J. Math. Soc. Japan
Vol. 41, No. 3, 1989

A formula for the logarithmic derivative of Selberg's zeta function

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(Received Dec. 7, 1987)
(Revised April 18, 1988)

1. Introduction.

Let M be a compact Riemann surface of genus $g\geqq 2$. The universal covering surface of M is conformally equivalent to the upper half plane H. Therefore M is the quotient of H by the discontinuous group Γ, consisting of, apart from the identity, hyperbolic transformations. We assume that the Gaussian curvature of M is normalized to be -1. It is known that the area $\mathcal{A}(M)$ of M is equal to $4\pi(g-1)$ via the Gauss-Bonnet theorem. Let T be a finite-dimensional unitary representation of Γ, and let χ be its character. Let Δ denote the Laplacian for H. Let $\{\lambda_n(\chi)\}_{n=0,1,2,\cdots}$ be the sequence of distinct eigenvalues corresponding to the problem $\Delta F+\lambda F=0$ on M, where the eigenfunction $F(x)$ is required to transform under Γ by $F(\gamma x)=T(\gamma)F(x)$. We denote by $m_n(\chi)$ the multiplicity of $\lambda_n(\chi)$. It is well known that the eigenvalues are all real and non-negative, and that the set of such eigenfunctions is complete in the space consisting of those measurable functions on H which transform in this manner, and square integrable over a fundamental domain of Γ. Associate with the sequence $0\leqq\lambda_0(\chi)<\lambda_1(\chi)<\cdots$ of eigenvalues, a sequence, consisting of those numbers $r_n(\chi)$ that satisfy the equation $\lambda_n(\chi)=1/4+r_n(\chi)^2$ $(n=0, 1, 2, \cdots)$. From this it follows that $r_n(\chi)$ is either real or pure imaginary. We choose and fix $r_n(\chi)$ so that when it is real, we have $r_n(\chi)\geqq 0$, and when it is pure imaginary, we have $\sqrt{-1}r_n(\chi)<0$. In the convention of notation, we put $\lambda_0(\chi)=0$ and $r_0(\chi)=\sqrt{-1}/2$, and we denote by $m_0(\chi)\geqq 0$ the multiplicity of the possible eigenvalue $\lambda_0(\chi)=0$ throughout this paper.

By assumption on Γ, each $\gamma\in\Gamma$ $(\gamma\neq e)$ is conjugate in $PSL(2, \boldsymbol{R})$ to a unique transformation of the form $z\mapsto e^{u_\gamma}z$, where u_γ is a positive real number. Clearly u_γ depends only on the conjugacy class. We will denote by $\{\gamma\}$ the conjugacy class corresponding to γ within Γ itself and by $\{\Gamma\}$ the set of all Γ-conjugacy classes in Γ. It is known that the numbers $\{u_\gamma ; \{\gamma\}\in\{\Gamma\}\setminus\{e\}\}$ are bounded away from zero. We choose and fix ε_0 so small that it is smaller than these numbers throughout the paper. An element $\gamma\in\Gamma$ $(\gamma\neq e)$ is called primitive if it can not be expressed as δ^j for some $j>1$ and $\delta\in\Gamma$. Also, we will denote by $\{\delta\}_p$ the primitive conjugacy class corresponding to primitive δ. It is well known that every γ ($\neq e$) is equal to a positive power of a unique primitive element δ. We define a positive integer $j(\gamma)$ by $\gamma=\delta^{j(\gamma)}$. Then we have $u_\gamma=j(\gamma)u_\delta$.

The Selberg zeta function is given by

$$Z_\Gamma(s,\chi)=\prod_{\{\delta\}_p}\prod_{n=0}^{\infty}\det(I-T(\delta)\exp\{-(s+n)u_\delta\}),$$

where the outer product is taken over all primitive conjugacy classes in Γ. Moreover the product converges absolutely, if $\mathrm{Re}\,s>1$. This zeta function has the following properties:

(A) $Z_\Gamma(s,\chi)$ is actually an entire function of order 2.

