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On Nelson processes with boundary condition

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0. Introduction.

In Nelson [**10**], it is shown that for each solution of a Schrödinger equation $\phi(x, t)$, $x \in \boldsymbol{R}^d$, $t \geqq 0$, with $\|\phi\|=1$, there exists a diffusion process which has probability density at time t given by $|\phi(x, t)|^2$, if ϕ has sufficient regularity. More precisely the following is shown in [**10**]: suppose that we are given a real valued function $V(x)$, $x \in \boldsymbol{R}^d$, and a complex valued function $\phi_0(x)$, $x \in \boldsymbol{R}^d$, with $\|\phi_0\|=1$, and consider the Schrödinger equation

$$i\frac{\partial\phi(x, t)}{\partial t}=\Big(-\frac{1}{2}\Delta+V(x)\Big)\phi(x, t), \quad t>0,\ x \in \boldsymbol{R}^d, \text{ with } \phi(x, 0)=\phi_0(x).$$

For such ϕ, let $\tilde{b}(x, t)=\mathrm{Im}\{\nabla\phi(x, t)/\phi(x, t)\}+\mathrm{Re}\{\nabla\phi(x, t)/\phi(x, t)\}$. Then, under the assumption that ϕ and $\tilde{b}$ are sufficiently regular, there exists a diffusion process $\{X_t\}$, $t \geqq 0$, with initial density $|\phi_0(x)|^2$ and generator $(1/2)\Delta+\tilde{b}(x, t)\cdot\nabla$ such that the probability density of this diffusion at time $t \geqq 0$ is $|\phi(x, t)|^2$. In addition, in [**10**] it is also shown that the diffusion process $\{X_t\}$, $t \geqq 0$, solves the second order stochastic differential equation $(1/2)(D_*D+DD_*)X_t=-\nabla V(X_t)$, where D and D_* are Nelson's forward and backward stochastic derivatives.

Generally speaking, the functions ϕ and $\tilde{b}$ may be singular, and the existence of the corresponding diffusion process must be studied carefully. Carlen [**1**, **2**], Meyer and Zheng [8] and Nagasawa [9] considered rigorously the construction of a diffusion process with a given initial density and a generator. In order to construct a Markovian propagator for the diffusion process, Carlen [**1**, **2**] used partial differential equation methods. Meyer and Zheng [8] restricted themselves to the case when $\phi(x, t)=\phi_0(x)$, $^\forall t \geqq 0$, and considered the construction of the diffusion process through the relationships between Markov processes and Dirichlet forms. We may consult Nagasawa [9] about recent developments of this area.

In Carlen [**1**], it is assumed that $V(x)$, $x \in \boldsymbol{R}^d$, is a Rellich class potential and $\|\nabla\phi_0\|^2<\infty$, and shown that there exists a diffusion process $\{X_t\}$, $t \geqq 0$, which has the probability density $|\phi(x, t)|^2$ at time $t \geqq 0$ and admits the representation of stochastic differential equation: $X_t=X_0+\int_0^t b(X_s, s)ds+B_t$, $t\geqq 0$, where $\{B_t\}$, $t\geqq 0$, is a standard Brownian motion process and b is a modification of $\tilde{b}$. In this paper we restrict ourselves to the case when $d=1$, one dimensional case, and consider the Schrödinger equation with boundary condition:

$$i\frac{\partial\psi(x, t)}{\partial t}=\left(-\frac{1}{2}\Delta+V(x)\right)\psi(x, t), \qquad t>0,\ x\in(\alpha, \beta),$$

with

$$\psi(x, 0)=\psi_0(x), \quad \text{and} \quad \psi'(\alpha, t)-\sigma_1\psi(\alpha, t)=\psi'(\beta, t)+\sigma_2\psi(\beta, t)=0, \quad t\geqq 0,$$

where $-\infty<\alpha<\beta<\infty$ and σ_1, $\sigma_2\geqq 0$ are given constants, V is a given Rellich class potential and $\psi_0\in H^2((\alpha, \beta))$, $\|\psi_0\|=1$. In Section 1 following Carlen [**1**], for each given V and ψ_0 we shall construct a Markovian propagator, and in Sections 2 and 3 we shall show that there exists a Markov process, which has probability density at time t given by $|\psi(x, t)|^2$, and that the Markov process has the representation:

$$X_t = X_0+\int_0^t b(X_s, s)ds+B_t+\int_0^t (I_{\{\alpha\}}(X_s)-I_{\{\beta\}}(X_s))d\xi_s, \quad t\geqq 0,$$

where $\{B_t\}$, $t\geqq 0$, is a standard Brownian motion process and $\{\xi_t\}$, $t\geqq 0$, is a continuous increasing process which increases only on $\{\alpha\}\cup\{\beta\}$, and I is the indicator function. Since $\|\psi_t\|=1$ when $\|\psi_0\|=1$ for a Rellich class potential V, the Markov process $\{X_t\}$, $t\geqq 0$, must be a conservative process. Under the condition that $\psi'(\alpha, t)-\sigma_1\psi(\alpha, t)=\psi'(\beta, t)+\sigma_2\psi(\beta, t)=0$, $|\psi(\alpha, t)|$ and $|\psi(\beta, t)|$ may be strictly positive. It is easy to expect that the conservative Markov process $\{X_t\}$, $t\geqq 0$, has the above representation.

1. Construction of Markovian propagator.

Let $D=(\alpha, \beta)\subset \boldsymbol{R}^1$, an open bounded interval of $\boldsymbol{R}^1$, and $\Gamma=\{\alpha\}\cup\{\beta\}$. We define two sets of complex-valued functions as follows:

$$\mathcal{D}(H_0)=\left\{\psi \mid \psi\in H^2(D),\ \frac{d}{dx}\psi(\alpha)-\sigma_1\psi(\alpha)=\frac{d}{dx}\psi(\beta)+\sigma_2\psi(\beta)=0\right\},$$

$$H^2(D)=\left\{\psi \mid \psi,\ \frac{d}{dx}\psi,\ \frac{d^2}{dx^2}\psi\in L^2(D)\right\},$$

where σ_1 and σ_2 are two given real numbers such that σ_1, $\sigma_2\geqq 0$. Let H_0 be the operator with domain $\mathcal{D}(H_0)$ such that

$$H_0\psi=-\frac{1}{2}\Delta\psi, \qquad \psi\in\mathcal{D}(H_0).$$

Then the operator H_0 is a self-adjoint operator on $\mathcal{D}(H_0)$, and $(H_0\psi, \psi)\geqq 0$, $^\forall\psi\in\mathcal{D}(H_0)$. Throughout this paper we assume the following.

ASSUMPTION 1. *The real valued function* $V(x)$, $x\in\bar{D}$, *is a Rellich class potential:* $^{\exists}a, b\geqq 0, (a<1)$,

$$\|V\psi\|_{L^2(D)}\leqq a\|H_0\psi\|_{L^2(D)}+b\|\psi\|_{L^2(D)},\qquad {}^{\forall}\psi\in\mathcal{D}(H_0).$$

PROPOSITION 1.1 (Kato-Rellich). *Suppose that* V *satisfies Assumption* 1, *then the operator* $H=H_0+V$ *is a self-adjoint operator with domain* $\mathcal{D}(H_0)$ *and satisfies*

$$((H_0+V)\psi, \psi)\geqq -\frac{b}{1-a}\|\psi\|^2_{L^2(D)},\qquad \psi\in\mathcal{D}(H_0).$$

LEMMA 1.2. *Suppose that we are given* $\psi_0\in\mathcal{D}(H_0)$. *Let* $\psi_t=e^{-itH}\psi_0$, $t\in\boldsymbol{R}^1$. *Then the following hold.*

(i) $^{\exists}C_2<\infty$, $^{\forall}t\in\boldsymbol{R}^1$,

$$\frac{1}{2}\|\psi_t'-\psi_0'\|^2+\lambda\|\psi_t-\psi_0\|^2\leqq C_2\|e^{-itH}(H+\lambda)\psi_0-(H+\lambda)\psi_0\|^2,\tag{1.1}$$

where $\lambda=(1-a+b)/(1-a)$ $(\geqq 1)$.

(ii) *For each fixed* $T<\infty$, *there exists an* $M_T<\infty$, *and*

$$\int_D\left\{\left(\mathrm{Re}\frac{\psi_t'}{\psi_t}\right)^2+\left(\mathrm{Im}\frac{\psi_t'}{\psi_t}\right)^2\right\}|\psi_t|^2dx\leqq M_T,\qquad {}^{\forall}t\in[0, T].\tag{1.2}$$

(iii) *For any real valued function* $f\in C^{1,1}(\bar{D}\times[s, t])$, $0\leqq s<t<+\infty$, $\int_D f(x, \tau)|\psi_\tau(x)|^2dx$ *is an absolutely continuous function on* $[s, t]$, *and*

$$\frac{d}{d\tau}\int_D f(x, \tau)|\psi_\tau(x)|^2dx=\int_D\dot{f}|\psi_\tau|dx+\frac{i}{2}\left\{\int_D(\Delta\psi_\tau)\bar{\psi}_\tau f dx-\int_D\psi_\tau(\Delta\bar{\psi}_\tau)fdx\right\}$$

$$=\int_D\dot{f}|\psi_\tau|^2dx+\int_D\left(\mathrm{Im}\frac{\psi_\tau'}{\psi_\tau}\right)f'|\psi_\tau|^2dx,\qquad a.e.\ \tau\in[s, t],\tag{1.3}$$

where we simply denote $f=f(x,\tau)$, $\dot{f}=(\partial/\partial\tau)f(x,\tau)$, $f'=(\partial/\partial x)f(x,\tau)$ *and* $\psi_\tau=\psi_\tau(x)$.

