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Generalized Faber expansions of hyperfunctions on analytic curves

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1. Introduction.

Generalized functions and hyperfunctions have been introduced and studied by many people using different approaches. In the work of Köthe [**11**], [**12**], Grothendieck [**6**], Gelfand and Silov [**3**], [**4**], Schwartz [**19**], Roumieu [**15**] and Lions and Magenes [**13**], they were viewed as continuous linear functionals acting on some test-function spaces which are usually the inductive limits of sequences of normed spaces. However, in the work of Sato [**17**], hyperfunctions were viewed more as algebraic objects pertaining to the boundary values of holomorphic functions than as continuous linear functionals. In the case of the real line $\boldsymbol{R}$, the notion of a hyperfunction in Sato's theory is very simple; a hyperfunction on $\boldsymbol{R}$ is defined by a holomorphic function on $\boldsymbol{C}-\boldsymbol{R}$ where $\boldsymbol{C}$ is the complex plane. And two such functions represent the same hyperfunction if and only if their difference is holomorphic on $\boldsymbol{C}$, hence on $\boldsymbol{R}$. More generally, if I is an open subset of $\boldsymbol{R}$ and V is an open subset of $\boldsymbol{C}$ containing I and in which I is relatively closed, then the module of hyperfunctions on I is defined as the quotient module $\mathcal{H}(V-I)/\mathcal{H}(V)$ where $\mathcal{H}(V-I)$ and $\mathcal{H}(V)$ are the complex modules of locally holomorphic functions on $V-I$ and V respectively.

Sato's hyperfunctions have been defined on more general sets in the complex plane such as curves and have also been generalized to higher dimensions using sheaf theory.

On the unit circle ∂D, hyperfunctions were first characterized by Köthe [**11**], [**12**] as continuous linear functionals acting on the linear space of holomorphic complex-valued functions on ∂D when provided with a certain locally convex topology. Using different approaches, Sato [**16**] and Johnson [8] were able to find a very interesting characterization of hyperfunctions on the unit circle in terms of Fourier series. They showed that $f(e^{i\theta})$ is a hyperfunction on ∂D if and only if $f(e^{i\theta})=\sum_{n=-\infty}^{\infty}c_n e^{in\theta}$ where $\limsup_{|n|\to\infty}{}^{|n|}\sqrt{|c_n|}\leqq 1$ and the series converges, of course, in the sense of hyperfunctions.

If one deforms the unit circle homotopically to a curve Γ, both Sato's and

Köthe's theories remain adequate to describe hyperfunctions on Γ, but unfortunately the Fourier series characterization ceases to make sense. No similar characterization to the one given by Fourier series seems to exist for hyperfunctions on Γ.

The aim of this paper is to derive the analogue of the Fourier series characterization of hyperfunctions on the unit circle for hyperfunctions on curves Γ that are conformal images of the unit circle. It will be shown that any hyperfunction on such a curve Γ can be expanded in a series of generalized Faber polynomials and their "conjugate" functions. As a consequence of this characterization, some of the local properties of these hyperfunctions, such as microanalyticity and singular spectrum, can also be described in terms of the coefficients of their generalized Faber expansions.

We divide the rest of this article into three main sections. Section 2 consists of two subsections A and B; in A we introduce the generalized Faber polynomials, their conjugate functions and some of their properties that will be used later on and in B we introduce hyperfunctions and some related concepts. In Section 3 we prove the main results and in Section 4 we give some examples.

2. Preliminaries.

A. The generalized Faber polynomials. Let $\boldsymbol{B}$ be an open, bounded subset of the complex z-plane with closure $\bar{\boldsymbol{B}}$ whose complement $\bar{\boldsymbol{B}}^c$ is a simply connected domain. Let $z=\chi(\omega)$ map the domain $|\omega|>\rho$ one-to-one conformally onto the domain $\bar{\boldsymbol{B}}^c$ such that $\chi(\infty)=\infty$. We denote the inverse function of $z=\chi(\omega)$ by $\omega=\phi(z)$. Let D_r denote the disc $\{\omega: |\omega|<r\}$ and ∂D_r its boundary $\{\omega: |\omega|=r\}$. We denote the image of ∂D_r $(r>\rho)$, when mapped by the function $z=\chi(\omega)$, by L_r, the bounded domain with boundary L_r by $\boldsymbol{B}_r$ and the boundary of the domain $\bar{\boldsymbol{B}}^c$ by L_ρ or simply by L.

Let the function

$$F(z)=\sum_{n=0}^{\infty} b_n z^n, \qquad b_n \neq 0$$

be analytic in $|z|<1$ and assume that F can be analytically continued to any point outside the unit disc $D=D_1$ by any path not passing through the points $z=0, 1, \infty$. If the same is true for the function

$$F_*(z)=\sum_{n=0}^{\infty} \frac{z^n}{b_n},$$

we say that $F(z)$ and $F_*(z)$ are adjoint.

Let

$$R(\omega)=\sum_{n=0}^{\infty} c_n \omega^{-n}, \qquad c_0 \neq 0$$

be analytic in the domain $|\omega|>\rho$ with $R(\omega)\neq 0$ thereon and, furthermore, let $\boldsymbol{B}$ contain the point $z=0$.

The generalized Faber polynomials $P_n(z)$ are defined by the generating function

$$F\left(\frac{z}{\chi(\omega)}\right)\frac{\omega\chi'(\omega)}{\chi(\omega)}R(\omega)=\sum_{n=0}^{\infty}\frac{P_n(z)}{\omega^n},\qquad |\omega|>\rho,\ z\in\bar{\boldsymbol{B}} \tag{2.1}$$

from which we immediately obtain for $n=0, 1, 2, \cdots$

$$P_n(z)=\frac{1}{2\pi i}\int_{|\omega|=r_1}F\left(\frac{z}{\chi(\omega)}\right)\frac{\chi'(\omega)}{\chi(\omega)}R(\omega)\omega^n d\omega,\qquad z\in\bar{\boldsymbol{B}}_r,\ \rho<r<r_1. \tag{2.2}$$

By substituting $\omega=\phi(u)$, we obtain

$$P_n(z)=\frac{1}{2\pi i}\int_{L_{r_1}}F\left(\frac{z}{u}\right)R(\phi(u))[\phi(u)]^n\frac{du}{u},\qquad z\in\bar{\boldsymbol{B}}_r,\ \rho<r<r_1. \tag{2.3}$$

