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Expansive homeomorphisms with the pseudo-orbit tracing property of n-tori

Dedicated to Professor Yukihiro Kodama on his 60th birthday

By Koichi HIRAIDE

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§1. Introduction.

The strongest useful equivalence for the study of orbit structures of homeomorphisms will be topological conjugacy. Our investigation will be within the context of the conjugacy problem for homeomorphisms with expansiveness and the pseudo-orbit tracing property (abbrev. POTP). The author proved in [12] that every compact surface which admits such homeomorphisms is the 2-torus and moreover that such a homeomorphism of the 2-torus is topologically conjugate to a hyperbolic toral automorphism. Thus it seems that orbit structures of homeomorphisms of the n-torus will be determined under the assumption of expansiveness and POTP. And so it will be natural to ask whether every homeomorphism with expansiveness and POTP of the n-torus is topologically conjugate to a hyperbolic toral automorphism. An answer of this problem is given as follows.

THEOREM. *Let $f: \boldsymbol{T}^n \to \boldsymbol{T}^n$ be a homeomorphism of the n-torus. If f is expansive and has* POTP, *then f is topologically conjugate to a hyperbolic toral automorphism.*

The notion of "c-map" is introduced for (self-) covering maps. The class of covering maps with this notion is wider than that of homeomorphisms having expansiveness and POTP. Recently in [2] N. Aoki and the author obtain some interesting results for c-maps, which relate to our theorem.

Let (X, d) be a metric space and $f: X \to X$ be a (self-) homeomorphism. We say that f is *expansive* if there is $c>0$ (called an *expansive constant*) such that if $x, y \in X$ and $x \neq y$ then $d(f^n(x), f^n(y))>c$ for some $n \in \boldsymbol{Z}$. A sequence $\{x_i\}_{i\in \boldsymbol{Z}}$ of X is a *δ-pseudo-orbit* of f if $d(f(x_i), x_{i+1})<\delta$ for all $i \in \boldsymbol{Z}$. A point $x \in X$ *ε-traces* a sequence $\{x_i\}_{i\in \boldsymbol{Z}}$ of X if $d(f^i(x), x_i)<\varepsilon$ for all $i \in \boldsymbol{Z}$. We say that f has *POTP* if for $\varepsilon>0$ there is $\delta>0$ such that every δ-pseudo-orbit of f is ε-traced by some point of X. Note that if X is compact, then expansiveness and POTP are independent of the metrics compatible with original topology, and preserved under topological conjugacy. For materials on topological dynamics on compact manifolds, the reader may refer to A. Morimoto [**16**].

For global analysis of our homeomorphisms, we prepare the notion of generalized foliations on topological manifolds.

Let M be a connected topological manifold without boundary and $\mathcal{F}$ be a family of subsets of M. We say that $\mathcal{F}$ is a *generalized foliation* on M if the following hold;

(1) $\mathcal{F}$ is a decomposition of M,

(2) each $L\in\mathcal{F}$ (called a *leaf*) is arcwise connected,

(3) if $x\in M$ then there exist non-trivial connected subsets D_x, K_x with $D_x\cap K_x=\{x\}$, a connected open neighborhood N_x of x in M and a homeomorphism $\phi_x: D_x\times K_x\to N_x$ (called a *local coordinate* (*around* x)) such that

(a) $\phi_x(x, x)=x$,

(b) $\phi_x(y, x)=y$ $(y\in D_x)$ and $\phi_x(x, z)=z$ $(z\in K_x)$,

(c) for any $L\in\mathcal{F}$ there is an at most countable set $B\subset K_x$ such that $N_x\cap L=\phi_x(D_x\times B)$.

Let $\mathcal{F}$ be a generalized foliation on M. For fixed $L\in\mathcal{F}$ let Q_L be a family of subsets of L defined as follows; $D\in Q_L$ if and only if there is an open subset O of M such that D is a connected component in $O\cap L$. The topology generated by Q_L is called the *leaf topology* of L.

If $x\in L$ and D_x is as in (3), then $D_x\subset L$ and D_x is open in L (with respect to Q_L). Moreover the relative topology of D_x by the leaf topology coincides with that of D_x by the topology of M. Note that the leaf topology has countable base.

If $f: M\to M$ is a homeomorphism such that $f(\mathcal{F})=\mathcal{F}$, then it is easily checked that for every $L\in\mathcal{F}$, $f: L\to f(L)$ is a homeomorphism (with respect to Q_L and $Q_{f(L)}$).

Let $\mathcal{F}$ and $\mathcal{F}'$ be generalized foliations on M. We say that $\mathcal{F}'$ is *transverse* to $\mathcal{F}$ if to $x\in M$ there exist non-trivial connected subsets D_x, D'_x with $D_x\cap D'_x=\{x\}$, a connected open neighborhood N_x of x in M (such a neighborhood N_x is called a *coordinate domain*) and a homeomorphism $\varphi_x: D_x\times D'_x\to N_x$ (called a *canonical coordinate* (*around* x)) such that

(a)′ $\varphi_x(x, x)=x$,

(b)′ $\varphi_x(y, x)=y$ $(y\in D_x)$ and $\varphi_x(x, z)=z$ $(z\in D'_x)$,

(c)′ for any $L\in\mathcal{F}$ there is an at most countable set $B'\subset D'_x$ such that $N_x\cap L=\varphi_x(D_x\times B')$,

(d)′ for any $L'\in\mathcal{F}'$ there is an at most countable set $B\subset D_x$ such that $N_x\cap L'=\varphi_x(B\times D'_x)$.

Let $\mathcal{F}'$ be transverse to $\mathcal{F}$. We denote by $L(x)$ and $L'(x)$ the leaves of $\mathcal{F}$ and $\mathcal{F}'$ through x respectively. Let N be a coordinate domain and write $D(x)$

and $D'(x)$ the connected components of x in $N\cap L(x)$ and $N\cap L'(x)$ respectively. For $x, y\in N$ it is not difficult to see that $D'(x)\cap D(y)$ is a single point. And so we can define a map

$$\gamma_N: N\times N \longrightarrow N$$

by $(x, y)\mapsto D'(x)\cap D(y)$, and have then γ_N is continuous and

$$\gamma_N(x, x)=x, \quad \gamma_N(x, \gamma_N(y, z))=\gamma_N(x, z),$$
$$\gamma_N(\gamma_N(x, y), z)=\gamma_N(x, z).$$

If N and $\tilde{N}$ are coordinate domains and if U is a coordinate domain such that $U\subset N\cap\tilde{N}$, then the following is checked from the definition

$$\gamma_U=\gamma_{N}|_{U\times U}=\gamma_{\tilde{N}}|_{U\times U}.$$

Let f be a homeomorphism of a metric space (X, d). For $x\in X$, define the *stable set* $W^s(x)$ and the *unstable set* $W^u(x)$ by

$$W^s(x)=\{y\in X: d(f^n(x), f^n(y))\to 0 \quad \text{as } n\to\infty\},$$
$$W^u(x)=\{y\in X: d(f^n(x), f^n(y))\to 0 \quad \text{as } n\to -\infty\}$$

and put

$$\mathcal{F}_f^\sigma=\{W^\sigma(x): x\in X\} \qquad (\sigma=s, u).$$

Then $\mathcal{F}_f^\sigma$ is a decomposition of X and $f(\mathcal{F}_f^\sigma)=\mathcal{F}_f^\sigma$.

For the proof of Theorem we need the following two propositions.

PROPOSITION A. *Let M be a closed topological manifold and $f: M\to M$ be a homeomorphism. If f is expansive and has* POTP, *then $\mathcal{F}_f^\sigma$ ($\sigma=s, u$) are generalized foliations on M and $\mathcal{F}_f^s$ is transverse to $\mathcal{F}_f^u$.*

When $\mathcal{F}$ is a generalized foliation on M and there exists a generalized foliation transverse to $\mathcal{F}$, the orientability for $\mathcal{F}$ will be defined (see § 4).

PROPOSITION B. *Let $f: M\to M$ be as in Proposition* A. *If the generalized foliation $\mathcal{F}_f^u$ is orientable, then there exists $l\in\mathbf{N}$ such that for any $m\geq l$ all the fixed points of f^m have the same fixed point index* 1 *or* -1.

If we establish Proposition B, then our theorem will be obtained by using skilfully the techniques of J. Franks [**9**], M. Brin and A. Manning [**5**] and the author [**12**].

§ 2. Geometric properties of homeomorphisms.

This section contains the proof of Proposition A and results about local behaviors of homeomorphisms with expansiveness and POTP.

Let M be a closed topological manifold and d be a metric for M. Let $f: M\to M$ be a homeomorphism. For $\varepsilon>0$ and $x\in M$ we define the *local stable set* $W_\varepsilon^s(x)$ and the *local unstable set* $W_\varepsilon^u(x)$ by

$$W_\varepsilon^s(x)=\{y\in M: d(f^n(x), f^n(y))\leqq\varepsilon,\ n\geqq 0\},$$
$$W_\varepsilon^u(x)=\{y\in M: d(f^n(x), f^n(y))\leqq\varepsilon,\ n\leqq 0\}.$$

Let f be expansive with expansive constant $c>0$. Then we know (cf. [**14**]) that

(I) *for every* $\varepsilon>0$ *there exists* $N>0$ *such that*

$$f^n W_c^s(x)\subset W_\varepsilon^s(f^n(x)),\quad f^{-n}W_c^u(x)\subset W_\varepsilon^u(f^{-n}(x))$$

for all $n\geqq N$ *and all* $x\in M$.

By using (I) it is easily checked that

$$W^s(x)=\bigcup_{n\geqq 0} f^{-n}W_\varepsilon^s(f^n(x)),\quad W^u(x)=\bigcup_{n\geqq 0} f^n W_\varepsilon^u(f^{-n}(x)) \tag{2.1}$$

for all $0<\varepsilon\leqq c$ and all $x\in M$.

If in addition f satisfies POTP, then the following (II), (III) and (IV) are proved in [**11**].

(II) *Let* $\varepsilon_0=c/4$. *Then there exists* $0<\delta_0<\varepsilon_0$ *such that if* $d(x, y)<\delta_0$, *then* $W_{\varepsilon_0}^s(x)\cap W_{\varepsilon_0}^u(y)$ *is a single point, which is denoted by* $\alpha(x, y)$.

(III) *Let* $0<\delta_0<\varepsilon_0$ *be as above and put*

$$\Delta(\delta_0)=\{(x, y)\in M\times M: d(x, y)<\delta_0\}.$$

Then $\alpha: \Delta(\delta_0)\to M$ *is a continuous map and*

$$\alpha(x, x)=x,\quad \alpha(x, \alpha(y, z))=\alpha(x, z),$$
$$\alpha(\alpha(x, y), z)=\alpha(x, z)$$

whenever the two sides of these relations are defined.

(IV) *For every* δ *with* $0<\delta<\delta_0/2$, *define*

$$W_{\varepsilon_0,\delta}^\sigma(x)=\{y\in W_{\varepsilon_0}^\sigma(x): d(x, y)<\delta\}\qquad(\sigma=s, u),$$
$$N_{x,\delta}=\alpha(W_{\varepsilon_0,\delta}^u(x)\times W_{\varepsilon_0,\delta}^s(x)).$$

Then there exist $0<\delta_1<\delta_0/2$ *and* $\rho_0>0$ *such that for every* $x\in M$

(a) N_{x,δ_1} *is open in* M *and* $\mathrm{diam}(N_{x,\delta_1})<\delta_0$,

(b) $\alpha: W_{\varepsilon_0,\delta_1}^u(x)\times W_{\varepsilon_0,\delta_1}^s(x)\to N_{x,\delta_1}$ *is a homeomorphism,*

(c) $N_{x,\delta_1}\supset B_{\rho_0}(x)$ *where* $B_{\rho_0}(x)$ *denotes the closed ball of radius* ρ_0 *centered at* x.

Let $0<\delta_1<\delta_0/2$ be as above. We denote by D_x^σ the connected component of x in $W_{\varepsilon_0,\delta_1}^\sigma(x)$ for $\sigma=s,\ u$, and define

$$N_x=\alpha(D_x^u\times D_x^s),\quad \alpha_x=\alpha_{|D_x^u\times D_x^s}: D_x^u\times D_x^s\to N_x.$$

Then it is easily checked that

$$(2.2)\qquad \begin{cases} \alpha_x(x,\ x)=x\,, \\ \alpha_x(y,\ x)=y\ (y\in D_x^u)\ \text{ and }\ \alpha_x(x,\ z)=z\ (z\in D_x^s). \end{cases}$$

Moreover we obtain the following (see [**12**] for the proof).

PROPOSITION 2.1 (Local product structure). *For every $x\in M$*

(a) *N_x is connected and open in M and* $\operatorname{diam}(N_x)<\delta_0$,

(b) *$\alpha_x: D_x^u\times D_x^s\to N_x$ is a homeomorphism,*

(c) *there exists $0<\rho<\varepsilon_0$ such that $N_x\supset B_\rho(x)$ for all $x\in M$,*

(d) *$D_x^\sigma\supsetneqq\{x\}$ for $\sigma=s,\ u$.*

LEMMA 2.2. *For $x,\ y\in M$ there exist at most countable sets $B'\subset D_x^s$ and $B\subset D_x^u$ such that*

(a) $N_x\cap W^u(y)=\alpha_x(D_x^u\times B')$,

(b) $N_x\cap W^s(y)=\alpha_x(B\times D_x^s)$.