$$\int_D f(x, t)|\psi_t(x)|^2dx-\int_D f(x, s)|\psi_s(x)|^2dx=\int_s^t\frac{d}{d\tau}\left\{\int_D f(x, \tau)|\psi_\tau(x)|^2dx\right\}d\tau.\tag{1.4}$$

(iv) *There exist jointly measurable functions* $\phi(x, t)$ *and* $\tilde{\phi}(x, t)$ *on* $\bar{D}\times[0, T]$, $T<\infty$, *such that for any* $t\in[0, T]$ $\phi(x, t)=\psi_t(x)$, $\tilde{\phi}(x, t)=\psi_t'(x)$, *a.e.* $x\in D$ *and* $\tilde{\phi}(\alpha, t)-\sigma_1\phi(\alpha, t)=\tilde{\phi}(\beta, t)+\sigma_2\phi(\beta, t)=0$. (1.5)

PROOF. First, we assume that the following inequality (1.6) holds, and prove Lemma 1.2. The validity of (1.6) will be shown at the end of this proof.

$$^{\exists}C_1, C_2>0;\ C_1\|\psi\|_1\leqq\|\psi\|_0\leqq C_2\|\psi\|_1,\qquad \forall\psi\in\mathcal{D}(H_0),\tag{1.6}$$

where

$$\|\psi\|_1=\|(H+\lambda)\psi\|,\qquad \|\psi\|_0=\|(H_0+\lambda)\psi\|.$$

For any $\psi\in\mathcal{D}(H_0)$[(1)], we have

(1) By Sobolev's lemma it holds that $\mathcal{D}(H_0)\subset C^{3/2}(\bar{D})$.

$$((H_0+\lambda)\phi,\ \phi)=\frac{1}{2}\|\phi'\|^2+\lambda\|\phi\|^2+\frac{1}{2}\{\sigma_1|\phi(\alpha)|^2+\sigma_2|\phi(\beta)|^2\}. \tag{1.7}$$

Let E_0 be the resolution of the identity for the self-adjoint operator H_0 on $\mathcal{D}(H_0)$. Since $\lambda\geqq 1$ and (1.6) holds, we have

$$\begin{aligned}((H_0+\lambda)\phi,\ \phi)&=\int_0^\infty(x+\lambda)d\|E_0(x)\phi\|^2\leqq\int_0^\infty(x+\lambda)^2d\|E_0(x)\phi\|^2\\&=\|(H_0+\lambda)\phi\|^2\leqq C_2\|(H+\lambda)\phi\|^2.\end{aligned} \tag{1.8}$$

From (1.7) and (1.8) we see that

$$\frac{1}{2}\|\phi'\|^2+\lambda\|\phi\|^2\leqq C_2\|(H+\lambda)\phi\|^2,\qquad \phi\in\mathcal{D}(H_0). \tag{1.9}$$

On the other hand from Proposition 1.1, the operator e^{-itH} is a strongly continuous unitary group on $L^2(D)$, and for any $\phi_0\in\mathcal{D}(H_0)$, $\phi_t=e^{-itH}\phi_0$ is an element of $\mathcal{D}(H_0)$. In the sequel we denote e^{-itH} by U_t. Since the operators U_t and H are commutative, we see that

$$\|(H+\lambda)(U_t\phi_0-\phi_0)\|=\|U_t(H+\lambda)\phi_0-(H+\lambda)\phi_0\|. \tag{1.10}$$

Thus if we let $\phi=\phi_t-\phi_0\in\mathcal{D}(H_0)$ in (1.9), then from (1.10) the desired inequality (1.1) follows.

Since U_t is strongly continuous on $L^2(D)$ and $(H_0+\lambda)\phi_0\in L^2(D)$ for $\phi_0\in\mathcal{D}(H_0)$, from (1.1) we see that the mapping $t\mapsto\phi_t'$ is a strongly continuous mapping from $[0, T]$ to $L^2(D)$. Hence, by Bochner-von Neumann measurability theorem (see for example p. 454 of [**6**]), the assertion (iv) follows. Obviously $\|\phi_t'\|$ is continuous in t, we have

$$\forall T<\infty,\quad \exists M_T<\infty,\quad \forall t\in[0, T],\qquad \|\phi_t'\|^2\leqq M_T. \tag{1.11}$$

Since the left hand side of (1.2) equals to $\|\phi_t'\|^2$, thus (1.2) follows from (1.11).

Now, we shall prove (iii). Suppose that $f\in C^{1,1}(\bar{D}\times[s, t])$, $0\leqq s<t<\infty$. Let

$$F(\tau)=\int_D f(x,\tau)|\phi_\tau(x)|^2dx,\qquad \tau\in[s, t].$$

Take any σ and τ, $s\leqq\sigma<\tau\leqq t$, then

$$\begin{aligned}|F(\tau)-F(\sigma)|&\leqq\left|\int_D f(x,\tau)\phi_\tau(\bar{\phi}_\tau-\bar{\phi}_\sigma)dx\right|+\left|\int_D f(x,\tau)\bar{\phi}_\sigma(\phi_\tau-\phi_\sigma)dx\right|\\&\quad+\left|\int_D(f(x,\tau)-f(x,\sigma))|\phi_\sigma|^2dx\right|\\&\leqq K\left(\int_D|\bar{\phi}_\tau-\bar{\phi}_\sigma|^2dx\right)^{1/2}+K\left(\int_D|\phi_\tau-\phi_\sigma|^2dx\right)^{1/2}+q(\tau,\sigma),\end{aligned}$$

where

$$K=\sup_{(x,\tau)\in\bar{D}\times[s,t]}|f(x,\tau)|,\qquad q(\tau,\sigma)=\sup_{x\in\bar{D}}|f(x,\tau)-f(x,\sigma)|.$$

For simplicity let $\tau-\sigma=\delta$, and denote by E the resolution of the identity for the self-adjoint operator H. Then from Proposition 1.1 we can write

$$\begin{aligned}\int_D|\psi_\tau-\psi_\sigma|^2dx &= \|U_\sigma(U_\delta-I)\psi_0\|^2 = \|(U_\delta-I)\psi_0\|^2\\ &=\int_{-\lambda+1}^{\infty}|e^{-i\delta x}-1|^2 d\|E(x)\psi_0\|^2 = \int_{-\lambda+1}^{\infty}(x\delta)^2\cos(\theta(x\delta))d\|E(x)\psi_0\|^2\\ &\leqq \delta^2\|H\psi_0\|^2, \qquad \text{for some } \theta(x\delta) \text{ such that } 0<\theta(x\delta)<x\delta.\end{aligned}$$

For $\int_D|\bar{\psi}_\tau-\bar{\psi}_\sigma|^2dx$ we have the same evaluation as above, and hence we can conclude that there exists a constant K' depending only on f and ψ_0 and

$$|F(\tau)-F(\sigma)| \leqq K'(\tau-\sigma)+q(\tau, \sigma), \qquad s\leqq\sigma<\tau\leqq t. \tag{1.12}$$

From (1.12) and the fact that $f\in C^{1,1}(\bar{D}\times[s, t])$, we see that $F(\tau)$, $\tau\in[s, t]$, is an absolutely continuous function of τ.

Next, we shall show that (1.3) holds. If (1.3) is true, then (1.4) follows from the absolute continuity of $F(t)$. In order to prove the first part of (1.3), it suffices to show that

$$\begin{aligned}\frac{d}{d\tau}\int_D f(x)|\psi_\tau(x)|^2dx = \frac{i}{2}\Big\{\int_D(\Delta\psi_\tau(x))\bar{\psi}_\tau(x)f(x)dx\\ -\int_D\psi_\tau(x)(\Delta\bar{\psi}_\tau(x))f(x)dx\Big\}, \quad \text{a.e. } \tau\in[s, t],\end{aligned} \tag{1.13}$$

for any $f\in C(\bar{D})$. If (1.13) is shown, then the other part of (1.3) follows by the integration by parts. We note that $(d/d\tau)\psi_\tau=\lim_{\delta\to0}(1/\delta)(\psi_{\tau+\delta}-\psi_\tau)=-iH\psi_\tau$, strongly, $^\forall\tau\in\boldsymbol{R}^1$, where $\psi_\tau=U_\tau\psi_0$, $\psi_0\in\mathcal{D}(H_0)$. Using this and Schwarz's inequality it is easy to see that

$$\left|\frac{1}{\delta}\Big(\int_D f|\psi_{\tau+\delta}|^2dx-\int_D f|\psi_\tau|^2dx\Big)-\int_D(-iH\psi_\tau)f\bar{\psi}_\tau dx-\int_D\psi_\tau f(\overline{iH\psi_\tau})dx\right| \longrightarrow 0$$

as $\delta\to0$ for any $f\in C(\bar{D})$. This is equivalent to (1.13).

