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Determination of the modulus of quadrilaterals by finite element methods

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Introduction.

In the present paper we aim to establish a method of finite element approximations by which we can determine the modulus of quadrilaterals on Riemann surfaces (cf. Mizumoto and Hara [**15**] for other treatment). Our method matches the abstract definition of Riemann surfaces, and also will offer a new technique of high practical use in numerical calculation not only for the case of Riemann surfaces but also for the case of plane domains.

Let Ω be a simply connected subdomain of a Riemann surface W whose closure $\bar{\Omega}$ is a compact bordered subregion of W. We assume that the boundary $\partial\Omega$ of Ω is a piecewise analytic curve. We assign four points p_1, p_2, p_3 and p_4 on $\partial\Omega$ (in positive orientation w.r.t. Ω), and the two opposite arcs C_0 (from p_1 to p_2) and C_1 (from p_3 to p_4). Then we say that a *quadrilateral* Q with opposite sides C_0 and C_1 is given.

We can conformally map the domain Ω onto a rectangular domain $R=\{w \mid 0<\operatorname{Re} w<1, 0<\operatorname{Im} w<M\}$ by a function $w=\mathfrak{f}(p)$ so that p_1, p_2, p_3 and p_4 are mapped to iM, 0, 1 and $1+iM$ respectively. Let $\mathfrak{F}$ be the class of all continuous functions v on $\bar{\Omega}$ with $v=0$ on C_0 and $v=1$ on C_1 which satisfy some restricted conditions (see §2.1). Then the modulus $M(Q)=M$ of the quadrilateral Q is uniquely determined by Q, and is given by

$$M(Q)=D(u)=\min_{v\in\mathfrak{F}} D(v) \qquad (u=\operatorname{Re}\mathfrak{f}(p)),$$

where by $D(v)$ we denote the Dirichlet integral of v. Next we assign the two opposite arcs $\tilde{C}_0$ (from p_2 to p_3) and $\tilde{C}_1$ (from p_4 to p_1) on $\partial\Omega$. Then a quadrilateral $\tilde{Q}$ with the opposite sides $\tilde{C}_0$ and $\tilde{C}_1$ is defined. We can easily see that $M(Q)=1/M(\tilde{Q})$. By making use of this relation Gaier [**9**] presented a method to obtain upper and lower bounds for the modulus $M(Q)$ in the case of some restricted domain Ω (e.g. a lattice domain, etc.) by the finite difference and

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finite element approximations, which originates from Opfer [16], [17]. We shall present a method to obtain fairly good upper and lower bounds for $M(Q)$ by our finite element approximation even in the case of a domain Ω with curvilinear boundary arcs, and with inner and corner singularities of high order.

It is characteristic of our method that we adopt ordinary triangular meshes and linear elements on a subregion of every fixed parametric disk, our approximating functions of $u=\operatorname{Re}\mathfrak{f}(p)$ satisfy the boundary conditions exactly even in the case of curvilinear boundary arcs, and express singular property exactly near inner and corner singularities. Hence the approximations of high precision of u are obtained, and the fairly good upper and lower bounds to $M(Q)$ can be evaluated exactly. It should be noted that we do not adopt any so-called refined or curvilinear mesh near singularities.

§1 is devoted to construction of triangulations K and K' of two kinds. K is a triangulation of $\bar{\Omega}$ and K' is a modification of K.

In §2, we introduce and investigate two classes of element functions on K and K': the *comparable class* $S=S(K)$ (with u) and the *computable class* $S'=S'(K')$. $S\subset\mathfrak{F}$ and S' is a collection of modifications $v'_h=F(v_h)$ of $v_h\in S$, where F defines a one-to-one mapping of S onto S'. $D(v'_h)$ can be numerically calculated. We shall investigate estimates of differences of $D(v_h)$ and $D(v'_h)$ (see Lemma 2.2).

The *finite element approximations* ω_h and u'_h of u in S and S' respectively are defined by the minimalities:

$$D(\omega_h)=\min_{v_h\in S} D(v_h) \quad \text{and} \quad D(u'_h)=\min_{v'_h\in S'} D(v'_h)$$

respectively. u'_h can be obtained by solving a system of linear equations. §3 is devoted to error estimates of ω_h and u_h for u. In Theorems 3.1 and 3.2, we obtain error estimates:

$$D(\omega_h-u)\leqq Ch^2 \quad \text{and} \quad D(u_h-u)\leqq C'h^2 \quad \text{resp.},$$

where C and C' are constants which depend only on the function u and the smallest value of interior angles of triangles. Further, in Theorem 3.2, we obtain an estimate for $D(u)$:

$$D(u)\leqq D(u'_h)+\varepsilon(u'_h),$$

where $\varepsilon(u'_h)$ is a quantity of $O(h^2)$ which can be numerically calculated.

Finally, in §4 we apply our results to numerical calculation of the modulus of quadrilaterals, and we shall show that calculation results for some concrete quadrilaterals are fairly good. With respect to the problems of this type, there have been some investigations by means of finite difference or finite element methods (cf. Gaier [9], Mizumoto [10], [11], [12], and Opfer [16], [17]).

With respect to treatment at boundary singularities, there have been some investigations (cf. Akin [**1**], Babuška [**2**], Babuška and Rosenzweig [**3**], Babuška, Szabo and Katz [**4**], Barnhill and Whiteman [**5**], Blackburn [**6**], Craig, Zhu and Zienkiewicz [**8**], Opfer and Puri [**18**], Rivara [**19**], Schatz and Wahlbin [**20**], [**21**], Thatcher [**23**], Tsamasphyros [**24**], Weisel [**25**], Whiteman and Akin [**26**], and Yserentant [**27**]).

§ 1. Triangulation.

1. Collection Φ of local mappings. Let Ω be a simply connected subdomain of a Riemann surface W whose closure $\bar{\Omega}$ is a compact bordered subregion of W. We assume that the boundary $\partial\Omega$ consists of a finite number of analytic arcs meeting at vertices p'_k $(k=1, \cdots, \kappa)$, and there exist parametric disks V_k $(k=1, \cdots, \kappa)$ with the centers p'_k and the local parameters $z=\psi_k(p)$ by which $V_k \cap \bar{\Omega}$ are mapped onto sectors $\{|z| \leqq r_k\} \cap \{0 \leqq \arg z \leqq \beta_k\}$ $(0<\beta_k \leqq 2\pi,\ \beta_k \neq \pi)$.

We assign four points p_1, p_2, p_3 and p_4 on $\partial\Omega$ (in the positive orientation with respect to Ω), and the two opposite arcs C_0 (from p_1 to p_2) and C_1 (from p_3 to p_4). Then we say that a *quadrilateral* Q *with opposite sides* C_0 *and* C_1 is given.

By $\Phi=\{z=\varphi_j(p), U_j;\ j=1, \cdots, m\}$ we denote a finite collection of local parameters $z=\varphi_j(p)$ $(j=1, \cdots, m)$ and parametric disks U_j $(j=1, \cdots, m)$ on W which satisfies the following conditions (i)$\sim$(v):

(i) By the mapping $z=\varphi_j(p)$ $(j=1, \cdots, m)$, U_j is mapped onto a disk $|z|<\rho_j$.

(ii) $\bar{\Omega}$ is covered by $\{U_j\}_{j=1}^m$.

(iii) If $U_j \cap U_k \neq \emptyset$, then there exists a constant $L(>1)$ such that for the mapping $\zeta=f(z)\equiv\varphi_k \circ \varphi_j^{-1}(z)$, $1/L<|df/dz|<L$ on $\varphi_j(U_j \cap U_k)$.

Let p_k $(k=5, \cdots, \nu)$ be the all vertices of $\partial\Omega$ which are defined as points of $\{p'_k\}_{k=1}^{\kappa}-\{p_1, p_2, p_3, p_4\}$.

(iv) Each U_j $(j=1, \cdots, m)$ contains at most one p_k $(k=1, \cdots, \nu)$ and if $p_k \in U_j$ then $\varphi_j(p_k)=0$.

(v) If $U_j \cap \partial\Omega \neq \emptyset$ and U_j does not contain any p_k $(k=1, \cdots, \nu)$, then $\varphi_j(U_j \cap \Omega)$ is a half disk $\{|z|<\rho_j\} \cap \{\operatorname{Im} z>0\}$. If U_j contains some p_k $(k=1, \cdots, \nu)$, then $\varphi_j(U_j \cap \Omega)$ is a sector $\{|z|<\rho_j\} \cap \{0<\arg z<\alpha_j\}$ $(0<\alpha_j \leqq 2\pi)$.

In the latter case of (v), if $p_k \neq p_1, p_2, p_3, p_4$ and $\alpha_j>\pi/2$, then by the mapping $\zeta=\{\varphi_j(p)\}^{\pi/\alpha_j}$, $U_j \cap \Omega$ is mapped onto a half disk $\{|\zeta|<\rho_j^{\pi/\alpha_j}\} \cap \{\operatorname{Im} \zeta>0\}$. In this case we define anew $z=\varphi_j(p)$ and ρ_j by $\zeta=\{\varphi_j(p)\}^{\pi/\alpha_j}$ and ρ_j^{π/α_j} respectively. Further, if U_j contains some p_k $(k=1, 2, 3, 4)$, then by the mapping $\zeta=\{\varphi_j(p)\}^{\pi/2\alpha_j}$, $U_j \cap \Omega$ is mapped onto a sector $\{|\zeta|<\rho_j^{\pi/2\alpha_j}\} \cap \{0<\arg \zeta<\pi/2\}$. In this case we define anew $z=\varphi_j(p)$ and ρ_j by $\zeta=\{\varphi_j(p)\}^{\pi/2\alpha_j}$ and $\rho^{\pi/2\alpha_j}$ respectively. Then, the local parameter $z=\varphi_j(p)$ is no longer conformal at the center of U_j except for the case when U_j contains some p_k $(k=1, 2, 3, 4)$ and $\alpha_j=\pi/2$.

2. Triangulation K associated to Φ. For the collection Φ of local parameters and parametric disks defined in § 1.1, and for a sufficiently small positive number h, we construct a triangulation $K=K^h$ of $\bar{\Omega}$ which satisfies the following conditions (i)$\sim$(v). This is called a *triangulation of $\bar{\Omega}$ with width h associated to Φ.*

(i) The points $p_1, \cdots, p_\nu$ are carriers of some 0-simplices of K.

(ii) K is the sum of subtriangulations $K_1, \cdots, K_m$ of K such that each 2-simplex of K belongs to one and only one K_j $(j=1, \cdots, m)$, and the carrier $|s|$ of each 2-simplex s of K_j is contained in U_j.

If a 1-simplex $e\in K_j$ does not belong to another K_k $(k\neq j)$, or a 1-simplex e belongs to $K_j\cap K_k$ $(j\neq k)$ and the mapping $\varphi_k\circ\varphi_j^{-1}$ is an affine transformation, then e is said to be *linear*. If each edge of a 2-simplex $s\in K_j$ is linear and $\varphi_j(s)$ is an ordinary triangle, then s is called a *natural simplex*.

(iii) Each 2-simplex $s\in K_j$ which has not a common edge with any 2-simplex of another K_k $(k\neq j)$, is a natural simplex.

A 2-simplex of K_k which has a common edge with a 2-simplex $s\in K_j$ $(j\neq k)$, is said to be an *adjoint* (simplex) of s and is denoted by s'.

(iv) For each pair of a 2-simplex $s\in K_j$ and its adjoint $s'\in K_k$ with a common edge e, either one of the following three cases (a), (b), (c) occurs.

(a) Both s and s' are natural simplices.

(b) $\varphi_j(s)$ is a curvilinear triangle such that $\varphi_j(e)$ is a strict concave arc w.r.t. $\varphi_j(s)$, $\varphi_k(s')$ is an ordinary triangle, and all edges of s and s' except for e are linear (cf. Figure 1). Then s is called a *minor simplex.* The case where s' is a minor simplex and s is its adjoint may also occur.

(c) $\varphi_j(s)$ is a curvilinear triangle such that $\varphi_j(e)$ is a strictly convex arc w.r.t. $\varphi_j(s)$, $\varphi_k(s')$ is an ordinary triangle, and all edges of s and s' except for e are linear (cf. Figure 2). Then s is called a *major simplex.* The case where s' is a major simplex and s is its adjoint may also occur.

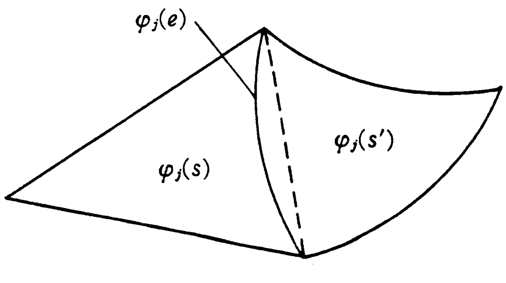


Figure 1. Minor simplex s and its adjoint s'.

