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Backward Itô's formula for sections of a fibered manifold

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1. Introduction.

When a stochastic differential equation on a C^{∞} manifold generates a stochastic flow of diffeomorphisms of the manifold, some geometric backward Itô's formulas related to the stochastic flow are known for tensor fields on the manifold [7], [**10**]. On the other hand, a forward (usual) Itô's formula was obtained for (local) sections of a fiber bundle [**1**]. It is, therefore, desirable to get a backward stochastic formula generalized for (local) sections of a fiber bundle or, more generally, of a fibered manifold.

The main purpose of the present paper is to obtain a backward stochastic formula, which we will also call backward Itô's formula, for (local) sections of a fibered manifold (Theorem 3.2) with the use of backward stochastic calculus ([**10**]). As a corollary, we obtain a backward Itô's formula for sections of a vector bundle (Corollary 4.1). Then, using this formula, we treat certain backward and forward differential equations for sections of a vector bundle (Corollaries 4.3 and 4.4).

Although the formula in Theorem 3.2 is applicable to C^{∞} sections of a general C^{∞} fibered manifold, we are chiefly concerned with sections of fiber bundles; we give applications to the study of the behavior (with respect to the initial-time parameter) of a time-dependent random C^{∞} distribution of a C^{∞} manifold and to backward and forward differential equations for second order (possibly degenerate) linear differential operators on C^{∞} functions on a C^{∞} manifold (§5). These applications are done by noting that a C^{∞} distribution of a C^{∞} manifold can be regarded as a section of a Grassmann bundle ([**3**]), and that each second order linear differential operator on C^{∞} functions on a C^{∞} manifold can be identified with a section of a certain vector bundle associated with the bundle of second order frames of the manifold.

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2. Preliminaries.

Let $(\boldsymbol{W}_0^k, \mathcal{F}, \mu)$ be the standard k-dimensional Wiener space: $\boldsymbol{W}_0^k$ is the space of all continuous paths $w: [0, \infty)\ni t \mapsto w(t)=(w^1(t), \cdots, w^k(t))\in \boldsymbol{R}^k$ such that $w(0)=0$ endowed with the topology of uniform convergence on every bounded interval, μ is the standard Wiener measure, and $\mathcal{F}$ is the completion of the Borel σ-field on $\boldsymbol{W}_0^k$. Thus $w(t)$ is the canonical realization of Brownian motion on the probability space. We set $w^0(t)\equiv t$. Since forward stochastic integrals are well-known, we recall some definitions on backward stochastic integrals for later use.

Let $T>0$. For $0\leqq s<t\leqq T$, let $\mathcal{F}_s^t(\subset \mathcal{F})$ be the least complete σ-field for which $w(u)-w(v)$, $s\leqq u\leqq v\leqq t$, are measurable. Then $\mathcal{F}_s^t \subset \mathcal{F}_{s'}^{t'}$ if $0\leqq s'\leqq s<t\leqq t' \leqq T$. If $h(u)$, $u\in[0, t]$, is a real-valued continuous backward semimartingale relative to $\mathcal{F}_u^t$, then for each $\lambda=0, 1, \cdots, k$, the backward Itô integral of $h(u)$ with respect to $w^\lambda(u)$ is defined by

$$\int_s^t h(u)\cdot \hat{d} w^\lambda(u) = \operatorname*{l.i.p.}_{|\Delta|\to 0} \sum_{i=0}^{m-1} h(t_{i+1})(w^\lambda(t_{i+1})-w^\lambda(t_i)),$$

where Δ: $s=t_0<\cdots<t_m=t$, $|\Delta|=\max_i|t_{i+1}-t_i|$, and l.i.p. denotes "limit in probability"; and the backward Stratonovich integral of $h(u)$ with respect to $w^\lambda(u)$ is defined by

$$\int_s^t h(u)\circ \hat{d} w^\lambda(u) = \operatorname*{l.i.p.}_{|\Delta|\to 0} \sum_{i=0}^{m-1} \frac{1}{2}(h(t_{i+1})+h(t_i))(w^\lambda(t_{i+1})-w^\lambda(t_i)).$$

We shall use freely concepts and notations in [**6**] and [**10**]. As for manifolds, we refer to [**3**], [**9**].

3. Backward Itô's formula for sections of a fibered manifold.

All *manifolds* in this paper are finite dimensional, σ-compact, and of class C^∞. For a C^∞ vector bundle V over a manifold N, we denote by $\Gamma(V)$ the space of C^∞ global sections of V. Every C^∞ vector field on N is regarded as an element of $\Gamma(TN)$, where TN denotes the tangent bundle over N.

Let E be a C^∞ fibered manifold over a manifold M, with projection π: $E\to M$; that is, π is a C^∞ surjective submersion and thus has maximal rank everywhere. The fiber $\pi^{-1}(x)$ over $x\in M$ is denoted by E_x.

Let $Y_\lambda(t)\in\Gamma(TE)$, $\lambda=0, 1, \cdots, k$, be time-dependent projectable C^∞ vector fields (with time-parameter $t\in[0, T]$) on E; that is, for each λ, there exists a (unique) time-dependent C^∞ vector field $X_\lambda(t)\in\Gamma(TM)$ on M such that the vector field $Y_\lambda(t)$: $E\ni q\mapsto Y_\lambda(t, q)\in T_qE$ (=the tangent space to E at q) is π-related to the vector field $X_\lambda(t)$: $M\ni x\mapsto X_\lambda(t, x)\in T_xM$;

$$\pi_{*,q}Y_\lambda(t, q) = X_\lambda(t, \pi(q)), \qquad q\in E,$$

where $\pi_{*,q}$: $T_qE\to T_{\pi(q)}M$ is the differential of π at q. We let all time-dependent vector fields in this paper be also C^∞ in the time-parameter t.

Consider following two (forward) stochastic differential equations on E and M, respectively, in the Stratonovich form:

$$d\eta_t = \sum_{\lambda=0}^{k} Y_\lambda(t, \eta_t)\circ dw^\lambda(t), \tag{3.1}$$

$$d\theta_t = \sum_{\lambda=0}^{k} X_\lambda(t, \theta_t)\circ dw^\lambda(t). \tag{3.2}$$

In the following, we *assume* that the (maximal) solution $\eta_{s,t}(q)$, $0\leqq s\leqq t<\tau_\eta(s, q)$, $\eta_{s,s}(q)=q$, $q\in E$, of the equation (3.1) is strictly conservative [that is, $\mu(\tau_\eta(s, q)=T$ for all $(s, q))=1$, where $\tau_\eta(s, q)$, $s\leqq\tau_\eta(s, q)\leqq T$, is the explosion time of $\eta_{s,t}(q)$] and generates a stochastic flow of (C^∞) diffeomorphisms of E, a.s.; thus

$$\eta_{s,u}(q) = \eta_{t,u}(\eta_{s,t}(q)), \qquad q\in E,\ s<t<u<T, \text{ a.s.}$$

The following lemma shows that $\eta_{s,t}$ induces a stochastic flow of diffeomorphisms of M.

