$ are symmetric sets at* 0 *in* X.

(i) and (ii) are the consequences of (L.10). (iii) follows from the definition of central spin together with (L.10). (iv) is clear from the definition.

For $\varepsilon\in(0, \alpha_0]$, put $K(\varepsilon)=W^s(\varepsilon)\oplus(W^c(\varepsilon)\cap H)$ and $W_n^u(\varepsilon)=\{x\in\sigma^n W^u(\varepsilon);\ x+W^u(\varepsilon)\subset\sigma^n W^u(\varepsilon)\}$ $(n\geqq 1)$. Then we have the following:

(L.13) *For every* $\varepsilon\in(0, \alpha_0]$,

(i) $\sigma K(\varepsilon)\subset K(\varepsilon)$,

(ii) $K(\varepsilon)\oplus\phi B^c(\varepsilon)=W^s(\varepsilon)\oplus W^c(\varepsilon)$,

(iii) $W_n^u(\varepsilon)\subset\sigma W_n^u(\varepsilon)\subset W_{n+1}^u(\varepsilon)$ $(n\geqq 1)$.

(L.14) *Assume that either (X, σ) is hyperbolic, or ergodic and has central spin. Then, for every $\varepsilon\in(0, 2\alpha_0/3)$ there is $M=M(\varepsilon)>0$ such that for every $n\geqq M$, $W_n^u(\varepsilon)+K(\varepsilon)\oplus\phi B(\varepsilon)=X$.*

From the proof (Step 1.2.1) of ([3], Proposition 1.2), for every $\varepsilon\in(0, 2\alpha_0/3)$ there is $M=M(\varepsilon)>0$ such that for every $n\geqq M$ and $x\in X$, $W_n^u(\varepsilon)\cap(x+K(\varepsilon)\oplus\phi B^c(\varepsilon))\neq\emptyset$. Since $K(\varepsilon)\oplus\phi B^c(\varepsilon)=W^s(\varepsilon)\oplus W^c(\varepsilon)$ is symmetric, we have $x\in W_n^u(\varepsilon)+K(\varepsilon)\oplus\phi B^c(\varepsilon)$.

(L.15) *Assume that (X, σ) is ergodic; then there exists a finite sequence $X=X_0\supset X_1\supset\cdots\supset X_n\supset X_{n+1}=\{0\}$ of σ-invariant subgroups such that each X_i is connected and either $(X_i/X_{i+1}, \sigma)$ is hyperbolic or ergodic and has central spin.*

By ([3], Theorem 3), there is a finite sequence $X=X_0\supset X_1\supset\cdots\supset X_n\supset X_{n+1}=\{0\}$ of σ-invariant subgroups such that each X_i is connected and $(X_i/X_{i+1}, \sigma)$ satisfies weak specification. By ([3], Theorems 1 and 2), if $(X_i/X_{i+1}, \sigma)$ satisfies weak specification, then $(X_i/X_{i+1}, \sigma)$ is either hyperbolic or ergodic and has central spin.

(L.16) *For $s\geqq1$, $(a_1, \cdots, a_s)\in\boldsymbol{R}^s$ and $N\geqq1$, there is an integer n with $1\leqq n\leqq N^s$ such that $|na_i-m_i|<1/N\,(1\leqq i\leqq s)$ for some $(m_1, \cdots, m_s)\in\boldsymbol{Z}^s$.*

This is shown as follows. For every $1\leqq n\leqq N^s$ there is $(m_1^{(n)}, \cdots, m_s^{(n)})\in\boldsymbol{Z}^s$ such that $na_i-m_i^{(n)}\in[0, 1)\,(1\leqq i\leqq s)$. If there is an n with $1\leqq n\leqq N^s$ such that for every $1\leqq i\leqq s$, $na_i-m_i^{(n)}\in[0, 1/N)$, (L.16) holds. For otherwise, we can find u and v with $1\leqq u<v\leqq N^s$ and j_i with $0\leqq j_i\leqq N-1$ such that for $1\leqq i\leqq s$, $ua_i-m_i^{(n)}, va_i-m_i^{(n)}\in[j_i/N, (j_i+1)/N)$. Put $n=v-u$ and $m_i=m_i^{(v)}-m_i^{(u)}$ $(1\leqq i\leqq s)$. Then we have $|na_i-m_i|<1/N\,(1\leqq i\leqq s)$.