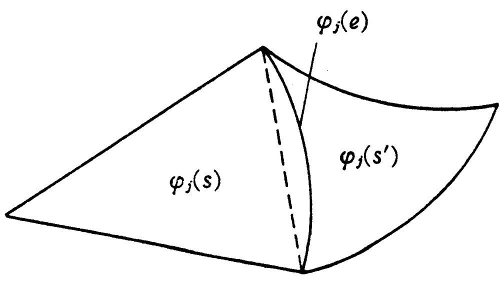


Figure 2. Major simplex s and its adjoint s'.

If s is a minor or major simplex of K_j, then it is assumed that $|s'|\subset U_j$ for its adjoint s'.

(v) For each 2-simplex $s\in K_j$ $(j=1, \cdots, m)$, $d(\varphi_j(s))\leqq h$, where throughout the present paper we denote the diameter of a region G by $d(G)$.

Next, we assume that for the fixed Φ the class of the triangulations $K=K^h$ satisfies the following condition (i') and (ii'):

(i') For each $j=1, \cdots, m$ the union of carriers of all minor and major simplices of K_j, and all their adjoints is contained in a closed subset R_j of $U_j\cap\bar{\Omega}$ which is independent of the individual triangulation K.

(ii') The number N of minor and major simplices of K satisfies the inequality:

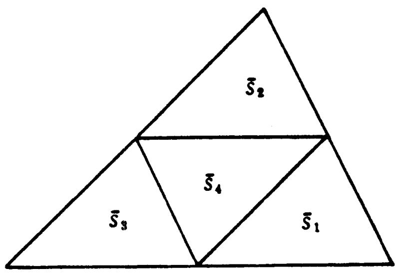
$$N\leqq M\cdot\frac{1}{h}, \tag{1.1}$$

where M is a constant which is independent of the individual triangulation K.

3. Normal subdivision of triangulation K. For a triangulation $K=K^h$ of $\bar{\Omega}$ with width h associated to Φ we can construct a subdivision $K^1=K^{1,h/2}$, called the *normal subdivision* of $K=K^h$ by the following procedure:

(i) K^1 is the sum of the subtriangulations $K_1^1, \cdots, K_m^1$ which are the subdivisions of $K_1, \cdots, K_m$ respectively which are defined in the following (ii), (iii).

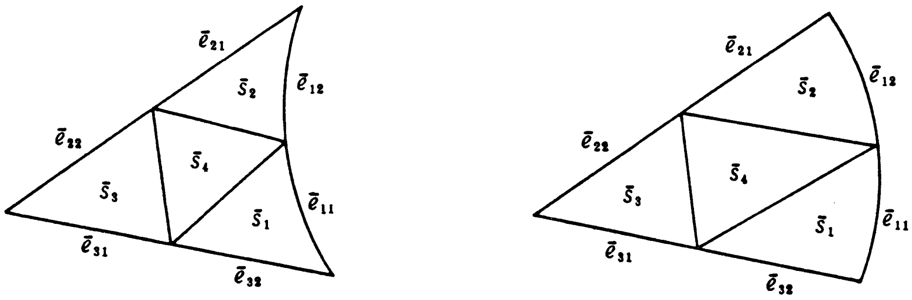
(ii) If $s\in K_j$ is a 2-simplex which is not minor or major, then s is subdivided to four 2-simplices s_1, s_2, s_3 and s_4 of K_j^1 so that $\varphi_j(s_1)$, $\varphi_j(s_2)$, $\varphi_j(s_3)$ and $\varphi_j(s_4)$ are mutually congruent ordinary triangles in Figure 3.



$\bar{a}=\varphi_j(a)$ (a : simplex)

Figure 3. Normal subdivision of 2-simplex which is not minor or major.

(iii) Let $s\in K_j$ and $s'\in K_k$ be a minor (or major) simplex and its adjoint, and let e_1, e_2 and e_3 be edges of s such that e_1 is the common edge of s and s'. We subdivide the edges e_1, e_2 and e_3 to two edges e_{11} and e_{12}, e_{21} and e_{22}, and e_{31} and e_{32} respectively so that $\varphi_k(e_{11})$ and $\varphi_k(e_{12})$, $\varphi_j(e_{21})$ and $\varphi_j(e_{22})$, and $\varphi_j(e_{31})$ and $\varphi_j(e_{32})$ have the same length respectively. Then we subdivide the simplex s to two minor (or major resp.) simplices s_1 and s_2 of K_j^1 and, two natural simplices s_3 and s_4 of K_j^1 so that e_{11}, e_{12}, e_{21}, e_{22}, e_{31} and e_{32} are edges



$\bar{a}=\varphi_j(a)$ (a : simplex)

Figure 4. Normal subdivision of minor and major simplices.

of s_1, s_2 and s_3 (cf. Figure 4). Here we note that such a subdivision is always possible if h is sufficiently small.

We can see that the normal subdivision $K^1=\sum_{j=1}^{m} K_j^1$ is a triangulation of $\bar{\Omega}$ with width $h/2+O(h^2)$ associated to Φ (cf. §1 of [**15**]).

4. Naturalized triangulation. For each minor (or major) simplex $s\in K_j$ we define the *naturalized simplex* $\natural s$ of s as the 2-simplex such that $|s|\subset|\natural s|$ ($|\natural s|\subset|s|$ resp.) and $\varphi_j(\natural s)$ is the ordinary triangle which has two common sides with $\varphi_j(s)$. Further we define a 2-simplex $\flat\ell=\flat\ell(s)$ ($\sharp\ell=\sharp\ell(s)$ resp.) with two edges whose carrier is the closed region $\overline{|\natural s|-|s|}$ ($\overline{|s|-|\natural s|}$ resp.). $\flat\ell(s)$ ($\sharp\ell(s)$ resp.) is called the *deficient* (*excessive* resp.) *lune* of s.

Each triple of a minor (or major) simplex $s\in K_j$, its adjoint $s'\in K_k$ and its deficient lune $\flat\ell$ (excessive lune $\sharp\ell$ resp.) is denoted by $(s, s', \flat\ell)$ ($(s, s', \sharp\ell)$ resp.), and is called a *triple for a minor* (*major* resp.) *simplex* s or *for a deficient* (*excessive* resp.) *lune* $\flat\ell$ ($\sharp\ell$ resp.) (cf. Figure 5), where it is always assumed that $|\flat\ell|\subset|s'|$ for each $(s, s', \flat\ell)$.

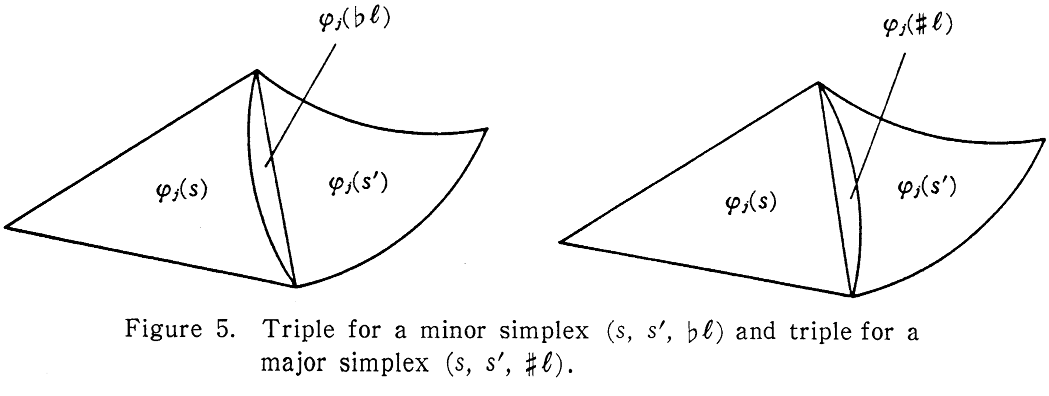


Figure 5. Triple for a minor simplex $(s, s', \flat\ell)$ and triple for a major simplex $(s, s', \sharp\ell)$.

For simplicity of notation, we also denote $\flat\ell=\flat\ell(s)$ or $\sharp\ell=\sharp\ell(s)$ by $\ell=\ell(s)$. If a minor or major simplex s is in K_j, then we say that $\ell=\ell(s)$ is a *lune of*

K_j and write $\ell \in K_j$.

Now we shall define the *naturalized triangulation K' associated to K.*

First, K'_j ($j=1, \cdots, m$) are defined as triangulations such that the collection of all 2-simplices of K'_j consists of all 2-simplices of K_j which are not minor or major, and of all naturalized simplices of minor and major ones of K_j. Then the triangulation K' is defined as the sum of K'_j ($j=1, \cdots, m$). We should note that K' is no longer a triangulation of $\bar{\Omega}$, and also is not an ordinary triangulation.

5. Parametrization of lunar domains. Let (s, s', ℓ) be an arbitrary triple for a deficient or excessive lune ℓ, and let e_1 and e_2 be two edges of ℓ such that $e_1 \subset \partial s$. Further, let

(1.2) $$z' = (1-t)z_1 + tz_2 \qquad (0 \leqq t \leqq 1)$$

and

(1.3) $$\zeta'' = (1-t)\zeta_1 + t\zeta_2 \qquad (0 \leqq t \leqq 1)$$

be parameter representations of the oriented segments $\varphi_j(-e_2)$ and $\varphi_k(e_1)$ respectively. The representation (1.3) induces a parameter representation of the curve $\varphi_j(e_1)$:

(1.4) $$z'' = g((1-t)\zeta_1 + t\zeta_2) \qquad (0 \leqq t \leqq 1),$$

where $z = g(\zeta) \equiv \varphi_j \circ \varphi_k^{-1}(\zeta)$. By (1.2) and (1.4) we obtain a parameter representation of the lunar domain $\varphi_j(\ell)$:

(1.5) $$\begin{aligned} z = z(t, \tau) &\equiv (1-\tau)z' + \tau z'' \\ &= (1-\tau)((1-t)z_1 + tz_2) + \tau g((1-t)\zeta_1 + t\zeta_2) \qquad (0 \leqq t \leqq 1,\ 0 \leqq \tau \leqq 1). \end{aligned}$$

6. Area of lune.

LEMMA 1.1. *Let (s, s', ℓ) be a triple for an arbitrary deficient or excessive lune l. Then, the estimate*

(1.6) $$A(\varphi_j(\ell)) \leqq \frac{h_1^3}{8}\left(\left|\frac{g''(\zeta_1)}{g'(\zeta_1)^2}\right| + O(h_1)\right)$$

holds, where throughout the present paper we denote the area of a region G by $A(G)$, $z = g(\zeta) \equiv \varphi_j \circ \varphi_k^{-1}(\zeta)$, $h_1 = d(\varphi_j(\ell))$ and ζ_1 is one of the vertices of the lunar domain $\varphi_k(\ell)$.

See Lemma 1.1 of [**15**] for the proof.

§2. Classes of functions.

1. Class $\mathfrak{F}$. By $\mathfrak{F}$ we denote the class of all continuous functions v on $\bar{\Omega}=\Omega\cup\partial\Omega$ with $v=0$ on C_0 and $v=1$ on C_1, for which the partial derivatives $\partial v/\partial x$ and $\partial v/\partial y$ with respect to the local parameter $z=x+iy$ exist and are continuous on Ω at most except for a finite number of rectifiable curves on Ω, and for which the Dirichlet integral

$$D(v)=D_{\Omega}(v)=\iint_{\Omega}\left(\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right)dxdy$$

is finite.

2. Subclass S of $\mathfrak{F}$. We define a subclass $S=S(K)$ of $\mathfrak{F}$, called the *comparable class* (with u), as the class of functions v_h which satisfy the following conditions (i)~(iv):

(i) $v_h \in \mathfrak{F}$.

(ii) If $s\in K_j$ $(j=1, \cdots, m)$ is a natural simplex, then

$$v_h = ax+by+c \qquad \text{on} \quad \varphi_j(s) \ (z=x+iy),$$

where a, b and c are constants.