LEMMA 3.1. *$\pi(\eta_{s,t}(q))$, $0\leqq s\leqq t<T$, does not depend on the choice of $q\in E_x$ for every $x\in M$, and the stochastic map $\theta_{s,t}$: $M\to M$ defined by*

$$\theta_{s,t}(x) := \pi(\eta_{s,t}(q)), \qquad q\in E_x,\ x\in M,$$

is the solution of (3.2). *Moreover, $\theta_{s,t}$ defines a stochastic flow of diffeomorphisms of M, a.s.*

PROOF. Since $\eta_{s,t}$ is strictly conservative and defines a stochastic flow of diffeomorphisms of E, a.s., the (maximal) solution of the adjoint equation

$$d\hat{\eta}_t = -\sum_{\lambda=0}^{k} Y_\lambda(t, \hat{\eta}_t)\circ dw^\lambda(t) \tag{3.3}$$

is also strictly conservative; cf. [**10**, p. 251, Theorem 9.2]. Since $\pi(\eta_{s,t}(q))$ satisfies

$$\begin{aligned} d_t\pi(\eta_{s,t}(q)) &= \sum_{\lambda=0}^{k} \pi_{*,\eta_{s,t}(q)}Y_\lambda(t, \eta_{s,t}(q))\circ dw^\lambda(t) \\ &= \sum_{\lambda=0}^{k} X_\lambda(t, \pi(\eta_{s,t}(q)))\circ dw^\lambda(t), \qquad s<t<T, \end{aligned}$$

where d_t denotes stochastic differential with respect to the parameter t, the uniqueness of the solution of (3.2) implies that $\eta_{s,t}(q)$ does not depend on the choice of $q\in E_x$ for each $x\in M$, and thus we can define a stochastic map $\theta_{s,t}$: $M\to M$ by setting

$$\theta_{s,t}(x) = \pi(\eta_{s,t}(q)), \qquad q\in E_x,\ x\in M.$$

Note that $\theta_{s,t}$ is the (maximal) solution of (3.2) and is strictly conservative.

In the same way, since the solution of (3.3) is strictly conservative, the solution of the adjoint equation

$$d\hat{\theta}_t = -\sum_{\lambda=0}^{k} X_\lambda(t, \hat{\theta}_t)\circ dw^\lambda(t)$$

is also strictly conservative. Therefore the solution $\theta_{s,t}$ of (3.2) defines a stochastic flow of diffeomorphisms of M, a. s. This completes the proof.

Let σ be a C^∞ (local) section of E over an open set $\mathrm{Dom}(\sigma)$ (=the domain of σ) of M, so that σ is a C^∞ map σ: $\mathrm{Dom}(\sigma)\to E$ such that $\pi\circ\sigma(x)=x$ for all $x\in\mathrm{Dom}(\sigma)$. We want to obtain a backward Itô's formula for the E_x-valued process $\sigma_{s,t}(x) := \eta_{s,t}^{-1}(\sigma(\theta_{s,t}(x)))$, $x\in\mathrm{Dom}(\sigma)$. Let $\sigma^*TE\to\mathrm{Dom}(\sigma)$ be the pull-back of the tangent bundle $TE\to E$ by σ. For fixed t, define $D_\lambda(t)\sigma\in\Gamma(\sigma^*TE)$ by (cf. [1])

$$(D_\lambda(t)\sigma)(x) := (D(X_\lambda(t), Y_\lambda(t))\sigma)(x) = \sigma_{*,x}X_\lambda(t, x) - Y_\lambda(t, \sigma(x)) \in T_{\sigma(x)}E,$$
$$x\in\mathrm{Dom}(\sigma),\ \lambda=0, 1, \cdots, k.$$

Let $\tilde{Y}_\lambda(t)\in\Gamma(T(TE))$ be the natural lift of $Y_\lambda(t)$ to TE, $\lambda=0, 1, \cdots, k$. For each C^∞ (local) section ζ of σ^*TE over an open set $\mathrm{Dom}(\zeta)$ $(\subset\mathrm{Dom}(\sigma))\subset M$, noting that $\zeta(x)\in T_{\sigma(x)}E\subset TE$ for $x\in\mathrm{Dom}(\zeta)$, we let $\zeta^*T(TE)\to\mathrm{Dom}(\zeta)$ be the pull-back of the vector bundle $T(TE)\to TE$ by ζ: $\mathrm{Dom}(\zeta)\to TE$, and define $\tilde{D}_\lambda(t)\zeta\in\Gamma(\zeta^*T(TE))$ by

$$(\tilde{D}_\lambda(t)\zeta)(x) := (\tilde{D}(X_\lambda(t), \tilde{Y}_\lambda(t))\zeta)(x) = \zeta_{*,x}X_\lambda(t, x) - \tilde{Y}_\lambda(t, \zeta(x)) \in T_{\zeta(x)}(TE),$$
$$x\in\mathrm{Dom}(\zeta),\ \lambda=0, 1, \cdots, k.$$

Moreover, for each C^∞ function f: $E\to\boldsymbol{R}$, define a C^∞ function G_f: $TE\to\boldsymbol{R}$ by

$$G_f(X) = \boldsymbol{d}f(X) = X[f], \qquad X\in TE,$$

where $\boldsymbol{d}f$ denotes the total derivative of f. Then we obtain the following backward Itô's formula.