§1. Results.

The following is the main result of our paper.

THEOREM 1. *Let X be a solenoidal group and σ be an automorphism of X. Then (X, σ) is ergodic (under the Haar measure) if and only if (X, σ) satisfies almost weak specification.*

Theorem 1 derives the following.

COROLLARY. *Let X be a compact metric abelian group and σ be an automorphism of X. Then (X, σ) is ergodic (under the Haar measure) if and only if (X, σ) satisfies almost weak specification.*

If we established Theorem 1, then the corollary is shown as follows. Clearly (X, σ) is ergodic if (X, σ) satisfies almost weak specification. Assume that (X, σ) is ergodic. Then X splits into a sum $X=X_1+X_2+X_3$ of σ-invariant subgroups with the notation of (L.1). And so (X_1, σ) satisfies almost weak specification by (1.4), and (X_3, σ) satisfies almost weak specification by (L.3). Use (L.5) for (X_2, σ). Then there is a sequence $X_2=X_{2,0}\supset X_{2,1}\supset\cdots\supset\bigcap_{n\geqq0}X_{2,n}=\{0\}$ of σ-

invariant subgroups such that each $X_2/X_{2,n}$ is solenoidal. Since (X_2, σ) is ergodic, each $(X_2/X_{2,n}, \sigma)$ is ergodic. By Theorem 1, $(X_2/X_{2,n}, \sigma)$ satisfies almost weak specification. Since the product system $(X_1\times X_2\times X_3, \sigma\times\sigma\times\sigma)$ satisfies almost weak specification, (X, σ) satisfies almost weak specification.

§ 2. Proof of Theorem 1.

As before let (G, γ) be the dual of (X, σ). Since X is solenoidal, we have $\mathrm{rank}(G)=\dim(X)=r<\infty$. Then X is expressed as

$$X=\phi(E^u\oplus E^s\oplus E^c)+\{F^-\oplus F^+\oplus H\}.$$

We prepare a sequence of lemmas leading to the proof of Theorem 1.

LEMMA 1. *If* $E^c\neq\{0\}$ *and* $a=\dim(E^c)$, *then* $\|\gamma^n x\|_c\leqq(n+1)^{a-1}\|x\|_c$ $(x\in E^c, n\geqq 0)$.

PROOF. E^c splits into a finite direct sum $E^c=E^{c_0}\oplus\cdots\oplus E^{c_k}$ of 1 or 2-dimensional subspaces which satisfy (a) and (b) in § 2. For $x\in E^{c_0}\oplus\cdots\oplus E^{c_k}$ and for $n\geqq 0$, $\gamma^n_{\restriction E^c}x$ splits into $\gamma^n_{\restriction E^c}x=x^n_0+\cdots+x^n_k$ with $x^n_i\in E^{c_i}$ $(1\leqq i\leqq k)$. By (b) we get $\|x^n_i\|_{c_i}\leqq\|x^{n-1}_i\|_{c_i}+\|x^{n-1}_{i+1}\|_{c_{i+1}}$ $(0\leqq i\leqq k-1, n\geqq 0)$ and $\|x^n_k\|_{c_k}=\|x^0_k\|_{c_k}$ $(n\geqq 0)$. It is checked that for every $n\geqq 0$, $\|x^n_i\|_{c_i}\leqq(n+1)^{k-i}\|x\|_c$ $(1\leqq i\leqq k)$. Indeed, $\|x^0_i\|_{c_i}\leqq\|x\|_c$ when $n=0$. Assume that the inequality is true for $n-1$. Then $\|x^n_i\|_{c_i}\leqq\|x^{n-1}_i\|_{c_i}+\|x^{n-1}_{i+1}\|_{c_i}\leqq n^{k-i}\|x\|_c+n^{k-i-1}\|x\|_c\leqq(n+1)^{k-i}\|x\|_c$ for $0\leqq i\leqq k-1$. Since $k\geqq a-1$, the conclusion is obtained.