(iii) Let $(s, s', \flat\ell)$ be a triple for a minor simplex s, and let e_1 and e_2 be two edges of $\flat\ell$ such that $-e_1\subset\partial s$. Then

$$v_h = ax+by+c \qquad \text{on} \quad \varphi_j(s),$$

$$v_h = \alpha\xi+\beta\eta+\gamma \qquad \text{on} \quad \varphi_k(s')-\varphi_k(\flat\ell),$$

and v_h is a harmonic function in $\flat\ell$ which satisfies the boundary conditions:

$$v_h = ax+by+c \qquad \text{on} \quad \varphi_j(e_1)$$

and

$$v_h = \alpha\xi+\beta\eta+\gamma \qquad \text{on} \quad \varphi_k(e_2),$$

where a, b, c, α, β and γ are constants, and

$$\zeta = f(z) \equiv \varphi_k\circ\varphi_j^{-1}(z) \qquad (z=x+iy,\ \zeta=\xi+i\eta).$$

(iv) Let $(s, s', \sharp\ell)$ be a triple for a major simplex s, and let e_1 and e_2 be two edges of $\sharp\ell$ such that $e_1\subset\partial s$. Then

$$v_h = ax+by+c \qquad \text{on} \quad \varphi_j(\natural s),$$

$$v_h = \alpha\xi+\beta\eta+\gamma \qquad \text{on} \quad \varphi_k(s'),$$

and v_h is a harmonic function in $\sharp\ell$ which satisfies the boundary conditions:

$$v_h = ax+by+c \qquad \text{on} \quad \varphi_j(e_2)$$

and

$$v_h = \alpha\xi+\beta\eta+\gamma \qquad \text{on} \quad \varphi_k(e_1),$$

where a, b, c, α, β and γ are constants, and $\zeta=\xi+i\eta$ is as in (iii).

3. Class S' of functions. Let K' be the naturalized triangulation associated to K. For each function $v_h \in S$, we define the *function v'_h on K' associated to v_h* as the function v'_h which satisfies the following conditions (i)~(iv):

(i) For each 2-simplex $s \in K'_j$ $(j=1, \cdots, m)$

$$v'_h = ax+by+c \qquad \text{on} \quad \varphi_j(s),$$

where a, b and c are constants.

(ii) If $s \in K$ is a natural simplex, then

$$v'_h = v_h \qquad \text{on} \quad |s|.$$

(iii) If $(s, s', \flat\ell)$ is a triple for a minor simplex s, then

$$v'_h = v_h \qquad \text{on} \quad |s| \cup |s'| - |\flat\ell|.$$

(iv) If $(s, s', \sharp\ell)$ is a triple for a major simplex s, then

$$v'_h = v_h \qquad \text{on} \quad |\natural s| \cup |s'|.$$

We should note that the function v'_h is defined just twice on each deficient lune $\flat\ell$, while it is never defined on any excessive lune $\sharp\ell$. In the former case, for each triple $(s, s', \flat\ell)$ we shall denote the function v'_h on $\natural s \in K'_j$ and $s' \in K'_k$ by $v'_{h,\natural s}$ and $v'_{h,s'}$ respectively.

The class of all functions v'_h associated to $v_h \in S$ is denoted by $S'=S'(K')$ and called the *computable class.* Let v'_h and ϕ'_h be two functions of S'. Then the *mixed Dirichlet integral* $D_{K'}(v'_h, \phi'_h)$ of v'_h and ϕ'_h is defined by

$$D(v'_h, \phi'_h) = D_{K'}(v'_h, \phi'_h) = \sum_{s\in K'} \iint_{|s|} \left(\frac{\partial v'_h}{\partial x}\frac{\partial \phi'_h}{\partial x} + \frac{\partial v'_h}{\partial y}\frac{\partial \phi'_h}{\partial y} \right) dxdy,$$

and the *Dirichlet integral* $D_{K'}(v'_h)$ of v'_h is defined by

$$D(v'_h) = D_{K'}(v'_h) = D_{K'}(v'_h, v'_h)^{1)},$$

where $D(v'_h)$ can be numerically calculated.

We see that $v'_h = F(v_h)$ defines a one-to-one mapping of S onto S'.

4. Finite element interpolations. Let v be a function of $\mathfrak{F}$. We define the *finite element interpolation $\hat{v}$ of v* in the class S as the function uniquely determined by the following conditions (i) and (ii):

1) We shall use the common notations $D(\ ,\ )$ and $D(\ )$ for both mixed and ordinary Dirichlet integrals of functions of the classes S and S'.

(i) $\hat{v} \in S$;

(ii) $\hat{v}(p) = v(p)$ at carrier p of each 0-simplex of K.

5. Harmonic functions on a lune.

LEMMA 2.1. *Let $\ell = \ell(s)$ be a deficient or excessive lune of K_j, let e_1 and e_2 be two edges of ℓ, and let q_1 and q_2 be two vertices of ℓ. Let v_1 and v_2 be the functions in the class C^1 on ℓ which satisfy the condition*

$$v_1(q_j) = v_2(q_j) \qquad (j=1, 2).$$

Further, let ϕ be the harmonic function in ℓ which satisfies the boundary conditions

$$\phi = v_i \quad on \quad e_i \quad (i=1, 2).$$

Then the inequalities

$$\text{(2.1)} \qquad D_\ell(\phi) \leqq \iint_{\varphi_j(\ell)} \max\left(\left(\frac{\partial v_1}{\partial x}\right)^2 + \left(\frac{\partial v_1}{\partial y}\right)^2, \left(\frac{\partial v_2}{\partial x}\right)^2 + \left(\frac{\partial v_2}{\partial y}\right)^2\right) dxdy$$
$$\leqq D_\ell(v_1) + D_\ell(v_2)$$

hold.

If we set $\sigma_1 = dv_1$ and $\sigma_2 = dv_2$, then the proof is reduced to one of Lemma 2.1 of [**15**].

6. Difference of Dirichlet integrals of v_h and v'_h.

LEMMA 2.2. *Let v_h be an arbitrary function of the class S and let $v'_h = F(v_h)$.*

(i) *The inequalities*

$$\text{(2.2)} \qquad D(v_h) \leqq D(v'_h) + \sum_{\sharp\ell \in K} D_{\sharp\ell}(v_h)$$
$$\leqq D(v'_h) + \sum_{j=1}^{m} \sum_{\sharp\ell \in K_j} A(\varphi_j(\sharp\ell)) \cdot \frac{(v'_h(q_2) - v'_h(q_1))^2}{|\varphi_j(q_2) - \varphi_j(q_1)|^2}$$
$$\cdot \max\left\{1, \frac{|\varphi_j(q_2) - \varphi_j(q_1)|^2}{|\varphi_k(q_2) - \varphi_k(q_1)|^2} \max_{\varphi_j(\sharp\ell)} |f'(z)|^2\right\}$$

hold, where

$$\frac{|\varphi_j(q_2) - \varphi_j(q_1)|^2}{|\varphi_k(q_2) - \varphi_k(q_1)|^2} \max_{\varphi_j(\sharp\ell)} |f'(z)|^2 \leqq 1 + \kappa h,$$

q_1 and q_2 are the vertices of $\sharp\ell$, and κ is a constant which depends only on the transformations $f(z) = \varphi_k \circ \varphi_j^{-1}(z)$.

(ii)

$$\text{(2.3)} \qquad D(v'_h) \leqq D(v_h) + \sum_{\flat\ell \in K} (D_{\flat\ell}(v'_{h,\natural s}) + D_{\flat\ell}(v'_{h,s'}))$$
$$= D(v_h) + \sum_{j=1}^{m} \sum_{\flat\ell \in K_j} (A(\varphi_j(\flat\ell)) \cdot (a^2 + b^2) + A(\varphi_k(\flat\ell)) \cdot (\alpha^2 + \beta^2)),$$

where for each triple $(s, s', \flat\ell)$ *the notations in* (iii) *of* § 2.2 *are preserved.*

If we set $\sigma_h = dv_h$ and $\sigma'_h = dv'_h$, then the proof is reduced to one of Lemma 2.2 of [**15**].

§ 3. Finite element approximations.

1. Formulation of problems. We can conformally map the domain Ω defined in § 1.1 onto a rectangular domain

$$R = \{w \mid 0 < \mathrm{Re}\, w < 1,\ 0 < \mathrm{Im}\, w < M\}$$

by a function $w = \mathfrak{f}(p)$ so that p_1, p_2, p_3 and p_4 are mapped to iM, 0, 1 and $1 + iM$ respectively. Then the *modulus* of the quadrilateral Q:

$$M(Q) = M$$

is uniquely determined by Q. Our aim is to determine $M(Q)$ by finite element method.

Now we assign the two opposite arcs $\tilde{C}_0$ (from p_2 to p_3) and $\tilde{C}_1$ (from p_4 to p_1) on $\partial\Omega$. Then a quadrilateral $\tilde{Q}$ with opposite sides $\tilde{C}_0$ and $\tilde{C}_1$ is defined. We see that the domain Ω can be conformally mapped onto a rectangular domain

$$\tilde{R} = \{w \mid 0 < \mathrm{Re}\, w < 1,\ 0 < \mathrm{Im}\, w < 1/M\}$$

by a function $w = \tilde{\mathfrak{f}}(p)$ so that p_2, p_3, p_4 and p_1 are mapped to i/M, 0, 1 and $1 + i/M$ respectively. Hence

$$M(\tilde{Q}) = \frac{1}{M(Q)}. \tag{3.1}$$

We characterize $M(Q)$ by a minimal property.

LEMMA 3.1. *Let* $u = \mathrm{Re}\,\mathfrak{f}$. *Then the equalities*

$$M(Q) = D(u) = \min_{v \in \mathfrak{F}} D(v) \tag{3.2}$$

hold. The minimum of the right hand side of (3.2) *is attained if and only if* $v = u$.

PROOF. By $*du$ we denote the conjugate differential of du. Then

$$*du = 0 \quad \text{along} \quad \partial\Omega - C_0 \cup C_1, \quad \text{and}$$

$$v - u = 0 \quad \text{on} \quad C_0 \cup C_1 \quad \text{for each } v \in \mathfrak{F}.$$

Hence

$$D(v - u, u) = \int_{\partial\Omega} (v - u) * du = 0. \tag{3.3}$$

This equality implies that

$$D(v) = D(u) + D(v-u) \geqq D(u).$$

In the last inequality, the equality holds if and only if $v=u$.

The first equality of (3.2) follows from the equalities

$$D(u) = \int_{\partial\Omega} u * du = \int_{C_1} * du = M.$$

We call u the *harmonic solution* in $\mathfrak{F}$. Our aim is to obtain finite element approximations of u in the classes S and S', and error estimates of them for u.

2. Finite element approximation ω_h in S. By ω_h we denote the function of S such that

$$D(\omega_h) = \min_{v_h \in S} D(v_h). \tag{3.4}$$

Since $S \subset \mathfrak{F}$, we see that

$$D(u) \leqq D(\omega_h). \tag{3.5}$$

We call ω_h the *finite element approximation of u in S.*

LEMMA 3.2. (i) *The function ω_h has the minimal property*

$$D(\omega_h - u) = \min_{v_h \in S} D(v_h - u), \tag{3.6}$$

where the minimum is attained if and only if $v_h = \omega_h$.

(ii) *The equality*

$$D(\omega_h - u) = D(\omega_h) - D(u) \tag{3.7}$$

holds.

PROOF. (i) First, by a method similar to (3.3), it is shown that

$$D(v_h - \omega_h, u) = 0 \quad \text{for each} \quad v_h \in S. \tag{3.8}$$

By (3.4), standard arguments imply that

$$D(\omega_h, v_h - \omega_h) = 0 \quad \text{for each} \quad v_h \in S. \tag{3.9}$$

From (3.8) and (3.9), it follows that

$$D(u - v_h) = D(u - \omega_h) + D(v_h - \omega_h) \geqq D(u - \omega_h).$$

In the last inequality, the equality holds if and only if $v_h = \omega_h$.

(ii) By (3.3) $D(\omega_h - u, u) = 0$ and thus (3.7) is obtained.

From (3.9) the following lemma immediately follows.

LEMMA 3.3. *The equality*

$$D(v_h-\omega_h)=D(v_h)-D(\omega_h) \tag{3.10}$$

holds for each $v_h \in S$.

3. Finite element approximation u'_h **in** S'. By u'_h we denote the function of S' such that

$$D(u'_h)=\min_{v'_h \in S'} D(v'_h). \tag{3.11}$$

We call u'_h the *finite element approximation of* u *in* S'. u'_h can be obtained by solving a system of linear equations.

LEMMA 3.4. *The equality*

$$D(v'_h-u'_h)=D(v'_h)-D(u'_h) \tag{3.12}$$

holds for each $v'_h \in S'$.

PROOF. By (3.11), standard arguments imply that

$$D(u'_h, v'_h-u'_h)=0 \qquad \text{for each} \quad v'_h \in S'. \tag{3.13}$$

This implies (3.12).

4. Lemma of Bramble and Zlámal. The following lemma is due to J.H. Bramble and M. Zlámal (cf. [7]).

LEMMA 3.5. *Let* Δ *be a closed triangle on the* z*-plane* $(z=x+iy)$ *with* $d(\Delta) \leqq h$ *and let* v *be a function of the class* C^2 *defined on* Δ *such that* $v=0$ *at each vertex of* Δ. *Then, the inequality*

$$\iint_\Delta \left(\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right)dxdy \leqq \frac{B}{\sin^2\theta} h^2 \iint_\Delta \left(\left(\frac{\partial^2 v}{\partial x^2}\right)^2+2\left(\frac{\partial^2 v}{\partial x \partial y}\right)^2+\left(\frac{\partial^2 v}{\partial y^2}\right)^2\right)dxdy \tag{3.14}$$

holds, where B *is an absolute constant and* θ *is the smallest interior angle of the triangle* Δ.