THEOREM 3.2 (Backward Itô's formula for sections of a fibered manifold). *Let $\eta_{s,t}$ and $\theta_{s,t}$ be as above. Let σ be a C^∞ (local) section of E defined on an open set* $\mathrm{Dom}(\sigma)\subset M$. *Put $\sigma_{s,t}=\eta_{s,t}^{-1}\circ\sigma\circ\theta_{s,t}$. Then for every $x\in\mathrm{Dom}(\sigma)$ and every C^∞ function f: $E\to\boldsymbol{R}$, it holds that*

$$f(\sigma_{s,t}(x)) - f(x) = \sum_{\lambda=0}^{k}\int_s^t (D_\lambda(u)\sigma_{u,t})(x)[f]\circ\hat{d}w^\lambda(u) \tag{3.4a}$$

$$= \sum_{\lambda=0}^{k}\int_s^t (D_\lambda(u)\sigma_{u,t})(x)[f]\cdot\hat{d}w^\lambda(u) + \frac{1}{2}\sum_{\alpha=1}^{k}\int_s^t (\tilde{D}_\alpha(u)(D_\alpha(u)\sigma_{u,t}))(x)[G_f]du \tag{3.4b}$$

$$= \sum_{\lambda=0}^{k} \int_s^t (D_\lambda(u)\sigma_{u,t})(x)[f] \cdot \hat{d}w^\lambda(u) + \frac{1}{2} \sum_{\alpha=1}^{k} \int_s^t \{X_\alpha(u, x)[(D_\alpha(u)\sigma_{u,t})[f]] - (D_\alpha(u)\sigma_{u,t})(x)[Y_\alpha(u)[f]]\} du, \quad \tau_{\theta,\sigma}(t, x) < s < t < T, \tag{3.4c}$$

where $\tau_{\theta,\sigma}(t, x)$ is the backward stopping time defined by

$$\tau_{\theta,\sigma}(t, x) := \sup\{u \in (0, t) ; \theta_{u,t}(x) \notin \mathrm{Dom}(\sigma)\}$$
$$(=0 \text{ if } \{u \in (0, t) ; \theta_{u,t}(x) \notin \mathrm{Dom}(\sigma)\} = \emptyset).$$

PROOF. We first note that $\eta_{s,t}^{-1}$ and $\theta_{s,t}$ satisfy backward equations

$$\hat{d}_s \eta_{s,t}^{-1} = \sum_{\lambda=0}^{k} Y_\lambda(s, \eta_{s,t}^{-1}) \circ \hat{d}w^\lambda(s)$$

and

$$\hat{d}_s \theta_{s,t}(x) = -\sum_{\lambda=0}^{k} (\theta_{s,t})_{*,x} X_\lambda(s, x) \circ \hat{d}w^\lambda(s),$$

respectively, where $\hat{d}_s$ denotes backward stochastic differential with respect to the parameter s (see [**10**, p. 251, Theorem 9.2, and p. 262, Theorem 1.3]). Therefore

$$\begin{aligned}
&\hat{d}_s(f(\sigma_{s,t}(x))) = \hat{d}_s(f \circ \eta_{s,t}^{-1} \circ \sigma \circ \theta_{s,t}(x)) \\
&= (\hat{d}_s(f \circ \eta_{s,t}^{-1} \circ \sigma))(\theta_{s,t}(x)) + (\hat{d}_s \theta_{s,t}(x))[f \circ \eta_{s,t}^{-1} \circ \sigma] \\
&= \sum_{\lambda=0}^{k} Y_\lambda(s, \eta_{s,t}^{-1} \circ \sigma \circ \theta_{s,t}(x))[f] \circ \hat{d}w^\lambda(s) - \sum_{\lambda=0}^{k} (\theta_{s,t})_{*,x} X_\lambda(s, x)[f \circ \eta_{s,t}^{-1} \circ \sigma] \circ \hat{d}w^\lambda(s) \\
&= \sum_{\lambda=0}^{k} \{Y_\lambda(s, \sigma_{s,t}(x)) - (\sigma_{s,t})_{*,x} X_\lambda(s, x)\}[f] \circ \hat{d}w^\lambda(s) \\
&= -\sum_{\lambda=0}^{k} (D_\lambda(s)\sigma_{s,t})(x)[f] \circ \hat{d}w^\lambda(s).
\end{aligned} \tag{3.5}$$

Since

$$\int_s^t \hat{d}_u(f(\sigma_{u,t}(x))) = f(\sigma_{t,t}(x)) - f(\sigma_{s,t}(x))$$

and $\sigma_{t,t}(x) = \sigma(x)$, (3.5) implies (3.4a).

Next, using (3.5), we have

$$\hat{d}_s(X_\lambda(s, x)[f \circ \sigma_{s,t}]) = \frac{\partial}{\partial v}(X_\lambda(v, x)[f \circ \sigma_{s,t}])\Big|_{v=s} \cdot \hat{d}s - \sum_{\nu=0}^{k} X_\lambda(s, x)[(D_\nu(s)\sigma_{s,t})[f]] \circ \hat{d}w^\nu(s) \tag{3.6}$$

and

$$\begin{aligned}
&\hat{d}_s(Y_\lambda(s, \sigma_{s,t}(x))[f]) = \hat{d}_s(Y_\lambda(s)[f](\sigma_{s,t}(x))) \\
&= \Big(\frac{\partial}{\partial s}(Y_\lambda(s)[f])\Big)(\sigma_{s,t}(x)) \cdot \hat{d}s - \sum_{\nu=0}^{k} (D_\nu(s)\sigma_{s,t})(x)[Y_\lambda(s)[f]] \circ \hat{d}w^\nu(s).
\end{aligned} \tag{3.7}$$

By (3.6) and (3.7), we have

$$\begin{aligned}
&\hat{d}_s((D_\lambda(s)\sigma_{s,t})(x)[f]) \\
&= \hat{d}_s(X_\lambda(s, x)[f\circ\sigma_{s,t}]) - \hat{d}_s(Y_\lambda(s, \sigma_{s,t}(x))[f]) \\
&= \left\{\frac{\partial}{\partial v}(X_\lambda(v, x)[f\circ\sigma_{s,t}])\Big|_{v=s} - \Big(\frac{\partial}{\partial s}(Y_\lambda(s)[f])\Big)(\sigma_{s,t}(x))\right\}\cdot\hat{d}s \\
&\quad - \sum_{\nu=0}^{k}\{X_\lambda(s, x)[(D_\nu(s)\sigma_{s,t})[f]] - (D_\nu(s)\sigma_{s,t})(x)[Y_\lambda(s)[f]]\}\circ\hat{d}w^\nu(s). \quad (3.8)
\end{aligned}$$