LEMMA 2. *Assume that* (X, σ) *is either hyperbolic or ergodic and has central spin. Then for every* $\varepsilon\in(0, \alpha_0/3)$ *there is a sequence* $\{N_\varepsilon(n)\}_{n=1}^\infty$ *of non-negative integers such that* $N_\varepsilon(n^p)/n\to 0$ *as* $n\to\infty$ *for all* $p\geqq 1$, *and* $W^u_m(\varepsilon)+K(\varepsilon)\oplus\phi B^c(\varepsilon/n)\supset\phi B^c(\varepsilon)$ *for* $n\geqq 1$ *and* $m\geqq N_\varepsilon(n)$.

PROOF. If $(\boldsymbol{R}^r, \gamma)$ is hyperbolic (i.e., $E^c=\{0\}$), by putting $N_\varepsilon(n)=0$ for $\varepsilon\in(0, \alpha_0/3)$ and $n\geqq 1$, the lemma holds.

It only remains to prove the lemma for the case when $(\boldsymbol{R}^r, \gamma)$ has central spin. To see this, use (L.14). Then there is $M>0$ such that $W^u_M(\varepsilon)+K(\varepsilon)\oplus\phi B^c(\varepsilon)\supset\phi B^c(2\varepsilon)$. Since $a=\dim(E^c)$, we can find points $t_1, \cdots, t_a\in B^c(3\varepsilon)$ such that $\{t_1, \cdots, t_a\}$ is linearly independent over $\boldsymbol{R}$ and

$$\phi(t_i)\in W^u_M(\varepsilon)+K(\varepsilon) \qquad (1\leqq i\leqq a). \tag{2.1}$$

Since γ has central spin, γ is an isometry on $(E^c, \|\cdot\|_c)$. By Dirichlet's theorem there is $L>0$ such that

$$\|(\gamma^L-I)x\|_c\leqq\frac{1}{2}\|x\|_c \qquad (x\in E^c) \tag{2.2}$$

where I denotes the identity map.

Since γ is aperiodic (by ergodicity of σ), $\gamma^L - I$ is one-to-one and so for some μ with $0<\mu<1/2$

$$\|(\gamma^L - I)x\|_c > \mu\|x\|_c \qquad (x\in E^c). \tag{2.3}$$

Notice that $\{(\gamma^L-I)^n t_i;\ 1\leqq i\leqq a\}(n>0)$ is linearly independent over $\boldsymbol{R}$. Define $A=\{s\in E^c;\ s=\sum_{i=1}^a a_i t_i,\ a_i\in \boldsymbol{Z},\ 1\leqq i\leqq a\}$ and put $\delta=\min\{\|s\|_c;\ 0\neq s\in A\}$. Then by (2.3)

$$\mu^n\delta < \min\{\|t\|_c;\ 0\neq t\in(\gamma^L-I)^n A\} \qquad (n>0). \tag{2.4}$$

Since $\Theta=(\gamma^L-I)^n A\cap B^c(3\varepsilon/2)$ is non-trivial, every element of Θ is expressed as $t=\sum_{i=1}^a n_i(\gamma^L-I)^n t_i$. Put $C_1=2^a a(3a\varepsilon/2\delta)^a$ and $C_2=\mu^{-a}$. Then we have the following Step 1.

Step 1. $\displaystyle\sum_{i=1}^a |n_i| \leqq C_1 C_2^n \qquad (n\geqq 2).$

Indeed, put $c_n=3a\varepsilon/2\delta\mu^n$. For $t\in\Theta$ $(t=\sum_{i=1}^a n_i(\gamma^L-I)^n t_i)$, if we have

$$|n_i| \leqq (c_n+1)^a \qquad (1\leqq i\leqq a), \tag{2.5}$$

then $\sum_{i=1}^a |n_i|\leqq a(c_n+1)^a = a\sum_{j=1}^a \binom{a}{j} c_n^j \leqq C_1 C_2^n$.

