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Quasi-periodicity of bounded solutions to some periodic evolution equations

Dedicated to Professor Hiroshi Fujita on the occasion of his sixtieth birthday

By Alain HARAUX and Mitsuharu ÔTANI

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Introduction.

Let H be a real Hilbert space with norm denoted by $\|\cdot\|$ and inner product by $\langle\ ,\ \rangle$. For any $t\in\boldsymbol{R}$, let $A(t): D[A(t)]\to H$ be a maximal monotone operator. We consider the evolution equation

$$u'(t)+A(t)u(t)\ni 0. \tag{0.1}$$

In the sequel, we denote by $(C_t)_{t\in\boldsymbol{R}}$ the closure in H of the domain $D[A(t)]$. It is well known that C_t is *convex* (cf. e.g. [**5**]).

Under several different types of technical assumptions, it is possible to define for any $s\in\boldsymbol{R}$ and any $x\in C_s$ a unique "weak" solution $u(t)$ of (0.1) on $[s, +\infty[$ such that $u(s)=x$. In general, u is not differentiable and is constructed by some approximation procedure (cf. e.g. [**1, 2, 4, 5, 6, 14, 17, 20**]).

In all the cases in which this construction is possible, u is given by the formula

$$\forall t\geqq s, \qquad u(t)=E(s, t)x \tag{0.2}$$

where $E(s, t): C_s\to H$ is defined for $t\geqq s$ and satisfies the following properties

$$\forall s\in\boldsymbol{R},\ \forall x\in C_s,\ \forall t\geqq s, \qquad E(s, t)x\in C_t. \tag{0.3}$$

$$\forall s\in\boldsymbol{R},\ \forall x\in C_s,\ \forall t_2\geqq t_1\geqq s, \qquad E(s, t_2)x=E(t_1, t_2)E(s, t_1)x. \tag{0.4}$$

$$\forall s\in\boldsymbol{R},\ \forall t\geqq s,\ \forall x\in C_s,\ \forall y\in C_s, \qquad \|E(s, t)x-E(s, t)y\|\leqq\|x-y\|. \tag{0.5}$$

Let J be a closed interval of $\boldsymbol{R}$. We say that a function $u\in C(J, H)$ is *a solution of* (0.1) *on* J if u satisfies

$$\forall s\in\boldsymbol{R},\ \forall t\in J,\ t\geqq s, \qquad u(t)=E(s, t)u(s). \tag{0.6}$$

We say that u is a *strong solution of* (0.1) *on* J if $u\in W^{1,1}(K, H)$ for any compact interval $K\subset J$ and for almost all $t\in K$, $u(t)\in D[A(t)]$ and $u'(t)\in -A(t)u(t)$.

In this paper, we are mainly interested in the case where $A(t)$ is periodic, i.e. for some $\tau>0$,

(0.7) $$A(t+\tau)=A(t) \quad \text{for almost all } t\in \boldsymbol{R}.$$

In the special case where $C_t=C_0$ for almost all $t\in \boldsymbol{R}$, in [**11**] it was established that any solution u of (0.1) such that

(0.8) $$u(\boldsymbol{R}) \text{ is precompact in } H$$

is in fact almost periodic: $\boldsymbol{R}\to H$. When $A(t)=\partial\varphi^t$, the subdifferential of a periodic convex function on H with variable domain, a similar result is established in [**16**] without conditions on the domain. Later in [**12**] it was shown that the method of [**11**] is applicable to the general case and can even be extended in a way to encompass the (non monotone) linear case when $H=\boldsymbol{R}^N$, $N\in\boldsymbol{N}$. In such a case, Floquet's theory (cf. e.g. [8]) shows that in fact u must be quasi-periodic: $\boldsymbol{R}\to\boldsymbol{R}^N$. The main objective of this paper is to establish a similar result when $H=\boldsymbol{R}^N$ and $A(t)$ is a general, τ-periodic, maximal monotone operator on H.

The paper is organized as follows.

In Section 1, we state the main results after recalling a preliminary result from [**11**]. For completeness we give a short proof of the preliminary result.

In Section 2-3, the main results are proved. The main tool is a structure theorem for the set of bounded solutions of some difference equations.

In Section 4, the main results are shown to be optimal by exhibiting various counterexamples. We also indicate some possible extensions and related research problems.

1. Main results.

Let H, $A(t)$ be as in the introduction. First of all we recall the following general property which is essentially contained in [**11**].

THEOREM 1. *Let $u(t)$ be a solution of* (0.1) *on* $\boldsymbol{R}$, *and assume that* (0.7)-(0.8) *are satisfied. Then u is almost periodic:* $\boldsymbol{R}\to H$.

Theorem 1 is an immediate consequence of the following result, established in [**12**].

PROPOSITION 1.1. *Let $u:\boldsymbol{R}\to H$ be any continuous function with $u(\boldsymbol{R})$ precompact in H and such that for some $\tau>0$ we have*

(1.1) *The function $t\mapsto\|u(t+m\tau)-u(t+n\tau)\|$ is non-increasing for all $(m, n)\in \boldsymbol{Z}\times\boldsymbol{Z}$. Then $u:\boldsymbol{R}\to H$ is almost periodic.*

SKETCH OF PROOFS. The hypotheses of Proposition 1.1 imply (cf. the argument p. 57, (29)→(31) of [**12**]) that $\bigcup_{m\in\boldsymbol{Z}}\{u(\cdot+m\tau)\}$ is precompact in

$C_B(\boldsymbol{R}, H)$. Then $u: \boldsymbol{R} \to H$ must be uniformly continuous, and a division argument shows that $\bigcup_{a \in \boldsymbol{R}} \{u(\cdot + a)\}$ is precompact in $C_B(\boldsymbol{R}, H)$.

To deduce the result of Theorem 1 from Proposition 1.1, it is then sufficient to note that as a consequence of (0.4)-(0.5) and (0.6), any solution of (0.1) on $\boldsymbol{R}$ satisfies (1.1).

We now formulate our main results.

THEOREM 2. *Let $N \in \boldsymbol{N} - \{0\}$, $H = \boldsymbol{R}^N$ and let $u: \boldsymbol{R} \to H$ be as in the statement of Proposition* 1.1. *Then either u is τ-periodic, or otherwise there exists a non-empty set $S = \{\lambda_1, \cdots, \lambda_k\} \subset]0, \pi/\tau]$ such that*

$$u(t) = v_0(t) + \sum_{j=1}^{k} \{v_j(t)\cos(\lambda_j t) + w_j(t)\sin(\lambda_j t)\} \qquad \text{for all } t \in \boldsymbol{R}, \tag{1.2}$$

where v_0, v_j, w_j are all continuous τ-periodic functions: $\boldsymbol{R} \to H$, with the same smoothness properties as u.

If we assume $(v_j, w_j) \neq (0, 0)$ for all $j \in \{1, \cdots, k\}$, then in fact

$$S = \left\{\lambda \in]0, \pi/\tau], \exists m \in \boldsymbol{Z}, \lim_{t \to +\infty} \frac{1}{t}\int_0^t e^{-i(\lambda + 2m\pi/\tau)s} u(s) ds \neq 0\right\} \tag{1.3}$$

and if we assume that S is ordered increasingly, then the decomposition (1.2) *is unique.*

By convention we assume that if u is not τ-periodic, (1.2) *is always meant with S as in* (1.3). *If u is τ-periodic we have $S = \emptyset$ and we set $k = 0$, $u = v_0$. Following this notation we have $k \leqq (N+1)/2$ and if $\pi/\tau \notin S$, then $k \leqq N/2$. Moreover if $\pi/\tau \notin S$, $\|u(t)\|$ is constant and $k = N/2$, then $v_0 = 0$.*

Theorem 2 implies the following structure theorem for the solutions of (0.1).