The case where $F(z)=1/(1-z)$ is of special importance to us; therefore, we shall denote the generalized Faber polynomials in this case by $\pi_n(z)$. For $\pi_n(z)$, equations (2.1), (2.2) and (2.3) take on the form

$$\frac{\omega\chi'(\omega)}{\chi(\omega)-z}R(\omega)=\sum_{n=0}^{\infty}\frac{\pi_n(z)}{\omega^n},\qquad |\omega|>\rho,\ z\in\bar{\boldsymbol{B}}, \tag{2.4}$$

$$\pi_n(z)=\frac{1}{2\pi i}\int_{|\omega|=r_1}\frac{\chi'(\omega)}{\chi(\omega)-z}R(\omega)\omega^n d\omega,\qquad z\in\bar{\boldsymbol{B}}_r,\ \rho<r<r_1 \tag{2.5}$$

and

$$\pi_n(z)=\frac{1}{2\pi i}\int_{L_{r_1}}\frac{R(\phi(u))[\phi(u)]^n}{u-z}du,\qquad z\in\bar{\boldsymbol{B}}_r,\ \rho<r<r_1. \tag{2.6}$$

If we set $R(\omega)=1$ in (2.4), we obtain the original Faber polynomials as introduced by Faber [**2**].

Let $f(z)$ be analytic in $\boldsymbol{B}$ and have the expansion

$$f(z)=\sum_{n=0}^{\infty}a_n z^n \tag{2.7}$$

in a neighbourhood of $z=0$, then it is easy to show that the function

$$\tilde{f}_*(z)=\sum_{n=0}^{\infty}a_n b_n z^n \tag{2.8}$$

is also analytic in $\boldsymbol{B}$ and

$$\tilde{f}_*(z)=\frac{1}{2\pi i}\int_L f(u)F\left(\frac{z}{u}\right)\frac{du}{u},\qquad z\in\boldsymbol{B} \tag{2.9}$$

$$f(z)=\frac{1}{2\pi i}\int_L \tilde{f}_*(u)F_*\left(\frac{z}{u}\right)\frac{du}{u},\qquad z\in\boldsymbol{B}\qquad \text{(see [23])}. \tag{2.10}$$

In particular, for fixed $\zeta\in\bar{\boldsymbol{B}}^c$, if we set $f(u)=\sum_{n=0}^{\infty}(1/b_n\zeta^n)u^n$ in (2.9), we obtain

$$\frac{\zeta}{\zeta-z}=\frac{1}{2\pi i}\int_L F_*\left(\frac{u}{\zeta}\right)F\left(\frac{z}{u}\right)\frac{du}{u}. \tag{2.11}$$

From (2.2), (2.5), (2.9) and (2.10) we deduce that

$$P_n(z)=\frac{1}{2\pi i}\int_L \pi_n(u)F\left(\frac{z}{u}\right)\frac{du}{u}, \qquad z\in \boldsymbol{B} \tag{2.12}$$

$$\pi_n(z)=\frac{1}{2\pi i}\int_L P_n(u)F_*\left(\frac{z}{u}\right)\frac{du}{u}, \qquad z\in \boldsymbol{B}. \tag{2.13}$$

Since $P_n(z)$ and $\pi_n(z)$ are polynomials, with some minor modification of the argument above, one can extend (2.12) and (2.13) to the case where $z\in\boldsymbol{B}_r$ $(r>\rho)$. By substituting $u=\chi(t)$ in (2.11) and using (2.1), we obtain

$$\frac{1}{\zeta-z}=\sum_{n=0}^{\infty}P_n(z)q_n(\zeta), \qquad z\in\boldsymbol{B}_r,\ \zeta\in\bar{\boldsymbol{B}}_r^c \tag{2.14}$$

where

$$q_n(\zeta)=\frac{1}{2\pi i}\int_{|t|=r}F_*\left(\frac{\chi(t)}{\zeta}\right)\frac{1}{\zeta R(t)t^{n+1}}dt \tag{2.15}$$

which is called the conjugate function of the polynomial $P_n(z)$. The conjugate function of the polynomial $\pi_n(z)$ will be denoted by $Q_n(\zeta)$.

The two sequences of functions $\{P_n(z)\}_{n=0}^{\infty}$ and $\{q_n(z)\}_{n=0}^{\infty}$ form a biorthogonal system of functions on any closed path γ containing $\bar{\boldsymbol{B}}$ in its interior, i.e.,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\gamma P_n(z)P_m(z)dz=0, && n,m=0,1,2,\cdots\\
&\frac{1}{2\pi i}\int_\gamma q_n(z)q_m(z)dz=0, && n,m=0,1,2,\cdots\\
&\frac{1}{2\pi i}\int_\gamma P_n(z)q_m(z)dz=\delta_{nm}, && n,m=0,1,2,\cdots.
\end{aligned} \tag{2.16}$$

In view of the biorthogonality, formula (2.14) and the estimates

$$\limsup_{n\to\infty}\sqrt[n]{|P_n(z)|}=r \quad\text{and}\quad \limsup_{n\to\infty}\sqrt[n]{|q_n(z)|}=\frac{1}{r} \tag{2.17}$$

which hold uniformly for $z\in L_r$ $(r>\rho)$, one can show that if $f(z)$ is analytic in a doubly-connected domain $\boldsymbol{B}_{r_1,r_2}$, $\rho<r_1<r_2<\infty$, bounded by the contour lines L_{r_1} and L_{r_2} and has singular points on these contour lines, then

$$f(z)=\sum_{n=0}^{\infty}a_nP_n(z)+\sum_{n=0}^{\infty}b_nq_n(z), \tag{2.18}$$

where

$$a_n=\frac{1}{2\pi i}\int_{L_r}f(\zeta)q_n(\zeta)d\zeta, \qquad b_n=\frac{1}{2\pi i}\int_{L_r}f(\zeta)P_n(\zeta)d\zeta, \qquad r_1<r<r_2 \tag{2.19}$$

and

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{r_2}, \qquad \limsup_{n\to\infty} \sqrt[n]{|b_n|} = r_1. \tag{2.20}$$

For more details on Faber polynomials, see [**2**], [**7**], [**9**], [**20**] and [**21**].

In the next subsection, we shall introduce some of the definitions and notations pertaining to hyperfunctions that will be used later on.