Noting that

$$\begin{aligned}
&\sum_{i=0}^{m-1}\frac{1}{2}(a_{i+1}+a_i)(b_{i+1}-b_i) \\
&\quad = \sum_{i=0}^{m-1}a_{i+1}(b_{i+1}-b_i) - \frac{1}{2}\sum_{i=0}^{m-1}(a_{i+1}-a_i)(b_{i+1}-b_i)
\end{aligned}$$

for $a_i, b_i\in \boldsymbol{R}$, $i=0, 1, \cdots, m$, and using (3.8) and backward Itô stochastic differentials, we rewrite (3.5) as

$$\begin{aligned}
&\hat{d}_s(f(\sigma_{s,t}(x))) \\
&= -\sum_{\lambda=0}^{k}(D_\lambda(s)\sigma_{s,t})(x)[f]\cdot\hat{d}w^\lambda(s) - \frac{1}{2}\sum_{\alpha=1}^{k}\{X_\alpha(s, x)[(D_\alpha(s)\sigma_{s,t})[f]] \\
&\quad -(D_\alpha(s)\sigma_{s,t})(x)[Y_\alpha(s)[f]]\}\cdot\hat{d}s. \quad (3.9)
\end{aligned}$$

On the other hand, for $\lambda, \nu=0, 1, \cdots, k$, we have

$$\begin{aligned}
&(\tilde{D}_\lambda(s)(D_\nu(s)\sigma_{s,t}))(x)[G_f] \\
&\quad = (D_\nu(s)\sigma_{s,t})_{*,x}X_\lambda(s, x)[G_f] - \tilde{Y}_\lambda(s, (D_\nu(s)\sigma_{s,t})(x))[G_f]. \quad (3.10)
\end{aligned}$$

The first term in the right hand side of (3.10) equals

$$X_\lambda(s, x)[G_f\circ(D_\nu(s)\sigma_{s,t})] = X_\lambda(s, x)[(D_\nu(s)\sigma_{s,t})[f]]. \quad (3.11)$$

If $\rho_{s,t}^{(\lambda)}$ [resp. $\tilde{\rho}_{s,t}^{(\lambda)}$] denotes the (time-dependent) flow of $Y_\lambda(\cdot)$ [resp. $\tilde{Y}_\lambda(\cdot)$], that is, $(d/dt)\rho_{s,t}^{(\lambda)}(\cdot)=Y_\lambda(t, \rho_{s,t}^{(\lambda)}(\cdot))$, $\rho_{s,s}^{(\lambda)}=$identity, and similar for $\tilde{\rho}_{s,t}^{(\lambda)}$, then for $Y\in TE$ it holds that

$$\begin{aligned}
\tilde{Y}_\lambda(s, Y)[G_f] &= \frac{d}{du}G_f(\tilde{\rho}_{s,u}^{(\lambda)}(Y))\Big|_{u=s} = \frac{d}{du}G_f((\rho_{s,u}^{(\lambda)})_*Y)\Big|_{u=s} \\
&= \frac{d}{du}((\rho_{s,u}^{(\lambda)})_*Y[f])\Big|_{u=s} = \frac{d}{du}Y[f\circ\rho_{s,u}^{(\lambda)}]\Big|_{u=s} \\
&= Y\left[\frac{d}{du}(f\circ\rho_{s,u}^{(\lambda)})\Big|_{u=s}\right] = Y[Y_\lambda(s)[f]].
\end{aligned}$$

Thus

$$\tilde{Y}_\lambda(s, (D_\nu(s)\sigma_{s,t})(x))[G_f] = (D_\nu(s)\sigma_{s,t})(x)[Y_\lambda(s)[f]]. \quad (3.12)$$

By (3.10), (3.11) and (3.12), we have

$$(\tilde{D}_\lambda(s)(D_\nu(s)\sigma_{s,t}))(x)[G_f]$$
$$= X_\lambda(s, x)[(D_\nu(s)\sigma_{s,t})[f]]-(D_\nu(s)\sigma_{s,t})(x)[Y_\lambda(s)[f]]. \tag{3.13}$$

Then (3.4b) and (3.4c) follow from (3.9) and (3.13). This completes the proof of the theorem.

4. Backward Itô's formula for sections of a vector bundle.

Consider next the case where E is a real vector bundle with standard fiber $\boldsymbol{R}^m$ associated with a principal fiber bundle $P(M, G, \pi_P)$ through a representation of G into $GL(m, \boldsymbol{R})$ ([**9**, Vol. I, p. 113]). Let R_a: $P\ni p\mapsto pa\in P$ denote the right-translation by $a\in G$. In this section, we assume the following:

(#) There exist time-dependent vector fields $A_\lambda(t)\in\Gamma(TP)$, $\lambda=0, 1, \cdots, k$, which are invariant by R_a for every $a\in G$ such that each $Y_\lambda(t)$ is induced from $A_\lambda(t)$ in a natural manner (cf. [**1**]).

Let $\rho_{s,t}^{(\lambda)}$ [resp. $\theta_{s,t}^{(\lambda)}$] be the (time-dependent) flow of $Y_\lambda(\cdot)$ [resp. $X_\lambda(\cdot)$]. Under the assumption (#), the map

$$\rho_{s,t}^{(\lambda)}|_{E_x}: E_x(=\pi^{-1}(x)) \longrightarrow \pi^{-1}(\theta_{s,t}^{(\lambda)}(x))$$

is $\boldsymbol{R}$-linear for every $\lambda=0, 1, \cdots, k$ and $x\in M$. Let $\sigma\in\Gamma(E)$, so that $\mathrm{Dom}(\sigma)= M$. Define $L_\lambda(s)\sigma\in\Gamma(E)$ by (cf. [**1**])

$$\begin{aligned}(L_\lambda(s)\sigma)(x) &:= (L(X_\lambda(s), Y_\lambda(s))\sigma)(x)\\ &= \lim_{t\to s}\frac{1}{t}\{(\rho_{s,t}^{(\lambda)})^{-1}\circ\sigma\circ\theta_{s,t}^{(\lambda)}(x)-\sigma(x)\}\in E_x, \qquad x\in M.\end{aligned}$$

We note that (cf. [**1**]) for a C^∞ function f: $E\to\boldsymbol{R}$,

$$(D_\lambda(s)\sigma)(x)[f] = \frac{d}{dt}f((\rho_{s,t}^{(\lambda)})^{-1}\circ\sigma\circ\theta_{s,t}^{(\lambda)}(x))\Big|_{t=s}, \qquad x\in M.$$

Backward Itô's formulas are known for $\theta_{s,t}$ acting on tensor fields and for stochastic parallel displacement of tensors; see [**10**, p. 279, Theorem 4.3, and p. 293, Theorem 6.4] (cf. [**7**]). The following corollary gives a backward Itô's formula for sections of a vector bundle.