THEOREM 3. *Assume $H = \boldsymbol{R}^N$ and* (0.7). *Then if* (0.1) *has a solution u with $u(\boldsymbol{R}^+)$ bounded; any solution of* (0.1) *is asymptotic, as $t \to +\infty$, to a quasi-periodic solution of* (0.1) *with $r \leqq N/2 + 1$ basic frequencies. Moreover any solution of* (0.1) *which is defined and bounded on $\boldsymbol{R}$ has the form* (1.2) *with $k \leqq (N+1)/2$ and if $\pi/\tau \notin S$, then $k \leqq N/2$. Finally if $0 \in A(t)0$ for all $t \in \boldsymbol{R}$, $\pi/\tau \notin S$, and $k = N/2$, then $v_0 = 0$.*

When $A(t)$ is a subdifferential, the following more precise result is available.

THEOREM 4. *Assume $H = \boldsymbol{R}^N$ and $A(t) = \partial\varphi^t$, where $\varphi^t: \boldsymbol{R}^N \to]-\infty, +\infty]$ is a l.s.c. proper convex function such that for some $\tau > 0$, $\varphi^{t+\tau} = \varphi^t$ for almost all $t \in \boldsymbol{R}$. Assume in addition that any solution* (0.1) *which is defined on $\boldsymbol{R}$ is a strong solution. Then the conclusions of Theorem* 3 *remain valid with N replaced by $(N-1)$.*

REMARK 5. a) Sufficient conditions are known in order for all solutions of (0.1) to be strong when $A(t)=\partial\varphi^t$, cf. e.g. [**1**, **14**, **20**].

b) In Section 4, it will be shown that the above estimates on r, k are essentially optimal.

2. Bounded solutions of some difference equations.

For the proof of the main results, we shall rely on the following proposition which extends to the case of a general Banach space some well-known results from the scalar theory of linear difference equations (cf. e.g. [**3**, **18**, **19**]).

PROPOSITION 2.1. *Let X be a complex Banach space, n a positive integer and $\{\alpha_r\}_{0\leq r\leq n-1}$ an n-tuple of complex numbers. Let $v\in C(\boldsymbol{R}, X)$ be a solution of*

$$v(t+n)=\sum_{r=0}^{n-1}\alpha_r v(t+r), \qquad \forall t\in\boldsymbol{R}. \tag{2.1}$$

Then there exists an integer $k\leqq n-1$, a finite sequence $\{\omega_j\}_{0\leq j\leq k}$ of complex numbers, a finite sequence $\{d_j\}_{0\leq j\leq k}$ of integers, and some 1-periodic continuous functions $z_{j,s}\colon \boldsymbol{R}\to X$ defined for $0\leqq j\leqq k$ and $0\leqq s\leqq d_j$ such that

$$\forall t\in\boldsymbol{R}, \qquad v(t)=\sum_{j=0}^{k}e^{\omega_j t}\sum_{s=0}^{d_j}t^s z_{j,s}(t). \tag{2.2}$$

In addition this decomposition is unique in the sense that $v=0$ implies $z_{j,s}=0$ for $0\leqq j\leqq k$ and $0\leqq s\leqq d_j$. The numbers $\{\omega_j\}_{0\leq j\leq k}$ are arbitrary complex solutions of

$$e^{n\omega}=\sum_{r=0}^{n-1}\alpha_r e^{r\omega}, \qquad \omega\in\boldsymbol{C}. \tag{2.3}$$

Finally for all $j\in\{0,\cdots,k\}$, d_j is the order of multiplicity of $\exp(\omega_j)$ *as a complex root of* (2.3).

PROOF. By induction on n.

1) If $n=1$ and $v\in C(\boldsymbol{R}, X)$ satisfies $v(t+1)=\alpha_0 v(t)$ for all t, then either $v\equiv 0$, or $\alpha_0\neq 0$. Setting $\alpha_0=\exp(\omega_0)$, we see that the function $w(t)=\exp(-\omega_0 t)v(t)$ satisfies

$$\forall t\in\boldsymbol{R}, \qquad w(t+1)=\exp(-\omega_0 t)[(\alpha_0)^{-1}v(t+1)]=w(t),$$

therefore w is a 1-periodic continuous function: $\boldsymbol{R}\to X$, and (2.2) is valid with $k=0=n-1$.

2) If $n>1$, assume that Proposition 2.1 is already proved for $n-1$ instead of n. Let $v\in C(\boldsymbol{R}, X)$ be any solution of (2.1): we show that v has the form (2.2) with k, ω_j and d_j as stated. In order to do this, we distinguish two cases

Case 1. $v\equiv 0$: then (2.2) holds true with $z_{j,s}=0$ for $0\leqq j\leqq k$ and $0\leqq s\leqq d_j$.

Case 2. $v\not\equiv 0$: then $\alpha_m\neq 0$ for some $m\in\{0,\cdots,n-1\}$. In particular the polynomial P defined by

$$P(\zeta)=\zeta^n-\sum_{r=0}^{n-1}\alpha_r\zeta^r$$

has at least a root $\zeta\neq 0$. We set $\zeta=e^{\omega}$, $\omega\in\boldsymbol{C}$ and we introduce $w(t):=e^{-\omega t}v(t)$. Then

$$w(t+n)=\sum_{r=0}^{n-1}\gamma_r w(t+r),\qquad \forall t\in\boldsymbol{R} \tag{2.4}$$

with

$$\gamma_r := e^{(r-n)\omega}\alpha_r \qquad \forall r\in\{0,\cdots,n-1\}. \tag{2.5}$$

In particular by the choice of ω we have

$$\sum_{r=0}^{n-1}\gamma_r=\zeta^{-n}\sum_{r=0}^{n-1}\alpha_r\zeta^r=1.$$

We now introduce

$$\forall s\in\{0,\cdots,n-1\},\qquad \beta_s:=\sum_{r=0}^{s}\gamma_r.$$

In particular we have $\beta_{n-1}=1$ and

$$\forall r\in\{0,\cdots,n-1\},\qquad \gamma_r=\beta_r-\beta_{r-1}.$$

Therefore (2.4) can be rewritten as

$$\begin{aligned} w(t+n) &= \sum_{r=1}^{n-1}(\beta_r-\beta_{r-1})w(t+r)+\beta_0 w(t)\\ &= \sum_{s=0}^{n-2}\beta_s[w(t+s)-w(t+s+1)]+\beta_{n-1}w(t+n-1).\end{aligned}$$

Since $\beta_{n-1}=1$ we obtain

$$\sum_{s=0}^{n-1}\beta_s[w(t+s)-w(t+s+1)]=0,\qquad \forall t\in\boldsymbol{R}.$$

Hence $\varphi(t):=w(t+1)-w(t)$ satisfies the lower order equation

$$\varphi(t+n-1)=-\sum_{s=0}^{n-2}\beta_s\varphi(t+s),\qquad \forall t\in\boldsymbol{R}. \tag{2.6}$$