PROOF. We prove (b). If this is done, then (a) is obtained in the same way. Take $z\in D_x^u$ and put $D_{x,z}^s=N_x\cap W_{\varepsilon_0}^s(z)$. Then it is checked that $D_{x,z}^s=\alpha_x(\{z\}\times D_x^s)$. Indeed, by the definitions of N_x and α_x, we have that $D_{x,z}^s\supset\alpha_x(\{z\}\times D_x^s)$. Conversely, if $w\in D_{x,z}^s$, then there are $u\in D_x^u$ and $v\in D_x^s$ such that $w=\alpha_x(u,\ v)$. Since $w\in W_{\varepsilon_0}^s(u)$ and $w\in W_{\varepsilon_0}^s(z)$, obviously $u,\ z\in W_{\varepsilon_0}^s(w)$. Since $u,\ z\in D_x^u\subset W_{\varepsilon_0}^u(x)$, by expansiveness $z=u$, and therefore $D_{x,z}^s\subset\alpha(\{z\}\times D_x^s)$.

By the above result and Proposition 2.1(b), we have

$$N_x=\bigcup_{z\in D_x^u} D_{x,z}^s\qquad \text{(disjoint union)}.$$

CLAIM 1. *If $D_{x,z}^s\cap W^s(y)\neq\emptyset$ for some $z\in D_x^u$, then $D_{x,z}^s\subset W^s(y)$.*

PROOF. Let $w\in D_{x,z}^s\cap W^s(y)\neq\emptyset$. Since $D_{x,z}^s\subset W_{\varepsilon_0}^s(z)$, by (I) there is $n\geqq 0$ such that $f^n(w)\in f^n(D_{x,z}^s)\subset W_{\varepsilon_0/3}^s(f^n(z))$. By the definition of $W^s(y)$, $f^n(w)\in W_{\varepsilon_0/3}^s(f^n(y))$ for sufficiently large $n\geqq 0$. Hence $f^n(D_{x,z}^s)\subset W_{\varepsilon_0}^s(f^n(y))$, and therefore $D_{x,z}^s\subset W^s(y)$ by (I).

Let $B=D_x^u\cap W^s(y)$. Then we have

$$N_x\cap W^s(y)=\bigcup_{z\in D_x^u} D_{x,z}^s\cap W^s(y)=\bigcup_{z\in B} D_{x,z}^s=\alpha_x(B\times D_x^s).$$

Hence it remains only to prove the following

CLAIM 2. *B is at most countable.*

PROOF. By (2.1), $W^s(y)=\bigcup_{n\geq 0} s_n(y)$ where $s_n(y)=f^{-n}W^s_{\varepsilon_0}(f^n(y))$. Fix $n\geqq 0$ and let $p\in s_n(y)$. Then $D^s_p\cap s_n(y)$ is open in $s_n(y)$ (under the relative topology of M). This is proved as follows. We set

$$K^u_p=\{u\in D^u_p:\ d(f^i(p),\ f^i(u))<\varepsilon_0,\ 0\leqq i\leqq n\}.$$

Obviously K^u_p is open in D^u_p, and so $L_p=\alpha_p(K^u_p\times D^s_p)$ is open in M by Proposition 2.1. Hence $L_p\cap s_n(y)$ is open in $s_n(y)$. If we establish that $L_p\cap s_n(y)=D^s_p\cap s_n(y)$, then our requirement is obtained.

To get the conclusion, let $a\in L_p\cap s_n(y)$. Then there are $u\in K^u_p$ and $v\in D^s_p$ such that $a=\alpha_p(u, v)$. Hence $a\in W^s_{\varepsilon_0}(u)$ and $a\in W^u_{\varepsilon_0}(v)$. Since $a,\ p\in s_n(y)$, obviously $f^n(a),\ f^n(p)\in W^s_{\varepsilon_0}(f^n(y))$, and so $f^n(a)\in W^s_{2\varepsilon_0}(f^n(p))$. Since $v\in W^s_{\varepsilon_0}(p)$, we have $f^n(v)\in W^s_{\varepsilon_0}(f^n(p))$, and hence $f^n(a)\in W^s_{3\varepsilon_0}(f^n(v))$. For $0\leqq i\leqq n$, we have

$$\begin{aligned} d(f^i(a),\ f^i(v)) &\leqq d(f^i(a),\ f^i(u))+d(f^i(u),\ f^i(p))+d(f^i(p),\ f^i(v)) \\ &\leqq \varepsilon_0+\varepsilon_0+\varepsilon_0=3\varepsilon_0 \end{aligned}$$

and hence $a\in W^s_{3\varepsilon_0}(v)\cap W^u_{\varepsilon_0}(v)=\{v\}$. Therefore $L_p\cap s_n(y)\subset D^s_p$. Since $p\in K^u_p$, $D^s_p=\alpha_p(\{p\}\times D^s_p)\subset L_p$ and so $D_p\cap s_n(y)=L_p\cap s_n(y)$.

Since $s_n(y)$ is compact, there is a finite set $\{p_i\}\subset s_n(y)$ such that $s_n(y)\subset\bigcup_i D^s_{p_i}\subset\bigcup_i W^s_{\varepsilon_0}(p_i)$. Let $z_1,\ z_2\in s_n(y)\cap D^u_x$. If $z_1,\ z_2\in W^s_{\varepsilon_0}(p_i)$ for some i, then $z_2\in W^s_{2\varepsilon_0}(z_1)\cap W^u_{\varepsilon_0}(x)=\{z_1\}$, and so $z_1=z_2$. This implies that $s_n(y)\cap D^u_x$ is finite. Since

$$B=W^s(y)\cap D^u_x=\Big(\bigcup_{n\geqq 0} s_n(y)\Big)\cap D^u_x=\bigcup_{n\geqq 0}(s_n(y)\cap D^u_x),$$

B is at most countable.

The conclusion of Proposition A is obtained in proving the following

PROPOSITION 2.3. *Under the above assumptions and notations*
(1) $\mathcal{F}^\sigma_f$ ($\sigma=s,\ u$) *are generalized foliations on* M,
(2) $\mathcal{F}^s_f$ *is transverse to* $\mathcal{F}^u_f$,
(3) *for any* $x\in M$, $\alpha_x: D^u_x\times D^s_x\to N_x$ *is a canonical coordinate around* x.

PROOF. Let ρ be as in Proposition 2.1. Then it follows that for every $x\in M$

$$W^\sigma_\rho(x)\subset D^\sigma_x \qquad (\sigma=s,\ u). \tag{2.3}$$

Indeed, since $W^s_\rho(x)\subset B_\rho(x)$ and $B_\rho(x)\subset N_x$ by Proposition 2.1(c), we have $y\in N_x$ when $y\in W^s_\rho(x)$. Since $N_x=\alpha(D^u_x\times D^s_x)$, obviously $y=\alpha(u, v)$ for some $u\in D^u_x$ and $v\in D^s_x$. Hence $\alpha(x, y)=\alpha(x, \alpha(u, v))=\alpha(x, v)$ by (II). Since $\rho<\varepsilon_0$, we have $W^s_\rho(x)\subset W^s_{\varepsilon_0}(x)$ and hence $y\in W^s_{\varepsilon_0}(x)$. By the definition of α, $y=\alpha(x, y)\in W^s_{\varepsilon_0}(x)\cap W^u_{\varepsilon_0}(y)$. Since $v\in D^s_x\subset W^s_{\varepsilon_0}(x)$, we have also $\alpha(x, v)=v$. From the above calculations we have $y=v\in D_x^s$. In the same way, the result for $\sigma=u$ is checked.

By (2.1) and (2.3) we have that for every $x\in M$

$$W^s(x)=\bigcup_{n\geq 0} f^{-n}D^s_{f^n(x)},\quad W^u(x)=\bigcup_{n\geq 0} f^n D^u_{f^{-n}(x)}.$$

Since D_x^σ $(\sigma=s, u)$ are arcwise connected (by Proposition 2.1(a), (b)), so are $W^\sigma(x)$. Hence by (2.2), Proposition 2.1 and Lemma 2.2 the conclusion is obtained.

Let $n=\dim M$. We take an atlas $\{(U_i, g_i)\}$ of M such that

(2.4) $\begin{cases} \textit{each } g_i: U_i\to \boldsymbol{R}^n \textit{ is a homeomorphism, } \{U_i\} \textit{ is} \\ \textit{finite and a Lebesgue number of } \{U_i\} \textit{ is } r_1>0. \end{cases}$

Let δ_0 be as in (II). Since δ_0 is taken sufficiently small, we can assume that $\delta_0<r_1$. Thus $\mathrm{diam}(N_x)\leqq r_1$ for all $x\in M$ (by Proposition 2.1(a)).

Let $\rho>0$ be as in Proposition 2.1 and choose a finite open cover $\{V_j\}$ of M such that

(2.5) $\begin{cases} \textit{each diameter of } V_j \textit{ is less than } \rho \textit{ and there} \\ \textit{are an open set } V'_j\supset V_j \textit{ and a homeomorphism} \\ h'_j: V'_j\to \boldsymbol{R}^n \textit{ such that } V_j=h_j'^{-1}(V^n) \textit{ where } V^n \\ \textit{denotes the unit open ball in } \boldsymbol{R}^n. \end{cases}$

Then we can take a Lebesgue number r_2 with $0<r_2<\rho$.

Since M is compact, by (III) there is $0<r_3<r_2$ such that for all $x\in M$

$$\alpha(B_{r_3}(x)\times\{x\})\subset B_{r_2}(x).$$

And we choose a new atlas $\{(W_i, h_i)\}$ of M such that

(2.6) $\begin{cases} \textit{each } h_i: W_i\to \boldsymbol{R}^n \textit{ is a homeomorphism, } \{W_i\} \textit{ is} \\ \textit{finite and each diameter of } W_i \textit{ is less than } r_3, \end{cases}$

and let $0<r_4<r_3$ be a Lebesgue number of $\{W_i\}$.

By (I) there is $l>0$ such that for $x\in M$ and $m\geqq l$

(2.7) $$f^m D_x^s\subset W^s_{r_4}(f^m(x)),\quad f^{-m}D_x^u\subset W^u_{r_4}(f^{-m}(x)).$$

Fix $m\geqq l$ and take $x\in \mathrm{Fix}(f^m)$. We write

$$K_x^s=f^m(D_x^s),\quad K_x^u=f^{-m}(D_x^u).$$

PROPOSITION 2.4. *For* $x\in\mathrm{Fix}(f^m)$ *the following hold;*

(a) $K_x^\sigma\subset B_{r_4}(x)$ $\quad(\sigma=s, u)$,

(b) K_x^σ *is an open subset of* $D_x^\sigma(x)$ $\quad(\sigma=s, u)$,

(c) *the diagram*

$$\begin{array}{ccc} K^u_x\times D^s_x & \xrightarrow{\quad f^m\times f^m \quad} & D^u_x\times K^s_x \\ {\scriptstyle \alpha_x}\Big\downarrow & & \Big\downarrow{\scriptstyle \alpha_x} \\ \alpha_x(K^u_x\times D^s_x) & \xrightarrow{\quad f^m \quad} & \alpha_x(D^u_x\times K^s_x) \end{array}$$

commutes.

PROOF. (a) is clear. Since $r_4<\rho$, by (2.3) we have $K^\sigma_x\subset D^\sigma_x$. Since D^σ_x is open in $W^\sigma(x)$ by Proposition 2.3 and $f^m: W^\sigma(x)\to W^\sigma(x)$ is a homeomorphism under the leaf topology, $K^\sigma(x)$ is open in $W^\sigma(x)$, and hence K^σ_x is open in D^σ_x. (b) was proved.

To show (c), take $(y, z)\in K^u_x\times D^s_x$. Then we have by the definition

$$a=\alpha_x(y, z)\in W^s_{\varepsilon_0}(y)\cap W^u_{\varepsilon_0}(z). \tag{2.8}$$

On the other hand, since

$$f^m(y)\in f^m(K^u_x)=D^u_x \tag{2.9}$$

$$f^m(z)\in K^s_x\subset D^s_x, \tag{2.10}$$

we have

$$b=\alpha_x(f^m(y), f^m(z))\in W^s_{\varepsilon_0}(f^m(y))\cap W^u_{\varepsilon_0}(f^m(z)). \tag{2.11}$$

Since for $0\leqq i\leqq m$

$$d(f^i(a), f^i(x))\leqq d(f^i(a), f^i(y))+d(f^i(y), f^i(x)),$$

by using (2.8) and (2.9) we have

$$d(f^i(a), f^i(x))\leqq 2\varepsilon_0 \qquad (0\leqq i\leqq m).$$

If $a'=f^{-m}(b)$, then

$$d(f^i(a'), f^i(x))\leqq d(f^i(a'), f^i(z))+d(f^i(z), f^i(x))$$

and hence from (2.11) and the fact that $z\in D^s_x\subset W^s_{\varepsilon_0}(x)$,

$$d(f^i(a'), f^i(x))\leqq 2\varepsilon_0 \qquad (0\leqq i\leqq m).$$

Hence we have $d(f^i(a), f^i(a'))\leqq 4\varepsilon_0$ for $0\leqq i\leqq m$. For $i\geqq m$, we have

$$\begin{aligned} d(f^i(a), f^i(a')) &= d(f^i(a), f^{i-m}(b)) \\ &\leqq d(f_i(a), f^i(y))+d(f^{i-m}\circ f^m(y), f^{i-m}(b)) \end{aligned}$$

and so by (2.8) and (2.11)

$$d(f^i(a), f^i(a'))\leqq 2\varepsilon_0 \qquad (i\geqq m).$$

From (2.8) and (2.11)

$$d(f^{i}(a),\ f^{i}(a'))\leqq d(f^{i}(a),\ f^{i}(z))+d(f^{i-m}\circ f^{m}(z),\ f^{i-m}(b))$$
$$\leqq 2\varepsilon_{0}\qquad (i\leqq 0).$$

Since $4\varepsilon_0=c$ is an expansive constant for f, we have $a=a'$ and the proof is completed.

§3. Lifting of the geometric properties.