We must prove (2.5) to get Step 1. Assume that $n_1\geqq (c_n+1)^a$ for some $n\geqq 2$ and $t=\sum_{i=1}^a n_i(\gamma^L-I)^n t_i\in\Theta$. We write $s=a-1$ and put $a_i=n_{i+1}/n_1$ $(1\leqq i\leqq a-1=s)$. Choose $N\in\boldsymbol{N}$ with $c_n<N\leqq c_n+1$. Then by (L.16) we get an integer m_1 with $1\leqq m_1\leqq N^{a-1}$ such that $|m_1 n_i/n_1 - m_i|<1/N<c_n^{-1}$ for $2\leqq i\leqq a$ and for some $(m_2,\cdots,m_a)\in\boldsymbol{Z}^{a-1}$. Since $m_1\neq 0$, we get $0\neq m_i(\gamma^L-I)^n t_i$ and since $t=\sum_{i=1}^a n_i(\gamma^L-I)^n t_i$, $m_i(\gamma^L-I)^n t_i$ and since $t=\sum_{i=1}^a n_i(\gamma^L-I)^n t_i$,

$$\begin{aligned}\|\sum_i m_i(\gamma^L-I)^n t_i\| &\leqq \|\sum_i m_i(\gamma^L-I)^n t_i-(m_1/n_1)t\|_c+\|(m_1/n_1)t\|_c\\ &< \sum_i c_n^{-1}\|(\gamma^L-I)^n t_i\|_c+(m_1/n_1)\|t\|_c.\end{aligned}$$

Since $t\in B^c(3\varepsilon/2)$, clearly $\|t\|_c\leqq 3\varepsilon/2$. Since $\|(\gamma^L-I)^n t_i\|_c<(1/2^n)\|t_i\|$ (by (2.3)) and $m_1/n_1<c_n^{-1}$ (because $m_1<N^{a-1}$ and $n_1>N^a$), we have $\|\sum_i m_i(\gamma^L-I)^n t_i\|_c< \delta\mu^n/2^n+\delta\mu^n/a<\delta\mu^n$ (because $a\geqq 2$ by ergodicity). Comparing this inequality with (2.4), we have $t=0$, which is impossible. Therefore $n_1\leqq(c_n+1)^a$. Repeat the same argument for n_i. Then we get $n_i\leqq(c_n+1)^a$ for $1\leqq i\leqq a$.

To get the conclusion of Lemma 2, we prepare the following Step 2.

Step 2. *Let ϕ be as in* (L.6). *For every $n\geqq 2$, we can find $D(n)$, $C(n)\in\boldsymbol{Z}^+$ such that* $\sup_n D(n)/n<\infty$, *$C(n)<D(n)$ and for every $m\geqq D(n)$*

$$W_m^u(\varepsilon)+K(\varepsilon)\supset \sigma^{C(n)}\phi\{(\gamma^L-I)^n A\cap B^c(3\varepsilon/2)\}.$$

Indeed, let λ_0 be as in (L.10) and let C_1 and C_2 be as in Step 1. Choose positive integers D_1, $D_2(n)$ satisfying $D_1\geqq -(\log\lambda_0)^{-1}\log C_1$ and $D_2(n)\geqq$

$-n(\log \lambda_0)^{-1}\log 2C_2$, and put $C(n)=D_1+D_2(n)$ for $n\geqq 2$. Fix $n\geqq 2$ and take $t\in (\gamma^L-I)^n A\cap B^c(3\varepsilon/2)$. Since $t=\sum_{i=1}^a n_i(\gamma^L-I)^n t_i$, we can easily calculate

$$\sigma^{C(n)}\phi(t)=\sigma^{C(n)}\phi\Big(\sum_{i=1}^a n_i(\gamma^L-I)^n t_i\Big)=\sum_{i=1}^a n_i((\sigma^L-I)^n\sigma^{C(n)}\phi(t_i)$$

$$=\sum_{i=1}^a n_i\sum_{j=0}^n \binom{n}{j}\sigma^{Lj+C(n)}\phi(t_i).$$

Since $\sum_{i=1}^a |n_i|\leqq C_1C_2^n$ (by Step 1) and $\sum_{j=0}^n\binom{n}{j}=2^n$, and since $W_M^u(\varepsilon)\subset\sigma W_M^u(\varepsilon)$ and $K(\varepsilon)\supset\sigma K(\varepsilon)$, we have $\sigma^{Lj+C(n)}\phi(t_i)\in\sigma^{Ln+C(n)}W_M^u(\varepsilon)+K(\varepsilon)$ for $0\leqq j\leqq n$ and $1\leqq i\leqq a$, and so $\sigma^{C(n)}\phi(t)\in\sigma^{Ln+C(n)}J_M^0+J^1$, where

$$J_M^0=\underbrace{W_M^u(\varepsilon)+\cdots+W_M^u(\varepsilon)}_{C_1(2C_2)^n}\quad\text{and}\quad J^1=\underbrace{K(\varepsilon)+\cdots+K(\varepsilon)}_{C_1(2C_2)^n}.$$