5. Pointwise estimate.

LEMMA 3.6. *Let* Δ *be a closed curvilinear triangle on the* z*-plane* $(z=x+iy)$ *with* $d(\Delta) \leqq h$ *which is the image of some 2-simplex* $s \in K_j$ $(j=1, \cdots, m)$ *by* $z=\varphi_j(p)$, *and let* v *be a function of the class* C^2 *defined on* Δ *such that* $v=0$ *at each vertex of* Δ. *Then,*

$$\left|\frac{\partial v}{\partial x}\right|, \left|\frac{\partial v}{\partial y}\right| \leqq h \cdot \frac{4}{\sin\theta} \max_{z \in \Delta}\left(\left|\frac{\partial^2 v}{\partial x^2}\right|+2\left|\frac{\partial^2 v}{\partial x \partial y}\right|+\left|\frac{\partial^2 v}{\partial y^2}\right|\right)(1+\kappa h)$$

on Δ, where θ is the smallest interior angle of the ordinary triangle which has common vertices with Δ, and κ is a constant which depends only on the transformations $f(z)=\varphi_k \circ \varphi_j^{-1}(z)$.

See Lemma 3.6 of [**15**] for the proof, and also refer to Theorem 3.1 of Strang and Fix [**22**].

6. Smoothness of u on $\bar{\Omega}$.

LEMMA 3.7. *Let u be the harmonic solution in $\mathfrak{F}$. Then $u\circ\varphi_j^{-1}$ $(j=1, \cdots, m)$ are of the class C^2 on $\varphi_j(U_j \cap \bar{\Omega})$ respectively.*

PROOF. (i) The case where U_j contains some $p_k (k=1, 2, 3, 4)$.

Let us assume that U_j contains p_1. The other cases are also similar. Then, $\varphi_j(p_1)=0$, $\varphi_j(U_j \cap \bar{\Omega})=\{|z|<\rho_j\}\cap\{0\leqq \arg z \leqq \pi/2\}$,

$$u\circ\varphi_j^{-1}=0 \quad \text{on} \quad \{z \mid \operatorname{Im} z=0,\ 0\leqq \operatorname{Re} z \leqq \rho_j\} \tag{3.15}$$

and

$$\frac{\partial}{\partial n} u\circ\varphi_j^{-1}=0 \quad \text{on} \quad \{z \mid \operatorname{Re} z=0,\ 0<\operatorname{Im} z \leqq \rho_j\}, \tag{3.16}$$

where by $\partial/\partial n$ we denote the inner normal derivative. By (3.15) and (3.16) we see that $u\circ\varphi_j^{-1}$ can be harmonically continued to $\varphi_j(U_j)=\{|z|<\rho_j\}$ and thus especially is of the class C^2 on $\varphi_j(U_j \cap \bar{\Omega})$.

(ii) The case where $\varphi_j(U_j\cap\bar{\Omega})=\{|z|<\rho_j\}\cap\{0\leqq \arg z\leqq \alpha_j\}$ and $\alpha_j \leqq \pi/2$.

Let $\mathfrak{g}$ be the function defined on $D=\{\operatorname{Im}\zeta>0\}\cap\{|\zeta|<\rho_j^{\pi/\alpha_j}\}$ by $\mathfrak{g}(\zeta)\equiv \mathfrak{f}\circ\varphi_j^{-1}(\zeta^{\alpha_j/\pi})$. Since $\operatorname{Re}\mathfrak{g}=$const. or $\operatorname{Im}\mathfrak{g}=$const. on $\{\operatorname{Im}\zeta=0\}\cap\{|\zeta|<\rho_j^{\pi/\alpha_j}\}$, $\mathfrak{g}$ is analytic on the closure $\bar{D}$. Then

$$\frac{d\mathfrak{f}\circ\varphi_j^{-1}(z)}{dz}=\frac{d\mathfrak{g}}{d\zeta}(z^{\pi/\alpha_j})\cdot\frac{\pi}{\alpha_j}z^{\pi/\alpha_j-1}$$

and

$$\frac{d^2\mathfrak{f}\circ\varphi_j^{-1}(z)}{dz^2}=\frac{d^2\mathfrak{g}}{d\zeta^2}(z^{\pi/\alpha_j})\cdot\left(\frac{\pi}{\alpha_j}\right)^2 z^{2(\pi/\alpha_j-1)}+\frac{d\mathfrak{g}}{d\zeta}(z^{\pi/\alpha_j})\cdot\frac{\pi}{\alpha_j}\left(\frac{\pi}{\alpha_j}-1\right)z^{\pi/\alpha_j-2}$$

on $\varphi_j(U_j\cap\bar{\Omega})$. Hence, $\alpha_j\leqq\pi/2$ implies that $d^2\mathfrak{f}\circ\varphi_j^{-1}(z)/dz^2$ is continuous on $\varphi_j(U_j\cap\bar{\Omega})$ and thus $u\circ\varphi_j^{-1}=\operatorname{Re}\mathfrak{f}\circ\varphi_j^{-1}$ is of the class C^2 on $\varphi_j(U_j\cap\bar{\Omega})$.

(iii) The cases except (i) and (ii).

Since $u\circ\varphi^{-1}=$const. or $\partial u\circ\varphi^{-1}/\partial n=0$ on $\varphi_j(U_j\cap C)=\{|z|<\rho_j\}\cap\{\operatorname{Im} z=0\}$, or $\varphi_j(U_j\cap C)=\emptyset$, $u\circ\varphi^{-1}$ is harmonic on $\varphi_j(U_j\cap\bar{\Omega})$.

7. Approximation by ω_h.

THEOREM 3.1. *Let u be the harmonic solution in $\mathfrak{F}$ defined in §3.1 and let*

ω_h be the finite element approximation of u in S. Then,

$$(3.17)\qquad D(\omega_h-u)\leqq\frac{h^2}{\sin^2\theta}\Big(B\sum_{j=1}^{m}\iint_{\varphi_j(K_j')}\Big(\Big(\frac{\partial^2u}{\partial x^2}\Big)^2+2\Big(\frac{\partial^2u}{\partial x\partial y}\Big)^2+\Big(\frac{\partial^2u}{\partial y^2}\Big)^2\Big)dxdy$$

$$+Ch^2\sum_{j=1}^{m}\max_{\varphi_j(R_j)}\Big(\Big(\frac{\partial^2u}{\partial x^2}\Big)^2+2\Big(\frac{\partial^2u}{\partial x\partial y}\Big)^2+\Big(\frac{\partial^2u}{\partial y^2}\Big)^2\Big)\Big),$$

where B and C are constants independent of the triangulation K and the function u, θ is the smallest value of interior angles of all triangles $\varphi_j(s)$ ($s\in K_j'$; $j=1,\cdots,m$), by $\varphi_j(K_j')$ we denote the image set by φ_j of the carrier of K_j', and R_j ($j=1,\cdots,m$) are the closed subsets of $U_j\cap\bar{\Omega}$ defined in (i′) of § 1.2.

PROOF. First, by (i) of Lemma 3.2,

$$(3.18)\qquad D(\omega_h-u)\leqq D(\hat{u}-u).$$

Hence it is sufficient to estimate $D(\hat{u}-u)$.

We have

$$(3.19)\qquad D(\hat{u}-u)=\sum_{j=1}^{m}\sum_{s\in K_j}D_s(\hat{u}-u).$$

Here we note that by the Lemma 3.7 $u\circ\varphi_j^{-1}$ ($j=1,\cdots,m$) is of the class C^2 on $\varphi_j(U_j\cap\bar{\Omega})$. Then, by Lemma 3.5,

$$(3.20)\qquad D_s(\hat{u}-u)\leqq\frac{B}{\sin^2\theta}h^2\iint_{\varphi_j(s)}\Big(\Big(\frac{\partial^2u}{\partial x^2}\Big)^2+2\Big(\frac{\partial^2u}{\partial x\partial y}\Big)^2+\Big(\frac{\partial^2u}{\partial y^2}\Big)^2\Big)dxdy$$

for each natural simplex s of K_j. For simplicity, we denote the right hand side of (3.20) by $I[\varphi_j(s)]$.

For a triple (s, s', ℓ) for a minor simplex s, we denote the function $\hat{u}'$ on $\natural s\in K_j'$ and $s'\in K_k'$ by $\hat{u}'_{\natural s}$ and $\hat{u}'_{s'}$ respectively. Then, by Lemma 2.1,

$$(3.21)\qquad D_\ell(\hat{u}-u)\leqq D_\ell(\hat{u}'_{\natural s}-u)+D_\ell(\hat{u}'_{s'}-u).$$

This inequality and Lemma 3.5 imply that

$$(3.22)\qquad D_{s+s'}(\hat{u}-u)\leqq D_{\natural s}(\hat{u}_{\natural s}-u)+D_{s'}(\hat{u}'_{s'}-u)\leqq I[\varphi_j(\natural s)]+I[\varphi_k(s')].$$

Let (s, s', ℓ) be a triple for a major simplex s. Then, by Lemma 3.5

$$(3.23)\qquad D_s(\hat{u}-u)\leqq I[\varphi_j(\natural s)]+D_\ell(\hat{u}-u)$$

and

$$(3.24)\qquad D_{s'}(\hat{u}-u)\leqq I[\varphi_k(s')].$$

Let

$$\hat{u}=ax+by+c\qquad\text{on}\quad\varphi_j(\natural s),\quad\text{and}$$

$$\hat{u}=\alpha\xi+\beta\eta+\gamma\qquad\text{on}\quad\varphi_k(s'),$$

where a, b, c, α, β and γ are constants. Then we define functions $\hat{u}_s$ and $\hat{u}_{s'+\ell}$ on s and $s'+\ell$ respectively by

$$\hat{u}_s = ax+by+c \qquad \text{on} \quad \varphi_j(s), \quad \text{and}$$

$$\hat{u}_{s'+\ell} = \alpha\xi+\beta\eta+\gamma \qquad \text{on} \quad \varphi_k(s'+\ell).$$

Then, by Lemma 2.1

$$D_\ell(\hat{u}-u) \leqq D_\ell(\hat{u}_s-u)+D_\ell(\hat{u}_{s'+\ell}-u). \tag{3.25}$$

Further, by Lemma 3.6

$$D_\ell(\hat{u}_s-u) \leqq A(\varphi_j(\ell))\cdot\frac{32h^2}{\sin^2\theta}\cdot\max_{\varphi_j(s)}\left(\left|\frac{\partial^2 u}{\partial x^2}\right|+2\left|\frac{\partial^2 u}{\partial x\partial y}\right|+\left|\frac{\partial^2 u}{\partial y^2}\right|\right)^2(1+\kappa h)^2 \tag{3.26}$$

and

$$D_\ell(\hat{u}_{s'+\ell}-u) \leqq A(\varphi_k(\ell))\cdot\frac{32h^2}{\sin^2\theta}\cdot\max_{\varphi_k(s'+\ell)}\left(\left|\frac{\partial^2 u}{\partial \xi^2}\right|+2\left|\frac{\partial^2 u}{\partial \xi\partial \eta}\right|+\left|\frac{\partial^2 u}{\partial \eta^2}\right|\right)^2(1+\kappa h)^2. \tag{3.27}$$

By (3.18)~(3.27), Lemma 1.1 and (1.1), the estimate (3.17) is obtained.

8. Approximation by u'_h.

THEOREM 3.2. *Let u be the harmonic solution in $\mathfrak{F}$ defined in* §3.1, *let u'_h be the finite element approximation of u in S' and let $u_h=F^{-1}(u'_h)$.*

(i) *The estimate*

$$D(u_h-u) \leqq \frac{h^2}{\sin^2\theta}\Big(A'\sum_{j=1}^{m}\iint_{\varphi_j(K'_j)}\Big(\Big(\frac{\partial^2 u}{\partial x^2}\Big)^2+2\Big(\frac{\partial^2 u}{\partial x\partial y}\Big)^2+\Big(\frac{\partial^2 u}{\partial y^2}\Big)^2\Big)dxdy$$
$$+B'h^2\sum_{j=1}^{m}\max_{\varphi_j(R_j)}\Big(\Big(\frac{\partial^2 u}{\partial x^2}\Big)^2+2\Big(\frac{\partial^2 u}{\partial x\partial y}\Big)^2+\Big(\frac{\partial^2 u}{\partial y^2}\Big)^2\Big)\Big)$$
$$+C'h^2\sum_{j=1}^{m}\max_{\varphi_j(R_j)}\Big(\Big(\frac{\partial u}{\partial x}\Big)^2+\Big(\frac{\partial u}{\partial y}\Big)^2\Big) \tag{3.28}$$

holds, where A', B' and C' are constants independent of the triangulation K and the function and other notations are the same as in Theorem 3.1.