COROLLARY 4.1 (Backward Itô's formula for sections of a vector bundle). *When E is a vector bundle with standard fiber $\boldsymbol{R}^m$ associated with a principal fiber bundle $P(M, G, \pi_P)$ through a representation of G into $GL(m, \boldsymbol{R})$, under the assumption (#), it holds that for $\sigma\in\Gamma(E)$ and $0\leqq s<t<T$,*

$$\sigma_{s,t}-\sigma = \sum_{\lambda=0}^{k}\int_s^t L_\lambda(u)\sigma_{u,t}\circ\hat{d}w^\lambda(u) \tag{4.1a}$$

$$= \sum_{\alpha=1}^{k}\int_s^t L_\alpha(u)\sigma_{u,t}\cdot\hat{d}w^\alpha(u)+\int_s^t\Big\{\frac{1}{2}\sum_{\alpha=1}^{k}(L_\alpha(u))^2+L_0(u)\Big\}\sigma_{u,t}du. \tag{4.1b}$$

PROOF. Let $\pi_{E^*}: E^* \to M$ denote the dual vector bundle of E. Let $\phi \in \Gamma(E^*)$. Define a C^∞ function $f_\phi: E \to \boldsymbol{R}$ by $f_\phi(q) = \langle \phi(\pi(q)), q \rangle$, $q \in E$, where $\langle\ ,\ \rangle$ stands for the canonical pairing between $E_x^* = \pi_{E^*}^{-1}(x)$ and E_x for any $x \in M$. Note that

$$(D_\lambda(u)\sigma_{u,t})(x)[f_\phi] = \frac{d}{dv} f_\phi((\rho_{u,v}^{(\lambda)})^{-1} \circ \sigma_{u,t} \circ \theta_{u,v}^{(\lambda)}(x))\Big|_{v=u}$$
$$= \frac{d}{dv} \langle \phi(x), (\rho_{u,v}^{(\lambda)})^{-1} \circ \sigma_{u,t} \circ \theta_{u,v}^{(\lambda)}(x) \rangle \Big|_{v=u} \quad [\text{by } \phi(\pi((\rho_{u,v}^{(\lambda)})^{-1} \circ \sigma_{u,t} \circ \theta_{u,v}^{(\lambda)}(x))) = \phi(x)]$$
$$= \langle \phi(x), (L_\lambda(u)\sigma_{u,t})(x) \rangle = f_\phi \circ (L_\lambda(u)\sigma_{u,t})(x). \tag{4.2}$$

Therefore by (3.4a) we have

$$\langle \phi, \sigma_{s,t} - \sigma \rangle = \Big\langle \phi, \sum_{\lambda=0}^{k} \int_s^t L_\lambda(u)\sigma_{u,t} \circ \hat{d} w^\lambda(u) \Big\rangle.$$

This proves (4.1a).

Next we notice that by (4.2)

$$X_\lambda(u, x)[(D_\nu(u)\sigma_{u,t})[f_\phi]] = X_\lambda(u, x)[f_\phi \circ (L_\nu(u)\sigma_{u,t})]$$
$$= (L_\nu(u)\sigma_{u,t})_{*,x} X_\lambda(u, x)[f_\phi]. \tag{4.3}$$

For $\lambda, \nu = 0, 1, \cdots, k$, fixing u, t, and x, we have

$$(D_\nu(u)\sigma_{u,t})(x)[Y_\lambda(u)[f_\phi]] = (D_\nu(u)\sigma_{u,t})(x)\Big[\frac{d}{dv}(f_\phi \circ \rho_{u,v}^{(\lambda)})\Big|_{v=u}\Big]$$
$$= \frac{\partial}{\partial v'} \frac{\partial}{\partial v} f_\phi(\rho_{u,v}^{(\lambda)} \circ (\rho_{u,v'}^{(\nu)})^{-1} \circ \sigma_{u,t} \circ \theta_{u,v'}^{(\nu)}(x))\Big|_{v=u,\, v'=u}$$
$$= \frac{\partial}{\partial v} \frac{\partial}{\partial v'} \langle \phi(\theta_{u,v}^{(\lambda)}(x)), \rho_{u,v}^{(\lambda)} \circ (\rho_{u,v'}^{(\nu)})^{-1} \circ \sigma_{u,t} \circ \theta_{u,v'}^{(\nu)}(x) \rangle \Big|_{v'=u,\, v=u}$$
$$= \frac{d}{dv} \langle \phi(\theta_{u,v}^{(\lambda)}(x)), \rho_{u,v}^{(\lambda)} \circ (L_\nu(u)\sigma_{u,t})(x) \rangle \Big|_{v=u} \quad [\text{by } \boldsymbol{R}\text{-linearity of } \rho_{u,v}^{(\lambda)}|_{E_x}]$$
$$= \frac{d}{dv} f_\phi(\rho_{u,v}^{(\lambda)} \circ (L_\nu(u)\sigma_{u,t})(x))\Big|_{v=u} \quad [\text{by } \theta_{u,v}^{(\lambda)}(x) = \pi(\rho_{u,v}^{(\lambda)} \circ (L_\nu(u)\sigma_{u,t})(x))]$$
$$= Y_\lambda(u, (L_\nu(u)\sigma_{u,t})(x))[f_\phi]. \tag{4.4}$$

From (4.3) and (4.4) it holds that

$$X_\lambda(u, x)[(D_\nu(u)\sigma_{u,t})[f_\phi]] - (D_\nu(u)\sigma_{u,t})(x)[Y_\lambda(u)[f_\phi]]$$
$$= (L_\nu(u)\sigma_{u,t})_{*,x} X_\lambda(u, x)[f_\phi] - Y_\lambda(u, (L_\nu(u)\sigma_{u,t})(x))[f_\phi]$$
$$= (D_\lambda(u)(L_\nu(u)\sigma_{u,t}))(x)[f_\phi] = \langle \phi(x), (L_\lambda(u)(L_\nu(u)\sigma_{u,t}))(x) \rangle, \tag{4.5}$$

the last equality being shown as in (4.2). Hence, by (3.4c), (4.2), and (4.5), we obtain (4.1b). This completes the proof.

For a better understanding of Corollary 4.1, we give another proof.