B. Hyperfunctions. Let G be an open subset of the Riemann sphere $\mathcal{O}$ and $\mathcal{H}(G)$ be the ring of locally holomorphic functions on G. When provided with the topology of uniform convergence on compact subsets of G, $\mathcal{H}(G)$ becomes a Frechet space which will also be denoted by $\mathcal{H}(G)$. Let K be a compact subset of the complex plane $\boldsymbol{C}$ and consider the system of open neighbourhoods $\{G_n\}_{n=1}^{\infty}$ of K defined by

$$G_n = \left\{z:\ \operatorname{dist}(z, K) < \frac{1}{n}\right\}.$$

Hence, $G_1 \supset G_2 \supset G_3 \cdots\cdots$ and consequently, $\mathcal{H}(G_1) \subset \mathcal{H}(G_2) \subset \mathcal{H}(G_3) \subset \cdots$ where $f(z) \in \mathcal{H}(G_n)$ defines an element in $\mathcal{H}(G_{n+1})$ by restriction, i.e., $f(z)|_{G_{n+1}}$. We define $\mathcal{H}(K)$ by

$$\mathcal{H}(K) = \bigcup_{n=1}^{\infty} \mathcal{H}(G_n),$$

i.e., $f(z) \in \mathcal{H}(K)$ if and only if it is locally holomorphic in some neighbourhood of K. We endow $\mathcal{H}(K)$ with the inductive limit topology induced by the topologies of $\{\mathcal{H}(G_n)\}_{n=1}^{\infty}$. It is known that the dual space $\mathcal{H}'(K)$ of $\mathcal{H}(K)$, when provided with the strong topology, is isomorphic to $\mathcal{H}_0(\boldsymbol{C} \setminus K) = \{f: f$ is holomorphic on $\boldsymbol{C} \setminus K$ and $f(\infty) = 0\}$; see [**10**].

Let I be an open subset of the real line $\boldsymbol{R}$ and V be a complex neighbourhood of I (V is an open subset of the complex plane $\boldsymbol{C}$ which contains I in its interior) such that I is relatively closed in V. The complex-module $\mathcal{B}(I)$ of hyperfunctions on I is defined as the quotient module

$$\mathcal{B}(I) = \mathcal{H}(V - I)/\mathcal{H}(V). \tag{2.21}$$

It can be shown [**17**] that this definition is independent of the neighbourhood V, hence for $V = \boldsymbol{C} - \partial I$ where ∂I is the boundary of I relative to $\boldsymbol{R}$, then

$$\mathcal{B}(I) \cong \mathcal{H}(\boldsymbol{C} - \bar{I})/\mathcal{H}(\boldsymbol{C} - \partial I), \tag{2.22}$$

where $\cong$ means isomorphic to. It is easy to see that $I \to \mathcal{B}(I)$ is a sheaf.

If I is relatively compact in $\boldsymbol{R}$, it follows that

$$\mathcal{B}(I) \cong \mathcal{H}'(\bar{I})/\mathcal{H}'(\partial I). \tag{2.23}$$

But if I is compact, then

$$\mathcal{B}(I) \cong \mathcal{H}'(I), \tag{2.24}$$

moreover for any $\tilde{f}\in\mathcal{H}'(I)$, there exists $f(z)\in H_0(\boldsymbol{C}-I)$ such that for any $\phi(z)\in\mathcal{H}(I)$

$$\langle\tilde{f},\phi\rangle=\frac{1}{2\pi i}\int_\gamma\phi(z)f(z)dz, \tag{2.25}$$

where γ is any simple closed path enclosing I and lying in the domain of analyticity of $\phi(z)$.

Let $C^{\pm}=\{z:\ \mathrm{Im}z\gtrless 0\}$, hence $\boldsymbol{C}=C^{+}+\boldsymbol{R}+C^{-}$. Similarly, for a complex neighbourhood V of I, we define $V^{\pm}=\{z\in V:\ \mathrm{Im}z\gtrless 0\}$. Since any hyperfunction f on I is represented by a holomorphic function $\phi(z)$ in $V-I$ (up to a holomorphic function on V), we may write $f(x)=[\phi(z)]_{z=x}$. Let

$$\varepsilon(z)=\begin{cases}1 & \text{if } z\in C^{+}\\ 0 & \text{if } z\in C^{-},\end{cases}\qquad \bar{\varepsilon}(z)=\begin{cases}0 & \text{if } z\in C^{+}\\ -1 & \text{if } z\in C^{-}\end{cases}$$

and $1(x)=[\varepsilon(z)]_{z=x}$ $(=[\bar{\varepsilon}(z)]_{z=x}$ since $\varepsilon-\bar{\varepsilon}$ is holomorphic everywhere). Thus, we can define the hyperfunctions $f(x+i0)$, $f(x-i0)$ as

$$f(x+i0)=[\varepsilon(z)\phi(z)]_{z=x}\quad\text{and}\quad f(x-i0)=[\bar{\varepsilon}(z)\phi(z)]_{z=x}$$

and hence, $f(x)=f(x+i0)-f(x-i0)$. Moreover, we can inject $\mathcal{H}(I)$ into $\mathcal{B}(I)$ via

$$f(z)\in\mathcal{H}(I)\longmapsto f(x)=[f(z)\varepsilon(z)]_{z=x}=f(x)\cdot 1(x)\in\mathcal{B}(I).$$

The complex-submodule $\mathcal{H}(I)$ of $\mathcal{B}(I)$ is called the module of holomorphic hyperfunction on I. It is easy to see that a hyperfunction $f(x)=[\phi(z)]_{z=x}$ on I is holomorphic if and only if

$$\phi(z)=\varepsilon(z)f_1(z)+\bar{\varepsilon}(z)f_2(z),\qquad z\in V'-I,$$

where $V'\subseteqq V$ is a complex neighbourhood of I and $f_1, f_2\in\mathcal{H}(V')$. A hyperfunction $f\in\mathcal{B}(I)$ is said to be upper (lower) semi-holomorphic if $f(x)$ is of the form $g(x+i0)$ $(g(x-i0))$ for some $g\in\mathcal{B}(I)$ or equivalently f is upper (lower) semi-holomorphic if and only if $f(x)=[\phi(z)]_{z=x}$ where $\phi(z)=\varepsilon(z)\phi_1(z)+\bar{\varepsilon}(z)f_2(z)$ $(\phi(z)=\varepsilon(z)f_1(z)+\bar{\varepsilon}(z)\phi_2(z))$, with $\phi_1(z)\in\mathcal{H}(V^{+})$, $f_2(z)\in\mathcal{H}(V')$ $(f_1(z)\in\mathcal{H}(V'),\ \phi_2(z)\in\mathcal{H}(V^{-}))$ where $V'\subseteq V$ is a complex neighbourhood of I. We denote the submodule of upper (lower) semi-holomorphic hyperfunctions on I by $\mathcal{B}^{+}(I)$ $(\mathcal{B}^{-}(I))$. Clearly, $f\in\mathcal{B}(I)$ is holomorphic if and only if it is upper and lower semi-holomorphic.