(ii) *The estimate*

$$D(u) \leqq D(u'_h)+\varepsilon(u'_h) \tag{3.29}$$

holds with

$$\varepsilon(u'_h) \equiv \sum_{j=1}^{m}\sum_{\sharp\ell\in K_j} A(\varphi_j(\sharp\ell))\cdot\frac{(u'_h(q_2)-u'_h(q_1))^2}{|\varphi_j(q_2)-\varphi_j(q_1)|^2}$$
$$\cdot\max\left\{1,\ \frac{|\varphi_j(q_2)-\varphi_j(q_1)|^2}{|\varphi_k(q_2)-\varphi_k(q_1)|^2}\max_{\varphi_j(\sharp\ell)}|f'(z)|^2\right\},$$

where q_1 and q_2 are the vertices of $\sharp\ell$, $f(z)\equiv\varphi_k\circ\varphi_j^{-1}(z)$, and $\varepsilon(u'_h)$ is a quantity of

$O(h^2)$ which can be numerically calculated.

PROOF. (i) First, note that

$$D(u_h-u) \leqq 2D(\omega_h-u)+2D(u_h-\omega_h). \tag{3.30}$$

From Lemmas 2.1, 2.2 and 3.3, and (3.11), it follows that

$$\begin{aligned}(3.31)\quad D(u_h-\omega_h) &= D(u_h)-D(\omega_h)\\ &\leqq D(u'_h)-D(\omega_h)+\sum_{\sharp\ell\in K} D_{\sharp\ell}(u_h) \leqq D(\omega'_h)-D(\omega_h)+\sum_{\sharp\ell\in K} D_{\sharp\ell}(u_h)\\ &\leqq \sum_{j=1}^{m}\sum_{\flat\ell\in K_j}(A(\varphi_j(\flat\ell))\cdot(a'^2+b'^2)+A(\varphi_k(\flat\ell))\cdot(\alpha'^2+\beta'^2))\\ &\quad+\sum_{j=1}^{m}\sum_{\sharp\ell\in K_j}(A(\varphi_j(\sharp\ell))\cdot(a^2+b^2)+A(\varphi_k(\sharp\ell))\cdot(\alpha^2+\beta^2)),\end{aligned}$$

where for each triple $(s, s', \flat\ell)$ for $\flat\ell\in K_j$

$$\omega'_h = a'x+b'y+c' \quad \text{on} \quad \varphi_j(\natural s) \quad \text{and}$$
$$\omega'_h = \alpha'\xi+\beta'\eta+\gamma' \quad \text{on} \quad \varphi_k(s'),$$

and for each triple $(s, s', \sharp\ell)$ for $\sharp\ell\in K_j$

$$u_h = ax+by+c \quad \text{on} \quad \varphi_j(\natural s) \quad \text{and}$$
$$u_h = \alpha\xi+\beta\eta+\gamma \quad \text{on} \quad \varphi_k(s')$$

with constants $a', b', c', \alpha', \beta', \gamma', a, b, c, \alpha, \beta$ and γ.

In the inequality (3.31), we have

$$\begin{aligned}(3.32)\quad A(\varphi_j(\flat\ell))\cdot(a'^2+b'^2) &= \frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(s))}D_s(\omega_h) \leqq 2\frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(s))}(D_s(\omega_h-u)+D_s(u))\\ &\leqq 2\frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(s))}D_s(\omega_h-u)+2A(\varphi_j(\flat\ell))\cdot\max_{\varphi_j(s)}\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right).\end{aligned}$$

Since we can easily verify that

$$A(\varphi_j(\natural s)) > \frac{h_1^2}{4}\sin\theta \qquad (h_1=d(\varphi_j(\natural s))),$$

by Lemma 1.1 we have

$$\frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(s))} = \frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(\natural s))-A(\varphi_j(\flat\ell))} \leqq \frac{h}{2\sin\theta}\left(\left|\frac{g''(\zeta_1)}{g'(\zeta_1)^2}\right|+O(h)\right) \tag{3.33}$$

with the notations in Lemma 1.1. (3.32) and (3.33) imply

$$\begin{aligned}(3.34)\quad &\sum_{j=1}^{m}\sum_{\flat\ell\in K_j}A(\varphi_j(\flat\ell))\cdot(a'^2+b'^2)\\ &\leqq \frac{Ch}{\sin\theta}\sum_{j=1}^{m}\sum_{\flat\ell\in K_j}D_s(\omega_h-u)+2\sum_{j=1}^{m}\sum_{\flat\ell\in K_j}A(\varphi_j(\flat\ell))\cdot\max_{\varphi_j(s)}\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right),\end{aligned}$$

where C is a constant depending only on the transformations of local parameters. Since similar estimates for other terms of the right hand side of (3.31) are obtained, from (3.31) it follows that

$$(3.35)\qquad D(u_h-\omega_h)\leqq \frac{Ch}{\sin\theta}D(u_h-u)+\frac{Ch}{\sin\theta}D(\omega_h-u)$$
$$+2\sum_{j=1}^{m}\sum_{\ell\in K_j}\Big(A(\varphi_j(\ell))\cdot\max_{\varphi_j(s)}\Big(\Big(\frac{\partial u}{\partial x}\Big)^2+\Big(\frac{\partial u}{\partial y}\Big)^2\Big)$$
$$+A(\varphi_k(\ell))\cdot\max_{\varphi_k(s')}\Big(\Big(\frac{\partial u}{\partial \xi}\Big)^2+\Big(\frac{\partial u}{\partial \eta}\Big)^2\Big)\Big).$$

(3.30), (3.35), Theorem 3.1, Lemma 1.1 and (1.1) imply the estimate (3.28).

(ii) Lemmas 3.2 (ii), 3.3 and 2.2 (i) imply the inequalities

$$D(u)\leqq D(\omega_h)\leqq D(u_h)$$
$$\leqq D(u'_h)+\sum_{j=1}^{m}\sum_{\sharp\ell\in K_j}A(\varphi_j(\sharp\ell))\frac{(u'_h(q_2)-u'_h(q_1))^2}{|\varphi_j(q_2)-\varphi_j(q_1)|^2}$$
$$\cdot\max\Big\{1,\ \frac{|\varphi_j(q_2)-\varphi_j(q_1)|^2}{|\varphi_k(q_2)-\varphi_k(q_1)|^2}\max_{\varphi_j(\sharp\ell)}|f'(z)|^2\Big\}.$$

9. Estimate of $D(u'_h-\hat{u}')$.

COROLLARY 3.1. *Let u and u'_h be the same as in Theorem 3.2, $\hat{u}$ be the finite element interpolation of u in the class S, and $\hat{u}'=F(\hat{u})$. Then, the estimate*

$$(3.36)\qquad D(u'_h-\hat{u}')\leqq A''h^2$$

holds, where A'' is a constant dependent only on u and θ in Theorem 3.1.

PROOF. First, by Lemma 2.2 (ii) and (3.33) we have

$$D(u'_h-\hat{u}')\leqq D(u_h-\hat{u})+\sum_{\flat\ell\in K}(D_{\flat\ell}(u'_{h,\natural s}-\hat{u}'_{\natural s})+D_{\flat\ell}(u'_{h,s'}-\hat{u}'_{s'}))$$
$$\leqq D(u_h-\hat{u})+\sum_{j=1}^{m}\sum_{\flat\ell\in K_j}\Big(\frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(s))}D_s(u_h-\hat{u})$$
$$+\frac{A(\varphi_k(\flat\ell))}{A(\varphi_k(s'))-A(\varphi_k(\flat\ell))}D_{s'}(u_h-\hat{u})\Big)$$
$$\leqq D(u_h-\hat{u})+\frac{Ch}{\sin\theta}\sum_{j=1}^{m}\sum_{\flat\ell\in K_j}(D_s(u_h-\hat{u})+D_{s'}(u_h-\hat{u}))$$
$$\leqq\Big(1+\frac{Ch}{\sin\theta}\Big)D(u_h-\hat{u})\leqq 2\Big(1+\frac{Ch}{\sin\theta}\Big)(D(u_h-u)+D(u-\hat{u})),$$

where C is the same constant as in (3.34). Then, the proof of Theorem 3.1 and Theorem 3.2 imply (3.36).

§ 4. Applications.

1. Modulus of a quadrilateral. By (3.1) and Lemma 3.1 the equalities

$$D(u) = M(Q) = \frac{1}{M(\tilde{Q})} = \frac{1}{D(\tilde{u})} \tag{4.1}$$

hold, where $u=\mathrm{Re}\,\mathfrak{f}$ and $\tilde{u}=\mathrm{Re}\,\tilde{\mathfrak{f}}$.

When we replace C_0 and C_1 by $\tilde{C}_0$ and $\tilde{C}_1$ respectively in the definition of the classes $\mathfrak{F}$, S and S' of functions, we obtain new classes $\tilde{\mathfrak{F}}$, $\tilde{S}$ and $\tilde{S}'$ corresponding to $\mathfrak{F}$, S and S' respectively. Let u_h' and $\tilde{u}_h'$ be the finite element approximations of u and $\tilde{u}$ in the spaces S' and $\tilde{S}'$ respectively. Then by (ii) of Theorem 3.2 we have the estimates

$$D(u) \leqq D(u_h') + \varepsilon(u_h') \tag{4.2}$$

and

$$D(\tilde{u}) \leqq D(\tilde{u}_h') + \varepsilon(\tilde{u}_h'). \tag{4.3}$$

By (4.1), (4.2) and (4.3) we have upper and lower bounds for the modulus $M(Q)$:

$$\frac{1}{D(\tilde{u}_h') + \varepsilon(\tilde{u}_h')} \leqq M(Q) \leqq D(u_h') + \varepsilon(u_h'). \tag{4.4}$$

2. Numerical example 1 (the example of Gaier [**9**]). Let Ω be the simply connected domain on the z-plane defined by

$$\Omega = \{z \mid 0<x<1,\ 0<y<1\} - \left\{z \,\middle|\, \frac{1}{2} \leqq x < 1,\ \frac{1}{2} \leqq y < 1\right\},$$

and let C_0 and C_1 be the boundary parts of Ω defined by

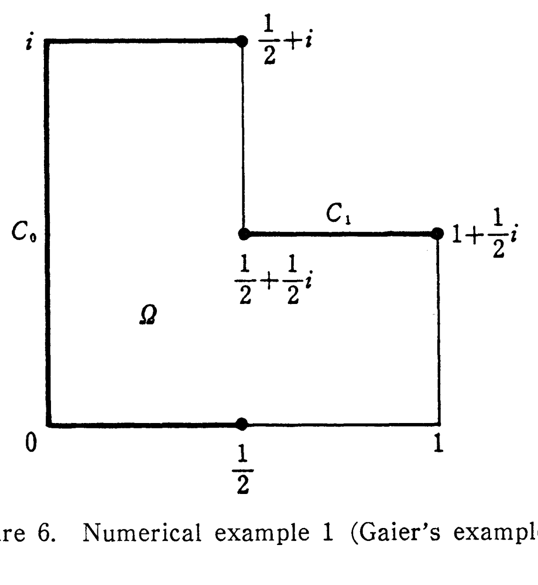


Figure 6. Numerical example 1 (Gaier's example).