By the induction hypothesis, we have

$$\forall t\in\boldsymbol{R},\qquad \varphi(t)=\sum_{j=0}^{h}e^{\nu_j t}\sum_{r=0}^{c_j}t^r y_{j,r}(t).$$

with $h\leqq n-2$ and some 1-periodic continuous functions $y_{j,r}:\boldsymbol{R}\to X$ defined for $0\leqq j\leqq h$ and $0\leqq r\leqq c_j$. For all $j\in\{0,\cdots,h\}$, c_j is the order of multiplicity of $\exp(\nu_j)$ as a complex root of the polynomial Q defined by the formula

$$Q(\zeta)=\zeta^{n-1}+\sum_{s=0}^{n-2}\beta_s\zeta^s.$$

The equation

$$\forall t \in \boldsymbol{R}, \qquad w(t+1)-w(t)=\sum_{j=0}^{h} e^{\nu_j t} \sum_{r=0}^{c_j} t^r y_{j,r}(t)$$

is readily solved to give

$$w(t)-\sum_{j=0}^{h} e^{\nu_j t} \sum_{r=0}^{d_j} t^r z_{j,r}(t) \qquad \text{is 1-periodic}$$

with: $d_j=c_j$ if $\exp(\nu_j)\neq 1$, $d_j=c_j+1$ if $\exp(\nu_j)=1$, where the functions $y_{j,r}: \boldsymbol{R}\to X$ are linear combinations of the $z_{j,s}$ for $0\leqq s\leqq r$ if $\exp(\nu_j)\neq 1$, $0\leqq s\leqq r+1$ if $\exp(\nu_j)=1$. On the other hand a straightforward calculation provides the identity

$$e^{-(n-1)\omega}P(\zeta)=(\zeta-e^{\omega})Q(e^{-\omega}\zeta),$$

from which the relationship between the zeroes of P and Q follows immediately. The induction proof can then be easily completed: we obtain formula (2.2) with $k=h$ or $h+1$, $\omega_j=\omega+\nu_j$ for $j\leqq h$ and $\omega_k=\omega$ when $Q(1)=0$ (in which case $k=h+1$ and ω has multiplicity>1).

The uniqueness of the decomposition (2.2) is clear as a consequence of the scalar case.

From the result of Proposition 2.1, it is relatively easy to establish the following:

PROPOSITION 2.2. *Let X be a complex Banach space, n a positive integer and $\{\alpha_r\}_{0\leq r\leq n-1}$ an n-tuple of complex numbers. Then a function $v\in C_B(\boldsymbol{R}, X)$ is a solution of* (2.1) *if, and only if*

$$\forall t\in \boldsymbol{R}, \qquad v(t)=\sum_{j=0}^{k} e^{i\lambda_j t} z_j(t), \tag{2.7}$$

where $\{\lambda_j\}_{0\leq j\leq k}$ is a set of solutions of

$$e^{in\lambda}=\sum_{r=0}^{n-1}\alpha_r e^{ir\lambda}, \qquad \lambda\in\boldsymbol{R}, \tag{2.8}$$

and $\{z_j\}_{0\leq j\leq k}$ is a finite sequence of 1*-periodic continuous functions: $\boldsymbol{R}\to X$.*

PROOF. It is quite straightforward to verify that in order for v to be bounded on $\boldsymbol{R}$, all the terms ω_j occuring in the decomposition formula (2.2) must be purely imaginary: therefore we set $\omega_j=i\lambda_j$. Then for a similar reason all monomials t^s with $s>0$ must be absent in the decomposition. Since this kind of argument is fairly standard we omit the details.

REMARK 2.3. Additional information may be obtained through decomposition formula (2.7). More specifically the following properties are useful.

a) Let $j\in\{0,\cdots,k\}$ be such that $z_j\neq 0$. Then we have

$$\exists m\in\boldsymbol{Z}, \qquad \lim_{t\to+\infty}\frac{1}{t}\int_0^t e^{-i(\lambda_j+2m\pi)s}v(s)ds\neq 0.$$

Indeed, from the hypothesis $z_j \neq 0$ we deduce

$$\exists m \in \boldsymbol{Z}, \qquad \lim_{t\to+\infty} \frac{1}{t}\int_0^t e^{-2im\pi s} z_j(s)ds =: c \neq 0 .$$

Then from (2.7) we derive

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t e^{-i(\lambda_j+2m\pi)s} v(s)ds = \lim_{t\to+\infty} \frac{1}{t}\int_0^t e^{-2im\pi s} z_j(s)ds = c \neq 0 .$$

b) It follows immediately from the method of proof of Proposition 2.1 that the functions $\{z_j\}_{0\leq j\leq k}$ are essentially as smooth as the function v. In particular if $v \in C^m(\boldsymbol{R}, X)$ for some $m \in \boldsymbol{N}$, then we have $z_j \in C^m(\boldsymbol{R}, X)$ for any $j \in \{0, \cdots, k\}$.

PROPOSITION 2.4 *Let Y be a real Banach space, n a positive integer and $\{\alpha_r\}_{0\leq r\leq n-1}$ an n-tuple of real numbers. Let $v \in C_B(\boldsymbol{R}, Y)$ be a solution of* (2.1). *Then either v is* 1*-periodic, or otherwise there exists a finite increasing sequence $\{\lambda_j\}_{1\leq j\leq k}$ with $0<\lambda_j\leq\pi$ for all j such that*

$$\forall t \in \boldsymbol{R}, \qquad v(t) = v_0(t) + \sum_{j=1}^{k} \{\cos(\lambda_j t) v_j(t) + \sin(\lambda_j t) w_j(t)\} , \tag{2.9}$$

where v_0, v_j and w_j are 1*-periodic continuous functions: $\boldsymbol{R}\to Y$. In addition all those functions are as smooth as v is and $\|v_j(t)\|+\|w_j(t)\|$ is not identically zero for any j. Moreover we have*

$$\bigcup_{1\leq j\leq k} \{\lambda_j\} = \left\{\lambda \in]0, \pi],\ \exists m \in \boldsymbol{Z},\ \lim_{t\to+\infty} \frac{1}{t}\int_0^t e^{-i(\lambda+2m\pi)s} v(s)ds \neq 0\right\} =: \Sigma . \tag{2.10}$$

Concerning the value of k we can distinguish 4 *cases as follows*

i) *If $\pi \in \Sigma$, $v_0=0$, then $k \leq (n+1)/2$.*
ii) *If $\pi \in \Sigma$, $v_0 \neq 0$, then $k \leq n/2$.*
iii) *If $\pi \notin \Sigma$, $v_0=0$, then $k \leq n/2$.*
iv) *If $\pi \notin \Sigma$, $v_0 \neq 0$, then $k \leq (n-1)/2$.*

PROOF. Let $X = Y\times Y \approx Y+iY$, endowed with the complex Banach structure such that $(a+ib)(y+iw) = ay-bw+i(aw+by)$ for all pairs $(a+ib, y+iw) \in \boldsymbol{C}\times X$. By applying Proposition 2.2 in X to the function $v: \boldsymbol{R}\to Y \approx Y\times\{0\}$ we obtain

$$v(t) = \sum_{j=0}^{k} e^{i\lambda_j t}(v_j(t)+iw_j(t))$$

where v_j, w_j are the "real" components of the functions $z_j \in C(\boldsymbol{R}, Y+iY)$ given by Proposition 2.2. Therefore, since v is real-valued:

$$v(t) = \sum_{j=0}^{k} \{\cos(\lambda_j t) v_j(t) - \sin(\lambda_j t) w_j(t)\} . \tag{2.11}$$

In this formula, by combining together the terms of order r such that $\lambda_r \pm \lambda_j \in 2\pi\boldsymbol{Z}$, we may decrease the value of k and achieve, after reordering the terms, the conditions $0=\lambda_0 < \cdots < \lambda_k \leq \pi$. Then by changing w_j to $(-w_j)$, (2.11) reduces to (2.9) with possibly $k=0$ or $v_0\equiv 0$. Finally we eventually reduce the value of k so that $\|v_j(t)\|+\|w_j(t)\|$ is not identically zero for any $j\in\{1, \cdots, k\}$. It is then immediate to derive (2.10) from Remark 2.3, a). Also the statement on the regularity of v_j, w_j follows at once from Remark 2.3, b).