The support of a hyperfunction is defined as the complement of the largest open set in which it vanishes. Thus, if $f(x)=[\phi(z)]_{z=x}\in\mathcal{B}(I)$ has support in a closed set $F\subset I$, then $\phi(z)\in\mathcal{H}(V-F)$, where V is a complex neighbourhood of I.

Let $f(x)=[\phi(z)]_{z=x}\in\mathcal{B}(I)$, where $\phi(z)$ is holomorphic in $V-I$ for some complex neighbourhood V of I. We define $\phi^{\pm}(z)$ as the restriction of $\phi(z)$ to $V^{\pm}$, i.e., $\phi^{\pm}(z)=\phi(z)|_{V^{\pm}}$.

$f(x)$ is said to be microanalytic at x_0+i0 (x_0-i0) if and only if $\phi^{+}(z)$ $(\phi^{-}(z))$

can be continued analytically in a neighbourhood of $x_0 \in I$. $f(x)$ is said to be analytic at $x_0 \in I$ if and only if it is microanalytic at $x_0 \pm i0$. The upper (lower) singular spectrum SS^+f (SS^-f) of $f \in \mathcal{B}(I)$ is the set of points $x_0 \in I$ such that f is not microanalytic at x_0+i0 (x_0-i0). The singular support of f is defined as sing. supp. $f = SS^+f \cup SS^-f$, which is the same as the complement in I of the set where f is analytic. Trivially, sing. supp. $f \subset$ supp. f.

EXAMPLES. 1) Let

$$f(x) = [\chi(z)]_{z=x} \qquad \text{where } \chi(z) = \begin{cases} \dfrac{1}{z}, & z \in C^+ \\ 0, & z \in C^- \end{cases}$$

then $SS^+f = \{0\}$, $SS^-f = \emptyset$, sing. supp. $f = \{0\}$, supp. $f = \boldsymbol{R}$.

2) $\delta(x) = \dfrac{-1}{2\pi i}\left[\dfrac{1}{z}\right]_{z=x}$.

Hence, $SS^+\delta = \{0\} = SS^-\delta$, sing. supp. $\delta = \{0\}$, supp. $\delta = \{0\}$.

Let $\mathcal{H}^+(V, I)$ ($\mathcal{H}^-(V, I)$) be the module of holomorphic functions in V^+ (V^-) which can be continued analytically across every point of I and set

$$\mathcal{H}(V, I) = \mathcal{H}^+(V, I) \cap \mathcal{H}^-(V, I).$$

Thus, $\mathcal{H}(V, I)$ is isomorphic to the module of all holomorphic functions $f(z)$ in $V-I$ such that $f|_{V^\pm} \in \mathcal{H}^\pm(V, I)$.

The modules of upper and lower microfunctions on I are defined by

$$C^+(I) = \mathcal{H}(V^+)/\mathcal{H}^+(V, I) \qquad \text{and} \qquad C^-(I) = \mathcal{H}(V^-)/\mathcal{H}^-(V, I).$$

It can be shown that this definition is independent of V and in addition

$$C^+(I) \cong \mathcal{B}(I)/\mathcal{B}^-(I) \qquad \text{and} \qquad C^-(I) \cong \mathcal{B}(I)/\mathcal{B}^+(I).$$

The module $C(I)$ of microfunctions on I, which is a refinement of the module of hyperfunctions on I, is defined by

$$C(I) = \mathcal{H}(V-I)/\mathcal{H}(V, I).$$

Again it can be shown that this definition is independent of V. With some easy arguments, one can show that

$$C(I) \cong \mathcal{B}(I)/\mathcal{H}(I) \qquad \text{and} \qquad C(I) \cong C^+(I) \oplus C^-(I).$$

Finally, let Γ be an oriented, simple analytic curve in a locally compact, analytic differential manifold X. Let V be an open neighbourhood of Γ in which Γ is relatively closed. Similar to $\boldsymbol{C}^\pm$, one can define $V^\pm$ such that $V = V^+ + \Gamma + V^-$. For every point $p \in \Gamma$ there exists an open neighbourhood V_p and a univalent holomorphic function $\phi_p \in \mathcal{H}(V_p)$ satisfying $\phi_p^{-1}(\boldsymbol{R}) = V_p \cap \Gamma$ and

$\phi_p^{-1}(C^{\pm})=V_p\cap V^{\pm}=V_p^{\pm}$. Therefore, all the previous notions pertaining to hyperfunctions and microfunctions on I can be carried over to Γ.

For more details on hyperfunctions, see [**14**], [**17**] and [**18**].

3. Characterization of hyperfunctions on analytic curves.

In this section we shall characterize hyperfunctions on simple closed analytic curves Γ in the complex plane in terms of their generalized Faber series representations. Any such a curve Γ can be viewed as the boundary of a domain $\boldsymbol{B}$ of the type described in Section 2.A. Therefore, we may write $\partial\boldsymbol{B}$ instead of Γ to emphasize the interrelation between hyperfunctions as objects defined on the boundary of the domain $\boldsymbol{B}$ and the holomorphic functions defined in the interior and the exterior of $\boldsymbol{B}$.

Since we are assuming that the boundary of $\boldsymbol{B}$ is analytic, the function $\chi(\omega)$ may be extended analytically to the domain $|\omega|>\rho_1$ for some $0<\rho_1<\rho$, and, thus, without loss of generality we may also assume that $R(\omega)\neq 0$ in this extended region.

THEOREM 1. *A function $\phi(\zeta)$ is in $\mathcal{H}(\partial\boldsymbol{B})$ if and only if it has a series expansion in terms of the generalized Faber polynomials in the form*

$$\phi(\zeta)=\sum_{n=0}^{\infty}a_nP_n(\zeta)+\sum_{n=0}^{\infty}b_nq_n(\zeta),\qquad \zeta\in\partial\boldsymbol{B}, \tag{3.1}$$

where

$$a_n=\frac{1}{2\pi i}\int_{\partial B}\phi(\zeta)q_n(\zeta)d\zeta,\qquad b_n=\frac{1}{2\pi i}\int_{\partial B}\phi(\zeta)P_n(\zeta)d\zeta \tag{3.2}$$

and

$$\limsup_{n\to\infty}\sqrt[n]{|a_n|}<\frac{1}{\rho},\qquad \limsup_{n\to\infty}\sqrt[n]{|b_n|}<\rho. \tag{3.3}$$