(B) $Z_\Gamma(s,\chi)$ satisfies the functional equation

$$Z_\Gamma(1-s,\chi)=\left\{\exp(-\chi(e)\mathcal{A}(M)\int_0^{s-1/2}z\tan(\pi z)dz)\right\}Z_\Gamma(s,\chi).$$

(C) $Z_\Gamma(s,\chi)$ has "trivial" zeros at $s=-k$ ($k=0,1,2,\cdots$), with multiplicity $2(g-1)(2k+1)\chi(e)$.

(D) The "nontrivial" zeros of $Z_\Gamma(s,\chi)$ are located at $s=1/2\pm\sqrt{-1}r_n(\chi)$, with multiplicity $m_n(\chi)$.

Let $\phi_\Gamma(s,\chi)$ be the logarithmic derivative of $Z_\Gamma(s,\chi)$. Namely, it is given by the absolutely and uniformly convergent series in any half plane $\mathrm{Re}\,s>1+\varepsilon$ ($\varepsilon>0$):

$$\phi_\Gamma(s,\chi)=\frac{1}{2}\sum_{\{\gamma\}\in\{\Gamma\}\setminus\{e\}}\chi(\gamma)j(\gamma)^{-1}u_\gamma\,\mathrm{cosech}\left(\frac{u_\gamma}{2}\right)\exp\left\{\left(\frac{1}{2}-s\right)u_\gamma\right\}.$$

Of course, above properties of the zeta function correspond to those of the logarithmic derivative of it. These properties are always derived from the Selberg trace formula. That is to say, the choice of the test function which we put into the trace formula is important. Suppose $F(r)$ is an even function, holomorphic in a strip of the form $\{r\in\boldsymbol{C};\,|\mathrm{Im}\,r|<1/2+\varepsilon\}$ for some positive ε, and satisfying a growth condition of the form $|F(r)|=O((1+|r|^2)^{-1-\varepsilon})$ uniformly in the strip. The trace formula then reads

$$\sum_{j=0}^{\infty}m_j(\chi)F(r_j(\chi))$$
$$=\frac{\chi(e)\mathcal{A}(M)}{4\pi}\int_{-\infty}^{\infty}F(r)r\tanh(\pi r)dr+\frac{1}{2}\sum_{\{\gamma\}\in\{\Gamma\}\setminus\{e\}}\chi(\gamma)j(\gamma)^{-1}u_\gamma\,\mathrm{cosech}\left(\frac{u_\gamma}{2}\right)F^*(u_\gamma)$$

where $F^*(u)=(1/2\pi)\int_{-\infty}^{\infty}F(r)\exp(\sqrt{-1}ru)dr$. It should be noted that both the series in the trace formula converge absolutely, uniformly with respect to χ.

The original choice of the test function which is taken by Selberg is as follows (cf. [2], [6], [8]):

$$F_s(r)=\frac{H(\sqrt{-1}s-\sqrt{-1}/2+r)}{s-1/2-\sqrt{-1}r}+\frac{H(\sqrt{-1}s-\sqrt{-1}/2-r)}{s-1/2+\sqrt{-1}r}$$

where $H(r)=\int_0^\infty g'(u)\exp(\sqrt{-1}ru)du$. Here the function g is an even, real-valued C^∞ function on $\boldsymbol{R}$ such that: (i) g vanishes in some neighborhood of zero, (ii) $g\equiv 1$ on $\{u\in\boldsymbol{R};\ |u|\geqq\varepsilon_0\}$ and (iii) $0\leqq g\leqq 1$.

Then the meromorphic continuation of $\phi_\Gamma(s,\chi)$ is given by

$$\phi_\Gamma(s,\chi)=\sum_{n=0}^\infty m_n(\chi)F_s(r_n(\chi))+\frac{1}{\pi}\chi(e)\mathcal{A}(M)\sum_{k=0}^\infty\frac{H(\sqrt{-1}(s+k))}{s+k}\Big(k+\frac{1}{2}\Big).$$

This formula is, of course, important, but in view of the proof of the functional equation it is somewhat troublesome.