Let M be a closed topological manifold and $\pi:\overline{M}\to M$ be a covering projection. Let d be a metric for M. Then we know (cf. [8]) that there exist a metric $\bar{d}$ for $\overline{M}$ and $r_0>0$ such that

(i) if $x,\ y\in\overline{M}$ and $\bar{d}(x,\ y)\leqq 2r_0$, then $d(\pi(x),\ \pi(y))=\bar{d}(x,\ y)$,

(ii) if $x\in\overline{M}$, $y\in M$ and $d(\pi(x),\ y)\leqq 2r_0$, then there is a unique $y'\in\overline{M}$ such that $y'\in\pi^{-1}(y)$ and $\bar{d}(x,\ y')=d(\pi(x),\ y)$,

(iii) all covering transformations are isometries,

(iv) $\bar{d}$ is complete.

If $\bar{B}_{r_0}(x)$ denotes the closed ball of radius r_0 centered at $x\in\overline{M}$, then $\pi_{|\bar{B}_{r_0}(x)}:\bar{B}_{r_0}(x)\to B_{r_0}(\pi(x))$ is an isometry (by (i) and (ii)).

Let $G(\pi)$ be the group of all covering transformations for π. Let $f:M\to M$ and $\bar{f}:\overline{M}\to\overline{M}$ be homeomorphisms such that $f\circ\pi=\pi\circ\bar{f}$. Then there is a group automorphism $\overline{A}:G(\pi)\to G(\pi)$ such that $\bar{f}\circ\beta=\overline{A}(\beta)\circ\bar{f}$ for all $\beta\in G(\pi)$, in which case $\overline{A}$ is denoted by $\bar{f}_{\#}$.

The following is easily checked (cf. [**16**]).

(v) $\bar{f}:\overline{M}\to\overline{M}$ is biuniformly continuous under $\bar{d}$.

By this fact there is $0<\eta_0<r_0$ such that if $d(x,\ y)<\eta_0$, then $\max\{\bar{d}(\bar{f}(x),\ \bar{f}(y)),\ \bar{d}(\bar{f}^{-1}(x),\ \bar{f}^{-1}(y))\}<r_0/2$.

LEMMA 3.1. *If f has* POTP, *then $\bar{f}$ has* POTP.

PROOF. Let $0<\varepsilon\leqq\eta_0$. Since f has POTP, there is $0<\delta\leqq r_0/2$ such that every δ-pseudo-orbit of f is ε-traced by some point of M. To show that $\bar{f}$ has POTP, let $\{x_i\}_{i\in Z}$ be a δ-pseudo-orbit of $\bar{f}$. Since $\delta<r_0$, we have that $\{\pi(x_i)\}_{i\in Z}$ is a δ-pseudo-orbit of f. Hence there is $x\in M$ which ε-traces $\{\pi(x_i)\}_{i\in Z}$. Since $\varepsilon\leqq\eta_0<r_0$, we can take $\bar{x}\in\overline{M}$ such that $\bar{x}\in\pi^{-1}(x)$ and $\bar{d}(\bar{x},\ x_0)<\varepsilon$. Then we have

$$\bar{d}(\bar{f}(\bar{x}),\ x_1)\leqq\bar{d}(\bar{f}(\bar{x}),\ \bar{f}(x_0))+\bar{d}(\bar{f}(x_0),\ x_1)\leqq r_0/2+\delta\leqq r_0,$$

and hence $\bar{d}(\bar{f}(\bar{x}),\ x_1)=d(f(x),\ \pi(x_1))<\varepsilon$. Inductively, $\bar{d}(\bar{f}^{i}(\bar{x}),\ x_i)<\varepsilon$ for $i\geqq 0$. In the same way, we have $\bar{d}(\bar{f}^{i}(\bar{x}),\ x_i)<\varepsilon$ for $i<0$, and therefore $\bar{f}$ has POTP.

For $x\in\bar{M}$ and $\varepsilon>0$, let $\bar{W}_{\varepsilon}^{s}(x)$ and $\bar{W}_{\varepsilon}^{u}(x)$ be the local stable and unstable sets of $\bar{f}$ respectively.

LEMMA 3.2. *For* $0<\varepsilon\leqq\eta_0$ *and* $x\in\bar{M}$, $\pi:\bar{W}_{\varepsilon}^{\sigma}(x)\to W_{\varepsilon}^{\sigma}(\pi(x))$ *is an isometry* ($\sigma=s, u$).

PROOF. If $y\in\bar{W}_{\varepsilon}^{s}(x)$ and $i\geqq 0$, then we have

$$\varepsilon\geqq\bar{d}(\bar{f}^{i}(x), \bar{f}^{i}(y))=d(f^{i}\circ\pi(x), f^{i}\circ\pi(y)),$$

and hence $\pi(y)\in W_{\varepsilon}^{s}(\pi(x))$. To see that $\pi_{|\bar{W}_{\varepsilon}^{s}(x)}$ is surjective, let $y\in W_{\varepsilon}^{s}(\pi(x))$. Then there is $y'\in\bar{M}$ such that $y'\in\pi^{-1}(y)$ and $\bar{d}(y', x)\leqq\varepsilon$. Since $\varepsilon\leqq\eta_0$, we have that $\bar{d}(\bar{f}(x), \bar{f}(y'))<r_0$ and so

$$\bar{d}(\bar{f}(x), \bar{f}(y'))=d(f\circ\pi(x), f(y))\leqq\varepsilon .$$

Inductively, we have $\bar{d}(\bar{f}^{i}(x), \bar{f}^{i}(y'))\leqq\varepsilon$ for $i\geqq 0$, and therefore $y'\in\bar{W}_{\varepsilon}^{s}(x)$. Since $\bar{W}_{\varepsilon}^{s}(x)\subset\bar{B}_{r_0}(x)$, we proved that $\pi:\bar{W}_{\varepsilon}^{s}(x)\to W_{\varepsilon}^{s}(\pi(x))$ is an isometry. In the same way, the conclusion for $\sigma=u$ is obtained.

LEMMA 3.3. *If f is expansive and $0<c\leqq\eta_0$ is an expansive constant for f, then $\bar{f}$ is expansive and c is an expansive constant for $\bar{f}$.*

PROOF. By Lemma 3.2 and the fact that c is an expansive constant for f, we have

$$\pi(\bar{W}_{c}^{s}(x)\cap\bar{W}_{c}^{u}(x))=\pi(\bar{W}_{c}^{s}(x))\cap\pi(\bar{W}_{c}^{u}(x))=W_{c}^{s}(\pi(x))\cap W_{c}^{u}(\pi(x))=\{\pi(x)\},$$

and hence $\bar{W}_{c}^{s}(x)\cap\bar{W}_{c}^{u}(x)=\{x\}$. This implies that $\bar{f}$ is expansive and c is an expansive constant for f.

By §2 (I) and Lemma 3.2 we have the following

LEMMA 3.4. *Under the assumptions in Lemma* 3.3, *for* $\varepsilon>0$ *there is* $N\geqq 0$ *such that*

$$\bar{f}^{n}\bar{W}_{c}^{s}(x)\subset\bar{W}_{\varepsilon}^{s}(\bar{f}^{n}(x)),\quad \bar{f}^{-n}\bar{W}_{c}^{u}(x)\subset\bar{W}_{\varepsilon}^{u}(\bar{f}^{-n}(x))$$

for all $n\geqq N$ *and all* $x\in\bar{M}$.

For $x\in\bar{M}$, let $\bar{W}^{s}(x)$ and $\bar{W}^{u}(x)$ be the stable and unstable sets of $\bar{f}$ respectively.

Lemma 3.4 ensures the following

LEMMA 3.5. *Under the assumptions in Lemma* 3.3

$$\bar{W}^{s}(x)=\bigcup_{n\geqq 0}\bar{f}^{-n}\bar{W}_{\varepsilon}^{s}(\bar{f}^{n}(x)),\quad \bar{W}^{u}(x)=\bigcup_{n\geqq 0}\bar{f}^{n}\bar{W}_{\varepsilon}^{u}(\bar{f}^{-n}(x))$$

for all $0<\varepsilon\leqq c$ *and all* $x\in\bar{M}$.

LEMMA 3.6. *For* $\beta\in G(\pi)$ *and* $x\in\overline{M}$

$$\beta\overline{W}^{\sigma}(x)=\overline{W}^{\sigma}(\beta(x)) \qquad (\sigma=s,\ u).$$

PROOF. If $y\in\overline{W}^{s}(x)$, then

$$\begin{aligned}\bar{d}(\bar{f}^{n}\circ\beta(x),\ \bar{f}^{n}\circ\beta(y)) &= \bar{d}(\bar{f}^{n}_{\#}(\beta)\circ\bar{f}^{n}(x),\ \bar{f}^{n}_{\#}(\beta)\circ\bar{f}^{n}(y))\\ &= \bar{d}(\bar{f}^{n}(x),\ \bar{f}^{n}(y))\rightarrow 0,\end{aligned}$$

and hence $\beta(y)\in\overline{W}^{s}(\beta(x))$. Conversely, $\beta(y)\in\overline{W}^{s}(\beta(x))$ implies $y\in\overline{W}^{s}(x)$. Therefore $\beta\overline{W}^{s}(x)=\overline{W}^{s}(\beta(x))$. In the same way, we have $\beta\overline{W}^{u}(x)=\overline{W}^{u}(\beta(x))$.

Hereafter let f is expansive and fix its expansive constant $0<c\leqq\eta_0$. Moreover let f have POTP. Let $0<\delta_0<\varepsilon_0=c/4$ be as in §2. If we set

$$\bar{\Delta}(\delta_0)=\{(x,\ y)\in\overline{M}\times\overline{M}:\bar{d}(x,\ y)<\delta_0\},$$

then $(\pi(x),\ \pi(y))\in\Delta(\delta_0)$ when $(x,\ y)\in\bar{\Delta}(\delta_0)$. Hence we have $\{\alpha(\pi(x),\ \pi(y))\}=W^{s}_{\varepsilon_0}(\pi(x))\cap W^{u}_{\varepsilon_0}(\pi(y))$. Since $d(\pi(x),\ \alpha(\pi(x),\ \pi(y)))\leqq\varepsilon_0<r_0$, there is a unique $\bar{\alpha}(x,\ y)\in\overline{M}$ such that $\bar{d}(x,\ \bar{\alpha}(x,\ y))\leqq\varepsilon_0$ and $\pi\circ\bar{\alpha}(x,\ y)=\alpha(\pi(x),\ \pi(y))$.

LEMMA 3.7.
(1) $\{\bar{\alpha}(x,\ y)\}=\overline{W}^{s}_{\varepsilon_0}(x)\cap\overline{W}^{u}_{\varepsilon_0}(y)$ *for all* $(x,\ y)\in\bar{\Delta}(\delta_0)$.
(2) $\bar{\alpha}:\bar{\Delta}(\delta_0)\rightarrow\overline{M}$ *is a continuous map and*

$$\bar{\alpha}(x,\ x)=x,\quad \bar{\alpha}(x,\ \bar{\alpha}(y,\ z))=\bar{\alpha}(x,\ z),$$

$$\bar{\alpha}(\bar{\alpha}(x,\ y),\ z)=\bar{\alpha}(x,\ z)$$

whenever the two sides of these relations are defined.

PROOF. Since $\varepsilon_0<\eta_0$ and $\varepsilon_0+\delta_0<r_0$, (1) is obtained from Lemma 3.2. (2) holds by §2 (III) and (1).

Let $x\in\overline{M}$ and let $D^{\sigma}_{\pi(x)}$ $(\sigma=s,\ u)$ and $N_{\pi(x)}$ be the subsets constructed in §2. We define

$$\begin{aligned}\overline{D}^{\sigma}_{x} &= (\pi_{|\bar{B}_{r_0}(x)})^{-1}(D^{\sigma}_{\pi(x)}) \qquad (\sigma=s,\ u)\\ \overline{N}_{x} &= (\pi_{|\bar{B}_{r_0}(x)})^{-1}(N_{\pi(x)}).\end{aligned}$$

Since $D^{\sigma}_{\pi(x)}$ is a connected component of $\pi(x)$ in $W^{\sigma}_{\varepsilon_0,\delta_1}(\pi(x))$, by Lemma 3.2 $\overline{D}^{\sigma}_{x}$ is a connected component of x in $\overline{W}^{\sigma}_{\varepsilon_0,\delta_1}(x)$. Since $N_{\pi(x)}=\alpha(D^{u}_{\pi(x)}\times D^{s}_{\pi(x)})$ and $\bar{\alpha}(x,\ y)=(\pi_{|\bar{B}_{r_0}(x)})^{-1}\circ\alpha(\pi(x),\ \pi(y))$, we have

$$\begin{aligned}\overline{N}_{x} &= (\pi_{|\bar{B}_{r_0}(x)})^{-1}\circ\alpha(D^{u}_{\pi(x)}\times D^{s}_{\pi(x)})\\ &= \bar{\alpha}((\pi_{|\bar{B}_{r_0}(x)})^{-1}(D^{u}_{\pi(x)})\times(\pi_{|\bar{B}_{r_0}(x)})^{-1}(D^{s}_{\pi(x)}))=\bar{\alpha}(\overline{D}^{u}_{x}\times\overline{D}^{s}_{x}),\end{aligned}$$

and so we can define

$$\bar{\alpha}_x : \bar{D}^u_x \times \bar{D}^s_x \longrightarrow \bar{N}_x$$

by $\bar{\alpha}_x = \bar{\alpha}_{|\bar{D}^u_x \times \bar{D}^s_x}$. Then the following is obtained from Proposition 2.1.

PROPOSITION 3.8 (Local product structure on $\bar{M}$). *For every* $x \in \bar{M}$

(a) $\bar{N}_x$ *is connected and open in* $\bar{M}$ *and* $\mathrm{diam}(\bar{N}_x) < \delta_0$,

(b) $\bar{\alpha}_x : \bar{D}^u_x \times \bar{D}^s_x \to \bar{N}_x$ *is a homeomorphism,*

(c) $\bar{N}_x \supset \bar{B}_\rho(x)$ *where* ρ *is as in Proposition* 2.1(c),

(d) $\bar{D}^\sigma_x \supsetneqq \{x\}$ $\quad (\sigma = s, u)$.