PROOF. Since $\partial\boldsymbol{B}$ is an analytic curve, the function $z=\chi(\omega)$ is holomorphic in the domain $|\omega|>\rho_1$ for some $0<\rho_1<\rho$ and hence in the z plane there will be an image L_{r^*} of the circle $|\omega|=r^*$, $\rho_1<r^*<\rho$, under the map $z=\chi(\omega)$. As before, we may denote the bounded domain with boundary L_{r^*} by $\boldsymbol{B}_{r^*}$. Let $\phi(\zeta)\in\mathcal{H}(\partial\boldsymbol{B})$. Since $\phi(\zeta)$ is locally holomorphic on $\partial\boldsymbol{B}$, then it is holomorphic in some neighbourhood V^* of $\partial\boldsymbol{B}$. Set $V=V^*\cap\bar{\boldsymbol{B}}_{r^*}^c$. Then, one can find two contours L_{r_1} and L_{r_2} lying entirely in the interior of V with $\rho_1<r^*<r_1<\rho<r_2$. By Cauchy's formula, for any ζ between L_{r_1} and L_{r_2}, in particular for $\zeta\in\partial\boldsymbol{B}$, we have

$$\phi(\zeta)=\frac{-1}{2\pi i}\int_{L_{r_1}}\frac{\phi(u)du}{u-\zeta}+\frac{1}{2\pi i}\int_{L_{r_2}}\frac{\phi(u)du}{u-\zeta}$$

which, when combined with (2.14) after a slight modification of its domain of validity, one obtains (3.1) and (3.2) with the integrals in (3.2) taken over L_{r_2}, L_{r_1} respectively. But, replacing L_{r_1}, L_{r_2} by $\partial\boldsymbol{B}$ is trivial since $\phi(\zeta)$, $P_n(\zeta)$ and $q_n(\zeta)$ are all holomorphic in V. As in (2.20), one has

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{r_2} < \frac{1}{\rho} \qquad \text{and} \qquad \limsup_{n\to\infty} \sqrt[n]{|b_n|} = r_1 < \rho\,.$$

Conversely, in view of (3.3) and the estimates (2.17), one can show that the series in (3.1) converges absolutely and uniformly on ∂B to a holomorphic function $\tilde{\phi}(\zeta)$. But from the uniqueness of the generalized Faber expansions, it follows that $\tilde{\phi}=\phi$.

COROLLARY 1. *Let $\phi(\zeta)\in\mathcal{H}(\partial\boldsymbol{B})$. Then, there exist two unique functions $\phi_1(z)$ and $\phi_2(z)$ such that $\phi_1(z)$ is holomorphic in a neighbourhood of $\bar{\boldsymbol{B}}$, $\phi_2(z)$ is holomorphic in a neighbourhood of $\boldsymbol{B}^c$ with $\phi_2(\infty)=0$ and*

$$\phi(\zeta) = \phi_1(z)|_{\partial\boldsymbol{B}}+\phi_2(z)|_{\partial\boldsymbol{B}}\,.$$

PROOF. Let $\phi_1(z)=\sum_{n=0}^{\infty}a_nP_n(z)$ and $\phi_2(z)=\sum_{n=0}^{\infty}b_nq_n(z)$. From (3.3) and the estimates (2.17), one can easily show that both $\phi_1(z)$ and $\phi_2(z)$ are holomorphic in the indicated regions. The uniqueness of ϕ_1 and ϕ_2 follows from the uniqueness of the generalized Faber representations.

We write $\phi_i(\zeta)=\phi_i(z)|_{\partial\boldsymbol{B}}$, $i=1, 2$ and hence,

$$\phi(\zeta) = \phi_1(\zeta)+\phi_2(\zeta), \qquad \zeta\in\partial\boldsymbol{B}$$

where $\phi_1(\zeta)$ can be extended analytically to a neighbourhood of $\bar{\boldsymbol{B}}$ and $\phi_2(\zeta)$ can be extended analytically to a neighbourhood of $\boldsymbol{B}^c$.

The next theorem gives a characterization of the dual space $\mathcal{H}'(\partial\boldsymbol{B})$ of $\mathcal{H}(\partial\boldsymbol{B})$.

THEOREM 2. *$f(\zeta)\in\mathcal{H}'(\partial\boldsymbol{B})$ if and only if it has the series expansion*

$$f(\zeta) = \sum_{n=0}^{\infty} c_nP_n(\zeta)+\sum_{n=0}^{\infty} d_nq_n(\zeta) \tag{3.4}$$

where

$$c_n = \langle f, q_n\rangle, \qquad d_n = \langle f, P_n\rangle \tag{3.5}$$

and

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} \leqq \frac{1}{\rho}, \qquad \limsup_{n\to\infty} \sqrt[n]{|d_n|} \leqq \rho\,. \tag{3.6}$$

Moreover, if $\phi(\zeta)\in\mathcal{H}(\partial\boldsymbol{B})$ with

$$\phi(\zeta) = \sum_{n=0}^{\infty} a_nP_n(\zeta)+\sum_{n=0}^{\infty} b_nq_n(\zeta),$$

then

$$\langle f, \phi\rangle = \sum_{n=0}^{\infty}(a_nd_n+b_nc_n)\,. \tag{3.7}$$

PROOF. From Theorem 1 and Corollary 1, it is clear that $\mathcal{H}(\partial\boldsymbol{B})$ is isomorphic to the direct sum of $\mathcal{H}(\bar{\boldsymbol{B}})$ and $\mathcal{H}_0(\boldsymbol{B}^c)$. Let us define

$$\mathcal{H}_1(\partial\boldsymbol{B}) = \{\phi_1(\zeta):\ \ \phi_1(\zeta)=\phi_1(z)|_{\partial B} \text{ for some } \phi_1(z)\in\mathcal{H}(\bar{\boldsymbol{B}})\}$$

and

$$\mathcal{H}_2(\partial\boldsymbol{B}) = \{\phi_2(\zeta):\ \ \phi_2(\zeta)=\phi_2(z)|_{\partial B} \text{ for some } \phi_2(z)\in\mathcal{H}_0(\boldsymbol{B}^c)\}.$$