More direct method is discovered by Hejhal ([4], [5]). His choice of the test function is defined by

$$F_s(r)=\frac{1}{r^2+(s-1/2)^2}-\frac{1}{r^2+(a-1/2)^2}\qquad(\mathrm{Re}\,s>1,\ \mathrm{Re}\,a>1).$$

Using this function he proved the following formula:

$$\begin{aligned}\frac{1}{2s-1}&\phi_\Gamma(s,\chi)\\&=\frac{1}{2a-1}\phi_\Gamma(a,\chi)+\sum_{n=0}^\infty m_n(\chi)\Big\{\frac{1}{r_n(\chi)^2+(s-1/2)^2}-\frac{1}{r_n(\chi)^2+(a-1/2)^2}\Big\}\\&\quad+\frac{\chi(e)\mathcal{A}(M)}{2\pi}\sum_{k=0}^\infty\Big(\frac{1}{a+k}-\frac{1}{s+k}\Big).\end{aligned}$$

All of the properties possessed by the zeta function are derived immediately from this formula.

The main purpose of this paper is to prove a new interesting formula for $\phi_\Gamma(s,\chi)$. The properties of the zeta function can be also derived from our formula immediately. Our choice of the test function is somewhat special one. We use two test functions which are essentially related to the inverse of the Plancherel measure of the upper half plane H. In this sense, I think one can extend this formula to the cases of symmetric spaces of rank one, but unfortunately, we must do them case by case. So we omit the description of their details. More generally, I think also that it is possible to extend our argument to the vector bundle cases (cf. [1], [9], [10]).

2. Main result.

The following formula holds:

$$(\mathbf{F}_\chi) \qquad \cot(\pi s)\phi_\Gamma\left(s+\frac{1}{2}, \chi\right) = -\pi m_0(\chi)+\pi(g-1)\chi(e)(2s-1)$$
$$-2\left(s^2-\frac{1}{4}\right)\sum_{n=1}^{\infty}\left\{\frac{n\phi_\Gamma(n+1/2, \chi)}{\pi(s^2-n^2)(1/4-n^2)}\right.$$
$$\left.+\frac{m_n(\chi)r_n(\chi)\coth(\pi r_n(\chi))}{(s^2+r_n(\chi)^2)(1/4+r_n(\chi)^2)}\right\}.$$

By this formula, we get the following theorem immediately.

THEOREM. *The series in the formula* $(\mathbf{F}_\chi)$ *converges absolutely, uniformly for* s *in any compact set disjoint from the numbers* $\{\pm\sqrt{-1}r_n(\chi)\}\cup\{\boldsymbol{Z}\setminus\{0\}\}$, *and defines a meromorphic function of* s *in the whole complex plane, thus gives us a meromorphic continuation of* $\phi_\Gamma(s+1/2, \chi)$. *Moreover the following functional equation holds:*

$$\phi_\Gamma\left(s+\frac{1}{2}, \chi\right)+\phi_\Gamma\left(-s+\frac{1}{2}, \chi\right) = 4\pi\chi(e)(g-1)s\tan(\pi s), \qquad s\in\boldsymbol{C}.$$

Also, the poles of $\phi_\Gamma(s+1/2, \chi)$ *are all simple, and are as follows:*

Pole	Residue	
$s=\pm\sqrt{-1}r_n(\chi)$	$m_n(\chi)$	$n\geqq 1$
$s=-\frac{1}{2}-n$	$2(g-1)(2n+1)\chi(e)$	$n\geqq 1$
$s=\frac{1}{2}$	$m_0(\chi)$	
$s=-\frac{1}{2}$	$m_0(\chi)+2(g-1)\chi(e)$.	