Finally let

$$P(\zeta)=\zeta^n-\sum_{r=0}^{n-1}\alpha_r\zeta^r.$$

Of course $P(\zeta)$ has at most n roots and if $P(\zeta_0)=0$, then also $P(\bar{\zeta}_0)=\overline{P(\zeta_0)}=0$. In particular if $|\zeta_0|=1$, then $1/\zeta_0=\bar{\zeta}_0$ and therefore the purely imaginary solutions of (2.3) occur by pairs of opposite numbers. We now consider the 4 cases.

i) If $\pi\in\Sigma$ and $v_0=0$, then the equation $P(e^{i\lambda})=0$ has at most $(n-1)/2$ real roots in $]0, \pi[$, hence in this case $k\leqq(n-1)/2+1=(n+1)/2$.

ii) If $\pi\in\Sigma$ and $v_0\neq 0$, then the equation $P(e^{i\lambda})=0$ has the roots 0 and π, therefore we have at most $(n-2)/2$ real roots in $]0, \pi[$, and in this case $k\leqq(n-2)/2+1=n/2$.

iii) If $\pi\notin\Sigma$, then $k\leqq\#\{\lambda\in]0, \pi[, P(e^{i\lambda})=0\}\leqq n/2$ in all cases, and in particular when $v_0=0$.

iv) If $\pi\notin\Sigma$ and $v_0\neq 0$, then $P(1)=0$ and therefore we find

$$k\leqq\#\{\lambda\in]0, \pi[, P(e^{i\lambda})=0\}\leqq(n-1)/2.$$

3. Proofs of the main results.

We start with the proof of Theorem 2, which will now follow rather easily from Theorem 1 and Proposition 2.4. As a preliminary step, we establish the following:

LEMMA 3.1. *Let $u:\boldsymbol{R}\to\boldsymbol{R}^N$ be any continuous function such that for some $\tau>0$ we have*

(3.1) $\quad \forall t\in\boldsymbol{R}, \qquad \|u(t)\|=\|u(0)\|,$

(3.2) $\quad \forall(m, n)\in\boldsymbol{Z}\times\boldsymbol{Z}, \forall t\in\boldsymbol{R}, \qquad \|u(t+m\tau)-u(t+n\tau)\|=\|u(m\tau)-u(n\tau)\|.$

Then u satisfies (1.2) *with k and λ_j as in the statement of Theorem* 2. *Moreover if $\pi/\tau\notin S$ and $k=N/2$, then $v_0=0$ in* (1.2).

PROOF. From (3.1)-(3.2) it follows immediately, by squaring and taking a suitable linear combination, that

(3.3) $\quad \forall(m, n)\in\boldsymbol{Z}\times\boldsymbol{Z}, \forall t\in\boldsymbol{R}, \qquad \langle u(t+m\tau), u(t+n\tau)\rangle=\langle u(m\tau), u(n\tau)\rangle.$

Since $u(\boldsymbol{R})\subset\boldsymbol{R}^N$, there exists a non trivial linear relation:

$$\sum_{r=0}^{N}a_r u(r\tau)=0.$$

Let $n=\mathrm{Sup}\{r\in\{0, \cdots, N\}, a_r\neq 0\}$. Then $n\leqq N$ and assuming $u\neq 0$, we must have $n\geqq 1$ and

$$u(n\tau)=\sum_{r=0}^{n-1}\alpha_r u(r\tau), \qquad \alpha_r:=-\frac{a_r}{a_n}.$$

It follows automatically, as a consequence of (3.3), that we have

$$\forall t\in \boldsymbol{R}, \qquad \|u(t+n\tau)-\sum_{r=0}^{n-1}\alpha_r u(t+r\tau)\|^2=\|u(n\tau)-\sum_{r=0}^{n-1}\alpha_r u(r\tau)\|^2=0.$$

Therefore:

$$\forall t\in \boldsymbol{R}, \qquad u(t+n\tau)=\sum_{r=0}^{n-1}\alpha_r u(t+r\tau).$$

Let $v(\theta):=u(\tau\theta)$ for all $\theta\in\boldsymbol{R}$. By applying Corollary 2.4 to $v(t)$ with $Y=H$, we obtain at once (1.2) with $k\leqq(n+1)/2\leqq(N+1)/2$. Moreover if $\pi/\tau\notin S$, then $\pi\notin\Sigma$ and therefore $k\leqq n/2\leqq N/2$, with $k<N/2$ whenever $v_0\neq 0$.

For the proof of Theorem 2, we need an additional observation which is the object of the next Lemma.

LEMMA 3.2. *Let* $u\in C_B(\boldsymbol{R}, \boldsymbol{R}^N)$ *and assume* (3.2). *Then there exists* $\omega\in C_B(\boldsymbol{R}, \boldsymbol{R}^N)$ *such that*

(3.4) $$\forall t\in\boldsymbol{R}, \qquad \omega(t+\tau)=\omega(t),$$

(3.5) $$\forall t\in\boldsymbol{R}, \qquad \|u(t)-\omega(t)\|=\|u(0)-\omega(0)\|.$$

PROOF. For all $p\in\boldsymbol{N}-\{0\}$ we define

$$\omega_p(t):=\frac{1}{p}\sum_{j=1}^{p}u(t+j\tau), \qquad \forall t\in\boldsymbol{R}.$$

It is clear that ω_p is such that $\|u(t)-\omega_p(t)\|$ and $\|\omega_p(t)-\omega_q(t)\|$ are constant for all $p, q\in\boldsymbol{N}-\{0\}$. Indeed setting $v_j(t)=u(t+j\tau)-u(t)$, we see that $\|v_j(t)\|$ and $\|v_j(t)-v_k(t)\|$ are constant for all $j, k\in\boldsymbol{N}-\{0\}$, therefore the products $\langle v_j(t), v_k(t)\rangle$ are also constant and since $\omega_p(t)-u(t)$ is a convex combination of $v_j(t)$ the property follows easily by expanding the squares of the norms. Let $p_n\to+\infty$ be such that $\omega_{p_n}(0)\to\omega^0$ in $\boldsymbol{R}^N$ as $n\to+\infty$. Then $\omega_{p_n}(t)$ converges uniformly on $\boldsymbol{R}$ to a limit $\omega(t)\in C_B(\boldsymbol{R}, \boldsymbol{R}^N)$ satisfying (3.5). Moreover

$$\|\omega_p(t)-\omega_p(t+\tau)\|=1/p\|u(t)-u(t+(p+1)\tau)\|\leqq C/p\longrightarrow 0$$

uniformly on $\boldsymbol{R}$ as $p\to+\infty$. Therefore ω also satisfies (3.4).