Last of all we shall show the validity of (1.6). Since $\lambda\geqq1$, we have

$$\|(H_0+\lambda)\psi\|^2\geqq\int_0^\infty|x|^2d\|E_0(x)\psi\|^2=\|H_0\psi\|^2, \quad \text{and}$$

$$\|(H_0+\lambda)\psi\|^2 \geqq \int_0^\infty|\lambda|^2d\|E_0(x)\psi\|^2 = \lambda^2\|\psi\|^2, \qquad \psi\in\mathcal{D}(H_0).$$

Thus, there exists a constant $C_1>0$ such that

$$\|(H_0+\lambda)\psi\| \geqq C_1(\|H_0\psi\|+\|\psi\|), \qquad ^\forall\psi\in\mathcal{D}(H_0). \tag{1.14}$$

Since there exists a constant $k\geqq1$, and $x^2\leqq k(x+\lambda)^2$ for any $x\in[-\lambda+1, \infty)$, we have

$$\|(H+\lambda)\psi\|^2 = k^{-1}\int_{-\lambda+1}^{\infty}k|x+\lambda|^2d\|E(x)\psi\|^2 \geqq k^{-1}\|H\psi\|^2, \quad \psi\in\mathcal{D}(H_0). \tag{1.15}$$

In addition, since $|x+\lambda|=x+\lambda\geqq 1$ for $x\geqq -\lambda+1$, we have $\|(H+\lambda)\psi\|^2\geqq\|\psi\|^2$, $\psi\in\mathcal{D}(H_0)$. Combining this and (1.15) we see that there exists a constant $C_2>0$ and

$$\|(H+\lambda)\psi\| \geqq C_2(\|H\psi\|+\|\psi\|), \qquad {}^\forall\psi\in\mathcal{D}(H_0). \tag{1.16}$$

On the other hand the following hold obviously:

$$\lambda(\|H_0\psi\|+\|\psi\|) \geqq \|(H_0+\lambda)\psi\|, \tag{1.17}$$

$$\lambda(\|H\psi\|+\|\psi\|) \geqq \|(H+\lambda)\psi\|, \qquad {}^\forall\psi\in\mathcal{D}(H_0). \tag{1.18}$$

From Proposition 1.1, the operators H_0 and H are closed operators, and hence $\mathcal{D}(H_0)$ becomes Banach spaces with two graph norms $\|\psi\|+\|H_0\psi\|$ and $\|\psi\|+\|H\psi\|$. From (1.14) and (1.17) we see that the norms $\|(H_0+\lambda)\psi\|$ and $\|\psi\|+\|H_0\psi\|$ are equivalent, and from (1.16) and (1.18) we see that $\|(H+\lambda)\psi\|$ and $\|\psi\|+\|H\psi\|$ are equivalent norms. Thus the space $X=\mathcal{D}(H_0)$ with norm $\|\psi\|_0$ and $Y=\mathcal{D}(H_0)$ with norm $\|\psi\|_1$ are Banach spaces. Obviously the identity mappings J: $X\to Y$ and J': $Y\to X$ are closed operators. By the closed graph theorem, J and J' are bounded operators. Hence we can conclude that $\|\cdot\|_0$ and $\|\cdot\|_1$ are equivalent norms, and (1.6) holds. The proof is complete.

Define

$$u(x, t) = \begin{cases} \mathrm{Re}\dfrac{\tilde{\phi}(x, t)}{\phi(x, t)} & \text{if } \phi(x, t)\neq 0, \\ 0 & \text{if } \phi(x, t)=0, \end{cases}$$

$$v(x, t) = \begin{cases} \mathrm{Im}\dfrac{\tilde{\phi}(x, t)}{\phi(x, t)} & \text{if } \phi(x, t)\neq 0, \\ 0 & \text{if } \phi(x, t)=0, \end{cases}$$

$$\rho(x, t) = |\phi(x, t)|^2, \qquad t\in[0, T],\ T<\infty.$$

If we exchange $|\psi_t|^2$, $\mathrm{Re}(\psi_t'/\psi_t)$ and $\mathrm{Im}(\psi_t'/\psi_t)$ for $\rho(x, t)$, $u(x, t)$ and $v(x, t)$ respectively, then the assertions (ii) and (iii) in Lemma 1.2 are also valid for this change. Note that $C_0^\infty(\bar{D}\times[0, T])$ is dense in $L^2(\bar{D}\times[0, T];\ \rho(x, t)dxdt)$. Let

$$b(x, t) = v(x, t)+u(x, t), \quad \text{and} \quad b^*(x, t) = v(x, t)-u(x, t).$$

Then from Lemma 1.2-(ii) it holds that

$${}^\forall T<\infty,\ {}^\exists M_T<\infty,\ \int_0^T\int_\alpha^\beta (b(x, t))^2\rho(x, t)dxdt \leqq 2TM_T, \tag{1.19}$$

and

$$\int_0^T\int_\alpha^\beta (b^*(x, t))^2\rho(x, t)dxdt \leqq 2TM_T. \tag{1.20}$$

Thus, we can choose sequences $\{b_n, n=1, 2, \cdots\}$ and $\{b_n^*, n=1, 2, \cdots\}$ so that $b_n\in C_0^\infty(\bar{D}\times[0, T])$ and $b_n^*\in C_0^\infty(\bar{D}\times[0, T])$, and

$$\lim_{n\to\infty}\int_0^T\int_\alpha^\beta |b_n-b|^2\rho(x, t)dxdt = 0, \tag{1.21}$$

$$\lim_{n\to\infty}\int_0^T\int_\alpha^\beta |b_n^*-b^*|^2\rho(x, t)dxdt = 0. \tag{1.22}$$

In the sequel we shall construct a Markov semi-group. To this end we will make use of the following well known properties for initial boundary value problems (see for example Itô [**5**] and Kannai [**7**]).

Let

$$F=\{f \mid f\in C^\infty(\bar{D}),\ f'(\alpha)=f'(\beta)=0\}.$$

PROPOSITION 1.3. *Let $T<\infty$, and let $\{b_n, n=1, 2, \cdots\}$ and $\{b_n^*, n=1, 2, \cdots\}$ be two sequences in $C_0^\infty(\bar{D}\times[0, T])$ satisfying* (1.21) *and* (1.22) *respectively. Then, for each $n=1, 2, \cdots$, $s\in[0, T]$ and $f\in F$, the following hold:*

(i) *there exists a unique $(T_{r,s}^{T,n}f)(x)=g(x, r)\in C^\infty(\bar{D}\times[0, s])$, such that*

$$\frac{\partial}{\partial r}g=\Big(-\frac{1}{2}\Delta-b_n\frac{\partial}{\partial x}\Big)g, \qquad (x, r)\in D\times(0, s),$$

$$g'(\alpha, r)=g'(\beta, r)=0, \quad r\in(0, s), \qquad g(x, s)=f(x), \quad x\in\bar{D},$$

$$\inf_{y\in\bar{D}}f(y)\leqq g(x, r)\leqq\sup_{y\in\bar{D}}f(y), \qquad (x, r)\in\bar{D}\times[0, s],$$

(ii) *there exists a unique $(P_{t,s}^{T,n}f)(x)=h(x, t)\in C^\infty(\bar{D}\times[s, T])$, such that*

$$\frac{\partial}{\partial t}h=\Big(\frac{1}{2}\Delta-b_n^*\frac{\partial}{\partial x}\Big)h, \qquad (x, t)\in D\times(s, T),$$

$$h'(\alpha, t)=h'(\beta, t)=0, \quad t\in(s, T), \qquad h(x, s)=f(x), \quad x\in\bar{D},$$

$$\inf_{y\in\bar{D}}f(y)\leqq h(x, t)\leqq\sup_{y\in\bar{D}}f(y), \qquad (x, t)\in\bar{D}\times[s, T].$$

Lemma 1.4 below is a version of Theorem 3.1 of Carlen [**1**] for our Nelson process problem with boundary condition. The proof of this lemma depends essentially on (1.26) and the following (i) and (ii) (see Lemma 1.2-(iii)):

(i) for $f\in C^1(\bar{D})$, $\displaystyle\frac{d}{dt}\int_D f(x)\rho(x, t)dx=\int_D v(x, t)f'(x)\rho(x, t)dx,$ a. e. $t\in\boldsymbol{R}_+$,

$$\int_D f(x)\Big(\frac{\partial}{\partial x}\rho(x, t)\Big)dx=\int_D 2u(x, t)f(x)\rho(x, t)dx, \qquad {}^\forall t\in\boldsymbol{R}_+,$$

(ii) $b_n\to b=v+u$, $b_n^*\to b^*=v-u$, as $n\to\infty$, in $L^2(\bar{D}\times[0, T]; \rho(x, t)dxdt)$.

LEMMA 1.4. *For each $T<\infty$, and any $s\in[0, T]$ and $f\in F$, let $f_n(x, t)=(P_{t,s}^{T,n}f)(x)\in C^\infty(\bar{D}\times[s, T])$ and $g_n(x, r)=(T_{r,s}^{T,n}f)(x)\in C^\infty(\bar{D}\times[0, s])$, defined in Proposition* 1.3-(ii) *and* (i) *respectively. Then the following hold.*

$$\int_s^T\int_D |f_n'(x, t)|^2\rho(x, t)dxdt\leqq(K\times\bar{M}_T^*)^2, \qquad {}^\forall n\in N, \tag{1.23}$$

$$\int_0^s\int_D |g_n'(x, r)|^2\rho(x, r)dxdr \leqq (K\times\bar{M}_T)^2, \qquad {}^\forall n\in N,$$

$$\int_s^T\int_D |f_n'-f_m'|^2\rho_t dxdt \leqq 4K\Big\{\Big(\int_s^T\int_D |b_n^*-b^*|^2\rho_t dxdt\Big)^{1/2}\times\Big(\int_s^T\int_D |f_n'|^2\rho_t dxdt\Big)^{1/2}$$
$$+\Big(\int_s^T\int_D |b_m^*-b^*|^2\rho_t dxdt\Big)^{1/2}\times\Big(\int_s^T\int_D |f_m'|^2\rho_t dxdt\Big)^{1/2}\Big\}, \quad {}^\forall n, m\in N, \tag{1.24}$$

$$\int_0^s\int_D |g_n'-g_m'|^2\rho_r dxdr \leqq 4K\Big\{\Big(\int_0^s\int_D |b_n-b|^2\rho_r dxdr\Big)^{1/2}\times\Big(\int_0^s\int_D |g_n'|^2\rho_r dxdr\Big)^{1/2}$$
$$+\Big(\int_0^s\int_D |b_m-b|^2\rho_r dxdr\Big)^{1/2}\times\Big(\int_0^s\int_D |g_m'|^2\rho_r dxdr\Big)^{1/2}\Big\}, \quad {}^\forall n, m\in N,$$