If for some n, *we have* $r_n(\chi)=0$, *then for that* n, *the residue at this pole is* $2m_n(\chi)$.

I do not know essential applications of the formula $(\mathbf{F}_\chi)$ except the above theorem. But, for example, letting s to $m+1/2$ $(m\in\boldsymbol{N})$, we have the following interesting identity:

$$\sum_{n=1}^{\infty}\left[\frac{n\phi_\Gamma(n+1/2, \chi)}{\pi\{(m+1/2)^2-n^2\}(1/4-n^2)}+\frac{m_n(\chi)r_n(\chi)\coth(\pi r_n(\chi))}{\{(m+1/2)^2+r_n(\chi)^2\}(1/4+r_n(\chi)^2)}\right]$$
$$=\frac{\pi}{m+1}\left\{(g-1)\chi(e)-\frac{m_0(\chi)}{2m}\right\}.$$

3. Proof of the formula ($\mathbf{F}_\chi$).

Our main tool is the Selberg trace formula. In what follows, we will be dealing with two test functions

$$^jF_t(r)=\pi r\left(r^2+\frac{1}{4}\right)^{-j}\coth(\pi r)\exp(-r^2t),\qquad t>0\quad (j=0,\,1)$$

in the trace formula. Since $^0F_t(\sqrt{-1}/2)=0$ and $^1F_t(\sqrt{-1}/2)=(\pi^2/2)\exp(t/4)$, we have

$$\begin{aligned}(1)\qquad &\pi\sum_{n=1}^{\infty}m_n(\chi)r_n(\chi)\left(r_n(\chi)^2+\frac{1}{4}\right)^{-j}\coth(\pi r_n(\chi))\exp(-r_n(\chi)^2t)\\ &\quad+j(\pi^2/2)m_0(\chi)\exp(t/4)-\frac{\chi(e)\mathcal{A}(M)}{4}\int_{-\infty}^{\infty}r^2\left(r^2+\frac{1}{4}\right)^{-j}\exp(-r^2t)dr\\ &=\sum_{\{\gamma\}\in\{\Gamma\}\setminus\{e\}}\varepsilon_{\gamma,\chi}{}^jF_t^*(u_\gamma)\qquad (j=0,\,1)\end{aligned}$$

where we put

$$\varepsilon_{\gamma,\chi}=\frac{1}{2}\chi(\gamma)j(\gamma)^{-1}u_\gamma\,\mathrm{cosech}\left(\frac{u_\gamma}{2}\right).$$

Now we need the following well known result.

LEMMA 1 (cf. [3]). *The series*

$$\sum_{r_n(\chi)\neq 0}\frac{m_n(\chi)}{|r_n(\chi)|^k}$$

converges if $k>2=\dim H$, *and diverges if* $k\leqq 2$.

We define $^j\theta_\chi(t)$ $(j=0,\,1)$ by the left side of (1). For $p>1/2$, multiply $^j\theta_\chi(t)$ by $t\exp(-p^2t)$ and integrate term by term with respect to t between $[0,\,\infty)$. The procedure can be justified by Lemma 1, and we obtain

$$(2_0)\qquad \int_0^\infty t^0\theta_\chi(t)\exp(-p^2t)dt=\pi\sum_{n=1}^{\infty}\frac{m_n(\chi)r_n(\chi)\coth(\pi r_n(\chi))}{(r_n(\chi)^2+p^2)^2}-\frac{\pi\chi(e)\mathcal{A}(M)}{8p},$$

$$\begin{aligned}(2_1)\qquad \int_0^\infty t^1\theta_\chi(t)\exp(-p^2t)dt&=\pi\sum_{n=1}^{\infty}\frac{m_n(\chi)r_n(\chi)\coth(\pi r_n(\chi))}{(r_n(\chi)^2+p^2)^2(r_n(\chi)^2+1/4)}\\ &\quad+\frac{\pi^2m_0(\chi)}{2(1/4-p^2)^2}-\frac{\pi\chi(e)\mathcal{A}(M)(p-1/2)^2}{8p(p^2-1/4)^2}.\end{aligned}$$

It is clear that (2_0) and (2_1) are valid for $\mathrm{Re}\,p>1/2$.