By (L.12, i) we get $\sigma^{-C(n)}W^u(\varepsilon)\subset W^u(\lambda_0^{C(n)}\varepsilon)=W^u(\varepsilon/(C_1(2C_2)^n))$, and so $\sigma^{-C(n)}W_M^u(\varepsilon)+\sigma^{-C(n)}W^u(\varepsilon)\subset\sigma^M W^u(\varepsilon/(C_1(2C_2)^n))$. Therefore $\sigma^{-C(n)}J_M^0+\sigma^{-C(n)}W^u(\varepsilon)\subset\sigma^M W^u(\varepsilon)$, i.e., $J_M^0\subset W_{M+C(n)}^u(\varepsilon)$. Since $\sigma^{n-m}W_m^u(\varepsilon)\subset W_n^u(\varepsilon)\,(n\geqq m)$ by (L.13, iii), we have $\sigma^{Ln+C(n)}J_M^0\subset W_{M+Ln+2C(n)}^u(\varepsilon)$. Using (L.12, ii), we get $\sigma^{C(n)}J^1\subset K(\varepsilon)$. Put $D(n)=M+Ln+2C(n)$ for $n\geqq 2$. Then from the above facts, we have $\sigma^{C(n)}\phi(t)\in W_m^u(\varepsilon)+K(\varepsilon)$ for every $m\geqq D(n)\,(n\geqq 2)$. The conclusion of Step 2 is obtained.

Now we are ready to prove the lemma. Let $J(n)$ be the integer part of $(\log 2)^{-1}(\log 3a+\log n)+1$ for $n\geqq 2$. Since $\|t_i\|_c<3\varepsilon$, by (2.2) we have $\|(\gamma^L-I)^{J(n)}t_i\|<\varepsilon/an$. Remark that $E^c=\text{span}\{(\gamma^L-I)^{J(n)}t_i;\ 1\leqq i\leqq a\}$ for $n\geqq 2$. For fixed $n\geqq 2$, $x\in E^c$ is expressed as $x=\sum_1^a(a_i+n)(\gamma^L-I)^{J(n)}t_i$ where $a_i\in[0,1)$ and $n_i\in\boldsymbol{Z}$. Recall that $A=\{s\in E^c;\ s=\sum_1^a n_it_i,\ n_i\in\boldsymbol{Z},\ 1\leqq i\leqq a\}$. Then we have

$$\min_{s\in(\gamma^L-I)^{J(n)}A}\|x-s\|_c<\sum_1^a a_i\|(\gamma^L-I)^{J(n)}t_i\|_c<\varepsilon/n, \tag{2.6}$$

and so $\{(\gamma^L-I)^{J(n)}A\cap B^c((1+1/n)\varepsilon)\}+B^c(\varepsilon/n)\supset B^c(\varepsilon)$. This follows from the fact that for every $x\in B^c(\varepsilon)$ there is $t\in(\gamma^L-I)^{J(n)}A$ such that $\|x-t\|_c<\varepsilon/n$ (by (2.6), and then $t\in(\gamma^L-I)^{J(n)}A\cap B^c((1+1/n)\varepsilon)$. Let $C(n)$ be as in Step 2. Then $\sigma^{C(J(n))}\phi\{(\gamma^L-I)^{J(n)}A\cap B^c(3\varepsilon/2)+B^c(\varepsilon/n)\}\supset\phi B^c(\varepsilon)$. From this and Step 2, we have $W_m^u(\varepsilon)+K(\varepsilon)\oplus\phi B^c(\varepsilon/n)\supset\phi B^c(\varepsilon)$ for $m\geqq D(J(n))$. We put $N_\varepsilon(n)=D(J(n))$ for $n\geqq 2$ and in particular $N_\varepsilon(1)=N_\varepsilon(2)$. Since $J(n^p)/n\to 0$ as $n\to\infty$ for all $p\geqq 1$ and $\sup_n D(n)/n<\infty$ (by Step 2), clearly $N_\varepsilon(n^p)/n\to 0$ as $n\to\infty$ for all $p\geqq 1$. The proof of Lemma 2 is completed.