$$\int_D |f_n(x, t)-f_m(x, t)|^2\rho(x, t)dx \leqq 4K^2\times\bar{M}_T^*\Big\{\Big(\int_s^t\int_D |b_n^*-b_m^*|^2\rho(x, \tau)dxd\tau\Big)^{1/2}$$
$$+\Big(\int_s^t\int_D |b_n^*+b_m^*-2b^*|^2\rho(x, \tau)dxd\tau\Big)^{1/2}\Big\}, \quad {}^\forall t\in[s, T],\ {}^\forall n, m\in N, \tag{1.25}$$

$$\int_D |g_n(x, r)-g_m(x, r)|^2\rho(x, r)dx \leqq 4K^2\times\bar{M}_T\Big\{\Big(\int_r^s\int_D |b_n-b_m|^2\rho(x, \tau)dxd\tau\Big)^{1/2}$$
$$+\Big(\int_r^s\int_D |b_n+b_m-2b|^2\rho(x, \tau)dxd\tau\Big)^{1/2}\Big\}, \quad {}^\forall r\in[0, s],\ {}^\forall n, m\in N,$$

where $\bar{M}_T^*=(4M_TT+2\bar{B}_T^*)^{1/2}+(4M_TT+2\bar{B}_T^*+4)^{1/2}$, $\bar{M}_T=(4M_TT+2\bar{B}_T)^{1/2}+(4M_TT+2\bar{B}_T+4)^{1/2}$,

$$\bar{B}_T^* = \sup_n\Big\{\int_0^T\int_D |b_n^*|^2\rho_t dxdt\Big\}, \qquad \bar{B}_T = \sup_n\Big\{\int_0^T\int_D |b_n|^2\rho_t dxdt\Big\},$$

M_T *is the constant in Lemma* 1.2-(ii), *and* $K=\sup_{x\in\bar{D}}|f(x)|$.

PROOF. $\rho(\alpha, t)$ and $\rho(\beta, t)$ may be strictly positive for $\phi_t\in\mathcal{D}(H_0)$. But, for $h\in F$ and $q\in C^1(\bar{D})$ we have the following integration by parts:

$$\int_D q'h'\rho(x, t)dx = -2\int_D q\Big(\Big(\frac{1}{2}\Delta+u\frac{d}{dx}\Big)h\Big)\rho(x, t)dx, \tag{1.26}$$
$${}^\forall q\in C^1(\bar{D}),\ {}^\forall h\in F,\ {}^\forall t\in[0, T].$$

Since Proposition 1.3-(ii) holds, for each fixed $t\in[s, T]$ we can take $q(x)=h(x)=f_n(x, t)$ in (1.26). If we note that f_n is the solution of the initial value problem in Proposition 1.3-(ii), then we have

$$\int_s^T\int_D f_n'f_n'\rho(x, t)dxdt = -2\int_s^T\int_D f_n\Big(\Big(\frac{1}{2}\Delta+u\frac{\partial}{\partial x}\Big)f_n\Big)\rho(x, t)dxdt$$
$$= -2\int_s^T\int_D f_n\Big(\Big(\frac{\partial}{\partial t}+b_n^*\frac{\partial}{\partial x}+u\frac{\partial}{\partial x}\Big)f_n\Big)\rho(x, t)dxdt. \tag{1.27}$$

On the other hand since $(f_n)^2\in C^\infty(\bar{D}\times[s, T])$, from (1.3) and (1.4) we have

$$-2\int_s^T\int_D f_n\dot{f}_n\rho(x, t)dxdt = 2\int_s^T\int_D f_nf_n'v\rho(x, t)dxdt+\int_D (f_n(x, s))^2\rho(x, s)dx$$
$$-\int_D (f_n(x, T))^2\rho(x, T)dx. \tag{1.28}$$

Inserting this into (1.27) and applying the quadratic formula (as is done in the derivation of (9) in Carlen [1]), we can derive (1.23).

Since (1.23) holds, and for each $t\in[s, T]$

$$W_{n,m}(\cdot, t)=f_n(\cdot, t)-f_m(\cdot, t)\in F, \quad \text{and} \quad W_{n,m}\in C^\infty(\bar{D}\times[s, T]),$$

from Lemma 1.2 and (1.26) the following formal calculations are valid.

$$\begin{aligned}\int_s^T\int_D|W'_{n,m}|^2\rho dxdt &= -2\int_s^T\int_D W_{n,m}\Big(\Big(\frac{1}{2}\Delta+u\frac{\partial}{\partial x}\Big)W_{n,m}\Big)\rho dxdt \\ &= -2\int_s^T\int_D W_{n,m}\Big(\frac{\partial}{\partial t}W_{n,m}+(u+b_n^*)f'_n-(u+b_m^*)f'_m\Big)\rho dxdt \\ &= 2\int_s^T\int_D W_{n,m}((b_m^*-b^*)f'_m-(b_n^*-b^*)f'_n)\rho dxdt+\|W_{n,m}\|_s^2-\|W_{n,m}\|_T^2,\end{aligned}$$

where $\|W\|_t^2=\int_D|W(x, t)|^2\rho(x, t)dx$, and here we used the similar formula as (1.28) for $W_{n,m}$. If we note that $W_{n,m}(x, s)=0$, and apply Schwarz's inequality to the above formula, then we get (1.24).

Since for each $t\in[s, T]$ $W_{n,m}(\cdot, t)\in F$, and $W_{n,m}\in C^\infty(\bar{D}\times[s, T])$, and since Lemma 1.2-(iii) and (1.23) hold, the formal calculations, which are made in the derivation of (11) in Carlen [1], can be carried out, and we can derive (1.25).

The proofs for the estimating formulas corresponding to g_n are very much similar to the ones corresponding to f_n, and hence omitted. The proof is complete.

Now, the following definition is valid.

DEFINITION 1. For each $T<\infty$ and any r, s, t, $0\leqq r\leqq s\leqq t\leqq T$, the operators $P_{t,s}^T$, $T_{r,s}^T$, $P_{s,T}^{T\prime}$, $T_{0,s}^{T\prime}$ are defined as follows:

$$P_{t,s}^T\colon F\ni f(x)\longmapsto(P_{t,s}^T f)(x)=\lim_{n\to\infty}(P_{t,s}^{T,n}f)(x)\in L^2(\bar{D};\ \rho(x, t)dx),$$

$$T_{r,s}^T\colon F\ni f(x)\longmapsto(T_{r,s}^T f)(x)=\lim_{n\to\infty}(T_{r,s}^{T,n}f)(x)\in L^2(\bar{D};\ \rho(x, r)dx),$$

$$\begin{aligned}P_{s,T}^{T\prime}\colon F\ni f(x)\longmapsto(P_{s,T}^{T\prime}f)(x, t) &= \lim_{n\to\infty}\frac{\partial}{\partial x}((P_{t,s}^{T,n}f)(x)) \\ &\in L^2(\bar{D}\times[s, T];\ \rho(x, t)dxdt),\end{aligned}$$

$$\begin{aligned}T_{0,s}^{T\prime}\colon F\ni f(x)\longmapsto(T_{0,s}^{T\prime}f)(x, r) &= \lim_{n\to\infty}\frac{\partial}{\partial x}((T_{r,s}^{T,n}f)(x)) \\ &\in L^2(\bar{D}\times[0, s];\ \rho(x, r)dxdr).\end{aligned}$$

Since, for each $t\in[0, T]$ and each $K_1, K_2\in[0, \infty)$ the set $\{g\,|\,g\in L^2(\bar{D};\ \rho(x, t)dx),\ -K_1\leqq g(x)\leqq K_2 \text{ a.e. } \rho(x, t)dx\}$ is a closed subset of $L^2(\bar{D};\ \rho(x, t)dx)$, from Proposition 1.3 and Lemma 1.4 we have the following.

LEMMA 1.5. *For any r, s, t, $0\leqq r\leqq s\leqq t\leqq T$, and $f\in F$,*

$\inf_{y\in\bar{D}} f(y) \leqq (P^T_{t,s}f)(x) \leqq \sup_{y\in\bar{D}} f(y)$, *a.e.* $x\in\bar{D}$ *with respect to* $\rho(x, t)dx$, *and*

$\inf_{y\in\bar{D}} f(y) \leqq (T^T_{r,s}f)(x) \leqq \sup_{y\in\bar{D}} f(y)$, *a.e.* $x\in\bar{D}$ *with respect to* $\rho(x, r)dx$.

We want to extend F, the domain of the operators $P^T_{t,s}$ and $T^T_{r,s}$, to $L^2(\bar{D}; \rho(x, s)dx)$. To this end we prepare Lemma 1.6.