PROOF OF THEOREM 2. The hypothesis (1.1) and Proposition 1.1 clearly imply (3.2). Let ω be as in Lemma 3.2 and set $u_1=u-\omega$. Then u_1 satisfies the hypotheses of Lemma 3.1 and therefore $u=u_1+\omega$ satisfies (1.2) with k, λ_j, v_j, w_j as in the statement of Theorem 2. Finally if $\|u(t)\|$ is constant, then by applying Lemma 3.1 directly to u, we find $v_0=0$ whenever $\pi/\tau\notin S$ and $k=N/2$.

PROOF OF THEOREM 3. It is sufficient to note that as a consequence of (0.4)-(0.6), any solution u of (0.1) on $\boldsymbol{R}$ satisfies automatically (1.1). If (0.1) has a solution bounded on $\boldsymbol{R}^+$, then it is classical, since $H=\boldsymbol{R}^N$, to construct a solution of (0.1) bounded on $\boldsymbol{R}$. Any such solution satisfies the conclusion of Theorem 2, and then the result on asymptotic behavior for all solutions follows classically (cf. e.g. [**12**], Theorem 8, p. 56).

Finally if $0\in A(t)0$ for all $t\in\boldsymbol{R}$, then $\|u(t)\|$ is constant for any almost periodic solution of (0.1). Theorem 3 is now an obvious consequence of Theorem 2.

PROOF OF THEOREM 4. Let u be a solution of (0.1) bounded on $\boldsymbol{R}$ and set

(3.6) $$\forall j\in\boldsymbol{Z},\quad \forall t\in\boldsymbol{R},\qquad u_j(t):=u(t+j\tau),$$

(3.7) $$\forall j\in\boldsymbol{Z},\qquad h_j(t):=-u_j'(t)\in A(t)u_j(t),\quad \text{a.e. on } \boldsymbol{R}.$$

It follows from Theorem 1 and (1.1) that

(3.8) $$\forall(j,r)\in\boldsymbol{Z}\times\boldsymbol{Z},\ \forall t\in\boldsymbol{R},\qquad \|u_j(t)-u_r(t)\|=\|u(j\tau)-u(r\tau)\|.$$

By using the trimonotonicity property of $A(t)$ as in [**9**], proof of Theorem 5, p. 213, it is classical to deduce from (3.7)-(3.8):

(3.9) $$\forall(j,r)\in\boldsymbol{Z}\times\boldsymbol{Z},\qquad h_j(t)\in A(t)u_r(t),\quad \text{a.e. on } \boldsymbol{R}.$$

Since $[A(t)]^{-1}(h_j(t))$ is convex in H, from (3.9) we deduce for all $p\in\boldsymbol{N}-\{0\}$

(3.10) $$\forall j\in\boldsymbol{Z},\qquad h_j(t)\in A(t)\omega_p(t),\quad \text{a.e. on } \boldsymbol{R},$$

with $\omega_p(t)$ as in the proof of Lemma 3.2. Since $A(t)\omega_p(t)$ is convex, (3.10) implies

(3.11) $$\forall p\in\boldsymbol{N}-\{0\},\qquad -\omega_p'(t)\in A(t)\omega_p(t),\quad \text{a.e. on } \boldsymbol{R}.$$

In particular, $\omega(t)$ is a solution of (0.1), and since by hypothesis any solution is a strong solution we have

(3.12) $$\omega\in W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}, H).$$

We now define $v:=u-\omega$ and

(3.13) $$\forall j\in\boldsymbol{Z},\ \forall t\in\boldsymbol{R},\qquad v_j(t):=u(t+j\tau)-\omega(t),$$

(3.14) $$B(t)\cdot:=A(t)\cdot+\omega'(t)=:\partial\phi^t\cdot,$$

(3.15) $$k_j(t):=h_j(t)+\omega'(t)=-v_j'(t).$$

Let also

(3.16) $$K_t=\mathrm{conv}\{v_0(t),\cdots,v_{N-1}(t)\}.$$

Since $[A(t)]^{-1}(h_j(t))$ is convex in H, from (3.9) we deduce

(3.17) $$\forall j\in\{0,\cdots,N\},\quad \text{for a.e. } t\in\boldsymbol{R},\ \forall w\in K_t,\qquad k_j(t)\in B(t)w.$$

Now assume that we have

(3.18) The vectors $v_r(0)$ are linearly independent in H for $r\in\{0, \cdots, N-1\}$.

From (3.5), (3.8) and the τ-periodicity of ω it follows that the vectors $v_r(t)$ are also linearly independent in H for $r\in\{0, \cdots, N-1\}$ and all $t\in\boldsymbol{R}$. In particular we obtain

(3.19) $$\forall t\in\boldsymbol{R}, \quad \Omega_t := \mathrm{Int}(K_t) \neq \emptyset .$$

For any $t\in\boldsymbol{R}$ satisfying (3.17), let $w_t\in\Omega_t$ and $\rho_t>0$ be such that $B(w_t, \rho_t)\subset K_t$. By (3.17) applied with $j=0$ we have

$$\forall z\in\boldsymbol{R}^N,\ \forall\xi\in\boldsymbol{R}^N \text{ with } \|\xi\| \leqq \rho_t,\ \phi^t(z)-\phi^t(w_t+\xi) \geqq \langle k_0(t), z-w_t-\xi\rangle .$$

By letting $z=w_t$ and applying (3.17) with $j=1$ we find

$$\langle k_0(t), \xi\rangle \geqq \phi^t(w_t+\xi)-\phi^t(w_t) \geqq \langle k_1(t), \xi\rangle, \qquad \forall\xi\in B(0, \rho_t).$$

Hence $k_1(t)=k_0(t)$ for a.e. $t\in\boldsymbol{R}$, in particular

(3.20) $$\forall t\in\boldsymbol{R}, \quad v(t+\tau)-v(t) = v(\tau)-v(0).$$

Since $v:\boldsymbol{R}\to H$ is bounded, (3.20) obviously implies $v(\tau)-v(0)=0$, which contradicts (3.18) when $N>1$. The case $N=1$ is irrelevant here since in such a case Theorem 2 already implies that $u:\boldsymbol{R}\to H$ is τ-periodic. Assuming $N>1$ we now deduce

$$v(t+n\tau) = \sum_{r=0}^{n-1} \alpha_r v(t+r\tau), \qquad \forall t\in\boldsymbol{R}$$

with $v:=u-\omega$ and $0<n\leqq N-1$. The rest of the proof is identical to that of the proof of Theorem 2, with N replaced by $N-1$.

4. Related results and counterexamples.

4.1. Linear and affine equations. Let us first consider the case where $H=\boldsymbol{R}^N$ and $A(t)\in L^1_{\mathrm{loc}}(\boldsymbol{R}, L(\boldsymbol{R}^N))$. In such a case, without positivity condition on $A(t)$, the equation (0.1) generates an evolution operator $E(s, t): H\to H$ defined for all $(s, t)\in\boldsymbol{R}\times\boldsymbol{R}$ such that the solution of (0.1) with $u(s)=\varphi$ is given by $u(t)=E(s, t)\varphi$ for all $t\in\boldsymbol{R}$. When $A(t)$ satisfies (0.7), the classical *Floquet theory* implies (cf. [8], Corollary 6.5, p. 101) that any solution u of (0.1) bounded on $\boldsymbol{R}$ is quasi-periodic. Indeed there exists a τ-periodic non-singular $P(t)\in C(\boldsymbol{R}, L(\boldsymbol{R}^N))$ such that the change of unknown:

$$u(t) = P(t)y(t), \qquad \forall t\in\boldsymbol{R}$$

reduces (0.1) to an autonomous equation: $y'(t)=By(t)$, $\forall t\in\boldsymbol{R}$.