ALTERNATIVE PROOF. Regard each $p\in P$ as the $\boldsymbol{R}$-linear map $p: \boldsymbol{R}^m\ni\xi\mapsto p\xi\in E_{\pi_P(p)}$. We associate with $\sigma\in\Gamma(E)$ the function $F_\sigma: P\to\boldsymbol{R}^m$ defined by $F_\sigma(p):=p^{-1}(\sigma(\pi_P(p)))$, $p\in P$ ([**9**, p. 116]). We have

LEMMA 4.2. $(L_\lambda(s)\sigma)(x)=p(A_\lambda(s, p)[F_\sigma])$, $\quad p\in\pi_P^{-1}(x)$, $x\in M$.

PROOF OF LEMMA 4.2. Let $\varphi_{s,t}^{(\lambda)}$ be the (time-dependent) flow of $A_\lambda(\cdot)$. Let $\rho_{s,t}^{(\lambda)}$ and $\theta_{s,t}^{(\lambda)}$ be as before. Then $\rho_{s,t}^{(\lambda)}(q)=(\varphi_{s,t}^{(\lambda)}(p)\circ p^{-1})(q)$, $\theta_{s,t}^{(\lambda)}(x)=\pi_P(\varphi_{s,t}^{(\lambda)}(p))$, $p\in\pi_P^{-1}(x)$, $x=\pi(q)$, $q\in E$. Therefore

$$F_\sigma\circ\varphi_{s,t}^{(\lambda)}(p)=(\varphi_{s,t}^{(\lambda)}(p))^{-1}(\sigma(\theta_{s,t}^{(\lambda)}(x)))=p^{-1}((\rho_{s,t}^{(\lambda)})^{-1}\circ\sigma\circ\theta_{s,t}^{(\lambda)}(x)),\quad p\in\pi_P^{-1}(x),\ x\in M.$$

Since $p^{-1}: E_x\to\boldsymbol{R}^m$ is $\boldsymbol{R}$-linear for $p\in\pi_P^{-1}(x)$ with $x\in M$, it holds that

$$p(A_\lambda(s, p)[F_\sigma])=p\Big(\frac{d}{du}(F_\sigma\circ\varphi_{s,u}^{(\lambda)}(p))\Big|_{u=s}\Big)=(L_\lambda(s)\sigma)(x),$$

which completes the proof of Lemma 4.2.

Let $\varphi_{s,t}(p)$ be the solution of

$$d\varphi_t=\sum_{\lambda=0}^{k}A_\lambda(t, \varphi_t)\circ dw^\lambda(t),\qquad \varphi_s=p\in P.$$

Then we have $\eta_{s,t}(q)=(\varphi_{s,t}(p)\circ p^{-1})(q)$, $\theta_{s,t}(x)=\pi_P(\varphi_{s,t}(p))$, $p\in\pi_P^{-1}(x)$, $x=\pi(q)$, $q\in E$ (cf. [1]). Since

$$\hat{d}_s\varphi_{s,t}(p)=-\sum_{\lambda=0}^{k}(\varphi_{s,t})_{*,p}A_\lambda(s, p)\circ\hat{d}w^\lambda(s),$$

it holds that

$$\hat{d}_sF_\sigma(\varphi_{s,t}(p))=-\sum_{\lambda=0}^{k}A_\lambda(s, p)[F_\sigma\circ\varphi_{s,t}]\circ\hat{d}w^\lambda(s) \tag{4.6}$$

and

$$\hat{d}_s(A_\lambda(s, p)[F_\sigma\circ\varphi_{s,t}])=\frac{\partial}{\partial v}(A_\lambda(v, p)[F_\sigma\circ\varphi_{s,t}])\Big|_{v=s}\cdot\hat{d}s$$
$$-\sum_{\nu=0}^{k}A_\lambda(s, p)[A_\nu(s)[F_\sigma\circ\varphi_{s,t}]]\circ\hat{d}w^\nu(s). \tag{4.7}$$

By (4.6) and (4.7) we have

$$\hat{d}_sF_\sigma(\varphi_{s,t}(p))=-\sum_{\lambda=0}^{k}A_\lambda(s, p)[F_\sigma\circ\varphi_{s,t}]\cdot\hat{d}w^\lambda(s)$$
$$-\frac{1}{2}\sum_{\alpha=1}^{k}A_\alpha(s, p)[A_\alpha(s)[F_\sigma\circ\varphi_{s,t}]]\cdot\hat{d}s. \tag{4.8}$$

Let $p\in\pi_P^{-1}(x)$, $x\in M$. Then

$$p(F_\sigma(\varphi_{s,t}(p)))=\sigma_{s,t}(x),$$
$$p(A_\lambda(s, p)[F_\sigma\circ\varphi_{s,t}])=p(A_\lambda(s, p)[F_{\sigma_{s,t}}])=(L_\lambda(s)\sigma_{s,t})(x),$$
$$p(A_\lambda(s, p)[A_\nu(s)[F_\sigma\circ\varphi_{s,t}]])=(L_\lambda(s)(L_\nu(s)\sigma_{s,t}))(x).$$

Hence by (4.6) and (4.8) we obtain (4.1a) and (4.1b). This completes the proof of Corollary 4.1.

Now it is easy to rewrite the examples in [**1**, § 4] in order to obtain their backward versions. We give here two examples of Corollary 4.1.

Example 1. Consider the case $E=T^p_q(M)$, the tensor bundle of type (p, q) over M. Let δ_λ, $\lambda=0, 1, \cdots, k$, be derivations of the tensor algebra of M, and recall that each δ_λ is uniquely decomposed in the form

$$\delta_\lambda = \mathcal{L}_{X_\lambda}+S_\lambda, \qquad X_\lambda\in\Gamma(TM), \quad S_\lambda\in\Gamma(T^1_1(M)),$$

where $\mathcal{L}_{X_\lambda}$ denotes Lie differentiation with respect to X_λ, and S_λ is regarded as a derivation ([**9**, Vol. I, p. 30, Proposition 3.3]). Take P to be the bundle of linear frames for TM over M. Define $Y_\lambda\in\Gamma(TE)$ by

$$Y_\lambda(q)[f] = \tilde{X}_\lambda(q)[f]+\frac{d}{d\varepsilon}f(q+\varepsilon S_\lambda(\pi(q))[q])\Big|_{\varepsilon=0}$$

for every $q\in E$ and C^∞ real function f on E, where $\tilde{X}_\lambda$ denotes the natural lift of X_λ to E and $S_\lambda(\pi(q))[\cdot]\in\mathrm{End}(E_{\pi(q)})$ stands for the restriction of the derivation S_λ to $E_{\pi(q)}$. (Note that $q+\varepsilon S_\lambda(\pi(q))[q]\in E_{\pi(q)}$.) Then each Y_λ is π-related to X_λ. Moreover, these $Y_\lambda(t):=Y_\lambda$ in fact satisfy the assumption (#) (cf. [**1**, § 4]), and for $\sigma\in\Gamma(E)$ we get a backward formula with $L_\lambda(u)\sigma_{u,t}=\delta_\lambda\sigma_{u,t}$.