LEMMA 1.6. *For each* $T<\infty$ *and any* s, $0\leqq s\leqq T$, *and* $f\in F$, *the following hold.*

$$\int_t^\tau\int_D |(P^{T'}_{s,T}f)(x, r)|^2\rho(x, r)dxdr+\|P^T_{\tau,s}f\|^2_\tau = \|P^T_{t,s}f\|^2_t, \tag{1.29}$$

$$\int_D(P^T_{\tau,s}f)(x)\rho(x, \tau)dx = \int_D(P^T_{t,s}f)(x)\rho(x, t)dx, \tag{1.30}$$

$$\|P^{T,n}_{\tau,s}f\|_\tau \leqq M^{*n}_K+\|P^{T,n}_{t,s}f\|_t, \quad {}^\forall t, \tau, \quad s\leqq t\leqq\tau\leqq T, \tag{1.31}$$

and

$$\int_t^\tau\int_D |(T^{T'}_{0,s}f)(x, r)|^2\rho(x, r)dxdr+\|T^T_{t,s}f\|^2_t = \|T^T_{\tau,s}f\|^2_\tau, \tag{1.32}$$

$$\int_D(T^T_{\tau,s}f)(x)\rho(x, \tau)dx = \int_D(T^T_{t,s}f)(x)\rho(x, t)dx, \tag{1.33}$$

$$\|T^{T,n}_{\tau,s}f\|_\tau \leqq M^n_K+\|T^{T,n}_{t,s}f\|_t, \quad {}^\forall t, \tau, \quad 0\leqq t\leqq\tau\leqq s, \tag{1.34}$$

where

$$M^{*n}_K = 2K\Big(\int_0^T\int_D |b^*_n-b^*|^2\rho_r\, dxdr\Big)^{1/2}, \quad M^n_K = 2K\Big(\int_0^T\int_D |b_n-b|^2\rho_r\, dxdr\Big)^{1/2}$$
$$\text{and} \quad K=\sup_{x\in\bar{D}}|f(x)|.$$

PROOF. Since $g_n(x, \tau)=(T^{T,n}_{\tau,s}f)(x)$ and $f_n(x, \tau)=(P^{T,n}_{\tau,s}f)(x)$ satisfy $g_n\in C^\infty(\bar{D}\times[0, s])$, $f_n\in C^\infty(\bar{D}\times[s, T])$, and $g_n(\cdot, \tau)\in F$ for each $\tau\in[0, s]$, $f_n(\cdot, \tau)\in F$ for each $\tau\in[s, T]$. Noting (1.26), (1.23) and Lemma 1.2-(iii), we can derive (1.29), (1.31), (1.32) and (1.34) through the same discussion which are made in the derivation of (13) and (15) in Carlen [**1**].

In order to see that (1.30) holds, we repeat the proof of (14) in Carlen [1] for our Nelson process problem with boundary condition. Since

$$\frac{1}{2}\int_D(\Delta f_n(x, \tau))\rho(x, \tau)dx = [f'_n(x, \tau)u(x, \tau)\rho(x, \tau)]^\beta_\alpha-\int_D f'_n(x, \tau)u(x, \tau)\rho(x, \tau)dx$$
$$= -\int_D f'_n(x, \tau)u(x, \tau)\rho(x, \tau)dx, \quad {}^\forall\tau\in[s, T],$$

from Lemma 1.2-(iii) we have

$$\frac{d}{d\tau}\int_D f_n\rho_\tau dx = \int_D\Big(\frac{1}{2}\Delta+u\frac{\partial}{\partial x}\Big)f_n\rho_\tau dx-\int_D(b^*_n-(v-u))\Big(\frac{\partial}{\partial x}f_n\Big)\rho_\tau dx$$
$$= -\int_D(b^*_n-(v-u))\Big(\frac{\partial}{\partial x}f_n\Big)\rho_\tau dx, \quad \text{a.e. } \tau\in[s, T].$$

Thus, from Lemma 1.2-(iii) and Schwarz's inequality we have

$$\left|\int_D f_n\rho_\tau dx-\int_D f_n\rho_t dx\right| \leqq \left(\int_t^\tau\int_D (b_n^*-(v-u))^2\rho_r\,dxdr\right)^{1/2}$$
$$\times\left(\int_t^\tau\int_D\left|\frac{\partial}{\partial x}f_n\right|^2\rho_r\,dxdr\right)^{1/2}.$$

From (1.22) and (1.23) we see that the right hand side of the above formula tends to 0 as $n\rightarrow\infty$, and hence from (1.25) we have (1.30).

The proof of (1.33) is very similar to the one of (1.30), and hence omitted. The proof is complete.

Set $t=s$ in (1.29) and $\tau=s$ in (1.32). Since $C_0^\infty(D)\subset F$ and $C_0^\infty(D)$ is dense in $L^2(\bar{D};\ \rho(x, s)dx)$, and by Lemma 1.4 $P_{t,s}^T$, $P_{s,T}^{T\prime}$, $T_{t,s}^T$ and $T_{0,s}^{T\prime}$ are linear operators, we can extend the domains of these operators.

DEFINITION 2. $P_{t,s}^T$, $T_{r,s}^T$, $P_{s,T}^{T\prime}$ and $T_{0,s}^{T\prime}$ are the continuous extensions of the operators in Definition 1:

$$P_{t,s}^T:\ L^2(\bar{D};\ \rho(x, s)dx)\longrightarrow L^2(\bar{D};\ \rho(x, t)dx),$$
$$T_{r,s}^T:\ L^2(\bar{D};\ \rho(x, s)dx)\longrightarrow L^2(\bar{D};\ \rho(x, r)dx),$$
$$P_{s,T}^{T\prime}:\ L^2(\bar{D};\ \rho(x, s)dx)\longrightarrow L^2(\bar{D}\times[s, T];\ \rho(x, \tau)dxd\tau),$$
$$T_{0,s}^{T\prime}:\ L^2(\bar{D};\ \rho(x, s)dx)\longrightarrow L^2(\bar{D}\times[0, s];\ \rho(x, \tau)dxd\tau),\qquad 0\leqq r\leqq s\leqq t\leqq T.$$

REMARK. By the construction of $P_{t,s}^T$, $T_{r,s}^T$, $P_{s,T}^{T\prime}$ and $T_{0,s}^{T\prime}$, it is obvious that $P_{t,s}^T=P_{t,s}^U$, $T_{r,s}^T=T_{r,s}^U$, $P_{s,T}^{T\prime}=\Theta_T P_{s,U}^{U\prime}$ and $T_{0,s}^{T\prime}=T_{0,s}^{U\prime}$ for $0\leqq r\leqq s\leqq t\leqq T$ with $T\leqq U$, where Θ_T is the projection from $L^2(\bar{D}\times[s, U];\ \rho(x, \tau)dxd\tau)$ to $L^2(\bar{D}\times[s, T];\ \rho(x, \tau)dxd\tau)$.

Noting the above Remark, from now on we shall omit the superscript T from $P_{t,s}^T$, $T_{r,s}^T$, $P_{s,T}^{T\prime}$ and $T_{0,s}^{T\prime}$.

We list Lemma 1.7 without proof, making use of Lemma 1.4, (1.31) and (1.34) this lemma can be proved through the completely same discussion which is made in proving Theorem 3.3 in Carlen [**1**].

LEMMA 1.7. *It holds that*

$$P_{t,r}=P_{t,s}\circ P_{s,r},\quad \textit{and}\quad T_{r,t}=T_{r,s}\circ T_{s,t},\qquad \textit{for}\quad 0\leqq r\leqq s\leqq t\leqq T.$$

2. Construction of Nelson processes.

Let $\Omega=\bar{D}^{[0,\infty)}$, and $\mathcal{F}'=(\mathcal{B}(\bar{D}))^{[0,\infty)}$, the product σ-field, where $\mathcal{B}(\bar{D})$ is the Borel σ-field of $\bar{D}$. Let X_t, $t\in[0, \infty)$, be the projection (t-configulation function on Ω),

$$X_t:\ \Omega\ni\omega\longmapsto X_t(\omega)\in\bar{D}.$$

For each finite subset U of $[0, \infty)$ let Ω_U denote the metrizable space $\bar{D}^U$.

From Lemma 1.5, Definition 2, Remark, Lemma 1.7 and Riesz-Markov theorem there exist tight probability measures P_U^* and P_U on Ω_U uniquely so that

$$E^{P_U^*}[F]=(f_n,\ P_{t_n,t_{n-1}}f_{n-1}P_{t_{n-1},t_{n-2}}f_{n-2}\cdots P_{t_2,t_1}f_1)_{t_n},\quad \text{and}$$
$$E^{P_U}[F]=(f_1,\ T_{t_1,t_2}f_2T_{t_2,t_3}f_3\cdots T_{t_{n-1},t_n}f_n)_{t_1},\quad \text{with}$$

$U=\{t_1, t_2, \cdots, t_n\}$, $0\leqq t_1\leqq t_2\leqq\cdots\leqq t_n<\infty$, and $F(\omega_U)=\prod_{i=1}^n f_i(X_{t_i}^U(\omega_U))$, $f_i\in C^\infty(\bar{D})$, $i=1, \cdots, n$, where $X_{t_i}^U$: $\Omega_U\ni\omega_U\mapsto X_{t_i}^U(\omega_U)$, the t_i-configulation function on Ω_U, $(f, g)_t=\int_D f(x)g(x)\rho(x, t)dx$, and $E^{P_U^*}$ and E^{P_U} denote the expectations with respect to the probability measures P_U^* and P_U respectively.

By Kolmogorov's extension theorem (see Corollary III-52 of Dellacherie and Meyer [**3**]), we can conclude that there exist probability measures P^* and P on $(\Omega, \mathcal{F}')$ uniquely such that

$$P^*q_U^{-1}=P_U^* \quad \text{and} \quad Pq_U^{-1}=P_U \qquad \text{for every finite}\quad U\subset[0, \infty),$$

where q_U is the projection of Ω onto Ω_U.

Now, let $\mathcal{F}^{P^*}$ and $\mathcal{F}^P$ be the completion of $\mathcal{F}'$ with respect to P^* and P respectively.