Let $\mathcal{F}$ be a generalized foliation on M. Let $\bar{\mathcal{F}}$ be a family of subsets of $\bar{M}$ defined as follows; $\bar{L} \in \bar{\mathcal{F}}$ if and only if there is $L \in \mathcal{F}$ such that $\bar{L}$ is an arc-wise connected component in $\pi^{-1}(L)$. Then it is easily checked that $\bar{\mathcal{F}}$ is a generalized foliation on $\bar{M}$ and for all $\bar{L} \in \bar{\mathcal{F}}$, $\pi(\bar{L}) \in \mathcal{F}$ and $\pi : \bar{L} \to \pi(\bar{L})$ is a covering map. We say that $\bar{\mathcal{F}}$ is the *lift* of $\mathcal{F}$ by π. If $\mathcal{F}'$ is a generalized foliation on M transverse to $\mathcal{F}$, then the lift $\bar{\mathcal{F}}'$ is transverse to $\bar{\mathcal{F}}$.

Since $f : M \to M$ is expansive and has POTP, we know that $\mathcal{F}^\sigma_f$ $(\sigma = s, u)$ are generalized foliations on M and $\mathcal{F}^s_f$ is transverse to $\mathcal{F}^u_f$ (see Proposition A). Note that

$$\mathcal{F}^\sigma_{\bar{f}} = \{\bar{W}^\sigma(x) : x \in \bar{M}\} \qquad (\sigma = s, u).$$

PROPOSITION 3.9. *Under the above assumptions and notations*

(1) $\mathcal{F}^\sigma_{\bar{f}}$ $(\sigma = s, u)$ *are generalized foliations on* $\bar{M}$,

(2) $\mathcal{F}^s_{\bar{f}}$ *is transverse to* $\mathcal{F}^u_{\bar{f}}$,

(3) *for any* $x \in \bar{M}$, $\bar{\alpha}_x : \bar{D}^u_x \times \bar{D}^s_x \to \bar{N}_x$ *is a canonical coordinate around* x. *Moreover for* $\sigma = s, u$ *the following hold;*

(4) $\mathcal{F}^\sigma_{\bar{f}}$ *is the lift of* $\mathcal{F}^\sigma_f$ *by* π,

(5) $\pi : \bar{W}^\sigma(x) \to W^\sigma(\pi(x))$ *is a homeomorphism for all* $x \in \bar{M}$.

PROOF. Use Lemmas 3.3, 3.4, 3.5 and 3.7 and Proposition 3.8 and run on the proofs of Lemma 2.2 and Proposition 2.3. Then we see that (1), (2) and (3) hold. To obtain (4) and (5), we show that $\pi : \bar{W}^s(x) \to W^s(\pi(x))$ is bijective.

Let $y \in W^s(\pi(x))$. Since $W^s_\rho(\pi(x)) \subset D^s_{\pi(x)}$ by (2.3), there is $n \geqq 0$ such that $f^n(y) \in D^s_{f^n(\pi(x))} = D^s_{\pi \circ \bar{f}^n(x)}$. Remark that $\bar{D}^s_{\bar{f}^n(x)} = (\pi_{|B_{r_0}(\bar{f}^n(x))})^{-1}(D^s_{\pi \circ \bar{f}^n(x)})$. Then we can take $y'_n \in \bar{D}^s_{\bar{f}^n(x)}$ such that $\pi(y'_n) = f^n(y)$. Let $y' = \bar{f}^{-n}(y'_n)$. Then $y' \in \bar{W}^s(x)$ and $\pi(y') = y$, and hence $\pi(\bar{W}^s(x)) \supset W^s(\pi(x))$. Conversely, by the definition $\pi(\bar{W}^s(x)) \subset W^s(\pi(x))$, and so $\pi(\bar{W}^s(x)) = W^s(\pi(x))$. Let $y_1, y_2 \in \bar{W}^s(x)$ and let $\pi(y_1) = \pi(y_2)$. Then $\bar{d}(\bar{f}^n(y_1), \bar{f}^n(y_2)) < r_0$ for some $n \geqq 0$. Since $\pi \circ \bar{f}^n(y_1) = f^n \circ \pi(y_1) = f^n \circ \pi(y_2) = \pi \circ \bar{f}^n(y_2)$, we have $\bar{f}^n(y_1) = \bar{f}^n(y_2)$ and hence $y_1 = y_2$. Therefore $\pi : \bar{W}^s(x) \to W^s(\pi(x))$ is bijective.

By the above result we have

$$\pi^{-1}(W^s(\pi(x))) = \bigcup_{x' \in \pi^{-1}(\pi(x))} \overline{W}^s(x') \qquad \text{(disjoint union)},$$

and hence by (1) each $\overline{W}^s(x')$ is an arcwise connected component in $\pi^{-1}(W^s(\pi(x)))$. Therefore $\mathcal{F}_{\bar{f}}^s$ is the lift of $\mathcal{F}_f^s$ by π. In the same way, we have $\mathcal{F}_{\bar{f}}^u$ is the lift of $\mathcal{F}_f^u$ by π. (5) follows from the fact that $\pi: \overline{W}^\sigma(x) \to W^\sigma(\pi(x))$ is bijective ($\sigma = s, u$).

§ 4. Orientability of generalized foliations.

Let X be a topological space which satisfies

$$H_i(X, X\setminus\{x\}) \cong \begin{cases} \boldsymbol{Z} & i=n \\ 0 & i \neq n \end{cases}$$

for all $x \in X$, and V_x be a connected open neighborhood of x in X. We say that V_x is a *canonical neighborhood* if there is $O_x \in H_n(X, X\setminus V_x)$ such that $i_{z*}(O_x)$ is a generator of $H_n(X, X\setminus\{z\})$ for all $z \in V_x$ where $i_z: (X, X\setminus V_x) \hookrightarrow (X, X\setminus\{z\})$ denotes the inclusion. Such an element O_x is called a *fundamental class* of $H_n(X, X\setminus V_x)$.

If, in particular, X is Hausdorff and $O'_x \in H_n(X, X\setminus V_x)$ is the other fundamental class, it follows from the definition of singular homology that either $i_{z*}(O'_x) = i_{z*}(O_x)$ for all $z \in V_x$, or $i_{z*}(O'_x) = -i_{z*}(O_x)$ for all $z \in V_x$.

Let M be a topological n-manifold without boundary. In this case we have always that for all $x \in M$, $H_i(M, M\setminus\{x\}) \cong \boldsymbol{Z}$ $(i=n)$ and $\cong 0$ $(i \neq n)$, and that for each $x \in M$, there is a canonical neighborhood of x in M.

LEMMA 4.1 (Bredon [4]). *Let X and Y be non-trivial topological spaces and $X \times Y$ denote the product topological space. If $X \times Y$ is a connected topological n-manifold without boundary, then*

(1) *there are $p, q > 0$ with $p+q=n$ such that*

$$H_i(X, X\setminus\{x\}) \cong \begin{cases} \boldsymbol{Z} & (i=p) \\ 0 & (i \neq p) \end{cases} \qquad (x \in X),$$

$$H_i(Y, Y\setminus\{y\}) \cong \begin{cases} \boldsymbol{Z} & (i=q) \\ 0 & (i \neq q) \end{cases} \qquad (y \in Y),$$

(2) *each point of X (resp. Y) has a canonical neighborhood in X (resp. Y).*

LEMMA 4.2. *Let M be a connected topological manifold without boundary and $\mathcal{F}$ be a generalized foliation on M. Then there exists $0 < p < \dim(M)$ such that for each leaf $L \in \mathcal{F}$*

$$H_i(L, L\setminus\{x\}) \cong \begin{cases} \boldsymbol{Z} & (i=p) \\ 0 & (i\neq p) \end{cases} \quad (x\in L).$$

PROOF. Let $\psi: D\times K\to N$ be a local coordinate. Then there is $0<p<\dim(M)$ such that

$$H_i(D, D\setminus\{y\}) \cong \begin{cases} \boldsymbol{Z} & (i=p) \\ 0 & (i\neq p) \end{cases}$$

for $y\in D$ (by Lemma 4.1). Take $z\in N$ and let $L\in\mathcal{F}$ be the leaf through z. Then there is $y\in D$ such that

$$H_i(L, L\setminus\{z\}) \cong H_i(D, D\setminus\{y\}) \quad (i\geqq 0).$$

For, if D' denotes the connected component of z in $N\cap L$, there is $z'\in K$ such that $\psi(\ , z'): D\to D'$ is a homeomorphism, from which

$$H_i(D', D'\setminus\{z\}) \cong H_i(D, D\setminus\{y\}) \quad (i\geqq 0)$$

where $z=\psi(y, z')$. Since D' is open in L and $D'\hookrightarrow L$ is a C^0 embedding, by applying excision isomorphism theorem we have

$$H_i(L, L\setminus\{z\}) \cong H_i(D', D'\setminus\{z\}) \quad (i\geqq 0)$$

and therefore the lemma holds.

Let $\mathcal{F}$ be a generalized foliation on M and p be as in Lemma 4.2. The natural number p is called the *dimension* of $\mathcal{F}$ and we write $p=\dim(\mathcal{F})$. If $\mathcal{F}'$ is a generalized foliation transverse to $\mathcal{F}$, then $\dim(\mathcal{F})+\dim(\mathcal{F}')=\dim(M)$ holds.

Assume that there exists a generalized foliation transverse to $\mathcal{F}$. From now on we introduce the orientability for $\mathcal{F}$. To do this, for any path $\omega: [0, 1]\to M$ we construct an isomorphism

$$\omega_*: H_p(L(\omega(0)), L(\omega(0))\setminus\{\omega(0)\}) \longrightarrow H_p(L(\omega(1)), L(\omega(1))\setminus\{\omega(1)\}),$$

where $L(x)$ denotes the leaf of $\mathcal{F}$ through x, such that the following hold;

(a) (constant path)$_*$=id

(b) $(\omega_1\cdot\omega_2)_* = \omega_{2*}\circ\omega_{1*}$

(c) if ω_1 is homotopic to ω_2 rel $\{0, 1\}$, then $\omega_{1*}=\omega_{2*}$.

Let $\varphi: D\times D'\to N$ be a canonical coordinate around some $a\in M$. Then N is expressed as the disjoint union $N=\bigcup_{x\in D'} D_x$ of subsets D_x where $D_x=\varphi(D\times\{x\})$ for $x\in D'$. We note that D_x is the connected component of x in $N\cap L(x)$. By Lemma 4.1(2) there is a canonical neighborhood of a in D which is denoted by V. Then $R=\varphi(V\times D')$ $(\subset N)$ is a coordinate domain. Obviously $R=\bigcup_{x\in D'} V_x$ where $V_x=\varphi(V\times\{x\})$ for $x\in D'$. Since $\varphi(a, x)=x$ and $\varphi(\ , x): D\to D_x$ is a homeomorphism, we see that $O_x=\varphi(\ , x)_*(O)\in H_p(D_x, D_x\setminus V_x)$ is a fundamental class if so is $O\in H_p(D, D\setminus V)$.

For $x, x' \in D'$ define a homeomorphism $\psi_{x,x'}: (D_x, D_x \setminus V_x) \to (D_{x'}, D_{x'} \setminus V_{x'})$ such that the diagram

$$\begin{array}{ccc} (D_x, D_x \setminus V_x) & \xrightarrow{\psi_{x,x'}} & (D_{x'}, D_{x'} \setminus V_{x'}) \\ & \nwarrow^{\varphi(\ ,\, x)} \qquad \nearrow^{\varphi(\ ,\, x')} & \\ & (D, D \setminus V) & \end{array}$$

commutes. Then $\psi_{x,x'*}(O_x) = O_{x'}$ holds.

For $z \in R$ choose $x_z \in D'$ such that $z \in V_{x_z} \subset D_{x_z}$. Since D_{x_z} is the connected component of z in $N \cap L(z)$, the composition map

$$j_{z*}: H_p(D_{x_z}, D_{x_z} \setminus V_{x_z}) \xrightarrow{i_{z*}} H_p(D_{x_z}, D_{x_z} \setminus \{z\}) \xrightarrow[\cong]{i'_{z*}} H_p(L(z), L(z) \setminus \{z\}) \cong \boldsymbol{Z}$$

sends the fundamental class O_{x_z} to a generator of $\boldsymbol{Z}$ where $i_z: (D_{x_z}, D_{x_z} \setminus V_{x_z}) \hookrightarrow (D_{x_z}, D_{x_z} \setminus \{z\})$ and $i'_z: (D_{x_z}, D_{x_z} \setminus \{z\}) \hookrightarrow (L(z), L(z) \setminus \{z\})$. And so for $z, w \in R$ we define an isomorphism

$$\Phi^w_{z,R}: H_p(L(z), L(z) \setminus \{z\}) \longrightarrow H_p(L(w), L(w) \setminus \{w\})$$

by $\Phi^w_{z,R}(j_{z*}(O_{x_z})) = j_{w*}(O_{x_w})$. It is clear that for $z, w, w' \in R$

(a) $\Phi^z_{z,R} = \mathrm{id}$,

(b) $\Phi^w_{z,R} = \Phi^w_{w',R} \circ \Phi^{w'}_{z,R}$,

(c) $(\Phi^w_{z,R})^{-1} = \Phi^z_{w,R}$.