In this case we recover (1.2) with

$$S = \{\lambda\in]0, \pi/\tau], i(\lambda+2m\pi/\tau) \text{ is an eigenvalue of } B \text{ for some } m\in\boldsymbol{Z}\}.$$

In the special case where $A(t)=A$ is *constant* with $A^*=-A$ and $\sigma(A)\subset i]0, \pi/\tau[$, the estimate $k\leqq N/2$ of Theorem 3 is optimal. More precisely:

If N is even, let $k=N/2$ and for all $z=(z_1, \cdots, z_k)\in \boldsymbol{C}^k \approx \boldsymbol{R}^N$, define $A(z):=(-i\lambda_1 z_1, \cdots, -i\lambda_k z_k)$. Then the general solution of (0.1) is given by (1.2) with $v_0=0$, $k=N/2$, the coefficients v_j and w_j being some constant vectors in $\boldsymbol{R}^N$. Since A is constant, it can be considered as τ-periodic for any $\tau>0$ and in particular by choosing $\tau<\pi/\lambda_k$, Theorem 3 provides the estimate $\#S\leqq N/2$. Here we have $\#S=k=N/2$ for the general solution.

If N is odd, setting $N=2k+1$ we can choose

$$A(z_1, \cdots, z_k; x) = (-i\lambda_1 z_1, \cdots, -i\lambda_k z_k; 0)$$
$$\text{for all } (z; x)=(z_1, \cdots, z_k; x)\in \boldsymbol{C}^k\times\boldsymbol{R}\approx\boldsymbol{R}^N .$$

The general solution of (0.1) is now given by (1.2) with $k=(N-1)/2$ (the largest integer$\leqq N/2$), with v_j and w_j some constant vectors in $\boldsymbol{R}^N$. Here $v_0=(0; c)$ for some $c\in\boldsymbol{R}$.

In the general case where $A(t)$ is time-dependent, the decomposition formula (1.2) becomes in fact rather sharp. Indeed, any vector function of the form (1.2) with C^1 components v_j and w_j which satisfy some non-degeneracy condition can be considered as a solution of some evolution equation $u'(t)=A(t)u(t)$, $t\in\boldsymbol{R}$. Even in the monotone framework it is possible to construct large families of functions of the form (1.2) which are actually solutions of some equations (0.1). For instance if $N=2k$ and we define a curve $z: \boldsymbol{R}\rightarrow\boldsymbol{C}^k\approx\boldsymbol{R}^N$ by the formula $z(t)=\{\rho_j \exp(it\lambda_j)\}_{1\leq j\leq k}$ where ρ_j, λ_j are some real numbers with $0<\lambda_1<\cdots<\lambda_k<\pi$, then for *any* 1-periodic matrix $P(t)\in C^1(\boldsymbol{R}, L(\boldsymbol{R}^N))$ such that

$$\forall t\in\boldsymbol{R}, \qquad P^*P(t) = PP^*(t) = \mathrm{Id}\,\boldsymbol{R}^N ,$$

the function $u(t)=P(t)z(t)$ is a solution of equation (0.1) with $A(t)$ 1-periodic and skew-symmetric for all $t\in\boldsymbol{R}$. Finally if we allow variable operators of the affine type $A(t)=A+h(t)$, then in (1.2) v_0 can be taken arbitrary in $C^1(\boldsymbol{R}, \boldsymbol{R}^N)$, since for A skew-symmetric, u a solution of the autonomous equation $u'+Au(t)=0$ and $v=v_0+u$, we have

$$v'(t)+Av(t) = v_0'(t)+Av_0(t) =: f(t), \qquad \text{a continuous 1-periodic function.}$$

4.2. The case $A(t)=\partial\varphi^t$. When $N=2$, Theorem 4 implies that all bounded solutions are periodic (only one basic frequency since $r\leqq 3/2$ implies $r=1$). In this case we have in fact

PROPOSITION 4.1. *When* $N=2$ *and* $A(t)$ *satisfies the hypotheses of Theorem 4, any solution* u *of* (0.1) *bounded on* $\boldsymbol{R}$ *is in fact* 2τ*-periodic.*

PROOF. Either u is τ-periodic and $S=\emptyset$. Or $S\neq\emptyset$ and $k=1$. In the second case $\pi/\tau\in S$ and therefore $S=\{\pi/\tau\}$. The result then follows at once.

REMARK 4.2. The result of Proposition 4.1 is optimal. Indeed, let $L(t)$ be the straight line through (0, 0) orthogonal to the vector $(-\sin t, \cos t)$. We consider $A(t)=\partial\varphi^t$ where φ^t is given by

$$\varphi^t(z)=0 \quad \text{if } z\in L(t), \qquad \varphi^t(z)=+\infty \quad \text{if } z\notin L(t).$$

Then $L(t)$, hence $A(t)$ is π-periodic. On the other hand, $u(t):=(\cos t, \sin t)$ is a 2π-periodic solution of $-u'(t)\in A(t)u(t)=\{L(t)\}^{\perp}$ which is in fact π-antiperiodic.

REMARK 4.3. The result of Theorem 4 is also optimal when $N\geqq 3$. Indeed for $N=3$ it is shown in [**15**] that in general bounded solutions are not all periodic. For N odd the example of [**15**] can be generalized as is shown below.

Let $H=\boldsymbol{R}^{2n+1}$, and denote a generic point of by $x=(x_0, x_1, \cdots, x_{2n})$. For each $k\in\{1, \cdots, n\}$, $\theta\in[0, \pi[$ and $t\in[0, 1/2n]$, we define an operator $R_k(\theta, t)$ from the hyperplane $X_0=\{0\}\times\boldsymbol{R}^{2n}$ into $\boldsymbol{R}^{2n+1}$ by

$$\forall x=(0, x_1, \cdots, x_{2n})\in X_0, \qquad R_k(\theta, t)x=(x_0(t), x_1(t), \cdots, x_{2n}(t))$$

with

$$\begin{cases} x_0(t)=r_k\sin 2n\pi t\sin(\theta-\alpha_k), \\ x_{2k-1}(t)=r_k[\cos\theta\cos(\theta-\alpha_k)+\sin\theta\sin(\theta-\alpha_k)\cos 2n\pi t], \\ x_{2k}(t)=r_k[\sin\theta\cos(\theta-\alpha_k)-\cos\theta\sin(\theta-\alpha_k)\cos 2n\pi t], \\ x_j(t)\equiv x_j \qquad \text{for } j\neq 0, 2k-1, 2k, \end{cases}$$

where

$$x_{2k-1}=r_k\cos\alpha_k, \qquad \text{and} \qquad x_{2k}=r_k\sin\alpha_k \qquad (0\leqq\alpha_k<2\pi).$$

$R_k(\theta, t)$ acts as the identity on the j-th coordinates for $j\neq 0, 2k-1, 2k$, and the transformation in $\boldsymbol{R}^3$ defined by $(0, x_{2k-1}, x_{2k})\mapsto(x_0(t), x_{2k-1}(t), x_{2k}(t))$ geometrically means the axial rotation with axis $l_\theta=\{(x_0, x_{2k-1}, x_{2k})\in\boldsymbol{R}^3, x_0=-x_{2k-1}\sin\theta+x_{2k}\cos\theta=0\}$ and angle $2n\pi t$. From this observation it easily follows that

(4.1) $R_k(\theta, t)$ is a linear isometry from the hyperplane X_0 onto the hyperplane $R_k(\theta, t)X_0$ for each $k\in\{1, \cdots, n\}$, $\theta\in[0, \pi[$ and $t\in[0, 1/2n]$. In particular $R_k(\theta, 1/2n)X_0=X_0$.