LEMMA 2.1. (i) *For any* $f\in C^{2,1}(\bar{D}\times[s, t])$, $0\leqq s\leqq t\leqq\infty$, *and* $g\in F$, *there exists a limit* $\lim_{n\to\infty}\int_s^t[(P_{\tau,s}^{t,n}g)(x)f'(x, \tau)\rho(x, \tau)]_\alpha^\beta d\tau=R_{g,f}(s, t)$, *and it holds that*

$$-\frac{1}{2}R_{g,f}(s, t)=\int_\alpha^\beta(P_{t,s}g)(x)f(x, t)\rho(x, t)dx-\int_\alpha^\beta g(x)f(x, s)\rho(x, s)dx$$
$$-\int_s^t\int_\alpha^\beta\Big\{(P_{\tau,s}g)(x)\Big(\frac{1}{2}\Delta+b(x, \tau)\frac{\partial}{\partial x}+\frac{\partial}{\partial\tau}\Big)f(x, \tau)\Big\}\rho(x, \tau)dxd\tau. \tag{2.1}$$

(ii) *For any* $f\in C^{2,1}(\bar{D}\times[0, t])$, $0\leqq t<\infty$, *any bounded non-negative measurable function* g *on* $\bar{D}$ *and any* $s\in[0, t]$, *it holds that*

$$E^{P^*}\Big[g(X_s)\Big\{f(X_t, t)-f(X_s, s)-\int_s^t\Big(\frac{1}{2}\Delta+b(X_\tau, \tau)\frac{\partial}{\partial x}+\frac{\partial}{\partial\tau}\Big)f(X_\tau, \tau)d\tau\Big\}\Big]$$
$$\begin{Bmatrix}\leqq\\ \geqq\end{Bmatrix}0 \quad if \quad \begin{cases}f'(\alpha, \tau)\leqq 0, & f'(\beta, \tau)\geqq 0, \quad {}^\forall\tau\in[0, t],\\ f'(\alpha, \tau)\geqq 0, & f'(\beta, \tau)\leqq 0, \quad {}^\forall\tau\in[0, t],\ \textit{respectively}.\end{cases} \tag{2.2}$$

PROOF. Since $(P_{\tau,s}^{t,n}g)(x)=g_n(x, \tau)\in C^\infty(\bar{D}\times[s, t])$ is a solution of the initial value problem in Proposition 1.3-(ii), and $f\in C^{2,1}(\bar{D}\times[s, t])$, by (1.3), (1.26) and the integration by parts argument we have

$$\frac{d}{d\tau}\int_\alpha^\beta g_n(x, \tau)f(x, \tau)\rho(x, \tau)dx=\int_\alpha^\beta\Big(\Big(\frac{1}{2}\Delta-b_{n\tau}^*\frac{\partial}{\partial x}\Big)g_n\Big)f_\tau\rho_\tau dx$$
$$+\int_\alpha^\beta v_\tau(g_n'f_\tau+g_nf_\tau')\rho_\tau dx+\int_\alpha^\beta g_n\dot{f}_\tau\rho_\tau dx$$

$$=-\frac{1}{2}[g_n(x,\tau)f'(x,\tau)\rho(x,\tau)]_\alpha^\beta+\int_\alpha^\beta\{Q(g_n,f)+W(g_n',f,b_n^*)\}\rho_\tau dx,$$
$$\text{a.e.}\quad \tau\in[s,t],$$

where Q and W are defined as follows;

$$Q(q,f)=q(x,\tau)\times\Big(\Big(\frac{1}{2}\Delta+b(x,\tau)\frac{\partial}{\partial x}+\frac{\partial}{\partial\tau}\Big)f(x,\tau)\Big),$$
$$W(q,f,h)=q(x,\tau)f(x,\tau)(v(x,\tau)-u(x,\tau)-h(x,\tau)),$$

for measurable functions q, h on $\bar{D}\times[s,t]$ and $f\in C^{2,1}(\bar{D}\times[s,t])$, and here we simply denote $f(x,\tau)$ by f_τ, $\rho(x,\tau)$ by ρ_τ and so on. Thus, from (1.4) we have

$$-\frac{1}{2}\int_s^t[g_n(x,\tau)f'(x,\tau)\rho(x,\tau)]_\alpha^\beta d\tau=\int_\alpha^\beta g_n(x,t)f_t\rho_t dx-\int_\alpha^\beta g_n(x,s)f_s\rho_s dx$$
$$-\int_s^t\int_\alpha^\beta\{Q(g_n,f)+W(g_n',f,b_n^*)\}\rho_\tau dxd\tau.$$

Since $g_n(x,s)=g(x)$ and Lemma 1.4 holds, if we let $n\to\infty$, then we get (2.1).

Since Lemma 1.4 holds, by Fubini's lemma it holds that

$$\text{right hand side of (2.1)}=\text{left hand side of (2.2)},\qquad {}^\forall f\in C^{2,1}(\bar{D}\times[0,t]),\quad {}^\forall g\in F. \tag{2.3}$$

On the other hand it holds that $(P_{\tau,s}^{t,n}g)(x)\geqq 0$, $(x,\tau)\in\bar{D}\times[s,t]$, for $g(x)\geqq 0$, $x\in\bar{D}$. Thus, for f such that $f'(\alpha,\tau)\leqq 0$ and $f'(\beta,\tau)\geqq 0$, from (2.1) and (2.3) the upper side of (2.2) follows. Similarly the lower side of (2.2) follows. Since F is dense in $L^2(\bar{D};\rho(x,s)dx)$, and Lemma 1.6 holds, we can conclude that (2.2) holds for any bounded non-negative measurable function g. The proof is complete.

LEMMA 2.2. *For any bounded measurable functions $f_1, f_2, \cdots, f_n$, and any $0\leqq t_1\leqq t_2\leqq\cdots\leqq t_n<\infty$, it holds that*

$$E^P[f_1(X_{t_1})\times f_2(X_{t_2})\times\cdots\times f_n(X_{t_n})]=E^{P^*}[f_1(X_{t_1})\times f_2(X_{t_2})\times\cdots\times f_n(X_{t_n})]. \tag{2.4}$$

And $P=P^*$ as a probability measure on $(\Omega,\mathcal{F}')$.

PROOF. First, we shall show that

$$E^P[f(X_t)g(X_s)]=E^{P^*}[f(X_t)g(X_s)],\qquad 0\leqq s\leqq t<\infty,\quad f,g\in F. \tag{2.5}$$

To this end we see that

$$E^{P^*}[f(X_t)g(X_s)]-E^{P^*}[f_k(X_s,s)g(X_s)]$$
$$=\int_s^t\int_\alpha^\beta(b(x,\tau)-b_k(x,\tau))f_k'(x,\tau)(P_{\tau,s}g)(x)\rho(x,\tau)dxd\tau, \tag{2.6}$$

where $f, g\in F$ and $f_k(x,\tau)=(T_{\tau,t}^{t,k}f)(x)\in C^\infty(\bar{D}\times[0,t])$. From Proposition 1.3-(i) and (2.1) it holds that $R_{g,f_k}(s, t)=0$, and hence we have

$$\int_\alpha^\beta (P_{t,s}g)f_k(x, t)\rho_t dx - \int_\alpha^\beta g f_k(x, s)\rho_s dx$$
$$= \int_s^t \int_\alpha^\beta (b_\tau - b_{k\tau}) f_k'(x, \tau)(P_{\tau,s}g)\rho_\tau dx d\tau, \tag{2.7}$$

and this is equivalent to (2.6).

From the definition of P and P^*, it holds that

$$E^{P^*}[h(X_s)] = \int_\alpha^\beta h(x)\rho(x, s)dx = E^P[h(X_s)],$$

for any $h \in L^1(\bar{D}; \rho(x, s)dx)$. Hence, we have the following evaluation:

$$\begin{aligned} &|E^{P^*}[f(X_t)g(X_s)] - E^P[f(X_t)g(X_s)]| \\ &= |E^{P^*}[f(X_t)g(X_s)] - E^P[(T_{s,t}f)(X_s)g(X_s)]| \\ &\leqq |E^{P^*}[f(X_t)g(X_s)] - E^P[f_k(X_s, s)g(X_s)]| \\ &\quad + |E^P[f_k(X_s, s)g(X_s)] - E^P[(T_{s,t}f)(X_s)g(X_s)]|. \end{aligned}$$

Since (2.6) and Lemma 1.4 hold, the first term of the right hand side tends to 0 as $k\to\infty$, and the second term also tends to 0 as $k\to\infty$. Hence we have verified the validity of (2.5) for $f, g \in F$.

Since Lemma 1.4 holds and F is dense in $L^2(\bar{D}; \rho(x, t)dx)$ and $L^2(\bar{D}; \rho(x, s)dx)$, (2.5) holds for any bounded measurable functions f and g. By induction we can show that (2.4) holds.

Noting the construction of P and P^*, we see that $P=P^*$ from (2.4). The proof is complete.

DEFINITION 3. Let $\mathcal{F}=\mathcal{F}^{P^*}=\mathcal{F}^P$, and let $X_t: \Omega \ni \omega \mapsto X_t(\omega) \in \bar{D}$, $t \in [0, \infty)$, by the projection. For each $t\in[0, \infty)$ let $\mathcal{G}_t=\sigma\{\{X_s, 0\leqq s\leqq t\}\cup\mathcal{N}\}$, the sub σ-field of $\mathcal{F}$ induced by X_s, $0\leqq s\leqq t$, containing the totality of P-null sets $\mathcal{N}$, and define $\mathcal{F}_t=\bigcap_{t'>t}\mathcal{G}_{t'}$, $t\in[0, \infty)$, the right continuous increasing family of sub σ-field of $\mathcal{F}$.