LEMMA 3. *Assume that (X,σ) is either hyperbolic or ergodic and has central spin. Then for every $\varepsilon\in(0,2\alpha_0/3)$, there is a sequence $\{M_\varepsilon(n)\}_{n=1}^\infty$ of positive integers such that for $p\geqq 1$ $M_\varepsilon(n^p)/n\to 0$ as $n\to\infty$ and for all $n\geqq 1$ and $m\geqq M_\varepsilon(n)$*

$$W_m^u(\varepsilon)+K(\varepsilon)\oplus\phi B^c(\varepsilon/n)=X.$$

PROOF. Take and fix $\varepsilon\in(0, 2\alpha_0/3)$. From (L.14) we have $W_M^u(\varepsilon/2)+K(\varepsilon/2)\oplus\phi B^c(\varepsilon/2)=X$ for some $M=M(\varepsilon/2)>0$. Let $\{N_\varepsilon(n)\}_{n=1}^\infty$ be as in Lemma 2. Then for $m\geqq N_{\varepsilon/2}(n)$,

$$\{W_M^u(\varepsilon/2)+K(\varepsilon/2)\}+\{W_m^u(\varepsilon/2)+K(\varepsilon/2)\oplus\phi B^c(\varepsilon/n)\}=X \tag{3.1}$$

by Lemma 2. Let $\lambda_0\in(0, 1)$ be as before, N be an integer with $\lambda_0^{-N}>2$ and put $M_\varepsilon(n)=N+\max\{M, N_{\varepsilon/2}(n)\}$. Clearly $M_\varepsilon(n^p)/n\to 0$ as $n\to\infty$ for all $p\geqq 1$. Since $\sigma^N W^u(\varepsilon)\supset W^u(\varepsilon)$, we have $W_m^u(\varepsilon/2)+W_m^u(\varepsilon/2)\subset W_{m+N}^u(\varepsilon)$ for all $m\geqq 1$. From (3.1) we have $X=W_m^u(\varepsilon)+K(\varepsilon)\oplus\phi B^c(\varepsilon/n)$ for $m\geqq M_\varepsilon(n)$.

LEMMA 4. *If (X, σ) is ergodic, then for every $\varepsilon\in(0, \alpha_0/3)$, there is a function L_ε: $\boldsymbol{Z}^+\to\boldsymbol{Z}^+$ such that for $p\geqq 1$, $L_\varepsilon(n^p)/n\to 0$ as $n\to\infty$ and such that for $n\geqq 1$, $m\geqq L_\varepsilon(n)$ and $x, y\in X$ there is $z\in y+K(\varepsilon)\oplus\phi B^c(\varepsilon/(n+1))$ such that $z+W^u(\varepsilon)\subset\sigma^m\{x+W^u(\varepsilon)\}$.*

PROOF. Since X is solenoidal, there are $n_0>0$ and a sequence $X=X_0\supset X_1\supset\cdots\supset X_{n_0-1}\supset X_{n_0}=\{0\}$ of σ-invariant subgroups which satisfy all the conditions of (L.15). Take and fix $\varepsilon\in(0, \alpha_0]$. For $0\leqq i\leqq n_0-1$ we put $W_n^u(\varepsilon)_i=W_n^u(\varepsilon)\cap X_i$ for $n\geqq 1$, $K(\varepsilon)_i=K(\varepsilon)\cap X_i$ and $\phi B^c(\varepsilon)_i=\phi B^c(\varepsilon)\cap X_i$. Since $(X_i/X_{i+1}, \sigma)$ is either hyperbolic, or ergodic and has central spin, we can use Lemma 3 for $(X_i/X_{i+1}, \sigma)$. Then there is a sequence $\{M_\varepsilon^{(i)}(n)\}_{n=1}^\infty$ of positive integers such that for $p\geqq 1$, $M_\varepsilon^{(i)}(n^p)/n\to 0$ as $n\to\infty$, and for $m\geqq M_\varepsilon^{(i)}(n)$

$$W_m^u(\varepsilon)_i+K(\varepsilon)_i\oplus\phi B^c(\varepsilon/n)_i+X_{i+1}=X_i.$$