Taking $b \in M$, for a canonical coordinate $\tilde{\varphi}: \tilde{D} \times \tilde{D}' \to \tilde{N}$ around b we can choose a canonical neighborhood $\tilde{V}$ of b in $\tilde{D}$. As we saw above, $\tilde{N} = \tilde{\varphi}(\tilde{D} \times \tilde{D}')$ and $\tilde{R} = \tilde{\varphi}(\tilde{V} \times \tilde{D}')$ are expressed as the disjoint unions $\tilde{N} = \bigcup_{y \in \tilde{D}'} \tilde{D}_y$ ($\tilde{D}_y = \tilde{\varphi}(\tilde{D} \times \{y\})$) and $\tilde{R} = \bigcup_{y \in \tilde{D}'} \tilde{V}_y$ ($\tilde{V}_y = \tilde{\varphi}(\tilde{V} \times \{y\})$). Let $\tilde{O} \in H_p(\tilde{D}, \tilde{D} \setminus \tilde{V})$ be a fundamental class and define $\tilde{O}_y = \tilde{\varphi}(\ ,\, y)_*(\tilde{O}) \in H_p(\tilde{D}_y, \tilde{D}_y \setminus \tilde{V}_y)$ for $y \in \tilde{D}'$. For $y, y' \in \tilde{D}'$, let $\tilde{\psi}_{y,y'} = \tilde{\varphi}(\ ,\, y') \circ \tilde{\varphi}(\ ,\, y)^{-1}$. Then $\tilde{\psi}_{y,y'*}(\tilde{O}_y) = \tilde{O}_{y'}$ holds.

For $z \in \tilde{R}$, choose $y_z \in \tilde{D}'$ such that $z \in \tilde{V}_{y_z} \subset \tilde{D}_{y_z}$. Then the composition map

$$\tilde{j}_{z*}: H_p(\tilde{D}_{y_z}, \tilde{D}_{y_z} \setminus \tilde{V}_{y_z}) \longrightarrow H_p(\tilde{D}_{y_z}, \tilde{D}_{y_z} \setminus \{z\}) \xrightarrow{\cong} H_p(L(z), L(z) \setminus \{z\}) \cong \boldsymbol{Z}$$

sends $\tilde{O}_{y_z}$ to a generator of $\boldsymbol{Z}$. For $z, w \in \tilde{R}$, as above we define

$$\Phi^w_{z,\tilde{R}}: H_p(L(z), L(z) \setminus \{z\}) \longrightarrow H_p(L(w), L(w) \setminus \{w\})$$

by $\Phi^w_{z,\tilde{R}}(\tilde{j}_{z*}(\tilde{O}_{y_z})) = \tilde{j}_{w*}(\tilde{O}_{y_w})$.

CLAIM. *Suppose* $R\cap\tilde{R}$ *is non-empty and* $w, z\in R\cap\tilde{R}$. *If there exists a path* $\omega: [0, 1]\to R\cap\tilde{R}$ *such that* $\omega(0)=z$ *and* $\omega(1)=w$, *then* $\Phi_{z,R}^{w}=\Phi_{z,\tilde{R}}^{w}$ *holds.*

PROOF. Since $\omega([0, 1])\subset N\cap\tilde{N}$ and N and $\tilde{N}$ are coordinate domains, there is a coordinate domain U such that $\omega([0, 1])\subset U$ and $U\subset N\cap\tilde{N}$. Then we have

$$\gamma_U=\gamma_{N|U\times U}=\gamma_{\tilde{N}|U\times U} \tag{4.1}$$

where γ_U, γ_N and $\gamma_{\tilde{N}}$ are as in §1. For $u\in U$, let E_u be the connected component of u in $U\cap L(u)$. Since $z, w\in U$, we have

$$\gamma_U(\cdot, w)_{|E_z}: E_z\longrightarrow E_w\,.$$

Since $U\subset N\cap\tilde{N}$, obviously $E_z\subset D_{x_z}\cap\tilde{D}_{y_z}$ and $E_w\subset D_{x_w}\cap\tilde{D}_{y_w}$. Remark that

$$\phi_{z,w}=\gamma_N(\cdot, w)_{|D_{x_z}}: D_{x_z}\longrightarrow D_{x_w}$$

and

$$\tilde{\phi}_{z,w}=\gamma_{\tilde{N}}(\cdot, w)_{|\tilde{D}_{y_z}}: \tilde{D}_{y_z}\longrightarrow \tilde{D}_{y_w}\,.$$

Then from (4.1) it follows that

$$\phi_{z,w|E_z}=\tilde{\phi}_{z,w|E_z}=\gamma_U(\cdot, w)_{|E_z}\,. \tag{4.2}$$

Since $\omega([0, 1])\subset R\cap\tilde{R}\cap U$, we can take a coordinate domain W such that $\alpha([0, 1])\subset W$ and $\mathrm{cl}(W)\subset R\cap\tilde{R}\cap U$. For $u\in W$, let F_u be the connected component of u in $W\cap L(u)$. Since $\mathrm{cl}(W)\subset U$, $\mathrm{cl}(F_u)\subset E_u$ for all $u\in W$. Hence by applying the excision theorem the isomorphisms induced by the inclusion maps are obtained;

$$H_p(E_z, E_z\setminus F_z) \xrightarrow[\cong]{k_{1*}} H_p(D_{x_z}, D_{x_z}\setminus F_z)=\Delta_1$$
$$H_p(E_z, E_z\setminus F_z) \xrightarrow[\cong]{\tilde{k}_{1*}} H_p(\tilde{D}_{y_z}, \tilde{D}_{y_z}\setminus F_z)=\tilde{\Delta}_1$$

$$H_p(E_w, E_w\setminus F_w) \xrightarrow[\cong]{k_{2*}} H_p(D_{x_w}, D_{x_w}\setminus F_w)=\Delta_2$$
$$H_p(E_w, E_w\setminus F_w) \xrightarrow[\cong]{\tilde{k}_{2*}} H_p(\tilde{D}_{y_w}, \tilde{D}_{y_w}\setminus F_w)=\tilde{\Delta}_2\,.$$

Since $W\subset R\cap\tilde{R}$, obviously $F_z\subset V_{x_z}\cap\tilde{V}_{y_z}$ and $F_w\subset V_{x_w}\cap\tilde{V}_{y_w}$, and hence we have the homomorphisms induced by the inclusion maps:

$$H_p(D_{x_z}, D_{x_z}\setminus V_{x_z})\xrightarrow{l_{1*}}\Delta_1$$
$$H_p(\tilde{D}_{y_z}, \tilde{D}_{y_z}\setminus\tilde{V}_{y_z})\xrightarrow{\tilde{l}_{1*}}\tilde{\Delta}_1$$

$$H_p(D_{x_w}, D_{x_w}\setminus V_{x_w}) \xrightarrow{l_{2*}} \Delta_2$$
$$H_p(\tilde{D}_{y_w}, \tilde{D}_{y_w}\setminus\tilde{V}_{y_w}) \xrightarrow{\tilde{l}_{2*}} \tilde{\Delta}_2 .$$

If $j_{z*}(O_{x_z})=\tilde{j}_{z*}(\tilde{O}_{y_z})\in H_p(L(z), L(z)\setminus\{z\})$, then by the commutativity of the inclusion maps we have

$$n_{z*}\circ k_{1*}^{-1}\circ l_{1*}(O_{x_z}) = n_{z*}\circ\tilde{k}_{1*}^{-1}\circ\tilde{l}_{1*}(\tilde{O}_{y_z})$$

where $n_z: (E_z, E_z\setminus F_z)\hookrightarrow(E_z, E_z\setminus\{z\})$. Since $k_{1*}^{-1}\circ l_{1*}(O_{x_z})$ and $\tilde{k}_{1*}^{-1}\circ\tilde{l}_{1*}(\tilde{O}_{y_z})$ are fundamental classes, for all $z'\in F_z$ we have

$$n_{z'*}\circ k_{1*}^{-1}\circ l_{1*}(O_{x_z}) = n_{z'*}\circ\tilde{k}_{1*}^{-1}\circ\tilde{l}_{1*}(\tilde{O}_{y_z}) ,$$

and hence by (4.2)

$$\phi_{z,w*}\circ n_{z'*}\circ k_{1*}^{-1}\circ l_{1*}(O_{x_z}) = \tilde{\phi}_{z,w*}\circ n_{z'*}\circ\tilde{k}_{1*}^{-1}\circ\tilde{l}_{1*}(\tilde{O}_{y_z}) .$$

Since $\phi_{z,w}$ and $\tilde{\phi}_{z,w}$ commute with the inclusion maps, it follows that for all $w'\in F_w$

$$m_{w'*}\circ k_{2*}^{-1}\circ l_{2*}\circ\phi_{z,w*}(O_{x_z}) = m_{w'}\circ\tilde{k}_{2*}^{-1}\circ\tilde{l}_{2*}\circ\tilde{\phi}_{z,w*}(\tilde{O}_{y_z}) ,$$

where $m_{w'}: (E_w, E_w\setminus F_w)\hookrightarrow(E_w, E_w\setminus\{w'\})$. Since $\phi_{z,w*}(O_{x_z})=O_{x_w}$ and $\tilde{\phi}_{z,w*}(\tilde{O}_{y_z}) =\tilde{O}_{y_w}$ and since $w\in F_w$, we have

$$m_{w*}\circ k_{2*}^{-1}\circ l_{2*}(O_{x_w}) = m_{w*}\circ\tilde{k}_{2*}^{-1}\circ\tilde{l}_{2*}(\tilde{O}_{y_w}) ,$$

and therefore

$$j_{w*}(O_{x_w}) = \tilde{j}_{w*}(\tilde{O}_{y_w})$$

by the commutativity of the inclusion maps. This implies that $\Phi^w_{z,R}=\Phi^w_{z,\tilde{R}}$.

Let $\omega: [0, 1]\to M$ be a path. Then we can find a sequence

$$s_0 = 0 \leqq s_1 \leqq s_2 \leqq \cdots \leqq s_n = 1$$

and a sequence $x_1, \cdots, x_n$ of points of M such that

$$\omega([s_{i-1}, s_i]) \subset R_{x_i}$$

where $R_{x_i}=\varphi_{x_i}(V_{x_i}\times D'_{x_i})$, $\varphi_{x_i}: D_{x_i}\times D'_{x_i}\to N_{x_i}$ is a canonical coordinate around x_i and V_{x_i} is a canonical neighborhood of x_i in D_{x_i}. Now we define an isomorphism

$$\omega_*: H_p(L(\omega(0)), L(\omega(0))\setminus\{\omega(0)\}) \longrightarrow H_p(L(\omega(1)), L(\omega(1))\setminus\{\omega(1)\})$$

by

$$\omega_* = \Phi^{\omega(1)}_{\omega(s_{n-1}),R_{x_n}}\circ\cdots\circ\Phi^{\omega(s_1)}_{\omega(0),R_{x_1}} .$$

It is not difficult to see that $\omega\mapsto\omega_*$ is well-defined and satisfies all our requirements.

From the definition of ω_* we see that the functor $\omega \mapsto \omega_*$ satisfies the following

LEMMA 4.3. *Let M_i be a connected topological manifold without boundary and $\mathcal{F}_i$ be a generalized foliation on M_i to which there exists a transverse generalized foliation $\mathcal{F}'_i$, where $i=1, 2$. If $f: M_1 \to M_2$ is a covering map and $\mathcal{F}_1$ (resp. $\mathcal{F}'_1$) is the lift of $\mathcal{F}_2$ (resp. $\mathcal{F}'_2$) by f, then for any path ω*

$$f_* \circ \omega_* = (f \circ \omega)_* \circ f_*$$

where f_* denotes the induced isomorphism $H_p(L, L \setminus \{x\}) \to H_p(f(L), f(L) \setminus \{f(x)\})$, $x \in L \in \mathcal{F}_1$ and $p = \dim(\mathcal{F}_1)$.

A generalized foliation $\mathcal{F}$ on M, to which there exists a transverse generalized foliation $\mathcal{F}'$, is called *orientable* if for $[\omega] \in \pi_1(M)$, ω_* is the identity. Let $\dim(\mathcal{F}) = p$. We call an *orientation* of $\mathcal{F}$ a family $\{O_x \in H_p(L, L \setminus \{x\}) : O_x$ is a generator and $x \in L \in \mathcal{F}\}$ if $\omega_*(O_{\omega(0)}) = O_{\omega(1)}$ for any path ω. If $\mathcal{F}$ is orientable, then there are exactly two orientations for $\mathcal{F}$.

LEMMA 4.4. *If $\mathcal{F}$ is non-orientable, then there exists a double covering map $\pi: \overline{M} \to M$ such that*

(1) *the lift $\overline{\mathcal{F}}$ is orientable,*

(2) *if $f: M \to M$ is a homeomorphism and if $f(\mathcal{F}) = \mathcal{F}$ and $f(\mathcal{F}') = \mathcal{F}'$, then there is a lift $\bar{f}: \overline{M} \to \overline{M}$ of f by π.*

PROOF. Let $\Lambda = \{[\omega] \in \pi_1(M) : \omega_* = \mathrm{id}\}$. Obviously Λ is a subgroup of $\pi_1(M)$ with index 2, and then there exist a topological manifold $\overline{M}$ and a double covering map $\pi: \overline{M} \to M$ such that $\pi_*(\pi_1(\overline{M})) = \Lambda$. Let $\overline{\mathcal{F}}$ be the lift of $\mathcal{F}$ by π. Since $\pi_* \circ \omega_* = (\pi \circ \omega)_* \circ \pi_*$ for $[\omega] \in \pi_1(\overline{M})$ (Lemma 4.3), we have $\pi_* \circ \omega_* = \pi_*$ because $[\pi \circ \omega] \in \Lambda$, and so $\omega_* = \mathrm{id}$. Thus $\overline{\mathcal{F}}$ is orientable. For $[\omega] \in \Lambda$ we have

$$f_* = f_* \circ \omega_* = (f \circ \omega)_* \circ f_*$$

and so $(f \circ \omega)_* = \mathrm{id}$, from which $[f \circ \omega] \in \Lambda$. This implies the existence of a lift $\bar{f}$ of f by π.

§ 5. Proof of Proposition B.

A topological space X is called an *Euclidean neighborhood retract* (abbrev. ENR) if there exist an open subset O of some Euclidean space $\boldsymbol{R}^n$ and continuous maps $i: X \to O$, $r: O \to X$ such that $r \circ i = \mathrm{id}$. Let X be an ENR and V be an open subset of X. Let $g: V \to X$ be a continuous map such that the fixed point set $\mathrm{Fix}(g)$ is compact. Then we define the fixed point index of g by

$$I(g) = \text{the fixed point index of } i\circ g\circ r$$

where $i: X\to O$ and $r: O\to X$ are as in the definition of ENR. For the details of fixed point indices the reader may refer to R. Brown [6] and A. Dold [7].