THEOREM 2.3. *It holds that*

$$P(X_t\in A|\mathcal{F}_s) = P(X_t\in A|X_s), \qquad a.s.\ P,\ \ 0\leqq s\leqq t<\infty,\ {}^\forall A\in\mathcal{B}(\bar{D}),$$
$$\text{Borel } \sigma\text{-field of } \bar{D}. \tag{2.8}$$

For each $f\in C^{2,1}(\bar{D}\times[0, \infty))$ *let*

$$Z_t^f = f(X_t, t) - \int_0^t \left(\frac{1}{2}\Delta + b(X_\tau, \tau)\frac{\partial}{\partial x} + \frac{\partial}{\partial \tau}\right) f(X_\tau, \tau)d\tau. \tag{2.9}$$

Then the $\mathcal{F}_t$*-adapted stochastic process* $\{Z_t^f\}$, $t\in[0, \infty)$, *on* $(\Omega, \mathcal{F}, P)$ *is a su-*

permartingale in class (DL)[2] *if* $f'(\alpha, t)\leqq 0$, $f'(\beta, t)\geqq 0$, $t\in[0, \infty)$, *and is a submartingale in class* (DL) *if* $f'(\alpha, t)\geqq 0$, $f'(\beta, t)\leqq 0$, $t\in[0, \infty)$.

PROOF. According to VI in Dellacherie and Meyer [**4**], we have

$$\lim_{\delta\downarrow 0} E^P[I_A(X_t)\,|\,\mathcal{G}_{s+\delta}] = E^P[I_A(X_t)\,|\,\mathcal{F}_s], \quad \text{a.s. } P,\ {}^\forall A\in\mathcal{B}(\bar{D}),\quad 0\leqq s<t<\infty, \tag{2.10}$$

where $I_A(x)$ is the indicator function;

$$I_A(x) = \begin{cases} 1 & \text{if } x\in A, \\ 0 & \text{if } x\notin A. \end{cases}$$

On the other hand, from Lemma 1.7 and the construction of P, we have

$$E^P[I_A(X_t)\,|\,\mathcal{G}_{s+\delta}] = E^P[I_A(X_t)\,|\,X_{s+\delta}] = (T_{s+\delta,t}I_A)(X_{s+\delta}), \quad \text{a.s. } P, \quad {}^\forall A\in\mathcal{B}(\bar{D}),\ 0\leqq s+\delta<t<\infty. \tag{2.11}$$

Noting (2.10) and (2.11), we can evaluate as follows:

$$\begin{aligned}
&E^P[\,|E^P[I_A(X_t)\,|\,\mathcal{F}_s]-(T_{s,t}I_A)(X_s)|\,] \\
&\leqq E^P[\,|E^P[I_A(X_t)\,|\,\mathcal{F}_s]-(T_{s+\delta,t}I_A)(X_{s+\delta})|\,]+E^P[\,|(T_{s+\delta,t}I_A)(X_{s+\delta})-(T_{s,t}I_A)(X_s)|\,] \\
&\leqq E^P[\,|E^P[I_A(X_s)\,|\,\mathcal{F}_s]-(T_{s+\delta,t}I_A)(X_{s+\delta})|\,] \\
&\quad +E^P[\,|(T_{s+\delta,t}I_A)(X_{s+\delta})-(T_{s+\delta,t}f)(X_{s+\delta})|\,] \\
&\quad +E^P[\,|(T_{s+\delta,t}f)(X_{s+\delta})-(T^{t;n}_{s+\delta,t}f)(X_{s+\delta})|\,] \\
&\quad +E^P[\,|(T^{t;n}_{s+\delta,t}f)(X_{s+\delta})-(T^{t;n}_{s;t}f)(X_{s+\delta})|\,] \\
&\quad +E^P[\,|(T^{t;n}_{s;t}f)(X_{s+\delta})-(T^{t;n}_{s;t}f)(X_s)|\,]+E^P[\,|(T^{t;n}_{s;t}f)(X_s)-(T_{s,t}f)(X_s)|\,] \\
&\quad +E^P[\,|(T_{s,t}f)(X_s)-(T_{s,t}I_A)(X_s)|\,], \qquad f\in F.
\end{aligned} \tag{2.12}$$

Since F is dense in $L^2(\bar{D};\ \rho(x, t)dx)$, and (1.34) holds, for any $\varepsilon>0$ we can choose $f\in F$ and the second and seventh terms in the right hand side of (2.12) become less than ε. Since Lemma 1.4 holds, for such $f\in F$, we can choose sufficiently large N, and the third and sixth terms in the right hand side of (2.12) are less than ε when $n\geqq N$. Since Proposition 1.3-(ii) holds, for each fixed f and n as above and fixed $s<t$, the fourth and fifth terms in the right hand side of (2.12) can be arbitrarily small for sufficiently small $\delta>0$. From (2.10), (2.11) and the dominated convergence theorem, the first term in the right hand side of (2.12) also can be arbitrarily small for sufficiently small $\delta>0$. Hence, we can conclude that $E^P[I_A(X_t)\,|\,\mathcal{F}_s]=E^P[I_A(X_t)\,|\,X_s]$. (2.8) is proved.

Since $P=P^*$, for $0\leqq s\leqq t<\infty$ (2.2) means that

$$E^P[g(X_s)(Z^f_t-Z^f_s)] \begin{Bmatrix} \leqq \\ = \\ \geqq \end{Bmatrix} 0 \quad \text{if} \begin{cases} f'(\alpha, \tau)\leqq 0, & f'(\beta, \tau)\geqq 0, \quad \tau\in[0, \infty), \\ f'(\alpha, \tau)\geqq 0, & f'(\beta, \tau)\leqq 0, \quad \tau\in[0, \infty), \end{cases} \tag{2.13}$$

(2) See [**4**].

for any bounded non-negative measurable function g. Since $Z_t^f - Z_s^f$ is $\sigma\{\{X_u; s\leqq u\leqq t\}\cup\mathfrak{N}\}$ measurable, and $\{X_t\}$, $t\in[0, \infty)$, satisfies (2.8), thus (2.13) shows the supermartingale and submartingale properties of $\{Z_t^f\}$, $t\in[0, \infty)$.

For $T<\infty$, let $\xi_t=\int_0^t b(X_\tau, \tau)d\tau$, $t\in[0, T]$, and let $\mathfrak{M}_T$ be the totality of any $\mathcal{F}_t$, $t\in[0, T]$, stopping times σ ($\leqq T$, a.s.). Then from (1.2) and Fubini's lemma we have

$$\sup_{\sigma\in\mathfrak{M}_T} E^P[|\xi_\sigma|] \leqq T(2M_T)^{1/2}, \tag{2.14}$$

and

$$E^P[|\xi_\sigma|I_A(\omega)] \leqq (P(A))^{1/2}\times\Big(T\cdot E^P\Big[\int_0^T (b(X_t, t))^2 dt\Big]\Big)^{1/2} \leqq T(P(A))^{1/2}(2M_T)^{1/2},$$
$$^\forall A\in\mathcal{F}, \quad ^\forall\sigma\in\mathfrak{M}_T. \tag{2.15}$$

From (2.15) we see that

$$\lim_{P(A)\to 0}\sup_{\sigma\in\mathfrak{M}_T} E^P[|\xi_\sigma|I_A]=0, \qquad A\in\mathcal{F}. \tag{2.16}$$

Since $f\in C^{2,1}(\bar{D}\times[0, \infty))$, from (2.14) and (2.16) it follows that $\{Z_t^f\}$, $t\in[0, \infty)$, is in class (DL). The proof is complete.

3. Existence of equivalent processes with continuous trajectories.

In this section we shall show the existence of continuous modification of $\{X_t\}$, $t\in[0, \infty)$, on $(\Omega, \mathcal{F}, P)$, and give the stochastic integral representation for the continuous process.

THEOREM 3.1. *For given $\psi_0\in\mathcal{D}(H_0)$, let $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ be the probability space defined in Section 2. Then there exists an $\mathcal{F}_t$-adapted continuous stochastic process $\{\tilde{X}_t\}$, $t\in[0, \infty)$, such that*

$$P(\tilde{X}_t=X_t) = 1, \qquad ^\forall t\in[0, \infty), \tag{3.1}$$

where X_t is the projection (t-configuration function on Ω).

In order to prove Theorem 3.1, we need the following lemma:

LEMMA 3.2. *Let $f\in F$. Then there exists a continuous process $\{\tilde{f}_t\}$, $t\in[0, \infty)$, such that*

$$P(\tilde{f}_t=f(X_t)) = 1, \qquad ^\forall t\in[0, \infty). \tag{3.2}$$

PROOF OF THEOREM 3.1. Suppose that Lemma 3.2 holds. Take $f\in F$, which is a strictly increasing function on D. If we set

$$\tilde{X}_t = f^{-1}(\tilde{f}_t), \qquad t\in[0, \infty),$$

then $\{\tilde{X}_t\}$, $t\in[0, \infty)$, satisfies the desired property. The proof is complete.

Now, we shall prove Lemma 3.2.