Example 2. Let E be a vector bundle endowed with a linear connection ∇^E. Let P be the frame bundle for E over M. For $\lambda=0, 1, \cdots, k$, let $Y_\lambda\in\Gamma(TE)$ be the horizontal lift of $X_\lambda\in\Gamma(TM)$ to E. Then we see that these Y_λ satisfy the assumption (#), and for $\sigma\in\Gamma(E)$ we obtain a backward formula with $L_\lambda(u)\sigma_{u,t}=\nabla^E_{X_\lambda}\sigma_{u,t}$.

Next we consider certain backward and forward differential equations (terminal and initial value problems) for sections of a vector bundle.

Corollary 4.3. *Under the same conditions as in Corollary* 4.1, *let* $\sigma\in\Gamma(E)$ *be of compact support. Then the expectation* $v_{s,t}(x)=\boldsymbol{E}[\sigma_{s,t}(x)]$ *is the solution of the terminal value problem of the backward differential equation*

$$\begin{cases} \dfrac{\partial v_{s,t}}{\partial s} = -\left\{\dfrac{1}{2}\displaystyle\sum_{\alpha=1}^{k}(L_\alpha(s))^2+L_0(s)\right\}v_{s,t} \qquad (0<s<t), \\ \lim\limits_{s\uparrow t} v_{s,t} = \sigma\,. \end{cases}$$

Proof. This follows from Corollary 4.1.

Corollary 4.4. *Under the same conditions as in Corollary* 4.1, *let* $\sigma\in\Gamma(E)$ *be of compact support. Let* $\hat{\eta}_{s,t}(q)$ *be the solution of the backward stochastic* *differential equation*

$$\hat{d}\hat{\eta}_s = -\sum_{\lambda=0}^{k} Y_\lambda(s, \hat{\eta}_s)\circ \hat{d}w^\lambda(s)$$

with the terminal condition $\hat{\eta}_t(q)=q\in E_x$, $x\in M$. *Put* $\hat{\sigma}_{s,t}=(\hat{\eta}_{s,t})^{-1}\circ\sigma\circ\hat{\theta}_{s,t}$, *where* $\hat{\theta}_{s,t}(x) := \pi(\hat{\eta}_{s,t}(q))$, $q\in E_x$, $x\in M$; *note that* $\hat{\theta}_{s,t}$ *is well-defined. Set* $\hat{v}_{s,t}=\boldsymbol{E}[\hat{\sigma}_{s,t}]$. *Then* $\hat{v}_{s,t}$ *is the solution of the initial value problem of the forward differential equation*

$$\begin{cases} \dfrac{\partial \hat{v}_{s,t}}{\partial t} = \left\{\dfrac{1}{2}\sum_{\alpha=1}^{k}(L_\alpha(t))^2 + L_0(t)\right\}\hat{v}_{s,t}, \\ \lim_{t\downarrow s}\hat{v}_{s,t} = \sigma. \end{cases} \tag{4.9}$$

PROOF. This is proved in the same way as in [**10**, pp. 297-298, Theorem 7.2 and Theorem 7.3]: Apply Corollary 4.1 to $\hat{\sigma}_{s,t}$, interchanging the forward and backward variables. Then

$$\hat{\sigma}_{s,t}-\sigma = \sum_{\alpha=1}^{k}\int_s^t L_\alpha(u)\hat{\sigma}_{s,u}\cdot dw^\alpha(u) + \int_s^t\left\{\frac{1}{2}\sum_{\alpha=1}^{k}(L_\alpha(u))^2 + L_0(u)\right\}\hat{\sigma}_{s,u}du.$$

Therefore we obtain

$$\boldsymbol{E}[\hat{\sigma}_{s,t}]-\sigma = \int_s^t\left\{\frac{1}{2}\sum_{\alpha=1}^{k}(L_\alpha(u))^2 + L_0(u)\right\}\boldsymbol{E}[\hat{\sigma}_{s,u}]du.$$

Then it follows that $\hat{v}_{s,t}=\boldsymbol{E}[\hat{\sigma}_{s,t}]$ is the solution of (4.9). This completes the proof.

5. Applications.

5.1. Time-dependent random C^∞ distribution. Let $L(M)$ be the bundle of linear frames for TM over M. Let r be an integer with $1\leq r<n=\dim M$. Let $G(n, r)$ be the Grassmann manifold of r-planes in $\boldsymbol{R}^n$, so that the general linear group $GL(n, \boldsymbol{R})$ acts on $G(n, r)$ on the left in a natural manner (cf. [**9**, Vol. II, p. 6]). Let $E(M, G(n, r), GL(n, \boldsymbol{R}), L(M))$ be the fiber bundle associated with $L(M)$ by this action. The bundle E is called the (unoriented) *Grassmann bundle* of r-planes over M ([**3**, pp. 48-49]). Since the total space E can be regarded as the collection of r-planes in the tangent spaces of M, the C^∞ (global) sections of E are in one-to-one correspondence with the C^∞ distributions of dimension r on M.

Using this correspondence, we define a time-dependent random C^∞ distribution of dimension r on M as follows: Let $\mathcal{D}$ be a C^∞ distribution of dimension r on M. Let σ be the C^∞ section of E corresponding to $\mathcal{D}$. Given vector fields X_λ, $\lambda=0, 1, \cdots, k$, on M, we take Y_λ in the equation (3.1) to be the natural lift of X_λ to E; each Y_λ is induced from the natural lift of X_λ to $L(M)$ in a natural manner. Let $\tilde{\theta}_{s,t}$ be the solution of the equation (3.1) with $\eta_s(q)=q$ for every $q\in E$. Define a time-dependent random C^∞ distribution $\mathcal{D}_{s,t}$ of dimension r on M with the terminal condition $\mathcal{D}_{t,t}=\mathcal{D}$ by specifying the corresponding time-dependent random cross section $\theta^{\#}_{s,t}\sigma := \tilde{\theta}^{-1}_{s,t}\circ\sigma\circ\theta_{s,t}$. The behavior of $\mathcal{D}_{s,t}$ with respect to the initial-time parameter s is just given by Theorem 3.2, where $\eta_{s,t}=\tilde{\theta}_{s,t}$, and now each $D_\lambda(u)$ is Lie differentiation with respect to X_λ in the sense of Salvioli [11] (see also [4], [5]; in [1], the notation $\hat{L}_{X_\lambda}$ is used to denote Lie differentiation with respect to X_λ in the sense of Salvioli).