Let $f: M\to M$ be as in Proposition B. By (I) of §2 there is $l>0$ such that (2.7) holds. Take and fix $m\geqq l$. Let $x\in\mathrm{Fix}(f^m)$. As in §2 we define D_x^σ and K_x^σ $(\sigma=s, u)$, and put

$$F_{x,s} = f^m{}_{|D_x^s}: D_x^s \to K_x^s \quad\text{and}\quad F_{x,u} = f^m{}_{|K_x^u}: K_x^u \to D_x^u .$$

Since x is in $D_x^s\cap K_x^u$, the point x is a fixed point of $F_{x,\sigma}$ $(\sigma=s, u)$. Since $D_x^s\subset W_{\varepsilon_0}^s(x)$ and $K_x^u\subset D_x^u\subset W_{\varepsilon_0}^u(x)$, by (I) of §2 we see that x is the only fixed point of $F_{x,\sigma}$. From Proposition 2.1(a) and (b) it follows that D_x^σ $(\sigma=s, u)$ are ENRs. By Proposition 2.4(b) K_x^σ is an open subset of D_x^σ. Hence, as above we can define the fixed point indices of the following maps;

$$F_{x,\sigma},\quad F_{x,\sigma}^{-1},\quad F_{x,u}\times F_{x,s},\quad F_{x,u}\times F_{x,s}^{-1},\quad F_{x,u}^{-1}\times F_{x,s}$$

where $\sigma=s, u$. We denote by $I_{f^m}(x)$ the fixed point index at x of f^m.

LEMMA 5.1. *For every* $x\in\mathrm{Fix}(f^m)$

$$I_{f^m}(x) = I(F_{x,u})\cdot I(F_{x,s}) .$$

PROOF. Combining Proposition 2.1 (a) and (b) with Proposition 2.4(b), we see that $\alpha_x(K_x^u\times D_x^s)$ is an open neighborhood of x in M. Hence $I_{f^m}(x)= I(F_{x,u}\times F_{x,s})$ by Proposition 2.4(c) and the property of fixed point index. Since $I(F_{x,u}\times F_{x,s})=I(F_{x,u})\cdot I(F_{x,s})$ by the property of fixed point index, the conclusion is obtained.

Let $n=\dim(M)$. Hereafter we fix an orientation of $\boldsymbol{R}^n$; $\{O_z\in H_n(\boldsymbol{R}^n, \boldsymbol{R}^n\setminus\{z\}): O_z$ is a generator and $z\in\boldsymbol{R}^n\}$. When V is an open subset of $\boldsymbol{R}^n$ and $z\in V$, we denote by i_V^z the inclusion $(V, V\setminus\{z\})\hookrightarrow(\boldsymbol{R}^n, \boldsymbol{R}^n\setminus\{z\})$. Obviously i_V^z induces an isomorphism

$$(i_V^z)_*:\ H_n(V, V\setminus\{z\}) \xrightarrow{\cong} H_n(\boldsymbol{R}^n, \boldsymbol{R}^n\setminus\{z\}) .$$

LEMMA 5.2. *For every* $x\in\mathrm{Fix}(f^m)$

(a) $I(F_{x,s}) = I(F_{x,u}^{-1}) = 1$,

(b) $I(F_{x,u})$ *is either* 1 *or* -1.

PROOF. (a): Let ρ and r_4 be as in §2 and $\{(W_i, h_i)\}$ be the atlas of (2.6). Since r_4 is a Lebesgue number of $\{W_i\}$, there is $(W, h)\in\{(W_i, h_i)\}$ such that $B_{r_4}(x)\subset W\subset B_\rho(x)\subset N_x$. By Proposition 2.4(a) we have $K_x^s\subset B_{r_4}(x)$. Define a homotopy $H: D_x^s\times[0, 1]\to D_x^s$ by

$$H(y, t) = P^s\circ\alpha_x^{-1}\circ h^{-1}(th\circ F_{x,s}(x))$$

where $P^s: D_x^u \times D_x^s \to D_x^s$ is the natural projection. Then it is easily checked that $H(y, 0)$ is a constant and $H(y, 1)=F_{x,s}(y)$. Since the image of H is contained in a compact set $P^s \circ \alpha_x^{-1}(B_\rho(x))$, obviously $\{y \in D_x^s : H(y, t)=y$ for some $t \in [0, 1]\}$ is compact, and hence $I(F_{x,s})=1$ by the property of fixed point index. In the same way, we have $I(F_{x,u}^{-1})=1$.

(b): Let $r_1>0$ and $\{(U_i, g_i)\}$ be as in (2.4). Since r_1 is a Lebesgue number and $\operatorname{diam}(N_x)<r_1$, there is $(U, g) \in \{(U_i, g_i)\}$ such that $N_x \subset U$. We define a C^0 embedding $\Psi: D^u \times D^s \to \boldsymbol{R}^n$ by $\Psi = g \circ \alpha_x$, and write

$$V = \Psi(K_x^u \times K_x^s) \qquad \text{and} \qquad a = \Psi(x, x).$$

Since V is open in $\boldsymbol{R}^n$, by the definition of fixed point index we have

$$G_* \circ (i_V^a)_*^{-1}(O_a) = I(F_{x,u} \times F_{x,s}^{-1}) O_0$$

where

$$G = (\mathrm{id} - \Psi \circ (F_{x,u} \times F_{x,s}^{-1}) \circ \Psi^{-1}) : (V, V \setminus \{a\}) \to (\boldsymbol{R}^n, \boldsymbol{R}^n \setminus \{0\}).$$

Let $\xi: \boldsymbol{R}^n \to \boldsymbol{R}^n$ be the homeomorphism defined by $z \mapsto -z$. Then it follows that

$$G = \xi \circ G_1 \circ [\Psi \circ (F_{x,u} \times F_{x,s}^{-1}) \circ \Psi^{-1}]$$

where

$$G_1 = (\mathrm{id} - \Psi \circ (F_{x,u}^{-1} \times F_{x,s}) \circ \Psi^{-1}) : (g(N_x), g(N_x) \setminus \{a\}) \to (\boldsymbol{R}^n, \boldsymbol{R}^n \setminus \{0\}).$$

Note that $\Psi \circ (F_{x,u} \times F_{x,s}^{-1}) \circ \Psi^{-1}$ is a homeomorphism from V onto $g(N_x)$. Then we have

$$I(F_{x,u} \times F_{x,s}^{-1}) = \pm I(F_{x,u}^{-1} \times F_{x,s})$$

since $G_{1*} \circ (i_{g(N_x)}^a)_*^{-1}(O_a) = I(F_{x,u}^{-1} \times F_{x,s}) O_0$ by the definition of fixed point index. By (a) $I(F_{x,u}^{-1} \times F_{x,s}) = I(F_{x,u}^{-1}) \cdot I(F_{x,s}) = 1$ and hence $\pm 1 = I(F_{x,u} \times F_{x,s}^{-1}) = I(F_{x,u}) \cdot I(F_{x,s}^{-1})$. Therefore $I(F_{x,u})$ is either 1 or -1.

We recall that $\mathcal{F}_f^\sigma$ $(\sigma = s, u)$ are generalized foliations on M and $\mathcal{F}_f^s$ is transverse to $\mathcal{F}_f^u$ (see Proposition A). Let T be a coordinate domain. As we saw in §1, a continuous map $\gamma_T: T \times T \to T$ is defined. For a C^0 embedding $\phi: T \to \boldsymbol{R}^n$ and fixed $x \in T$, we define a continuous map $R_{T,\phi}^x(\cdot): \phi(T) \to \boldsymbol{R}^n$ by

$$R_{T,\phi}^x(z) = \phi \circ \gamma_T(x, \phi^{-1}(z)) - \phi \circ \gamma_T(\phi^{-1}(z), x). \tag{5.1}$$

It is easily checked that $R_{T,\phi}^x(z)=0$ if and only if $z=\phi(x)$, and hence we have

$$R_{T,\phi}^x : (\phi(T), \phi(T) \setminus \{\phi(x)\}) \longrightarrow (\boldsymbol{R}^n, \boldsymbol{R}^n \setminus \{0\}).$$

LEMMA 5.3. *Let D be a connected open subset of M and $\phi: D \to \boldsymbol{R}^n$ be a C^0 embedding. Then there exists $k \in \boldsymbol{Z}$ such that for any coordinate domain T in D and for any $x \in T$*

$$R_{T,\phi *}^x \circ (i_{\phi(T)}^{\phi(x)})_*^{-1}(O_{\phi(x)}) = k O_0.$$

PROOF. Since T is open, for $x \in T$ there is $\delta > 0$ such that $\phi(T) \supset B_\delta$ where $B_\delta = \{z \in \boldsymbol{R}^n : \|\phi(x) - z\| \leqq \delta\}$. Let $V_{\delta/2} = \{z \in \boldsymbol{R}^n : \|\phi(x) - z\| < \delta/2\}$. Then there is $\varepsilon > 0$ such that

$$\|R^x_{T,\phi}(z)\| \geqq \varepsilon \qquad (z \in B_\delta \setminus V_{\delta/2}).$$

Since γ_T, ϕ and ϕ^{-1} are continuous, we can find $0 < \beta < \delta/2$ such that for $y \in T$ with $\phi(y) \in B_\beta$

$$\|R^x_{T,\phi}(z) - R^y_{T,\phi}(z)\| \leqq \varepsilon/2 \qquad (z \in B_\delta).$$

Define $H: B_\delta \times [0, 1] \to \boldsymbol{R}^n$ by

$$H(z, t) = (1-t) R^x_{T,\phi}(z) + t R^y_{T,\phi}(z).$$

Then we have

$$\begin{aligned} \|H(z, t)\| &= \|R^x_{T,\phi}(z) + t(R^y_{T,\phi}(z) - R^x_{T,\phi}(z))\| \\ &\geqq \|R^x_{T,\phi}(z)\| - t\|R^y_{T,\phi}(z) - R^x_{T,\phi}(z)\| \\ &\geqq \varepsilon - t\varepsilon/2 > 0 \qquad (z \in B_\delta \setminus V_{\delta/2}), \end{aligned}$$

and hence $R^x_{T,\phi}$ and $R^y_{T,\phi}$ are homotopic continuous maps from $(B_\delta, B_\delta \setminus V_{\delta/2})$ to $(\boldsymbol{R}^n, \boldsymbol{R}^n \setminus \{0\})$. Hence

$$(R^x_{T,\phi|B_\delta})_* = (R^y_{T,\phi|B_\delta})_* : H_n(B_\delta, B_\delta \setminus V_{\delta/2}) \longrightarrow H_n(\boldsymbol{R}^n, \boldsymbol{R}^n \setminus \{0\}),$$

and so

$$R^x_{T,\phi*} = R^y_{T,\phi*} : H_n(\phi(T), \phi(T) \setminus V_{\delta/2}) \longrightarrow H_n(\boldsymbol{R}^n, \boldsymbol{R}^n \setminus \{0\}).$$

Since $\phi(x), \phi(y) \in V_{\delta/2}$, obviously

$$\begin{array}{ccl} & \overset{\cong}{\nearrow} & H_n(\phi(T), \phi(T) \setminus \{\phi(x)\}) \\ H_n(\phi(T), \phi(T) \setminus V_{\delta/2}) & & \\ & \underset{\cong}{\searrow} & H_n(\phi(T), \phi(T) \setminus \{\phi(y)\}), \end{array}$$

and therefore

$$R^x_{T,\phi*} \circ (i^{\phi(x)}_{\phi(T)})^{-1}_* (O_{\phi(x)}) = R^y_{T,\phi*} \circ (i^{\phi(y)}_{\phi(T)})^{-1}_* (O_{\phi(y)}).$$

Since T is connected, there is $k \in \boldsymbol{Z}$ such that for every $x \in T$

$$R^x_{T,\phi*} \circ (i^{\phi(x)}_{\phi(T)})^{-1}_* (O_{\phi(x)}) = k O_0.$$

Let T, T' be coordinate domains in D. If $T \cap T' \neq \emptyset$, then there exists a coordinate domain T'' in $T \cap T'$. In this case we have $\gamma_{T''} = \gamma_{T | T'' \times T''} = \gamma_{T' | T'' \times T''}$, which means that for $x \in T''$

$$R^x_{T,\phi*} \circ (i^{\phi(x)}_{\phi(T)})^{-1}_* (O_{\phi(x)}) = R^x_{T',\phi*} \circ (i^{\phi(x)}_{\phi(T')})^{-1}_* (O_{\phi(x)}).$$

Since D is connected, the conclusion of the lemma is obtained.