The maps $\phi_1(z)\mapsto\phi_1(z)|_{\partial B}$, $\phi_2(z)\mapsto\phi_2(z)|_{\partial B}$ are surjective by definition and injective because of the analyticity of $\phi_1(z)$ and $\phi_2(z)$. If we provide $\mathcal{H}_1(\partial\boldsymbol{B})$, $\mathcal{H}_2(\partial\boldsymbol{B})$ with their natural topology inherited from $\mathcal{H}(\bar{\boldsymbol{B}})$ and $\mathcal{H}_0(\boldsymbol{B}^c)$, we obtain that

$$\mathcal{H}(\partial\boldsymbol{B}) = \mathcal{H}_1(\partial\boldsymbol{B})\oplus\mathcal{H}_2(\partial\boldsymbol{B}) \tag{3.8}$$

and hence

$$\mathcal{H}'(\partial\boldsymbol{B}) = \mathcal{H}_1'(\partial\boldsymbol{B})\oplus\mathcal{H}_2'(\partial\boldsymbol{B}) \cong \mathcal{H}'(\bar{\boldsymbol{B}})\oplus\mathcal{H}_0'(\boldsymbol{B}^c). \tag{3.9}$$

Since $\mathcal{H}(\bar{\boldsymbol{B}})$ can be characterized as the space of all functions $\phi_1(z)=\sum_{n=0}^{\infty}a_nP_n(z)$ with $\limsup_{n\to\infty}\sqrt[n]{|a_n|}=1/r_1$, for some r_1 with $\rho<r_1$ and $\mathcal{H}_0(\boldsymbol{B}^c)$ as the space of all functions $\phi_2(z)=\sum_{n=0}^{\infty}b_nq_n(z)$ with $\limsup_{n\to\infty}\sqrt[n]{|b_n|}=r_2$ for some r_2 with $r_2<\rho$, it follows from Köthe's duality theorem [**10**] and (2.20) that $\mathcal{H}'(\bar{\boldsymbol{B}})$ is isomorphic to the space of all holomorphic functions $f_1(z)=\sum_{n=0}^{\infty}d_nq_n(z)$ with $\limsup_{n\to\infty}\sqrt[n]{|d_n|}\leqq\rho$, when provided with the topology of uniform convergence on compact subsets of $\bar{\boldsymbol{B}}^c$, i.e., isomorphic to $\mathcal{H}_0(\bar{\boldsymbol{B}}^c)$. Moreover, if $\tilde{f}_1\in\mathcal{H}'(\bar{\boldsymbol{B}})$, then there exists a holomorphic function $f_1(z)$ as above such that for any $\phi_1(z)\in\mathcal{H}(\bar{\boldsymbol{B}})$

$$\langle\tilde{f}_1, \phi_1\rangle = \frac{1}{2\pi i}\int_{\gamma}f_1(z)\phi_1(z)dz = \frac{1}{2\pi i}\int_{\gamma}\Big(\sum_{n=0}^{\infty}d_nq_n(z)\Big)\Big(\sum_{m=0}^{\infty}a_nP_n(z)\Big)dz$$
$$= \sum_{n=0}^{\infty}a_nd_n < \infty$$

because of (2.16), where γ is any closed contour encircling $\bar{\boldsymbol{B}}$ but lying between $\partial\boldsymbol{B}$ and L_{r_1}. The last series converges absolutely since $\limsup_{n\to\infty}\sqrt[n]{|a_nd_n|}\leqq\rho/r_1<1$. Similarly, one can show that $\mathcal{H}'(\boldsymbol{B}^c)$ is isomorphic to the space of all holomorphic functions $f_2(z)=\sum_{n=0}^{\infty}c_nP_n(z)$ with $\limsup_{n\to\infty}\sqrt[n]{|c_n|}\leqq1/\rho$, when provided with the topology of uniform convergence on compact subsets of $\boldsymbol{B}$. Moreover, if $\tilde{f}_2\in\mathcal{H}'(\boldsymbol{B}^c)$, then there exists a holomorphic function $f_2(z)$ as above such that for any $\phi_2(z)\in\mathcal{H}(\boldsymbol{B}^c)$

$$\langle\tilde{f}_2, \phi_2\rangle = \frac{1}{2\pi i}\int_{\gamma_1}f_2(z)\phi_2(z)dz = \sum_{n=0}^{\infty}b_nc_n < \infty,$$

where γ_1 is any closed contour encircling $\boldsymbol{B}^c$ in its exterior but lying between $\partial\boldsymbol{B}$ and L_{r_2}.

Consider the sequence $f_{1,N}(\zeta)=\sum_{n=0}^{N}d_nq_n(\zeta)$, $\zeta\in\partial\boldsymbol{B}$, $N=0, 1, 2, \cdots$. Each member of this sequence defines a continuous linear functional on $\mathcal{H}_1(\partial\boldsymbol{B})$. Since

$$\lim_{N\to\infty}\frac{1}{2\pi i}\int_{\partial B} f_{1,N}(\zeta)\phi_1(\zeta)d\zeta = \lim_{N\to\infty}\sum_{n=0}^{N} a_n d_n = \sum_{n=0}^{\infty} a_n d_n$$
$$= \frac{1}{2\pi i}\int_{\gamma} f_1(z)\phi_1(z)dz,$$

it is easy to see that the sequence $\{f_{1,N}(\zeta)\}_{N=0}^{\infty}$ converges in $\mathcal{H}_1'(\partial\boldsymbol{B})$ to some element which we shall denote by $f_1(\zeta)$. Therefore, we may identify $\mathcal{H}_1'(\partial\boldsymbol{B})$ with $\mathcal{H}(\bar{\boldsymbol{B}}^c)$ via $f_1(\zeta)=\sum_{n=0}^{\infty} d_n q_n(\zeta) = \lim_{z\to\zeta}\sum_{n=0}^{\infty} d_n q_n(z) = \lim_{z\to\zeta} f_1(z)$, $\zeta\in\partial\boldsymbol{B}$. Similar results hold for $f_2(\zeta)$ and $f_2(z)$. Therefore, in view of (3.9), the theorem is now proved except for (3.5) which will follow from (3.7) and (2.16).

We shall say that $f_i(z)$ is the analytic extension of $f_i(\zeta)$, $i=1, 2$.

COROLLARY 2. *$f(\zeta)$ is a hyperfunction on $\partial\boldsymbol{B}$, i.e., $f(\zeta)\in\mathcal{B}(\partial\boldsymbol{B})$ if and only if $f(\zeta)=\sum_{n=0}^{\infty} c_n P_n(\zeta)+\sum_{n=0}^{\infty} d_n q_n(\zeta)$ with*

$$\limsup_{n\to\infty}\sqrt[n]{|c_n|}\leqq\frac{1}{\rho}, \qquad \limsup_{n\to\infty}\sqrt[n]{|d_n|}\leqq\rho\,.$$

Moreover,

$$f(\zeta)=\lim_{z\to\zeta}(f_1(z)-f_2(z)), \qquad \zeta\in\partial\boldsymbol{B}$$

where $f_1(z)$ is holomorphic in $\bar{\boldsymbol{B}}^c$ with $f_1(\infty)=0$, $f_2(z)$ is holomorphic in $\boldsymbol{B}$, and the limit is taken in the sense of $\mathcal{H}'(\partial B)$.