(4.2) For each $x\in X_0$, $k\in\{1, \cdots, n\}$ and $\theta\in[0, \pi[$, $R_k(\theta, t)$ is a C^∞ function of $t\in[0, 1/2n]$.

Now we fix a vector $\Theta=(\theta_1, \cdots, \theta_n)\in\boldsymbol{R}^n$ with $0<\theta_k<\pi$ for $k\in\{1, \cdots, n\}$ and we introduce for all $t\in[0, 1]$ the operator $S(\Theta, t)$ from X_0 into $\boldsymbol{R}^{2n+1}$ defined by

$$S(\Theta, t)=S_{k+1}(\theta_{k+1}, t-k/n)S_k(\theta_k, 1/n)\cdots S_1(\theta_1, 1/n)$$
$$\text{for } 1\leqq k\leqq n-1 \text{ and } k/n\leqq t\leqq(k+1)/n,$$

where

$$S_k(\theta_k, t)=\begin{cases} R_k(0, t) & \text{if } 0\leqq t\leqq 1/2n \\ R_k(\theta_k, t-1/2n)R_k(0, 1/2n) & \text{if } 1/2n\leqq t\leqq 1/n. \end{cases}$$

Then we extend $S(\Theta, t)$ to the whole line $\boldsymbol{R}$ by the formulas

$$S(\Theta, t) = S(\Theta, t-n)[S(\Theta, 1)]^n, \qquad n \leqq t \leqq n+1,\ \forall n \in \boldsymbol{N}$$
$$[S(\Theta, 1)]^n = \{[S(\Theta, 1)]^{-1}\}^{-n} \qquad \forall n \in \boldsymbol{Z},\ n<0.$$

Let us denote the hyperplane $S(\Theta, t)X_0$ by $X(t)$, then it easily follows from (4.1), (4.2) and the definition of $S(\Theta, t)$ that

(4.3) $X(t+n) = X(t), \quad \forall n \in \boldsymbol{Z}, \forall t \in \boldsymbol{R};$ In particular $X(n) = X_0, \quad \forall n \in \boldsymbol{Z}.$

(4.4) $S(\Theta, t)$ is a linear isometry from the hyperplane X_0 onto $X(t)$.

In addition for any $x=(0, r_1\cos\alpha_1, r_1\sin\alpha_1, \cdots, r_n\cos\alpha_n, r_n\sin\alpha_n) \in X_0$ (with $0 \leqq \alpha_p < 2\pi$ for $p \in \{1, \cdots, n\}$), we have

(4.5)
$$S(\Theta, m)x = (0, r_1\cos(\alpha_1+2m\theta_1), r_1\sin(\alpha_1+2m\theta_1), \cdots, r_n\cos(\alpha_n+2m\theta_n), r_n\sin(\alpha_n+2m\theta_n)).$$

(4.6) For each $x \in X_0$, $S(\Theta, t)x$ is a Lipschitz continuous function on $\boldsymbol{R}$. Moreover the right (resp. left) derivative $(d^+/dt)S(\Theta, t)x$ (resp. $(d^-/dt)S(\Theta, t)x$) exists for every $t \in \boldsymbol{R}$ and

(4.7) $\forall t \in \boldsymbol{R}, (d^+/dt)S(\Theta, t)x \in \{X(t)\}^\perp$ (resp. $(d^-/dt)S(\Theta, t)x \in \{X(t)\}^\perp$).

It is clear that for all $x \in X_0$, the function $u(t) := S(\Theta, t)x$ is a bounded strong solution of (0.1) with $A(t)=\partial\varphi^t$ where φ^t is given by

$$\varphi^t(z) = 0 \quad \text{if } z \in X(t), \qquad \varphi^t(z) = +\infty \quad \text{if } z \notin X(t).$$

However in general, when the numbers θ_p and 2π are linearly independent over $\boldsymbol{Q}$, the function $u(t)$ is quasi-periodic with $n+1$ independent frequencies $\{2\pi, \theta_1, \cdots, \theta_n\}$. Indeed let e_k be the unit vector whose k-th component is 1, and set $u_j(t)=S(\Theta, t)e_{2j-1}$. Then we have

(4.8)
$$u_j(m) = \cos 2m\theta_j e_{2j-1} + \sin 2m\theta_j e_{2j}, \qquad \forall m \in \boldsymbol{Z}.$$

In particular we get $u_j(2)+u_j(0)=2\cos(2\theta_j)u_j(1)$. Since $S(\Theta, t)$ is an isometry, we deduce

(4.9)
$$u_j(t+2)+u_j(t) = 2\cos(2\theta_j)u_j(t+1), \qquad \forall t \in \boldsymbol{R}.$$

It then follows from Proposition 2.4 that there exist some 1-periodic functions $z_j(t)$, $v_j(t)$ and $w_j(t)$ such that

(4.10)
$$u_j(t) = z_j(t)+\cos(2\theta_j t)v_j(t)+\sin(2\theta_j t)w_j(t), \qquad \forall t \in \boldsymbol{R}.$$

Assuming that θ_j/π irrational we deduce

(4.11)
$$z_j(0) = 0, \qquad v_j(0) = e_{2j-1}, \qquad w_j(0) = e_{2j}.$$

On the other hand by using the formulas defining $S(\Theta, t)$ we obtain rather easily

(4.12)
$$u_j(m+(2j-1)/n) = \cos 2m\theta_j e_{2j-1} - \sin 2m\theta_j e_{2j}, \qquad \forall m \in \boldsymbol{Z}$$

Since θ_j/π is assumed irrational we deduce

(4.13) $z_j((2j-1)/n)=0, \quad v_j((2j-1)/n)=e_{2j-1}, \quad w_j((2j-1)/n)=-e_{2j}.$

As a consequence the 1-periodic functions v_j and w_j are linearly independent, therefore if the numbers θ_j and 2π are linearly independent over $\boldsymbol{Q}$, then the solution $u(t)$ of (0.1) defined by

$$u(t):=\sum_{j=1}^{n} u_j(t), \qquad \forall t\in\boldsymbol{R}$$

cannot be quasi-periodic with n basic frequencies. This remark finishes the optimality proof in the odd-dimensional case $H=\boldsymbol{R}^{2n+1}$. Finally in the even-dimensional case $H=\boldsymbol{R}^{2n+2}$, it suffices to repeat the argument above in some $(2n+1)$-dimensional subspace.