Then for $n\geqq 1$ and $m\geqq M_{\varepsilon/n_0}^{(i)}(n+1)$

$$W_m^u(\varepsilon/n_0)_i+K(\varepsilon/n_0)_i\oplus\phi B^c(\varepsilon/n_i(n+1))_i+X_{i+1}=X_i. \tag{4.1}$$

Choose $C>0$ with $\lambda_0^{-C}>n_0$ and put $L_\varepsilon(n)=C+\max\{M_{\varepsilon/n_0}^{(i)}(n+1);\ 0\leqq i\leqq n_0\}$ for $n\geqq 1$. Clearly $L_\varepsilon(n^p)/n\to 0$ as $n\to\infty$ for all $p\geqq 1$. Since $\sigma^C W^u(\varepsilon/n_0)\supset W^u(\varepsilon)$, we have $W_m^u(\varepsilon)\supset\sum_{i=0}^{n_0-1}W_{m-C}^u(\varepsilon/n_0)_i$ for $m\geqq L_\varepsilon(n)$. It is clear that $K(\varepsilon)\supset\sum_{i=0}^{n_0-1}K(\varepsilon/n_0)_i$ and $\phi B^c(\varepsilon/(n+1))\supset\sum_{i=0}^{n_0-1}\phi B^c(\varepsilon/(n+1))_i$, and so by (4.1)

$$\begin{aligned}&W_m^u(\varepsilon)+K(\varepsilon)\oplus\phi B^c(\varepsilon/(n+1))\\&\supset\sum_{i=0}^{n_0-1}\{W_{m-C}^u(\varepsilon/n_0)_i+K(\varepsilon/n_0)_i\oplus\phi B^c(\varepsilon/(n+1))_i\}=X.\end{aligned}$$

Hence for $n\geqq 0$, $m\geqq L_\varepsilon(n)$ and $x, y\in X$, there is $z\in y+K(\varepsilon)\oplus\phi B^c(\varepsilon/(n+1))$ with $z+W^u(\varepsilon)\subset\sigma^m\{x+W^u(\varepsilon)\}$ since $X=\sigma^m x+W_m^u(\varepsilon)+K(\varepsilon)\oplus\phi B^c(\varepsilon/(n+1))\ni y$ and $K(\varepsilon)\oplus\phi B^c(\varepsilon/(n+1))$ is symmetry (by (L.12, iv)).

Now we are ready to prove Theorem 1.

PROOF OF THEOREM 1. Let $\varepsilon\in(0, 2\alpha_0/3)$ and L_ε: $\boldsymbol{Z}^+\to\boldsymbol{Z}^+$ as in Lemma 4.

With the notation $a=\dim(E^c)$, we define a function M_ε: $\boldsymbol{Z}^+\to\boldsymbol{Z}^+$ by $M_\varepsilon(n)=L_\varepsilon(n^a)$ $(n\geqq 1)$. Clearly $M_\varepsilon(n)/n\to 0$ as $n\to\infty$, and for every $n>0$, $m\geqq M_\varepsilon(n)$ and $x, y\in X$, there is $z\in y+K(\varepsilon)\oplus\phi B^c(\varepsilon/(n+1)^a)$ with $z+W^u(\varepsilon)\subset\sigma^m\{x+W^u(\varepsilon)\}$ (by Lemma 4). Applying Lemma 1, we get $\gamma^i B^c(\varepsilon/(n+1)^a)\subset B^c(\varepsilon(i+1)^{a-1}/(n+1)^a)$ for $i\geqq 0$, so that for $0\leqq i\leqq n$

$$\text{(1)}\qquad \sigma^i z\in\sigma^i y+\sigma^i K(\varepsilon)\oplus\phi B^c(\varepsilon(i+1)^{a-1}/(n+1)^a)\subset\sigma^i y+K(\varepsilon)\oplus\phi B^c(\varepsilon).$$