PROOF OF PROPOSITION B. By Lemmas 5.1 and 5.2 it is enough to show that $I(F_{u,x})$ is a constant for $x\in\mathrm{Fix}(f^m)$. Let $p=\dim(\mathcal{F}_f^u)$. Since $\mathcal{F}_f^u$ is orientable, there exists an orientation of $\mathcal{F}_f^u$;

$$\{O_z^u\in H_p(W^u(z), W^u(z)\setminus\{z\}):\ O_z^u \text{ is a generator and } z\in M\}.$$

Since $f^m(\mathcal{F}_f^\sigma)=\mathcal{F}_f^\sigma$ $(\sigma=s, u)$, by Lemma 4.3 there is a constant $\varepsilon=\pm1$ such that

$$f_*^m O_z^u=\varepsilon O_{f^m(x)}^u \qquad ({}^\forall z\in M). \tag{5.2}$$

For $x\in\mathrm{Fix}(f^m)$, define the inclusion maps

$$i_x:\ (K_x^u, K_x^u\setminus\{x\})\hookrightarrow(W^u(x), W^u(x)\setminus\{x\}),$$
$$j_x:\ (D_x^u, D_x^u\setminus\{x\})\hookrightarrow(W^u(x), W^u(x)\setminus\{x\}).$$

Then i_x and j_x are C^0 embeddings (Propositions 2.3 and 2.4(b)), and so the induced isomorphisms are obtained;

$$i_{x*}:\ H_p(K_x^u, K_x^u\setminus\{x\})\xrightarrow{\cong}H_p(W^u(x), W^u(x)\setminus\{x\}),$$
$$j_{x*}:\ H_p(D_x^u, D_x^u\setminus\{x\})\xrightarrow{\cong}H_p(W^u(x), W^u(x)\setminus\{x\}).$$

By (5.2) we have

$$F_{x,u*}^{-1}\circ j_{x*}^{-1}(O_x^u)=\varepsilon i_{x*}^{-1}(O_x^u) \qquad ({}^\forall x\in\mathrm{Fix}(f^m)). \tag{5.3}$$

For fixed $x\in\mathrm{Fix}(f^m)$, let $\phi: N_x\to\boldsymbol{R}^n$ be a C^0 embedding. We consider continuous maps $i=\phi_{|D_x^u}$ and

$$r=P^u\circ\alpha_x^{-1}\circ\phi^{-1}:\ \phi(N_1)\longrightarrow K_x^u$$

where $P^u: K_x^u\times D_x^s\to K_x^u$ denotes the natural projection and $N_1=\alpha_x(K_x^u\times D_x^s)\subset N_x$. Then we have

$$G_1=(\mathrm{id}-i\circ F_{x,u}\circ r):\ (\phi(N_1), \phi(N_1)\setminus\{\phi(x)\})\longrightarrow(\boldsymbol{R}^n, \boldsymbol{R}^n\setminus\{0\})$$

and by the definition

$$G_{1*}\circ(i_{\phi(N_1)}^{\phi(x)})_*^{-1}(O_{\phi(x)})=I(F_{x,u})O_0. \tag{5.4}$$

Let $\Psi=\phi\circ\alpha_x: D_x^u\times D_x^s\to\phi(N_x)$ and define

$$K_\phi=G_1\circ\Psi\circ(F_{x,u}^{-1}\times\mathrm{id})\circ\Psi^{-1}.$$

Then we have

$$K_\phi:\ (\phi(N_x), \phi(N_x)\setminus\{\phi(x)\})\longrightarrow(\boldsymbol{R}^n, \boldsymbol{R}^n\setminus\{0\}).$$

By Künneth formula there is a generator $O_x^s\in H_{n-p}(D_x^s, D_x^s\setminus\{x\})$ such that $\Psi_*^{-1}\circ(i_{\phi(N_x)}^{\phi(x)})_*^{-1}(O_{\phi(x)})=j_{x*}^{-1}O_x^u\times O_x^s$, and hence

$$(5.5)\qquad \begin{aligned} K_{\phi *}\circ(i^{\phi(x)}_{\phi(N_x)})^{-1}_*(O_{\phi(x)}) &= G_{1*}\circ\Psi_*\circ(F^{-1}_{x,u*}\times\mathrm{id})\circ\Psi^{-1}_*\circ(i^{\phi(x)}_{\phi(N_x)})^{-1}_*(O_{\phi(x)}) \\ &= G_{1*}\circ\Psi_*\circ(F^{-1}_{x,u*}\times\mathrm{id})(j^{-1}_{x*}(O^u_x)\times O^s_x) \\ &= G_{1*}\circ\Psi_*(\varepsilon i^{-1}_{x*}(O^u_x)\times O^s_x) \qquad \text{(by (5.3))} \\ &= \varepsilon G_{1*}\circ(i^{\phi(x)}_{\phi(N_1)})^{-1}_*(O_{\phi(x)}) \\ &= \varepsilon I(F_{x,u})O_0 \qquad \text{(by (5.4))}. \end{aligned}$$

By the definition of G_1 it follows that K_ϕ is of the form

$$(5.6)\qquad K_\phi(z)=\phi\circ\alpha_x\circ(F^{-1}_{x,u}\times\mathrm{id})\circ\alpha_x^{-1}\circ\phi^{-1}(z)-\gamma(z)$$

where $\gamma(z)=\phi\circ P^u\circ\alpha_x^{-1}\circ\phi^{-1}(z)$.

Let $\{(W_i, h_i)\}$ be the atlas of M such that (2.6) holds. Since r_4 is a Lebesgue number of $\{W_i\}$, there is $(W, h)\in\{(W_i, h_i)\}$ such that

$$F^{-1}_{x,u}(D^u_x)=K^u_x\subset B_{r_4}(x)\subset W\subset B_{r_3}(x).$$

Hence we define $\beta:\phi(N_x)\times[0,1]\to\boldsymbol{R}^n$ by

$$\beta(z,t)=th\circ F^{-1}_{x,u}\circ P^u\circ\alpha_x^{-1}\circ\phi^{-1}(z)+(1-t)h(x)$$

and next $H:\phi(N_x)\times[0,1]\to\boldsymbol{R}^n$ by

$$H(z,t)=\phi\circ\alpha_x(P^u\circ\alpha_x^{-1}\circ h^{-1}\circ\beta(z,t),\ P^s\circ\alpha_x^{-1}\circ\phi^{-1}(z))-\gamma(z).$$

Let $\{V_j\}$ be the finite cover of M as in (2.5). Since r_2 is a Lebesgue number of $\{V_j\}$, we can take $V\in\{V_j\}$ such that $B_{r_2}(x)\subset V$.

CLAIM. (1) *$H:(\phi(N_x),\phi(N_x)\setminus\phi(V))\times[0,1]\to(\boldsymbol{R}^n,\boldsymbol{R}^n\setminus\{0\})$,*
(2) *H is a homotopy between $R^x_{N_x,\phi}$ and K_ϕ.*

PROOF. (1): Let $t\in[0,1]$. For $z\in\phi(N_x)\setminus\phi(V)$, put $z_1=h^{-1}\circ\beta(z,t)$. Since $W\subset B_{r_3}(x)$, it follows that $z_1\in B_{r_3}(x)$, and $H(z,t)$ is expressed as

$$H(z,t)=\phi\circ\alpha_x(P^u\circ\alpha_x^{-1}(z_1),\ P^s\circ\alpha_x^{-1}\circ\phi^{-1}(z))-\gamma(z).$$

Since $P^u\circ\alpha_x^{-1}(z_1)\in B_{r_2}(x)$ by the choice of r_3 and $\phi^{-1}(z)\notin V\supset B_{r_2}(x)$, it is checked that

$$P^s\circ\alpha_x^{-1}\circ\phi^{-1}(z)\neq x\Longrightarrow H(z,t)\neq 0,$$
$$P^s\circ\alpha_x^{-1}\circ\phi^{-1}(z)=x\Longrightarrow z\in\phi(D^u)\ \text{ and so }\ H(z,t)=\phi\circ P^u\circ\alpha_x^{-1}(z_1)-z\neq 0,$$

from which (1) holds.

(2): Let $z\in\phi(N_x)$. Then we can calculate that

$$\begin{aligned} H(z,0)&=\phi\circ\alpha_x(x,\ P^s\circ\alpha_x^{-1}\circ\phi^{-1}(z))-\gamma(z) \\ &=\phi\circ P^s\circ\alpha_x^{-1}\circ\phi^{-1}(z)-\phi\circ P^u\circ\alpha_x^{-1}\circ\phi^{-1}(z) \\ &=\phi\circ\gamma_{N_x}(x,\phi^{-1}(z))-\phi\circ\gamma_{N_x}(\phi^{-1}(z),x)=R^x_{N_x,\phi} \end{aligned}$$

and next that

$$\begin{aligned}H(z, 1) &= \phi\circ\alpha_x(P^u\circ\alpha_x^{-1}\circ F_{x,u}^{-1}\circ P^u\circ\alpha_x^{-1}\circ\phi^{-1}(z),\ P^s\circ\alpha_x^{-1}\circ\phi^{-1}(z))-\gamma(z)\\ &= \phi\circ\alpha_x(F_{x,u}^{-1}\circ P^u\circ\alpha_x^{-1}\circ\phi^{-1}(z),\ P^s\circ\alpha_x^{-1}\circ\phi^{-1}(z))-\gamma(z)\\ &\qquad\qquad\qquad (\text{since } P^u\circ\alpha_x^{-1}(y)=y \quad \text{for } y\in D_x^u)\\ &= \phi\circ\alpha_x\circ(F_{x,u}^{-1}\times\mathrm{id})\circ\alpha_x^{-1}\circ\phi^{-1}(z)-\gamma(z)\\ &= K_\phi(z) \qquad (\text{by } (5.6)).\end{aligned}$$

By Claim we have

$$K_{\phi*}=R_{N_x,\phi*}^x:\quad H_n(\phi(N_x),\ \phi(N_x)\setminus\phi(V))\longrightarrow H_n(\boldsymbol{R}^n,\ \boldsymbol{R}^n\setminus\{0\}).$$

Since $V\subset B_\rho(x)\subset N_x$, by the choice of V it follows that V is a canonical neighborhood of $\phi(x)$ in $\phi(N_x)$, and hence

$$K_{\phi*}=R_{N_x,\phi*}^x:\quad H_n(\phi(N_x),\ \phi(N_x)\setminus\{\phi(x)\})\longrightarrow H_n(\boldsymbol{R}^n,\ \boldsymbol{R}^n\setminus\{0\}).$$

Therefore, by (5.5) we can conclude that

$$(5.7)\qquad R_{N_x,\phi*}^x\circ(i_{\phi(N_x)}^{\phi(x)})_*^{-1}(O_{\phi(x)})=\varepsilon I(F_{x,u})O_0 \qquad ({}^\forall x\in\mathrm{Fix}(f^m)).$$

Let $x,\ y\in\mathrm{Fix}(f^m)$. Then we can find an open subset D of M with $x,\ y\in D$ such that D is homeomorphic to $\boldsymbol{R}^n$. Obviously for $z\in D$ there exists a coordinate domain T_z around z such that $T_z\subset D\cap N_z$. Let $\{(U_i,\ g_i)\}$ be the atlas of M such that (2.4) holds. Since r_1 is a Lebesgue number of $\{U_i\}$ and $\mathrm{diam}(N_z)\leqq r_1$, there is $(U,\ g)\in\{(U_i,\ g_i)\}$ such that $N_z\subset U$. Take $\delta>0$ such that $g^{-1}(V_\delta(g(z)))\subset T_z$ where $V_\delta(g(z))=\{y\in\boldsymbol{R}^n:\ \|y-g(z)\|<\delta\}$. Since $T_z\subset N_z\subset U$, we can construct a C^0 embedding $\tau_z: N_z\to g^{-1}(V_\delta(g(z)))$ such that $\tau_z=\mathrm{id}$ on $g^{-1}(V_{\delta/2}(g(z)))$. Let $\phi: D\to\boldsymbol{R}^n$ be a homeomorphism and put $\phi_z=\phi\circ\tau_z$. Then $\phi_z: N_z\to\boldsymbol{R}^n$ is a C^0 embedding and $\phi_z=\phi$ on $g^{-1}(V_{\delta/2}(g(z)))$, which implies that

$$R_{N_z,\phi_z*}^z\circ(i_{\phi_z(N_z)}^{\phi(z)})_*^{-1}(O_{\phi(z)})=R_{T_z,\phi*}^z\circ(i_{\phi(T_z)}^{\phi(z)})_*^{-1}(O_{\phi(z)}).$$

Combining this fact with (5.7), we have

$$R_{T_x,\phi*}^x\circ(i_{\phi(T_x)}^{\phi(x)})_*^{-1}(O_{\phi(x)})=\varepsilon I(F_{x,u})O_0$$

and

$$R_{T_y,\phi*}^y\circ(i_{\phi(T_y)}^{\phi(y)})_*^{-1}(O_{\phi(y)})=\varepsilon I(F_{y,u})O_0.$$

By Lemma 5.3 we have $\varepsilon I(F_{x,u})=\varepsilon I(F_{y,u})$, and therefore $I(F_{x,u})=I(F_{y,u})$.

§6. Proof of Theorem.

Let $(X,\ d)$ be a compact metric space and $f: X\to X$ be a homeomorphism. We denote by $\mathrm{Per}(f)$ the set of all periodic points of f and by $\Omega(f)$ the nonwandering set of f. The following is checked in N. Aoki [1].

PROPOSITION 6.1. *If f is expansive and has* POTP, *then the following hold;*

(1) $\mathrm{Per}(f)$ *is dense in* $\Omega(f)$,

(2) $\Omega(f)=\Omega_1\cup\cdots\cup\Omega_s$ *where each* Ω_i *is a pairwise disjoint closed subset such that* $f(\Omega_i)=\Omega_i$ *and* $f_{|\Omega_i}$ *is topologically transitive, and*

$$\Omega_i = X_{i,1}\cup\cdots\cup X_{i,n_i},$$

where each $X_{i,j}$ *is a pairwise disjoint closed subset such that* $f(X_{i,j})=X_{i,j+1}$ $(X_{i,n_i+1}=X_{i,1})$ *and* $f^{n_i}{}_{|X_{i,j}}: X_{i,j}\to X_{i,j}$ *is topologically mixing,*

(3) $f_{|\Omega(f)}$ *is not topologically transitive if* $\Omega(f)\subsetneqq X$.

Let $\boldsymbol{T}^n$ be the n-torus $\boldsymbol{R}^n/\boldsymbol{Z}^n$ and $\pi:\boldsymbol{R}^n\to\boldsymbol{T}^n$ be the covering projection. We note that if $f:\boldsymbol{T}^n\to\boldsymbol{T}^n$ is a continuous map, then there is a unique group endomorphism $A:\boldsymbol{T}^n\to\boldsymbol{T}^n$ homotopic to f (cf. [**17**]). Define a metric d for $\boldsymbol{T}^n$ by

$$d(x, y)=\min\{\|x'-y'\|:\ x'\in\pi^{-1}(x),\ y'\in\pi^{-1}(y)\}.$$

Then it is easily checked that the Euclidean metric satisfies (i)~(iv) of §3 for d.