4.3. The case of quasi-periodic $A(t)$. It has been shown in [**11**], that the result of Theorem 1 is no longer valid, even for $H=\boldsymbol{R}^2$, if (0.7) is replaced by an almost periodicity assumption on $A(t)$. In this section we show that even if $A(t)$ is linear, quasi-periodic with 2 basic frequencies, the bounded solutions can fail to be almost periodic. The counterexample is based on the following generalization of [**11**], Remark 1.3, (b), pp. 477–478.

PROPOSITION 4.4. *Let* $\{\varepsilon_k\}_{k\in N}$ *be an infinite sequence of positive real numbers such that*

(4.8) $$\forall k\in\boldsymbol{N}, \qquad \varepsilon_{k+1}\leqq(1/2)\varepsilon_k.$$

Let us define

(4.9) $$\forall t\in\boldsymbol{R}, \qquad h(t)=\sum_{k\geq 0}\varepsilon_k\sin(\varepsilon_k t)\cos(\varepsilon_k t).$$

Then $h:\boldsymbol{R}\to\boldsymbol{R}$ *is an almost periodic function for which the only almost periodic solution* u *of*

(4.10) $$\forall t\in\boldsymbol{R}, \qquad u'(t)=ih(t)u(t)$$

is the trivial solution $u\equiv 0$.

PROOF. The solutions of (4.10) are given by

(4.11) $$u(t)=\exp(iH(t))u_0$$

with

(4.12) $$\forall t\in\boldsymbol{R}, \qquad H(t)=\frac{1}{2}\sum_{k\geq 0}\sin^2(\varepsilon_k t).$$

Let us establish

(4.13) the function $\exp(iH(t))$ is *not* almost periodic: $\boldsymbol{R}\to\boldsymbol{C}$.

According to a classical result (cf. e.g. [8], Lemma 6.7, p. 104), if $\exp(iH(t))$ is almost periodic: $\boldsymbol{R}\to\boldsymbol{C}$, there exists $\alpha\in\boldsymbol{R}$ such that $H(t)-\alpha t$ is almost periodic: $\boldsymbol{R}\to\boldsymbol{R}$. In particular we have

(4.14) $$H(t)-\alpha t \text{ is bounded on } \boldsymbol{R}.$$

Let us show that (4.14) is impossible for all $\alpha\in\boldsymbol{R}$. In order to do this we define

(4.15) $$\forall k\in\boldsymbol{N}, \qquad T_k=\pi/\varepsilon_k.$$

It follows in particular from hypothesis (4.8) that

(4.16) $$\exists c>0,\ \forall k\in\boldsymbol{N}, \qquad T_k\geqq c2^k.$$

We also define

(4.17) $$M_k:=2\int_0^{T_k}H(t)dt.$$

Therefore

(4.18) $$M_k=\sum_{r=0}^{k}\int_0^{T_k}\sin^2(\varepsilon_k t)dt+\sum_{k+1}^{\infty}\int_0^{T_k}\sin^2(\varepsilon_k t)dt=:P_k+Q_k.$$

It follows from the definition of T_k that for all $r\geqq k+1$, $\varepsilon_r T_k\leqq(\pi/2)2^{-r+k+1}$ and therefore

$$\int_0^{T_k}\sin^2(\varepsilon_r t)dt\leqq T_k\times\varepsilon_r T_k\leqq\frac{\pi}{2}T_k 2^{-r+k+1}.$$

In particular we find

(4.19) $$0<Q_k\leqq(\pi/2)T_k(1+1/2+\cdots)=\pi T_k.$$

For $0\leqq r\leqq k$ we have the formula

$$\int_0^{T_k}\sin^2(\varepsilon_k t)dt=\int_0^{T_k}\left[\frac{1-\cos(2\varepsilon_r t)}{2}\right]dt=\frac{T_k}{2}-\frac{\sin(2\varepsilon_r T_k)}{4\varepsilon_r},$$

and since $\varepsilon_r\geqq\varepsilon_k=\pi/T_k$ we deduce

$$T_k\left(\frac{1}{2}-\frac{1}{4\pi}\right)\leqq\int_0^{T_k}\sin^2(\varepsilon_r t)dt\leqq T_k\left(\frac{1}{2}+\frac{1}{4\pi}\right)$$

and in particular we obtain

(4.20) $$\forall k\in\boldsymbol{N}, \qquad [(k+1)/4]T_k\leqq P_k\leqq(k+1)T_k.$$

First of all since M_k/T_k tends to infinity as $k\to+\infty$, (4.14) cannot be satisfied with $\alpha=0$. On the other hand, if (4.14) is satisfied with $\alpha\neq0$, then necessarily $\alpha>0$ and we must have

(4.21) $$M_k\geqq\alpha T_k^2/4 \qquad \text{for all } k \text{ large enough.}$$

On the other hand (4.19) and (4.20) imply

(4.22) $$M_k\leqq 2kT_k \qquad \text{for all } k \text{ large enough.}$$

Therefore (4.21)-(4.22) imply with $C=8/\alpha$:

(4.23) $$T_k\leqq Ck \qquad \text{for all } k \text{ large enough.}$$

Now (4.23) contradicts (4.16): therefore (4.14) is impossible and (4.13) follows.

REMARK 4.5. Let λ be any irrational positive number. Then there exists a sequence of pairs $(m_k, n_k)\in\boldsymbol{N}\times\boldsymbol{N}$ such that

(4.24) $\forall k \in \boldsymbol{N}, \quad 0 < m_{k+1} - n_{k+1}\lambda \leqq (1/2)(m_k - n_k\lambda)$.

Let $\varepsilon_k := (m_k - n_k\lambda)$ and $h(t)$ as in (4.9): then we have $h(t) = h^*(t, \lambda t)$ where h^* is a 2π-periodic function in both variables given by

$$h^*(x, y) = \sum_{k \geqq 0} \varepsilon_k \sin(m_k x - n_k y) \cos(m_k x - n_k y).$$

Therefore h is quasi-periodic with 2 basic frequencies 1 and λ, and Proposition 4.4 shows that the result of Theorem 1 is no longer valid even for $H = \boldsymbol{R}^2$ and $A(t)$ quasi-periodic with 2 basic frequencies.

REMARK 4.6. For a quasi-autonomous equation in $\boldsymbol{R}^2$ of the general form $u'(t) + Au(t) \ni f(t)$ with f almost periodic: $\boldsymbol{R} \to \boldsymbol{R}^2$, it has been shown in [13] that all bounded trajectories are almost periodic. The following problems seem to be of some interest for future investigation.

1) What happens if f is assumed to be quasi-periodic: $\boldsymbol{R} \to \boldsymbol{R}^2$?

2) What about the equation in $\boldsymbol{R}^2$: $u'(t) + A(t)u(t) \ni f(t)$ with $A(t)$ periodic and f almost periodic (resp. quasi-periodic): $\boldsymbol{R} \to \boldsymbol{R}^2$?

3) What happens for a quasi-autonomous equation in $\boldsymbol{R}^N$ of the form $u'(t) + Au(t) \ni f(t)$ with f almost periodic: $\boldsymbol{R} \to \boldsymbol{R}^N$ when $N \geqq 3$?

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Alain HARAUX
Analyse Numérique, T. 55-65, 5ème étage
Université Pierre et Marie Curie
4 Place Jussieu, 75230 Paris Cedex 05
France

Mitsuharu ÔTANI
Department of Mathematics
Faculty of Science
Tokai University
Hiratsuka, Kanagawa
Japan