For every $k\geqq 1$, $x_1, \cdots, x_k\in X$ and a sequence of integers $a_1\leqq b_1<\cdots<a_k\leqq b_k$ with $a_i-b_{i-1}\geqq M_\varepsilon(b_i-a_i)$ $(2\leqq i\leqq k)$. Letting $z_1=\sigma^{a_1}x_1$, we can find a sequence of $k-1$ points $z_2, \cdots, z_k$ such that for $1\leqq i\leqq k-1$, $z_{i+1}\in\sigma^{a_{i+1}}x_i+K(\varepsilon)\oplus\phi B^c(\varepsilon/(b_{i+1}-a_{i+1}+1)^a)$ and $z_{i+1}+W^u(\varepsilon)\subset\sigma^{a_{i+1}-b_i}\{\sigma^{b_i-a_i}z_i+W^u(\varepsilon)\}$. This is easily obtained using (1). We now have for $1\leqq i\leqq k-1$,

$$\begin{aligned}\sigma^{-a_i}\{z_i+\sigma^{-(b_i-a_i)}W^u(\varepsilon)\} &\supset \sigma^{-a_{i+1}}\{z_{i+1}+W^u(\varepsilon)\}\\ &\supset \sigma^{-a_{i+1}}\{z_{i+1}+\sigma^{-(b_{i+1}-a_{i+1})}W^u(\varepsilon)\},\end{aligned}$$

which yields

$$\bigcap_1^k \sigma^{-a_i}\{z_i+\sigma^{-(b_i-a_i)}W^u(\varepsilon)\}=\sigma^{-a_k}\{z_k+\sigma^{-(b_k-a_k)}W^u(\varepsilon)\}.$$

Take a point x from the last set. Then for $a_i\leqq j\leqq b_i$ $(1\leqq i\leqq k)$ we get

$$\begin{aligned}\sigma^j x &\in \sigma^{j-a_i}z_i+\sigma^{-(b_i-a_i)}W^u(\varepsilon)\\ &\subset \sigma^j x_i+\{\sigma^{j-a_i}(K(\varepsilon)\oplus\phi B^c((j-a_i+1)^a\varepsilon/(b_i-a_i+1)^a)\oplus\sigma^{-(b_i-a_i)}W^u(\varepsilon)\}.\end{aligned}$$

This shows that (X, σ) satisfies almost weak specification.

References

[1] N. Aoki, A simple proof of Bernoullicity of ergodic automorphisms on compact abelian groups, Israel J. Math., **38** (1981), 189-198.

[2] N. Aoki, Zero-dimensional automorphisms with specification, Monatsh. Math., **95** (1983), 1-17.

[3] N. Aoki, M. Dateyama and M. Komuro, Solenoidal automorphisms with specifications, Monatsh. Math., **93** (1982), 79-110.

[4] M. Dateyama, Invariant measures for homeomorphisms with weak specification, Tokyo J. Math., **4** (1981), 389-397.

[5] M. Dateyama, Invariant measures for homeomorphisms with almost weak specification, Lecture Notes in Math., **1021**, Springer, 1983, pp. 93-96.

[6] M. Denker, C. Grillenberger and K. Sigmund, Ergodic Theory on Compact Spaces, Lecture Notes in Math., **527**, Springer, 1976.

[7] B. Marcus, A note on periodic points for ergodic toral automorphisms, Monatsh. Math., **89** (1980), 121-129.

[8] J. Miles and K. Thomas, Generalized torus are Bernoulli, Studies in Ergodic Theory and Probability, vol. 2, Academic Press, New York, 1978, pp. 231-249.

[9] L. Pontrjagin, Topological Groups, Gordon & Breach, New York-London-Paris, 1966.

[10] K. Sigmund, Generic properties of invariant measures for axiom A diffeomorphisms, Invent. Math., **11** (1970), 99-109.

[11] K. Sigmund, On dynamical systems with the specification property, Trans. Amer. Math. Soc., **190** (1974), 285-299.

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