5.2. Backward and forward differential equations for second order linear differential operators on C^∞ functions on a manifold. Let $L^2(M)(M, G^2(n))$ be the bundle of second order frames over an n-dimensional manifold M with structure group $G^2(n)=G^2(n, \boldsymbol{R})$; see [8, p. 36 and p. 139] (cf. [1], [4], [5]). We take $L^2(M)$ as P in the assumption (#) in § 4. Let $\boldsymbol{F}=(\boldsymbol{R}^n\circledcirc\boldsymbol{R}^n)\oplus\boldsymbol{R}^n\oplus\boldsymbol{R}$ ($\cong\boldsymbol{R}^m$ with $m=n(n+1)/2+n+1$), where $\boldsymbol{R}^n\circledcirc\boldsymbol{R}^n$ is the symmetric tensor product of $\boldsymbol{R}^n$ with itself. We define a left action of $G^2(n)$ on $\boldsymbol{F}$ by

$$(s^i_j;\ s^i_{jk})(a^{ij};\ b^i;\ c)=\Big(\sum_{k,l=1}^n s^i_k s^j_l a^{kl}\ ;\ \sum_{j,k=1}^n s^i_{jk}a^{jk}+\sum_{j=1}^n s^i_j b^j;\ c\Big),$$

where $(s^i_j;\ s^i_{jk})$ (with $s^i_{jk}=s^i_{kj}$) and $(a^{ij};\ b^i;\ c)$ (with $a^{ij}=a^{ji}$), $i, j, k=1, \cdots, n$, are natural coordinates in $G^2(n)$ and $\boldsymbol{F}$, respectively. Then we obtain a vector bundle $E(M, \boldsymbol{F}, G^2(n), L^2(M))$, with standard fiber $\boldsymbol{F}$ and structure group $G^2(n)$, which is associated with $L^2(M)$.

The C^∞ (global) sections of the vector bundle E are in one-to-one correspondence with the second order (possibly degenerate) linear differential operators on $\boldsymbol{R}$-valued C^∞ functions on M. Each $\sigma\in\Gamma(E)$ is expressed locally as

$$\Big(x^i;\ \Big[\Big(\frac{\partial^2}{\partial x^i\partial x^j};\ \frac{\partial}{\partial x^i}\Big),\ (a^{ij}(x);\ b^i(x);\ c(x))\Big]\Big),\quad (i, j=1, \cdots, n),$$

and the second order (possibly degenerate) linear differential operator $\mathcal{A}_\sigma$ corresponding to σ is expressed locally as

$$\mathcal{A}_\sigma=\sum_{i,j=1}^n a^{ij}(x)\frac{\partial^2}{\partial x^i\partial x^j}+\sum_{i=1}^n b^i(x)\frac{\partial}{\partial x^i}+c(x).$$

We identify σ with $\mathcal{A}_\sigma$.

Take each A_λ in the assumption (#) (§ 4) to be the natural lift of X_λ to $L^2(M)$. Then each Y_λ is the natural lift $\hat{X}_\lambda$ of X_λ to E. Writing the solution $\eta_{s,t}$ of (3.1) as $\tilde{\theta}_{s,t}$, we define $\theta^{\#}_{s,t}\sigma:=\tilde{\theta}^{-1}_{s,t}\circ\sigma\circ\theta_{s,t}$. It holds from Corollary 4.1 that $\theta^{\#}_{s,t}\sigma$ satisfies

$$\theta^{\#}_{s,t}\sigma-\sigma=\sum_{\alpha=1}^k\int_s^t \mathcal{L}_{X_\alpha}(\theta^{\#}_{u,t}\sigma)\cdot\hat{d}w^\alpha(u)+\int_s^t\Big\{\frac{1}{2}\sum_{\alpha=1}^k(\mathcal{L}_{X_\alpha})^2+\mathcal{L}_{X_0}\Big\}\theta^{\#}_{u,t}\sigma du, \quad (5.1)$$

where each $\mathcal{L}_{X_\lambda}$: $\Gamma(E)\to\Gamma(E)$, $\lambda=0, 1, \cdots, k$, denotes Lie differentiation with respect to X_λ (see [1, § 4], [12]).

Let $\sigma\in\Gamma(E)$ be of compact support. Then from (5.1) we see that the expectation $v_{s,t}(x):=\boldsymbol{E}[(\theta_{s,t}^{\#}\sigma)(x)]$ is the solution of the following terminal value problem for second order (possibly degenerate) linear differential operators on $\boldsymbol{R}$-valued C^{∞} functions on M (see Corollary 4.3):

$$\frac{\partial v_{s,t}}{\partial s}=-\left\{\frac{1}{2}\sum_{\alpha=1}^{k}(\mathcal{L}_{X_\alpha})^2+\mathcal{L}_{X_0}\right\}v_{s,t}, \qquad \lim_{s\uparrow t} v_{s,t}=\sigma .$$

Next, consider the solution $\hat{\eta}_{s,t}(q)$ of the backward stochastic differential equation

$$\hat{d}\hat{\eta}_s=-\sum_{\lambda=0}^{k}\hat{X}_\lambda(s,\hat{\eta}_s)\circ\hat{d}w^\lambda(s)$$

with the terminal condition $\hat{\eta}_t(q)=q\in E_x$ with $x\in M$. Let $\hat{v}_{s,t}$ be as in Corollary 4.4 (with $Y_\lambda=\hat{X}_\lambda$). Then $\hat{v}_{s,t}$ is the solution of the following initial value problem for second order (possibly degenerate) linear differential operators on $\boldsymbol{R}$-valued C^{∞} functions on M:

$$\frac{\partial \hat{v}_{s,t}}{\partial t}=\left\{\frac{1}{2}\sum_{\alpha=1}^{k}(\mathcal{L}_{X_\alpha})^2+\mathcal{L}_{X_0}\right\}\hat{v}_{s,t}, \qquad \lim_{t\downarrow s} \hat{v}_{s,t}=\sigma .$$

NOTE. In [2, §3], nonstandard analysis, especially a hyperfinite random walk on a uniform Loeb probability space, has been used for getting nonstandard backward Itô's formulas for (local) sections of fiber bundles.

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