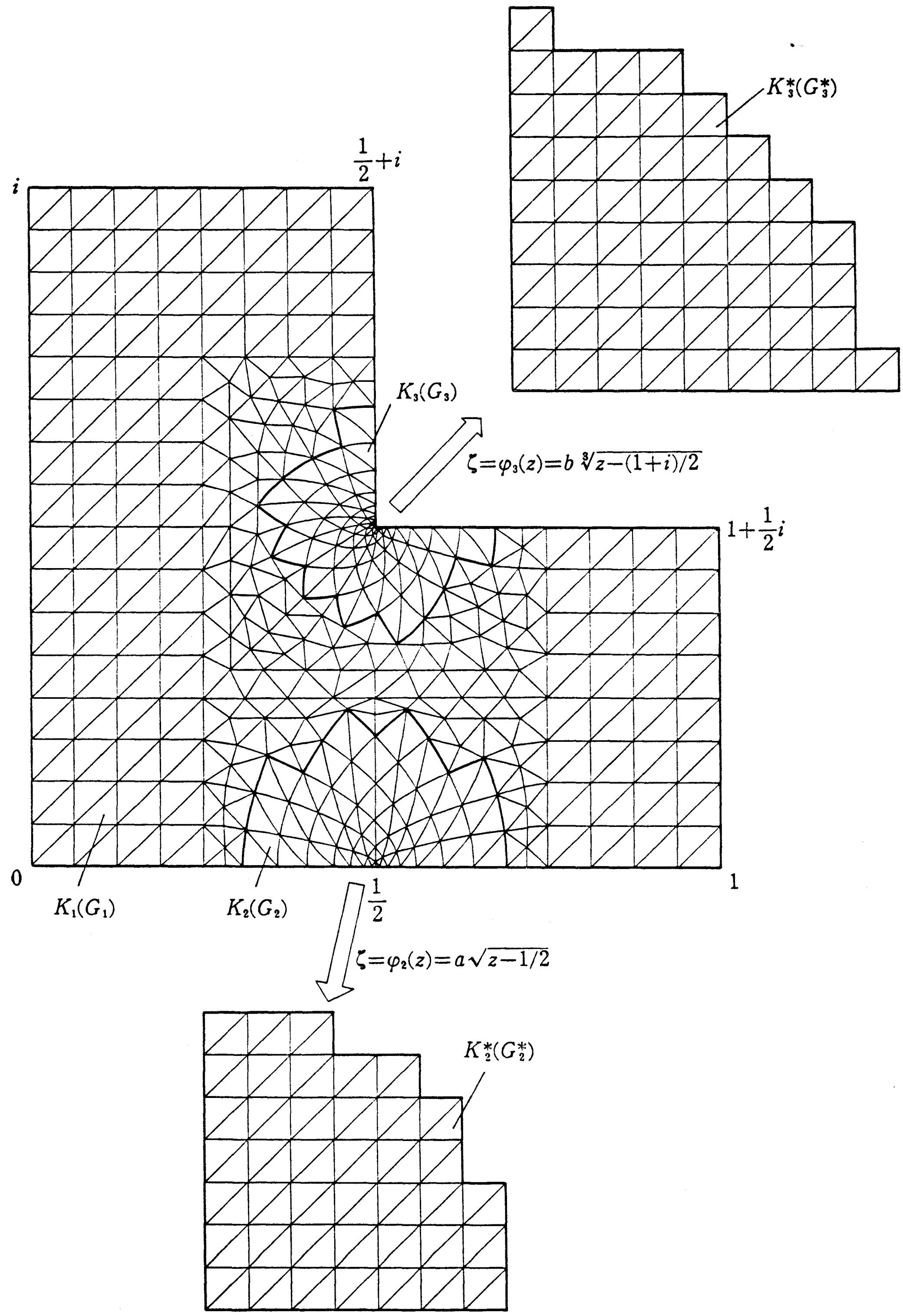


Figure 7. Triangulation of example 1.

$$C_0 = \left\{z \,\middle|\, 0 \leqq x \leqq \frac{1}{2},\ y=0\right\} \cup \{z \mid x=0,\ 0 \leqq y \leqq 1\} \cup \left\{z \,\middle|\, 0 \leqq x \leqq \frac{1}{2},\ y=1\right\}$$

and

$$C_1 = \left\{z \,\middle|\, \frac{1}{2} \leqq x \leqq 1,\ y=\frac{1}{2}\right\}$$

respectively, where $z=x+iy$. Let Q be the quadrilateral with the two opposite sides C_0 and C_1 (cf. Figure 6). We aim to obtain good upper and lower approximate values of the modulus of Q.

We construct a triangulation of the closed region $\bar{\Omega}$ as in Figure 7. The closed regions G_2 and G_3 are mapped onto the regions G_2^* and G_3^* respectively by the local parameters $\zeta=\varphi_2(z)=a\sqrt{z-1/2}$ and $\zeta=\varphi_3(z)=b\sqrt[3]{z-(1+i)/2}$ ($a=1$ and $b=e^{-\pi i/6}$) respectively, where a and b are so determined that $|d\zeta/dz|=1$ on $|z-1/2|=1/4$ and $|z-(1+i)/2|=1/\sqrt{27}$ respectively. We construct ordinary triangulations K_2^* and K_3^* of G_2^* and G_3^* as in Figure 7 respectively. By K_2 and K_3 we denote the image triangulations of K_2^* and K_3^* by the mappings φ_2^{-1} and φ_3^{-1} respectively. The triangulation K_1 of the region $G_1=\overline{\Omega-(G_2\cup G_3)}$ in Figure 7 is so constructed that each 2-simplex s of K_1 is natural or minor according as $|s|\cap|K_2+K_3|=\emptyset$ or $|s|\cap|K_2+K_3|\neq\emptyset$, where if some intersection is a point then it is interpreted to be vacuous, and the local parameter $\varphi_1(z)$ of K_1 is the identity mapping $\varphi_1(z)\equiv z$.

Let u and $\tilde{u}$ be the functions on the present Ω defined in § 4.1, and let u_h' and $\tilde{u}_h'$ be the finite element approximations of u and $\tilde{u}$ respectively in the classes $S'(K')$ and $\tilde{S}'(K')$ respectively, where K' is the naturalized triangulation associated to the present K. To attain our aim it is sufficient to make numerical calculations of u_h' and $\tilde{u}_h'$ (cf. Mizumoto and Hara [**13**], [**14**] for the calculation method).

Table 1 shows the exact value of the modulus $M(Q)$ (see Gaier [**9**] for the calculation method), Gaier's computation results and the values of our finite element approximations. Furthermore, computation results for the normal subdivision K^1 (see Figure 8) of the present K are shown. We note that $\varepsilon(u_h')=\varepsilon(\tilde{u}_h')=0$ in the present example. It can be said that the both of upper and lower bounds of $M(Q)$ by our method are much closer to the exact value than those by Gaier.

3. Numerical example 2 (the case of a Riemann surface). Let $D_1=\{z\mid |z|<\infty\}-\{z\mid 0\leqq x<\infty,\ y=0\}$ and C_0 be the upper boundary part of D_1 lying on $\{z\mid 1\leqq x<\infty,\ y=0\}$, where $z=x+iy$. Let $D_2=\{z\mid |z|<1\}-\{z\mid 0\leqq x<1,\ y=0\}$ and let C_1 be the boundary part of D_2 defined by $C_1=\{z\mid |z|=1,\ y\geqq 0\}$. Let Ω be the simply-connected covering surface obtained by connecting D_1 and D_2 crosswise along the segment $\{z\mid 0\leqq x<1,\ y=0\}$ (cf. Figure 9). Let Q be the quadrilateral with the opposite sides C_0 and C_1. By symmetricity of Q we

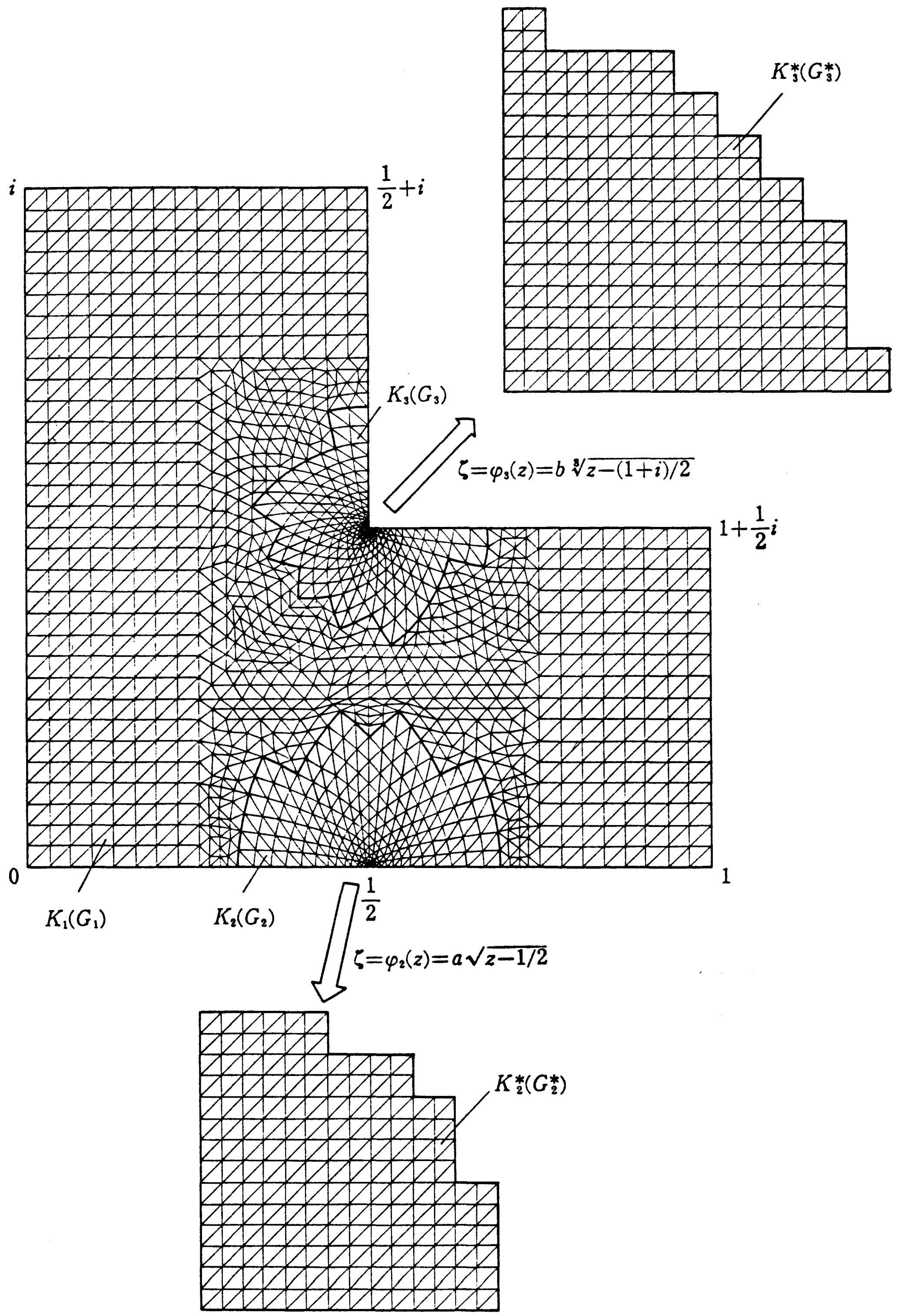


Figure 8. Normal subdivision of example 1.

Table 1. Modulus $M(Q)$ of example 1 (the example of Gaier [**9**]).

Exact value	$M(Q)=D(u)=1.279262$		
Gaier's computation results (Gaier [**9**])	$h=2^{-4}$	Upper bound$=1.49435$ (0.21509) Lower bound$=1.09543$ (−0.18383)	
	$h=2^{-7}$	Upper bound$=1.32659$ (0.04733) Lower bound$=1.23368$ (−0.04558)	
Our computation results	Original triangulation ($h=2^{-4}$)		
	Upper bound	$D(u'_h)+\varepsilon(u'_h)$ $=1.28396+0$ $=1.28396$ (0.00470)	$D(u'_h-\hat{u}')$ $=1.65238\times10^{-4}$
	Lower bound	$\dfrac{1}{D(\tilde{u}'_h)+\varepsilon(\tilde{u}'_h)}$ $=\dfrac{1}{0.783599+0}$ $=1.27616$ (−0.00310)	$D(\tilde{u}'_h-\hat{\tilde{u}}')$ $=5.26377\times10^{-5}$
	Normal subdivision ($h=2^{-5}$)		
	Upper bound	$D(u'_h)+\varepsilon(u'_h)$ $=1.28046+0$ $=1.28046$ (0.00120)	$D(u'_h-\hat{u}')$ $=1.51604\times10^{-5}$
	Lower bound	$\dfrac{1}{D(\tilde{u}'_h)+\varepsilon(\tilde{u}'_h)}$ $=\dfrac{1}{0.782185+0}$ $=1.27847$ (−0.00079)	$D(\tilde{u}'_h-\hat{\tilde{u}}')$ $=4.77743\times10^{-6}$

( ): Deviation from exact value.

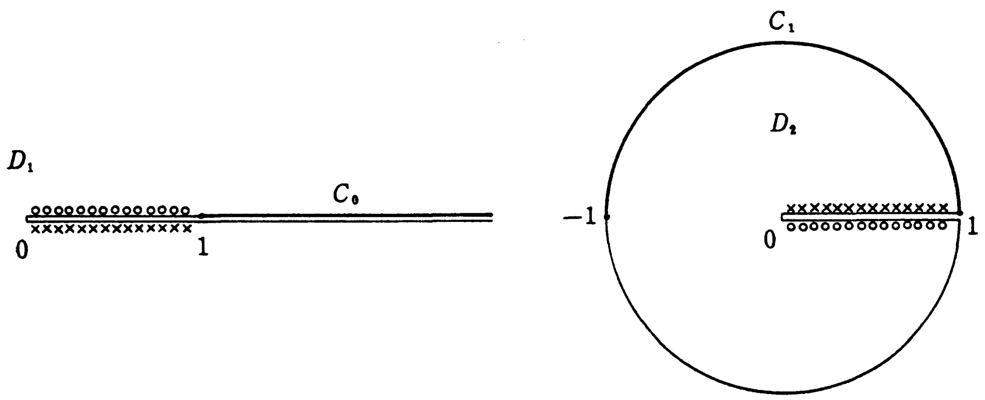


Figure 9. Numerical example 2 (the case of a Riemann surface).

immediately see that $M(Q)=1$. We aim to obtain good upper and lower approximate values of $M(Q)$. The present example is one which exhibits remarkable validity of our method. Namely, it is shown that an unbounded covering surface over the z-plane with many inner and corner singularities of high order, and with a curvilinear boundary is dealt with by our local treatment method without use of any global conformal mapping.