PROOF OF LEMMA 3.2. Let $0\leqq s\leqq t\leqq T<\infty$. From (2.1) we have

$$E^P[g(X_s)(f(X_t)-f(X_s))]$$
$$=\int_s^t\int_\alpha^\beta\Big(\frac{1}{2}\Delta+b(x,\tau)\frac{\partial}{\partial x}\Big)f(x)(P_{\tau,s}g)(x)\rho(x,\tau)dxd\tau,\qquad f,g\in F, \tag{3.3}$$

here we used the fact that $P=P^*$. Using (3.3), after simple calculation we have

$$E^P[|f(X_t)-f(X_s)|^4]$$
$$=E^P[\{(f(X_t))^4-(f(X_s))^4\}+6(f(X_s))^2\{(f(X_t))^2-(f(X_s))^2\}$$
$$-4f(X_s)\{(f(X_t))^3-(f(X_s))^3\}-4(f(X_s))^3\{f(X_t)-f(X_s)\}]$$
$$=4\int_s^t\int_\alpha^\beta b_\tau f'(f^3-P_{\tau,s}(f^3))\rho_\tau dxd\tau+12\int_s^t\int_\alpha^\beta b_\tau f'f(P_{\tau,s}(f^2)-fP_{\tau,s}f)\rho_\tau dxd\tau$$
$$+2\int_s^t\int_\alpha^\beta f''(f^3-P_{\tau,s}(f^3))\rho_\tau dxd\tau+6\int_s^t\int_\alpha^\beta f''f(P_{\tau,s}(f^2)-fP_{\tau,s}f)\rho_\tau dxd\tau$$
$$+6\int_s^t\int_\alpha^\beta(f')^2(f^2+P_{\tau,s}(f^2)-2fP_{\tau,s}f)\rho_\tau dxd\tau$$
$$=4\varepsilon_1+12\varepsilon_2+2\varepsilon_3+6\varepsilon_4+6\varepsilon_5,\qquad f\in F.$$

Here, we set especially $b_1^*=0$, and evaluate ε_1 as follows:

$$|\varepsilon_1|\leqq\Big|\int_s^t\int_\alpha^\beta b_\tau f'(f^3-P^1_{\tau,s}(f^3))\rho_\tau dxd\tau\Big|$$
$$+\Big|\int_s^t\int_\alpha^\beta b_\tau f'(P^1_{\tau,s}(f^3)-P_{\tau,s}(f^3))\rho_\tau dxd\tau\Big|=\varepsilon_{11}+\varepsilon_{12}. \tag{3.4}$$

Let $K_1=\sup_{x\in\bar D}|f'(x)|$, then, noting (1.2) we have

$$|\varepsilon_{12}|\leqq K_1\Big(\int_s^t\int_\alpha^\beta|b_\tau|^2\rho_\tau dxd\tau\Big)^{1/2}\Big(\int_s^t\int_\alpha^\beta|P^1_{\tau,s}(f^3)-P_{\tau,s}(f^3)|^2\rho_\tau dxd\tau\Big)^{1/2}$$
$$\leqq K_1(2M_T)^{1/2}(t-s)^{1/2}\Big(\int_s^t\int_\alpha^\beta|P^1_{\tau,s}(f^3)-P_{\tau,s}(f^3)|^2\rho_\tau dxd\tau\Big)^{1/2}. \tag{3.5}$$

And, from (1.2) and (1.25) we have

$$\int_\alpha^\beta|P^1_{\tau,s}(f^3)-P_{\tau,s}(f^3)|^2\rho_\tau dx\leqq 8(K_2)^2\overline{M}{}^*_T\Big(\int_s^\tau\int_\alpha^\beta(b_\tau^*)^2\rho_u dxdu\Big)^{1/2}$$
$$\leqq 8(K_2)^2\overline{M}{}^*_T(2M_T(\tau-s))^{1/2},\qquad\text{where }\ K_2=\sup_{x\in\bar D}|f^3(x)|. \tag{3.6}$$

Hence, from (3.5) and (3.6) we see that there exists a constant K_3, depending only on T and f such that

$$|\varepsilon_{12}|\leqq K_3(t-s)^{5/4},\qquad 0\leqq s<t\leqq T. \tag{3.7}$$

On the other hand, from Schwarz's inequality we have

$$|\varepsilon_{11}|\leqq K_1(2M_T)^{1/2}(t-s)^{1/2}\Big(\int_s^t\int_\alpha^\beta|f^3-P^1_{\tau,s}(f^3)|^2\rho_\tau dxd\tau\Big)^{1/2}. \tag{3.8}$$

But, from Proposition 1.3, we see that there exists a constant K_4 depending only on T and f, such that

$$\int_\alpha^\beta |f^3-P^1_{\tau,s}(f^3)|^2\rho_\tau dx \leqq K_4|\tau-s|^2, \qquad \text{for any} \quad 0\leqq s<\tau\leqq T.$$

Hence, from (3.8) we see that there exists a constant K_5 depending only on T and f, and

$$|\varepsilon_{11}| \leqq K_5(t-s)^2, \qquad \text{for any} \quad 0\leqq s<t\leqq T. \tag{3.9}$$

Combining (3.7) and (3.9), we can conclude that there exists a constant K_6 depending only on T and f, and

$$|\varepsilon_1| \leqq K_6(t-s)^{5/4}, \qquad \text{for any} \quad 0\leqq s<t\leqq T. \tag{3.10}$$

Similarly, we can show that $|\varepsilon_2|$, $|\varepsilon_3|$, $|\varepsilon_4|$ and $|\varepsilon_5|$ are dominated by $(t-s)^{5/4}$ times some constants, which depend only on T and f. Consequently we see that there exists a constant K depending only on T and f so that

$$E^P[|f(X_t)-f(X_s)|^4] \leqq K(t-s)^{5/4}, \qquad \text{for any} \quad 0\leqq s<t\leqq T.$$

Hence, according to Kolmogorov's continuous modification theorem, we have the desired result. The proof is complete.

For $f\in C^2(\bar{D})$, let

$$\tilde{Z}^f_t = f(\tilde{X}_t)-\int_0^t\left(\frac{1}{2}\Delta+b(\tilde{X}_u, u)\frac{\partial}{\partial x}\right)f(\tilde{X}_u)du, \qquad t\in[0, \infty),$$

and let $\{Z^f_t\}$, $t\in[0, \infty)$, be the stochastic process defined by (2.9). Then, from (1.2), Fubini's lemma and (3.1), we can prove that

$$E^P[|\tilde{Z}^f_t-Z^f_t|] = 0, \qquad {}^\forall t\in[0, \infty),$$

in other words,

$$P(\tilde{Z}^f_t=Z^f_t) = 1, \qquad {}^\forall t\in[0, \infty).$$

Thus, from Theorem 2.3, the stochastic process $\{\tilde{Z}^f_t\}$, $t\in[0, \infty)$, on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, is a continuous submartingale in class (DL), if $f\in C^2(\bar{D})$ with $f'(\alpha)\geqq 0$, $f'(\beta)\leqq 0$. Hence, by Doob-Meyer decomposition theorem(3), there exists an integrable nondecreasing, non-anticipating continuous process ξ^f: $[0, \infty)\times\Omega\to[0, \infty)$ such that $\xi^f_0=0$ and $\tilde{Z}^f_t-\xi^f_t$ is an $\mathcal{F}_t$-martingale. In our problem, the coefficient $b(x,t)$ may neither continuous nor bounded, but the submartingale $\{\tilde{Z}^f_t\}$, $t\in[0, \infty)$ is continuous and belongs to class (DL), thus following Stroock and Varadhan [**11**], we may derive assertions similar to Lemma 2.2, 2.3, 2.4, 2.5 and Theorem 2.4 in [**11**].

The following Proposition 3.3 can be proved by the similar way as Theorem 2.4 in [**11**].

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(3) See [**4**], [**12**].

PROPOSITION 3.3. *There exists a continuous, non-decreasing, non-anticipating stochastic process* $\{\xi_t\}$, $t\in[0, \infty)$, *on* $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ *uniquely, such that*

$$\xi_0 = 0, \qquad E^P[\xi_t] < \infty, \qquad \xi_t = \int_0^t I_\Gamma(\tilde{X}_u)d\xi_u,$$

and

$$f(\tilde{X}_t)-\int_0^t I_D(\tilde{X}_u)\Big(\frac{1}{2}\Delta+b(\tilde{X}_u, u)\frac{\partial}{\partial x}\Big)f(\tilde{X}_u)du$$
$$-\int_0^t(f'(\tilde{X}_u)I_{\{\alpha\}}(\tilde{X}_u)-f'(\tilde{X}_u)I_{\{\beta\}}(\tilde{X}_u))d\xi_u, \qquad t\in[0, \infty),$$

is an $\mathcal{F}_t$*-martingale for all* $f\in C^2(\bar{D})$, *where* I *is the indicator function and* $\Gamma=\{\alpha\}\cup\{\beta\}$.

THEOREM 3.4. *On the probability space* $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, *let*

$$B_t = \tilde{X}_t-\tilde{X}_0-\int_0^t b(\tilde{X}_u, u)du-\int_0^t(I_{\{\alpha\}}(\tilde{X}_u)-I_{\{\beta\}}(\tilde{X}_u))d\xi_u,$$

then $\{B_t\}$, $t\in[0, \infty)$, *is an* $\mathcal{F}_t$*-Brownian motion process such that* $P(B_0=0)=1$.

PROOF. We denote $I_{\{\alpha\}}(x)-I_{\{\beta\}}(x)$ by $\gamma(x)$. From Proposition 3.3, it follows that

$$M_t = \tilde{X}_t-\tilde{X}_0-\int_0^t I_D(\tilde{X}_u)b(\tilde{X}_u, u)du-\int_0^t\gamma(\tilde{X}_u)d\xi_u, \qquad t\in[0, \infty), \tag{3.12}$$

$$N_t = (\tilde{X}_t)^2-\int_0^t I_D(\tilde{X}_u)(1+2b(\tilde{X}_u, u)\tilde{X}_u)du-\int_0^t 2\tilde{X}_u\gamma(\tilde{X}_u)d\xi_u, \qquad t\in[0, \infty), \tag{3.13}$$

are continuous $\mathcal{F}_t$-martingales.

On the other hand from (3.12) and Itô's formula(4), we have

$$(\tilde{X}_t)^2 = 2\int_0^t\tilde{X}_u dM_u+2\int_0^t I_D(\tilde{X}_u)\tilde{X}_u b(\tilde{X}_u, u)du+2\int_0^t\gamma(\tilde{X}_u)\tilde{X}_u d\xi_u$$
$$+\int_0^t d\langle M\rangle_u, \qquad t\in[0, \infty). \tag{3.14}$$

Since the decomposition of $(\tilde{X}_t)^2$ is unique, comparing (3.13) and (3.14), we have $\langle M\rangle_u=u$. Thus by Theorem 3.6 in [**12**], the continuous martingale $\{M_t\}$, $t\in[0, \infty)$, is an $\mathcal{F}_t$-Brownian motion process. The proof is complete.

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