COROLLARY 3. *Let $f(\zeta)=\sum_{n=0}^{\infty} c_n P_n(\zeta)+\sum_{n=0}^{\infty} d_n q_n(\zeta)\in\mathcal{H}'(\partial\boldsymbol{B})$. Then, its analytic representation (indicatrix of Fantappie) $f(z)=(1/2\pi i)\langle f(\zeta), 1/(\zeta-z)\rangle$ is given by*

$$f(z)=\frac{1}{2\pi i}\langle f(\zeta), \frac{1}{\zeta-z}\rangle=\begin{cases}\sum_{n=0}^{\infty} c_n P_n(z) & \text{if } z\in\boldsymbol{B}\\ \sum_{n=0}^{\infty} d_n q_n(z) & \text{if } z\in\bar{\boldsymbol{B}}^c.\end{cases}$$

PROOF. This follows from (2.14) and (2.16).

Having characterized $\mathcal{H}(\partial\boldsymbol{B})$ and $\mathcal{H}'(\partial\boldsymbol{B})$ in terms of the generalized Faber expansions, one now can imitate Johnson's proofs in the case of the unit circle to reconstruct the theory of hyperfunctions on $\partial\boldsymbol{B}$ in a different way using sequence spaces. For example, assuming without loss of generality that $\rho\geqq 1$, we provide $\mathcal{H}_1(\partial\boldsymbol{B})$ with the following new topology:

Let A be the class of all sequences $\alpha=\{\alpha_n\}_{n=0}^{\infty}$ which satisfy

$$\alpha_n\geqq\alpha_{n+1}>0 \quad \text{for all } n \quad \text{and} \quad \lim_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n}=1,$$

then we take the collection of sets

$$V(\alpha)=\Big\{\phi_1(\zeta)=\sum_{n=0}^{\infty} a_n P_n(\zeta)\in\mathcal{H}_1(\partial\boldsymbol{B}):\ |a_n|\leqq\alpha_n \text{ for all } n\Big\}$$

for all $\alpha\in A$ as a base for the neighbourhood system at the origin. That $\{V(\alpha)\}_{\alpha\in A}$, indeed, forms a neighbourhood system at the origin and that the topology generated by it makes $\mathcal{H}_1(\partial\boldsymbol{B})$ a complete, nonmetrizable Montel space can be proved as in [8]. Similar neighbourhood systems, hence similar topologies, can also be defined for $\mathcal{H}_2(\partial\boldsymbol{B})$, $\mathcal{H}_1'(\partial\boldsymbol{B})$ and $\mathcal{H}_2'(\partial\boldsymbol{B})$; see [8].

Another advantage of our representation of hyperfunctions on $\partial\boldsymbol{B}$ as series of generalized Faber polynomials is that several analytic properties of hyperfunctions on $\partial\boldsymbol{B}$ can be characterized in terms of the coefficients of the expansions. For example, analogous to the concept of upper and lower hyperfunctions on $I\subset\boldsymbol{R}$, we shall say that a hyperfunction $f(\zeta)=\sum_{n=0}^{\infty}d_nq_n(\zeta)+\sum_{n=0}^{\infty}c_nP_n(\zeta)=f_1(\zeta)+f_2(\zeta)$ on $\partial\boldsymbol{B}$ is outer if

$$\limsup_{n\to\infty}\sqrt[n]{|c_n|}<\frac{1}{\rho},\qquad \limsup_{n\to\infty}\sqrt[n]{|d_n|}\leqq\rho$$

and inner if

$$\limsup_{n\to\infty}\sqrt[n]{|c_n|}\leqq\frac{1}{\rho},\qquad \limsup_{n\to\infty}\sqrt[n]{|d_n|}<\rho\,.$$

Thus, f is holomorphic on $\partial\boldsymbol{B}$ if and only if it is both outer and inner. To be consistent with the notation of Section 2, we may denote the analytic extensions of $f_1(\zeta)$, $f_2(\zeta)$ also by $f^+(z)$ and $f^-(z)$ respectively. Let N_ζ denote the unit normal vector to $\partial\boldsymbol{B}$ at the point $\zeta\in\partial\boldsymbol{B}$. We say that $f(\zeta)\in\mathcal{B}(\partial\boldsymbol{B})$ is microanalytic at $\zeta+ON_\zeta$ ($\zeta-ON_\zeta$) if and only if $f^+(z)$ ($f^-(z)$) can be continued analytically in a neighbourhood of $\zeta\in\partial\boldsymbol{B}$. Analogous to the upper (lower) singular spectrum of a hyperfunction on $I\subset R$, one can define the outer and inner singular spectra SS^+f, SS^-f of a hyperfunction f on $\partial\boldsymbol{B}$ and then define the singular support of f as their union (cf. [**14**]).

In the next theorem we characterize some microanalytic properties of hyperfunctions on $\partial\boldsymbol{B}$ in terms of the coefficients of their generalized Faber expansions.

THEOREM 3. i) *Let $\{c_n\}_{n=0}^{\infty}$ be a sequence of complex numbers such that $\lim_{n\to\infty}c_n/c_{n+1}=\rho e^{i\sigma^*}$. Then, the hyperfunction*

$$f(\zeta)=\sum_{n=0}^{\infty}c_nP_n(\zeta)+\sum_{n=0}^{\infty}d_nq_n(\zeta)\in\mathcal{B}(\partial\boldsymbol{B})$$

is not microanalytic at $\zeta^*-ON_{\zeta^*}$ where $\zeta^*=\chi(\rho e^{i\sigma^*})$.

ii) *Let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of positive integers such that*

$$\lim_{n\to\infty}\sqrt[\lambda_n]{|c_n|}=\rho\qquad\textit{and}\qquad\lim_{n\to\infty}\frac{\lambda_n}{n}=\infty\,.$$

Then, every point of $\partial\boldsymbol{B}$ is in the inner singular spectrum SS^-f of the hyperfunction

$$f(\zeta)=\sum_{n=0}^{\infty}c_nP_{\lambda_n}(\zeta)+\sum_{n=0}^{\infty}d_nq_n(\zeta)\,.$$

Hence, sing. supp f=supp. $f=\partial \boldsymbol{B}$.