We construct a triangulation of the bordered region $\bar{\Omega}$ as in Figures 10 and 11. In Figure 10, the closed regions $G_1\cup G_2\cup\cdots\cup G_5$, $G_6\cup G_7$ and G_9 are mapped onto the regions $G_1^*\cup G_2^*\cup\cdots\cup G_5^*$, $G_6^*\cup G_7^*$ and G_9^* respectively by the mappings $\zeta=\varphi_1(z)=(1/4)\cdot\log z$, $\zeta=\varphi_6(z)=1/z$ and $\zeta=\varphi_9(z)=\sqrt{z}$ respectively. Further, the regions G_3^*, G_4^*, G_5^* and G_7^* are mapped onto the regions G_3^{**}, G_4^{**}, G_5^{**} and G_7^{**} respectively by the mappings $Z=\phi_3(\zeta)=\sqrt[3]{\zeta}$, $Z=\phi_4(\zeta)=e^{-\pi i/6}\cdot\sqrt[3]{\zeta-\pi i/2}$, $Z=\phi_5(\zeta)=e^{-\pi i/4}\cdot\sqrt{\zeta-3\pi i/4}$ and $Z=\phi_7(\zeta)=\sqrt{2}\sqrt[4]{\zeta}$ respectively. Let $\varphi_3(z)=\phi_3\circ\varphi_1(z)$, $\varphi_4(z)=\phi_4\circ\varphi_1(z)$, $\varphi_5(z)=\phi_5\circ\varphi_1(z)$ and $\varphi_7(z)=\phi_7\circ\varphi_6(z)$. We note that $|d\varphi_1/dz|=1$ on $|z|=1/4$, $|d\phi_3/d\zeta|=1$ on $|\zeta|=1/\sqrt{27}$, $|d\phi_4/d\zeta|=1$ on $|\zeta-\pi i/2|=1/\sqrt{27}$, $|d\phi_5/d\zeta|=1$ on $|\zeta-3\pi i/4|=1/4$, $|d(\varphi_6\circ\varphi_1^{-1})/d\zeta|=1$ on $\mathrm{Re}\,\zeta=(1/4)\log 4$, $|d\phi_7/d\zeta|=1$ on $|\zeta|=1/4$ and $|d\varphi_9/dz|=1$ on $|z|=1/4$. We construct ordinary triangulations K_3^{**}, K_4^{**}, K_5^{**}, K_7^{**} and K_9^* of G_3^{**}, G_4^{**}, G_5^{**}, G_7^{**} and G_9^* as in Figure 11 respectively. By K_3, K_4, K_5, K_7 and K_9 we denote the image triangulations of K_3^{**}, K_4^{**}, K_5^{**}, K_7^{**} and K_9^* by the mappings φ_3^{-1}, φ_4^{-1}, φ_5^{-1}, φ_7^{-1} and φ_9^{-1} respectively, and the local parameters of K_3, K_4, K_5, K_7 and K_9 are $Z=\varphi_3(z)$, $Z=\varphi_4(z)$, $Z=\varphi_5(z)$, $Z=\varphi_7(z)$ and $\zeta=\varphi_9(z)$ respectively. The triangulations K_1 and K_2 of G_1 and G_2 respectively in Figure 11 are so constructed that each 2-simplex s of K_1 and K_2 is natural or minor according as $|s|\cap|K_3+K_4+K_5|=\emptyset$ or $|s|\cap|K_3+K_4+K_5|\neq\emptyset$, where the local parameter of K_1+K_2 is $\zeta=\varphi_1(z)$. Also the triangulation K_6 of G_6 is so constructed that each 2-simplex s of K_6 is natural, minor or major according as $|s|\cap|K_1+K_7|=\emptyset$, $|s|\cap|K_7|\neq\emptyset$ or $|s|\cap|K_1|\neq\emptyset$, where the local parameter of K_6 is $\zeta=\varphi_6(z)$. Further, the triangulation K_8 of G_8 is so constructed that each 2-simplex s of K_8 is natural, minor or major according as $|s|\cap|K_1+K_2+K_9|=\emptyset$, $|s|\cap|K_9|\neq\emptyset$ or $|s|\cap|K_1+K_2|\neq\emptyset$, where the local parameter of K_8 is the identity mapping $\varphi_8(z)\equiv z$.

Let u and $\tilde{u}$ be the functions on the present Ω defined in § 4.1, and let u'_h and $\tilde{u}'_h$ be the finite element approximations of u and $\tilde{u}$ respectively in the classes $S'(K')$ and $\tilde{S}'(K')$ respectively, where K' is the naturalized triangulation associated to the present K. To attain our aim it is sufficient to make numerical calculations of u'_h and $\tilde{u}'_h$.

Now the function u is obtained by the following procedure. Let Δ be the rectangular domain

$$\Delta=\{W\,|\,0<\mathrm{Re}\,W<1,\ 0<\mathrm{Im}\,W<1\},$$

and let Γ_0 and Γ_1 be the boundary parts of Δ defined by

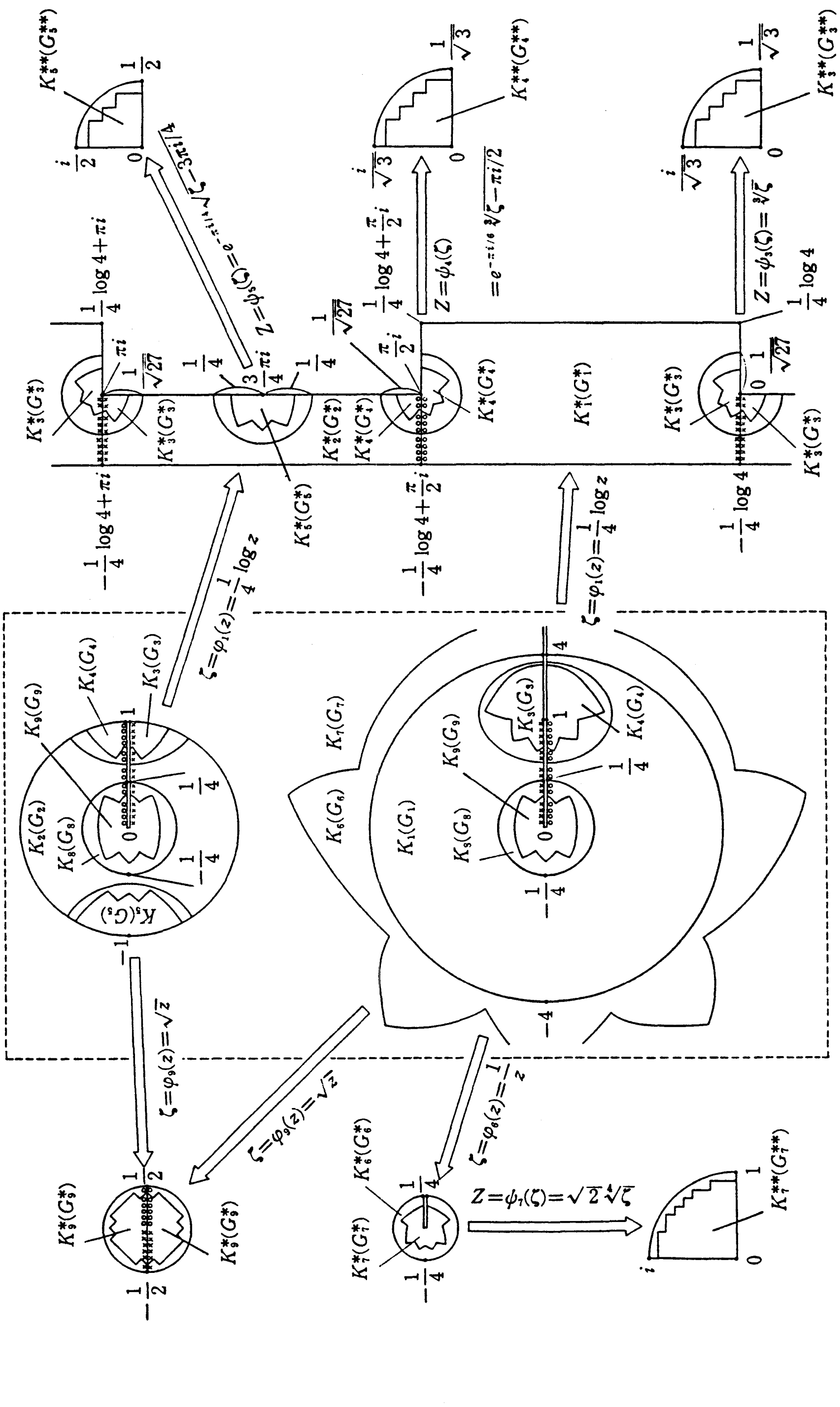


Figure 10. Local parameters of example 2.

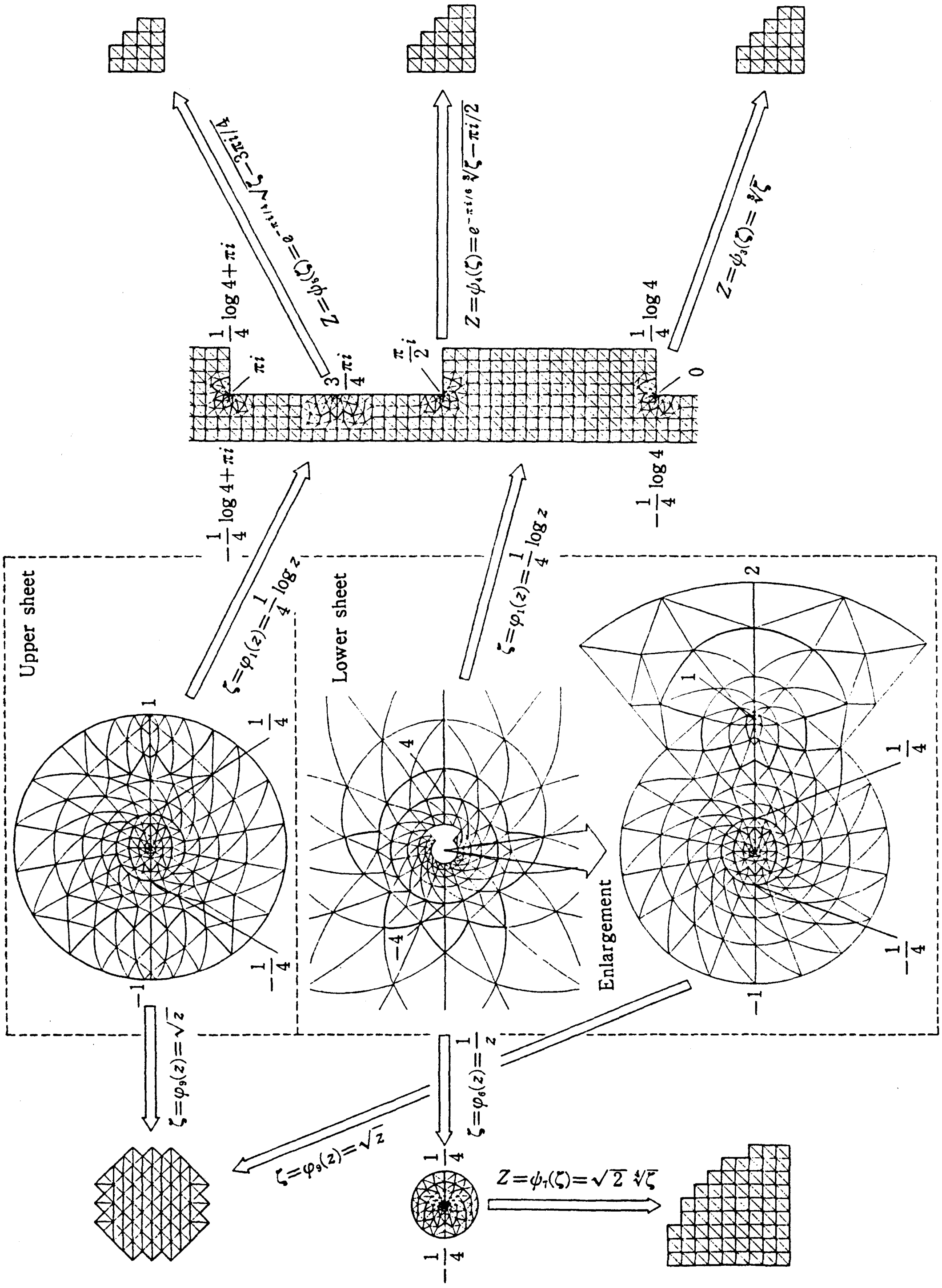


Figure 11. Triangulation of example 2. (Note: The scales are identical in all figures except the lower sheet extended to infinity.)