Observe that, because of the identity (1) we have

$$\int_0^\infty t^j\theta_\chi(t)\exp(-p^2t)dt=\int_0^\infty\sum_{\{\gamma\}\in\{\Gamma\}\setminus\{e\}}\varepsilon_{\gamma,\chi}{}^jF_t^*(u_\gamma)t\exp(-p^2t)dt\qquad (j=0,\,1).$$

We will calculate these integrals. At the first place, we state the following estimates.

LEMMA 2. *For any ε satisfying $0<|\varepsilon|<1/4$, there exists a positive constant C such that*

$$\sup_{u>0}\left|{}^{j}F_t^*(u)\exp\left(\left(\frac{1}{2}+\varepsilon\right)u\right)\right| \leqq C\max\left(\frac{1}{t}, \frac{1}{\sqrt{t}}\right)\exp\left(t\left(\frac{1}{2}+\varepsilon\right)^2\right) \qquad (j=0, 1)$$

for an arbitrary positive number t.

The proof is similar to that of the Paley-Wiener theorem with respect to the rapidly decreasing functions in the usual theory of Fourier transformation. So we omit it.

Let ε be a sufficiently small positive number. Then by Lemma 2, we have

$$\begin{aligned} |{}^{j}\theta_{\chi}(t)| &\leqq C\max\left(\frac{1}{t}, \frac{1}{\sqrt{t}}\right)\exp\left(t\left(\frac{1}{2}+\varepsilon\right)^2\right)\sum_{\{\gamma\}\in\{\Gamma\}\setminus\{e\}}|\varepsilon_{\gamma,\chi}|\exp\left(-\left(\frac{1}{2}+\varepsilon\right)u_{\gamma}\right) \\ &\leqq C\max\left(\frac{1}{t}, \frac{1}{\sqrt{t}}\right)\exp\left(t\left(\frac{1}{2}+\varepsilon\right)^2\right)\chi(e)\phi_{\Gamma}(1+\varepsilon\cdot\mathbf{1}) \qquad (j=0, 1), \end{aligned} \tag{3}$$

where $\mathbf{1}$ stands for the trivial character of Γ.

Suppose that p satisfies the condition $1>p>1/2+2\varepsilon$ for sufficiently small ε. Then, thanks to the estimate (3) and Lebesgue's dominated convergence theorem, we observe

$$\begin{aligned} &\int_0^{\infty} t^{j}\theta_{\chi}(t)\exp(-p^2t)dt \\ &= \sum_{\{\gamma\}\in\{\Gamma\}\setminus\{e\}}\varepsilon_{\gamma,\chi}\frac{1}{2\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty}\pi r\left(r^2+\frac{1}{4}\right)^{-j}\coth(\pi r)\exp(-r^2t)\exp(\sqrt{-1}ru_{\gamma})dr\right\} \\ &\qquad\qquad \times t\exp(-p^2t)dt \qquad (j=0, 1). \end{aligned}$$

Since the integrals

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_0^{\infty}\left|\pi r\left(r^2+\frac{1}{4}\right)^{-j}\coth(\pi r)\exp(\sqrt{-1}ru)t\exp(-(r^2+p^2)t)\right|dtdr \\ &= \int_{-\infty}^{\infty}\frac{\pi r\coth(\pi r)}{(p^2+r^2)^2(1/4+r^2)^j}dr \qquad (j=0, 1) \end{aligned}$$

converge for $u>0$, we have

$$\int_0^{\infty} t^{j}\theta_{\chi}(t)\exp(-p^2t)dt = \sum_{\{\gamma\}\in\{\Gamma\}\setminus\{e\}}\varepsilon_{\gamma,\chi}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi r\coth(\pi r)}{(p^2+r^2)^2(1/4+r^2)^j}\exp(\sqrt{-1}ru_{\gamma})dr \qquad (j=0, 1)$$

by means of Fubini's theorem.