PROOF. Recall that ([**1**], p. 376) under the hypothesis of i) the power series $g(\omega)=\sum_{n=0}^{\infty} c_n\omega^n$ has a singular point at $\omega^*=\rho e^{i\sigma^*}$ and under the hypothesis of ii) the circle $|\omega|=\rho$ is the natural boundary for the power series $\tilde{g}(\omega)=\sum_{n=0}^{\infty} c_n\omega^{\lambda_n}$, i.e., $\tilde{g}(\omega)$ can not be continued anywhere beyond $|\omega|=\rho$.

i) Let $g(\omega)=\sum_{n=0}^{\infty} c_n\omega^n$, $f_2(z)=f^-(z)=\sum_{n=0}^{\infty} c_nP_n(z)$ and $\tilde{f}_*^-(z)=\sum_{n=0}^{\infty} c_n\pi_n(z)$, where $\pi_n(z)$ are given by (2.5). From (2.12) and (2.13), one can easily show that

$$f^-(z)=\frac{1}{2\pi i}\int_{\Gamma}\tilde{f}_*^-(u)F\left(\frac{z}{u}\right)\frac{du}{u}, \qquad z\in\boldsymbol{B}$$

$$\tilde{f}_*^-(z)=\frac{1}{2\pi i}\int_{\Gamma}f^-(u)F_*\left(\frac{z}{u}\right)\frac{du}{u}, \qquad z\in\boldsymbol{B}$$

and then apply Hadamard's multiplication of singularities argument ([**23**]), to show that $f^-(z)$ and $\tilde{f}_*^-(z)$ have exactly the same singular points. From the formula ([**21**])

$$\omega^n=\frac{1}{2\pi i}\int_{|t|=r}\frac{\pi_n(\chi(t))}{(t-\omega)R(t)}dt, \qquad |\omega|<r<\rho$$

and (2.5), after modifying its domain of validity, one obtains

$$g(\omega)=\frac{1}{2\pi i}\int_{|t|=r}\frac{\tilde{f}_*^-(\chi(t))}{(t-\omega)R(t)}dt, \qquad |\omega|<r<\rho$$

and

$$\tilde{f}_*^-(z)=\frac{1}{2\pi i}\int_{|\omega|=r}\frac{\chi'(\omega)}{\chi(\omega)-z}R(\omega)g(\omega)d\omega, \qquad z\in\boldsymbol{B}_r,\ r<\rho$$

respectively. Of course, r must be chosen so that the circle $|\omega|=r$ is in the domain of definition of $\chi(\omega)$, which is the case if $r^*<r<\rho$; for the definition of r^*, see the proof of Theorem 1. Upon using Hadamard's argument once more, one can show that $g(\omega)$ has a singular point at $\omega=\omega^*$ if and only if $\tilde{f}_*^-(z)$ has one at $z=z^*$ where $z^*=\chi(\omega^*)$. Therefore, $g(\omega)$ has a singular point at $\omega=\omega^*$ if and only if $f^-(z)$ has one at $z=z^*$ where $z^*=\chi(\omega^*)$. The proof is now complete since from the first part of the proof $g(\omega)$ has a singularity at $\omega^*=\rho e^{i\sigma^*}$.

ii) The proof is similar to (i).

For more details on Hadamard's argument and its applications, we refer the reader to [**5**] and to a recent article by the author, Freund and Görlich [**24**] where a more detailed version of the above argument is given.

4. Examples.

1) In (2.1), let $\rho=1$, $R(\omega)=1$, $F(u)=1/(1-u)$ and $z=\chi(\omega)=\omega$. Then, it is easy

to see that $P_n(z)=z^n$ and $q_n(z)=1/z^{n+1}$. Therefore, $f(\zeta)=f(e^{i\sigma})$ is a hyperfunction on the unit circle if and only if

$$f(e^{i\sigma})=\sum_{n=0}^{\infty}c_n e^{in\sigma}+\sum_{n=0}^{\infty}d_n e^{-i(n+1)\sigma}=\sum_{n=-\infty}^{\infty}c_n e^{in\sigma}$$

with $\limsup_{|n|\to\infty}{}^{|n|}\sqrt{|c_n|}\leqq 1$, where $d_n=c_{-(n+1)}$.

This result was previously obtained by Sato [**16**] and Johnson [8].

2) In [**24**], we have shown that for $z=\chi(\omega)=1/2(\omega+1/\omega)$, $F(u)=1/(1-u)^{\lambda}$, $\lambda\geqq -1/2$, $\lambda\neq 0$ and $R(\omega)=\omega^{2\lambda}/(\omega^2+1)^{\lambda-1}(\omega^2-1)$, the generalized Faber polynomials are the Gegenbauer (ultraspherical) polynomials $C_n^{\lambda}(z)$ normalized by $C_n^{\lambda}(1)=\binom{n+2\lambda-1}{n}$. Therefore, $f(\zeta)$ is a hyperfunction on the ellipse $E=\{z: |z+1|+|z-1|=\rho+1/\rho,\ \rho>1\}$, which is the image of the circle $|\omega|=\rho$ under $\chi(\omega)$, if and only if

$$f(\zeta)=\sum_{n=0}^{\infty}c_n C_n^{\lambda}(\zeta)+\sum_{n=0}^{\infty}d_n q_n^{\lambda}(\zeta),\qquad \zeta\in E \tag{4.1}$$

with

$$\limsup_{n\to\infty}{}^{n}\sqrt{|c_n|}\leqq\frac{1}{\rho},\qquad \limsup_{n\to\infty}{}^{n}\sqrt{|d_n|}\leqq\rho$$

where $(z^2-1)^{1/2-\lambda}q_n^{\lambda}(z)$ are the Gegenbauer functions of the second kind normalized by

$$q_n^{\lambda}(z)=\left[\frac{\Gamma(2\lambda)}{2^{\lambda}\Gamma(\lambda+1/2)}\right]^2\frac{(2n+2\lambda)\Gamma(n+1)}{\Gamma(n+2\lambda)}\int_{-1}^{1}(1-t^2)^{\lambda-1/2}\frac{C_n^{\lambda}(t)}{z-t}\,dt,$$

(see [**22**]).

In the interesting case where $\rho=1$ and the ellipse E degenerates to the interval $[-1, 1]$, the representation (4.1) is no longer valid. This case was studied earlier by the author in [**26**] where he obtained, by using different techniques, a characterization of hyperfunctions on $[-1, 1]$ in terms of series of Gegenbauer polynomials. Similar characterizations for hyperfunctions on any finite closed interval $[a, b]$, in terms of series of a more general class of orthogonal polynomials on $[a, b]$ have also been obtained by the author and G. Walter in [**25**].

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