and

$$\Gamma_0 = \{W \mid 0 \leqq \mathrm{Im}\,W \leqq 1,\ \mathrm{Re}\,W = 0\}$$

$$\Gamma_1 = \{W \mid 0 \leqq \mathrm{Im}\,W \leqq 1,\ \mathrm{Re}\,W = 1\}.$$

The conformal map $W=\mathfrak{f}(p)$ such that Ω is conformally mapped onto Δ so that C_0 and C_1 are mapped onto Γ_0 and Γ_1 respectively, is constructed by the composition of the following mappings, and then $u=\mathrm{Re}\,\mathfrak{f}(p)$:

(i) $w=\sqrt{z}$;

(ii) $\zeta=\left(\dfrac{w-1}{w+1}\right)^{2/3}$;

(iii) $\dfrac{Z-Z_1}{Z-Z_2}\cdot\dfrac{Z_3-Z_2}{Z_3-Z_1}=\dfrac{\zeta-\zeta_1}{\zeta-\zeta_2}\cdot\dfrac{\zeta_3-\zeta_2}{\zeta_3-\zeta_1}$,

where $\zeta_1=0$, $\zeta_2=-1$, $\zeta_3=1$, $Z_1=1$, $Z_2=-1$ and $Z_3=1/k$ with $1/k=3+2\sqrt{2}$;

(iv) $W=-\dfrac{1}{2K}\left(\displaystyle\int_0^Z \frac{dZ}{\sqrt{(1-Z^2)(1-k^2Z^2)}}-(K+iK')\right)$,

where $K=K(k)$ and $K'=K'(k)$ are the complete elliptic integrals.

Table 2 shows the values of our finite element approximations. Furthermore,

Table 2. Modulus $M(Q)$ of example 2 (the case of a Riemann surface).

Exact value	$M(Q)=D(u)=1.0$		
Finite element approximations	Original triangulation ($h=0.141421$)		
	Upper bound	$D(u'_h)+\varepsilon(u'_h)$ $=1.00484+0.103287\times10^{-2}$ $=1.00587$ (0.00587)	$D(u'_h-\hat{u}')$ $=3.53832\times10^{-4}$
	Lower bound	$\dfrac{1}{D(\tilde{u}'_h)+\varepsilon(\tilde{u}'_h)}$ $=\dfrac{1}{1.00484+0.103287\times10^{-2}}$ $=0.994164$ (-0.005836)	$D(\tilde{u}'_h-\hat{u}')$ $=3.53824\times10^{-4}$
	Normal subdivision ($h=0.0707107$)		
	Upper bound	$D(u'_h)+\varepsilon(u'_h)$ $=1.00128+0.255952\times10^{-3}$ $=1.00154$ (0.00154)	$D(u'_h-\hat{u}')$ $=3.42089\times10^{-5}$
	Lower bound	$\dfrac{1}{D(\tilde{u}'_h)+\varepsilon(\tilde{u}'_h)}$ $=\dfrac{1}{1.00128+0.255957\times10^{-3}}$ $=0.998466$ (-0.001534)	$D(\tilde{u}'_h-\hat{u}')$ $=3.42716\times10^{-5}$

( ): Deviation from exact value.

computation results for the normal subdivision K^1 of the present K are shown. It can be said that the both of upper and lower bounds of $M(Q)$ are close to the exact values.

4. **Numerical example 3** (the case of an unbounded domain; cf. § 4.3 of [**15**]). Let $\Omega=\{z|y>0\}$, and let C_0 and C_1 be the boundary parts of Ω defined by $C_0=\{z|-3\leqq x\leqq -1, y=0\}$ and $C_1=\{z|1\leqq x\leqq 3, y=0\}$ respectively, where $z=x+iy$. Let Q be the quadrilateral with the two opposite sides C_0 and C_1 (cf. Figure 12). We obtain good upper and lower approximate values of the modulus of Q. See § 4.3 of [**15**] for the details. Table 3 shows the exact value of the modulus $M(Q)$ which can be calculated by making use of a complete elliptic integral, and the values of our finite element approximations.

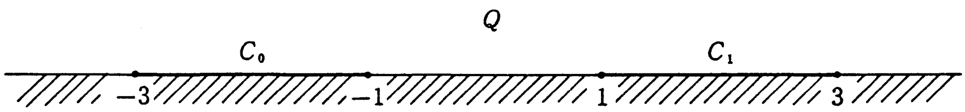


Figure 12. Numerical example 3 (the case of an unbounded domain).

Table 3. Modulus $M(Q)$ of example 3 (the case of an unbounded domain).

<table>
<tr><td>Exact value</td><td colspan="3">$M(Q)=D(u)=0.781701$</td></tr>
<tr><td rowspan="6">Finite element approximations</td><td colspan="3">Original triangulation ($h=0.213758$)</td></tr>
<tr><td>Upper bound</td><td>$D(u'_h)+\varepsilon(u'_h)$
$=0.782184+0.429347\times 10^{-3}$
$=0.782613$ (0.000912)</td><td>$D(u'_h-\hat{u}')$
$=1.41568\times 10^{-5}$</td></tr>
<tr><td>Lower bound</td><td>$\dfrac{1}{D(\tilde{u}'_h)+\varepsilon(\tilde{u}'_h)}$
$=\dfrac{1}{1.280878+0.150405\times 10^{-5}}$
$=0.780714$ (−0.000987)</td><td>$D(\tilde{u}'_h-\hat{u}')$
$=3.77307\times 10^{-5}$</td></tr>
<tr><td colspan="3">Normal subdivision ($h=0.106879$)</td></tr>
<tr><td>Upper bound</td><td>$D(u'_h)+\varepsilon(u'_h)$
$=0.781968+0.107413\times 10^{-3}$
$=0.782075$ (0.000374)</td><td>$D(u'_h-\hat{u}')$
$=1.25553\times 10^{-6}$</td></tr>
<tr><td>Lower bound</td><td>$\dfrac{1}{D(\tilde{u}'_h)+\varepsilon(\tilde{u}'_h)}$
$=\dfrac{1}{1.279506+0.381486\times 10^{-6}}$
$=0.781551$ (−0.000150)</td><td>$D(\tilde{u}'_h-\hat{u}')$
$=3.37903\times 10^{-6}$</td></tr>
</table>

( ): Deviation from exact value.

5. Numerical example 4 (the case of a curvilinear domain; cf. § 4.4 of [**15**]). Let

$$\Omega = \left\{ z \,\middle|\, \frac{x^2}{16} + \frac{y^2}{15} < 1,\ y > 0 \right\},$$

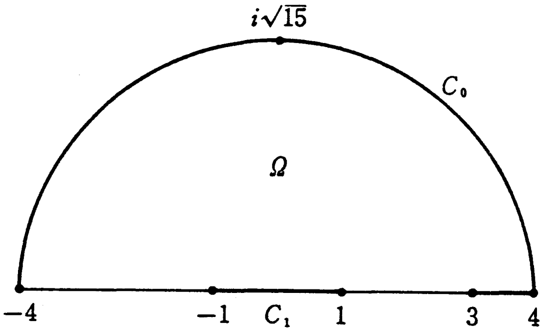


Figure 13. Numerical example 4 (the case of a curvilinear domain: quadrilateral Q).

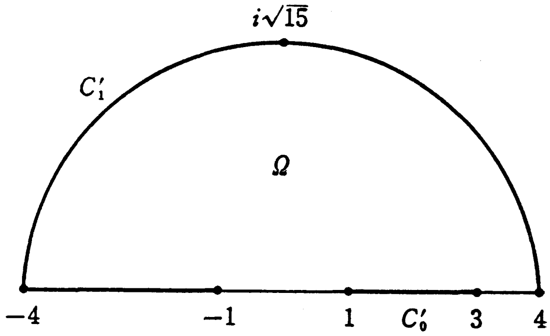


Figure 14. Numerical example 4 (the case of a curvilinear domain: quadrilateral Q').

Table 4. Modulus $M(Q)$ of example 4 (the case of a curvilinear domain).

Exact value	$M(Q)=D(u)=1.539330$		
Finite element approximation	Original triangulation ($h=0.138840$)		
	Upper bound	$D(u_h')+\varepsilon(u_h')$ $=1.540588+0.572262\times10^{-4}$ $=1.540645$ (0.00132)	$D(u_h'-\hat{u}')$ $=1.33022\times10^{-4}$
	Lower bound	$\dfrac{1}{D(\tilde{u}_h')+\varepsilon(\tilde{u}_h')}$ $=\dfrac{1}{0.649700+0.225117\times10^{-3}}$ $=1.538639$ (−0.00069)	$D(\tilde{u}_h'-\hat{\tilde{u}}')$ $=1.39974\times10^{-5}$
	Normal subdivision ($h=0.069420$)		
	Upper bound	$D(u_h')+\varepsilon(u_h')$ $=1.539652+0.142916\times10^{-4}$ $=1.539666$ (0.00034)	$D(u_h'-\hat{u}')$ $=3.47448\times10^{-5}$
	Lower bound	$\dfrac{1}{D(\tilde{u}_h')+\varepsilon(\tilde{u}_h')}$ $=\dfrac{1}{0.649652+0.558093\times10^{-4}}$ $=1.539153$ (−0.00018)	$D(\tilde{u}_h'-\hat{\tilde{u}}')$ $=1.19267\times10^{-6}$

( ) : Deviation from exact value.

and let C_0 and C_1 be the boundary parts of Ω defined by

$$C_0=\{z\,|\,3\leqq x\leqq 4,\ y=0\}\cup\left\{z\,\middle|\,\frac{x^2}{16}+\frac{y^2}{15}=1,\ y\geqq 0\right\}$$

and

$$C_1=\{z\,|\,-1\leqq x\leqq 1,\ y=0\}$$

respectively, where $z=x+iy$. Let Q be the quadrilateral with the opposite sides C_0 and C_1 (cf. Figure 13).

Further, let C_0' and C_1' be the boundary parts of Ω defined by

$$C_0'=\{z\,|\,1\leqq x\leqq 3,\ y=0\}$$

and

$$C_1'=\{z\,|\,-4\leqq x\leqq -1,\ y=0\}\cup\left\{z\,\middle|\,\frac{x^2}{16}+\frac{y^2}{15}=1,\ y\geqq 0\right\}$$

respectively, where $z=x+iy$. Let Q' be the quadrilateral with the opposite sides C_0' and C_1' (cf. Figure 14).

We obtain good upper and lower approximate values of the modulus of Q and Q'. See §4.4 of [**15**] for the details. Tables 4 and 5 show the exact

Table 5. Modulus $M(Q')$ of example 4 (the case of a curvilinear domain).

Exact value	$M(Q')=D(u)=1.839350$		
Finite element approximation	Original triangulation ($h=0.138840$)		
	Upper bound	$D(u_h')+\varepsilon(u_h')$ $=1.841976+0.351532\times10^{-3}$ $=1.842328$ (0.00298)	$D(u_h'-\hat{u}')$ $=5.86445\times10^{-5}$
	Lower bound	$\dfrac{1}{D(\tilde{u}_h')+\varepsilon(\tilde{u}_h')}$ $=\dfrac{1}{0.544588+0.145580\times10^{-3}}$ $=1.835760$ (-0.00359)	$D(\tilde{u}_h'-\hat{u}')$ $=2.73084\times10^{-5}$
	Normal subdivision ($h=0.069420$)		
	Upper bound	$D(u_h')+\varepsilon(u_h')$ $=1.840016+0.875764\times10^{-4}$ $=1.840104$ (0.00075)	$D(u_h'-\hat{u}')$ $=5.22641\times10^{-6}$
	Lower bound	$\dfrac{1}{D(\tilde{u}_h')+\varepsilon(\tilde{u}_h')}$ $=\dfrac{1}{0.543904+0.361871\times10^{-4}}$ $=1.838437$ (-0.00091)	$D(\tilde{u}_h'-\hat{u}')$ $=3.00439\times10^{-6}$

( ): Deviation from exact value.

values of the modulus $M(Q)$ and $M(Q')$ respectively (see § 4.4 of [**15**] for the calculation method) and the values of our finite element approximations.

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