Using the well known representation of $\pi r\coth(\pi r)$ by partial fractions, we obtain

$$\frac{\pi r\coth(\pi r)}{(p^2+r^2)^2} = \frac{1}{(p^2+r^2)^2}+2\sum_{n=1}^{\infty}\left\{\frac{n^2}{(n^2-p^2)^2(r^2+p^2)}\right.$$

$$-\frac{p^2}{(n^2-p^2)(r^2+p^2)^2}-\frac{n^2}{(n^2-p^2)^2(r^2+n^2)}\Big\}.$$

Using Lebesgue's theorem again, we see that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi r\coth(\pi r)}{(p^2+r^2)^2}\exp(\sqrt{-1}ru)dr=\frac{1}{4p^3}(pu+1)\exp(-pu)$$

$$+2\sum_{n=1}^{\infty}\frac{n^2}{(n^2-p^2)^2\cdot 2p}\exp(-pu)-2\sum_{n=1}^{\infty}\frac{p^2}{(n^2-p^2)\cdot 4p^3}(pu+1)\exp(-pu)$$

$$-2\sum_{n=1}^{\infty}\frac{n^2}{(n^2-p^2)^2\cdot 2n}\exp(-nu)=\frac{1}{4p}\left\{\frac{1}{p}+\sum_{n=1}^{\infty}\frac{2p}{p^2-n^2}\right\}u\exp(-pu)$$

$$-\frac{1}{4p}\frac{d}{dp}\left\{\frac{1}{p}+\sum_{n=1}^{\infty}\frac{2p}{p^2-n^2}\right\}\exp(-pu)-\sum_{n=1}^{\infty}\frac{n}{(n^2-p^2)^2}\exp(-nu)$$

$$=-\frac{\pi}{4p}\frac{d}{dp}\{\cot(\pi p)\exp(-pu)\}-\sum_{n=1}^{\infty}\frac{n}{(n^2-p^2)^2}\exp(-nu)$$

for $u>0$. Obviously, these manipulations are valid for p satisfying $1>\mathrm{Re}\,p>1/2+2\varepsilon$. Also, we can get the following identity in the same way as above.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi r\coth(\pi r)}{(p^2+r^2)^2(1/4+r^2)}\exp(\sqrt{-1}ru)dr=\frac{\pi}{4p(p^2-1/4)}\frac{d}{dp}\{\cot(\pi p)\exp(-pu)\}$$

$$-\frac{\pi\cot(\pi p)}{2(p^2-1/4)^2}\exp(-pu)+\sum_{n=1}^{\infty}\frac{n}{(n^2-p^2)^2(n^2-1/4)}\exp(-nu).$$

Note the fact that $\psi_\Gamma(p+1/2,\chi)$ is given by the absolutely and uniformly convergent series in any half plane $\mathrm{Re}\,p>1/2+2\varepsilon$:

$$\psi_\Gamma\Big(p+\frac{1}{2},\chi\Big)=\sum_{\{\gamma\}\in\{\Gamma\}\setminus\{e\}}\varepsilon_{\gamma,\chi}\exp(-pu_\gamma).$$

Hence we can differentiate $\psi_\Gamma(p+1/2,\chi)$ term by term with respect to p in that half plane. Recall that the constant ε_0 satisfies $0<\varepsilon_0\leqq u_\gamma$ for all $\{\gamma\}\in\{\Gamma\}\setminus\{e\}$. Therefore we have the following estimate:

$$(4)\qquad \left|\psi_\Gamma\Big(n+\frac{1}{2},\chi\Big)\right|\leqq\chi(e)\left|\psi_\Gamma\Big(n+\frac{1}{2},1\Big)\right|\leqq\chi(e)\psi_\Gamma\Big(\frac{3}{2},1\Big)\exp\{-\varepsilon_0(n-1)\}\qquad(n\in\boldsymbol{N}).$$

These facts guarantee that the following manipulations are legitimate:

$$\int_0^\infty t^0\theta_\chi(t)\exp(-p^2t)dt$$

$$(5_0)\qquad =\sum_{\{\gamma\}\in\{\Gamma\}\setminus\{e\}}\varepsilon_{\gamma,\chi}\left[-\frac{\pi}{4p}\frac{d}{dp}\{\cot(\pi p)\exp(-pu_\gamma)\}-\sum_{n=1}^{\infty}\frac{n}{(n^2-p^2)^2}\exp(-nu_\gamma)\right]$$

$$=-\frac{\pi}{4p}\frac{d}{dp}\left\{\cot(\pi p)\psi_\Gamma\Big(\frac{1}{2}+p,\chi\Big)\right\}-\sum_{n=1}^{\infty}\frac{n}{(n^2-p^2)^2}\psi_\Gamma\Big(\frac{1}{2}+n,\chi\Big),$$

$$(5_1)\quad \int_0^\infty t^1\theta_\chi(t)\exp(-p^2t)dt = \frac{\pi}{4p(p^2-1/4)}\frac{d}{dp}\Big\{\cot(\pi p)\phi_\Gamma\Big(\frac{1}{2}+p,\ \chi\Big)\Big\}$$
$$-\frac{\pi\cot(\pi p)}{2(p^2-1/4)^2}\phi_\Gamma\Big(\frac{1}{2}+p,\ \chi\Big)$$
$$+\sum_{n=1}^\infty \frac{n}{(n^2-p^2)^2(n^2-1/4)}\phi_\Gamma\Big(\frac{1}{2}+n,\ \chi\Big).$$

On account of (2_j) and (5_j) $(j=0,\ 1)$ respectively, we obtain

$$(6_0)\quad \frac{d}{dp}\Big\{\cot(\pi p)\phi_\Gamma\Big(\frac{1}{2}+p,\ \chi\Big)\Big\}$$
$$=\frac{1}{2}\chi(e)\mathcal{A}(M)-4p\sum_{n=1}^\infty\Big\{\frac{m_n(\chi)r_n(\chi)\coth(\pi r_n(\chi))}{(r_n(\chi)^2+p^2)^2}+\frac{n\phi_\Gamma(1/2+n,\ \chi)}{\pi(n^2-p^2)^2}\Big\},$$

$$(6_1)\quad \frac{d}{dp}\Big\{\cot(\pi p)\phi_\Gamma\Big(\frac{1}{2}+p,\ \chi\Big)\Big\}-\frac{2p\cot(\pi p)\phi_\Gamma(1/2+p,\ \chi)}{p^2-1/4}$$
$$=-\frac{\chi(e)\mathcal{A}(M)(p-1/2)^2}{2(p^2-1/4)}+\frac{2\pi m_0(\chi)p}{p^2-1/4}$$
$$+4p\Big(p^2-\frac{1}{4}\Big)\sum_{n=1}^\infty\Big\{\frac{m_n(\chi)r_n(\chi)\coth(\pi r_n(\chi))}{(r_n(\chi)^2+p^2)^2(r_n(\chi)^2+1/4)}+\frac{n\phi_\Gamma(1/2+n,\ \chi)}{\pi(n^2-p^2)^2(1/4-n^2)}\Big\}.$$

Thanks to the estimate (4) and Lemma 1, the series in the identities (6_j) $(j=0,\ 1)$ converge absolutely, uniformly for p in any compact set disjoint from the numbers $\{\pm\sqrt{-1}r_n(\chi)\}\cup\{\boldsymbol{Z}\setminus\{0\}\}\cup\{\pm 1/2\}$, and define meromorphic functions of p in the whole complex plane, thus give us meromorphic continuations of the left sides of identities (6_j) $(j=0,\ 1)$, respectively. Hence, subtracting (6_1) from (6_0) we have a desired result.

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