PROPOSITION 6.2. *Let* $f:\boldsymbol{T}^n\to\boldsymbol{T}^n$ *be a homeomorphism and* $A:\boldsymbol{T}^n\to\boldsymbol{T}^n$ *be the group automorphism homotopic to* f. *If* f *is expansive and has* POTP, *then* A *is hyperbolic.*

PROOF. By taking the double cover of $\boldsymbol{T}^n$ if necessary, we may assume that the generalized foliation $\mathcal{F}_f^u$ is orientable (see Proposition 3.9 and Lemmas 3.1, 3.3 and 4.4). Then by Proposition B there is $l\in\boldsymbol{N}$ such that for $m\geqq l$ all fixed points of f^m have the same fixed point index 1 or -1.

Since f is expansive and has POTP, by Lemma 6.1(2) there is a decomposition

$$\Omega(f)=\bigcup_{i=1}^{s}\bigcup_{j=1}^{n_i}X_{i,j}$$

such that $f^{n_i}(X_{i,j})=X_{i,j}$ and $f^{n_i}{}_{|X_{i,j}}$ is topologically mixing. Let $g=f^{ln_1\cdots n_s}$. Then $g:\boldsymbol{T}^n\to\boldsymbol{T}^n$ is expansive and has POTP, and moreover it is homotopic to $B=A^{ln_1\cdots n_s}$. It is enough to show that B is hyperbolic. To do this, let $\lambda_1,\cdots,\lambda_n$ be all eigenvalues of $g_*=B_*: H_1(\boldsymbol{T}^n;\boldsymbol{R})\to H_1(\boldsymbol{T}^n;\boldsymbol{R})$. Then the Lefschetz number of g^m $(m\in\boldsymbol{N})$ is given by

$$\lambda(g^m)=\lambda(B^m)=\prod_{i=1}^{n}(1-\lambda_i^m).$$

Since all fixed points of g^m have the same index 1 or -1, we have

$$N(g^m)=\Big|\sum_{x\in\mathrm{Fix}(g^m)}I_{g^m}(x)\Big|=|I(g^m)|$$

where $N(g^m)$ and $I(g^m)$ denote the number of fixed points of g^m and the fixed point index of g^m respectively. Hence by the Lefschetz formula

$$N(g^m)=\prod_{i=1}^{n}|1-\lambda_i^m| \qquad (^\forall m\in \boldsymbol{N}). \tag{6.1}$$

Let $c>0$ be an expansive constant for g. Then there is $\varepsilon>0$ such that any ε-pseudo-orbit of g is $c/3$-traced by some point of $\boldsymbol{T}^n$. Since g is topologically mixing on $X_{i,j}$, there is $k>0$ such that $g^k(K)\cap K'\neq\emptyset$ for any ε-balls K and K' in $X_{i,j}$. Let $x\in X_{i,j}$ be a fixed point of g^m and choose $y\in X_{i,j}$ such that $d(x, y)<\varepsilon$ and $d(x, g^k(y))<\varepsilon$. Then it is clear that

$$x,\ g(x),\ \cdots,\ g^{m-1}(x),\ y,\ g(y),\ \cdots,\ g^{k-1}(y),\ x$$

is a $(m+k)$-periodic ε-pseudo-orbit. Let $z_x\in \boldsymbol{T}^n$ be its $c/3$-tracing point. By expansiveness we see that $g^{m+k}(z_x)=z_x$ and $z_x\neq z_y$ for $x, y\in \mathrm{Fix}(g^m)$ with $x\neq y$. Therefore $N(g^m)\leqq N(g^{m+k})$ and so by (6.1)

$$\prod_{i=1}^{n}|1-\lambda_i^m| \leqq \prod_{i=1}^{n}|1-\lambda_i^{m+k}| \qquad (^\forall m\in \boldsymbol{N}). \tag{6.2}$$

Combining (6.1) with the fact that $\mathrm{Per}(g)\neq\emptyset$ (Lemma 6.1(1)), it follows that λ_i is not a root of unity. To obtain that $|\lambda_i|\neq 1$ for $1\leqq i\leqq n$, assume that $|\lambda_i|=1$ $(1\leqq i\leqq s)$, $|\lambda_i|<1$ $(s+1\leqq i\leqq t)$ and $|\lambda_i|>1$ $(t+1\leqq i\leqq n)$. Then by (6.2) we have

$$\frac{\prod_{i=s+1}^{t}|1-\lambda_i^m|\prod_{i=t+1}^{n}|\lambda_i^{-m-k}-\lambda_i^{-k}|}{\prod_{i=s+1}^{t}|1-\lambda_i^{m+k}|\prod_{i=t+1}^{n}|\lambda_i^{-m-k}-1|} \leqq \frac{\prod_{i=1}^{s}|1-\lambda_i^{m+k}|}{\prod_{i=1}^{s}|1-\lambda_i^m|}. \tag{6.3}$$

Obviously the left hand side of (6.3) tends to $\prod_{i=t+1}^{n}|\lambda_i^{-k}|$ as $m\to\infty$. Since $|\lambda_i|=1$ and λ_i is not a root of unity $(1\leqq i\leqq s)$, we can find a subsequence $\{m_j\}$ such that $\lambda_i^{m_j}\to\lambda_i^{-k}$ as $j\to\infty$. Therefore the right hand side of (6.3) tends to 0, thus contradicting.

COROLLARY 6.3. *If a homeomorphism* $f: \boldsymbol{T}^n\to \boldsymbol{T}^n$ *is expansive and has* POTP, *then* f *has fixed points.*

PROOF. By Proposition 6.2 we see that $f_*: H_1(\boldsymbol{T}^n; \boldsymbol{R})\to H_1(\boldsymbol{T}^n; \boldsymbol{R})$ is hyperbolic, and so $\lambda(f)\neq 0$. Hence $\mathrm{Fix}(f)\neq\emptyset$ by the fixed point theorem.

COROLLARY 6.4. *Let* $f: \boldsymbol{T}^n\to \boldsymbol{T}^n$ *be a homeomorphism and* $A: \boldsymbol{T}^n\to \boldsymbol{T}^n$ *be the group automorphism homotopic to* f. *If* f *is expansive and has* POTP *and if* $x_0\in \boldsymbol{T}^n$ *is a fixed point of* f, *then there exists a continuous map* $h: \boldsymbol{T}^n\to \boldsymbol{T}^n$ *homotopic to the identity map such that* $h(x_0)=e$ *and* $A\circ h=h\circ f$ *where* e *denotes the identity of* $\boldsymbol{T}^n$.

PROOF. Since A is hyperbolic by Proposition 6.2, this follows from Proposition 2.1 of [**10**].

LEMMA 6.5. *Let* $f: \boldsymbol{T}^n \to \boldsymbol{T}^n$ *be a homeomorphism and* $\bar{f}: \boldsymbol{R}^n \to \boldsymbol{R}^n$ *be a lift of* f *by* π. *If* f *is expansive and has* POTP, *then* $\bar{f}$ *has exactly one fixed point.*

PROOF. Let $A: \boldsymbol{T}^n \to \boldsymbol{T}^n$ be the group automorphism homotopic to f and $\bar{A}: \boldsymbol{R}^n \to \boldsymbol{R}^n$ be the linear automorphism which covers A. Then by Proposition 6.2, $\bar{A}$ is hyperbolic, and hence

$$\mu = \inf_{\|x\|=1} \|\bar{A}(x) - x\| > 0 .$$

Since A is homotopic to f, it is easily checked that

$$K = \sup_{x \in \boldsymbol{R}^n} \|\bar{f}(x) - \bar{A}(x)\| < \infty .$$

Choose $r>0$ such that $\mu r > K$. Then for $x \in \boldsymbol{R}^n$ with $\|x\| \geqq r$ we have

$$\|x - \bar{f}(x)\| \geqq \|x - \bar{A}(x)\| - \|\bar{A}(x) - \bar{f}(x)\| \geqq \mu r - K > 0 , \tag{6.4}$$

and so $\mathrm{Fix}(\bar{f}) \subset B_r(0)$ where $B_r(0) = \{x \in \boldsymbol{R}^n : \|x\| \leqq r\}$. By this fact the fixed point index $I(\bar{f})$ can be defined.

Let $H: \boldsymbol{R}^n \times [0, 1] \to \boldsymbol{R}^n$ be defined by $H(x, t) = t\bar{A}(x) + (1-t)\bar{f}(x)$. As (6.4) we would be able to show that $\{x \in \boldsymbol{R}^n : H(x, t) = x \text{ for some } 0 \leqq t \leqq 1\} \subset B_r(0)$, and so $I(\bar{A}) = I(\bar{f})$. Since $\bar{A}$ is hyperbolic, clearly $I(\bar{A}) = \pm 1$ and therefore $I(\bar{f}) \neq 0$. This implies that $\mathrm{Fix}(\bar{f}) \neq \emptyset$.

Next let us show that $\mathrm{Fix}(\bar{f})$ is exactly one point. By taking the double cover of $\boldsymbol{T}^n$ if necessary, we may assume the generalized foliation $\mathcal{F}_f^u$ is orientable. Then by Proposition B there is $m \in \boldsymbol{N}$ such that all fixed points of f^m has the same index 1 or -1. Since $\pi \circ \bar{f}^m = f^m \circ \pi$, all fixed points of $\bar{f}^m$ also has the same index 1 or -1, and hence

$$N(\bar{f}^m) = |\sum_{x \in \mathrm{Fix}(\bar{f}^m)} I_{\bar{f}^m}(x)| = |I(\bar{f}^m)| = |I(\bar{A}^m)| = 1$$

which ensures that our assertion holds.

PROPOSITION 6.6. *If a homeomorphism* $f: \boldsymbol{T}^n \to \boldsymbol{T}^n$ *is expansive and has* POTP, *then* $\Omega(f) = \boldsymbol{T}^n$.

PROOF. We may assume $\mathcal{F}_f^u$ is orientable (take the double cover of $\boldsymbol{T}^n$ if necessary). Then by Proposition B there is $l \in \boldsymbol{N}$ such that for $m \geqq l$ all fixed points of f^m has the same index 1 or -1. To obtain the conclusion, assuming $\Omega(f) \subsetneqq \boldsymbol{T}^n$, put $g = f^l$. Then $\Omega(g) \subsetneqq \boldsymbol{T}^n$. Let $B: \boldsymbol{T}^n \to \boldsymbol{T}^n$ be the group automorphism homotopic to g. Since g is expansive and has POTP, by Proposition 6.2 B is hyperbolic and hence

$$N(g^m) = |I(g^m)| = |I(B^m)| = N(B^m) \qquad (^\forall m \in \boldsymbol{N}) . \tag{6.5}$$

By Corollaries 6.3, 6.4 there is a continuous map $h: \boldsymbol{T}^n \to \boldsymbol{T}^n$ homotopic to the identity map such that $B\circ h=h\circ g$. For fixed $m\in\boldsymbol{N}$ let $x\in\mathrm{Fix}(B^m)$. Then $h^{-1}(x)$ is a g^m-invariant closed set.

CLAIM 1. $g^m{}_{|h^{-1}(x)}: h^{-1}(x)\to h^{-1}(x)$ *is expansive and has* POTP.

PROOF. By the definition it is easily checked that $g^m{}_{|h^{-1}(x)}$ is expansive. To show $g^m{}_{|h^{-1}(x)}$ has POTP, letting $c>0$ be an expansive constant for B^m, we choose $\varepsilon>0$ such that $d(x, y)\leqq\varepsilon$ implies $d(h(x), h(x))\leqq c$. Then there is $\delta>0$ such that any δ-pseudo-orbit of g^m is ε-traced. Let $\{x_i\}_{i\in\boldsymbol{Z}}\subset h^{-1}(x)$ be a δ-pseudo-orbit of g^m and let $z\in\boldsymbol{T}^n$ be its ε-tracing point. Then $\{h(x_i)\}_{i\in\boldsymbol{Z}}$ is c-traced by $h(z)$. Since $h(x_i)=x$, by expansiveness we have $h(z)=x$ and so $z\in h^{-1}(x)$, which means that $g_{|h^{-1}(x)}$ has POTP.

Let $\bar{x}\in\pi^{-1}(x)$ and let $\bar{h}: \boldsymbol{R}^n\to\boldsymbol{R}^n$ be a lift of h. Then it is easily checked that $\pi(\bar{h}^{-1}(\bar{x}))=h^{-1}(x)$.

CLAIM 2. $\pi: \bar{h}^{-1}(\bar{x})\to h^{-1}(x)$ *is injective.*

PROOF. If $a, b\in\bar{h}^{-1}(\bar{x})$ and $\pi(a)=\pi(b)$, there is $\alpha\in G(\pi)$ such that $b=\alpha(a)$. Since h is homotopic to the identity map, we have $\bar{h}_{\#}(\alpha)=\alpha$, and hence

$$\bar{x}=\bar{h}(b)=\bar{h}(\alpha(a))=\bar{h}_{\#}(\alpha)\circ\bar{h}(a)=\alpha(\bar{x})$$

which means that α is the identity map. Therefore $a=b$ and so the claim holds.

By Claim 2 we can find a lift $\tilde{g}: \boldsymbol{R}^n\to\boldsymbol{R}^n$ of g^m by π such that $\tilde{g}(\bar{h}^{-1}(\bar{x}))=\bar{h}^{-1}(\bar{x})$. Then by Claim 1 it follows that $\tilde{g}_{|\bar{h}^{-1}(\bar{x})}: \bar{h}^{-1}(\bar{x})\to\bar{h}^{-1}(\bar{x})$ is expansive and has POTP. If $h^{-1}(x)$ is not one point, then so is $\bar{h}^{-1}(\bar{x})$. By Proposition 6.1(1), (3), $\tilde{g}_{|\bar{h}^{-1}(\bar{x})}$ has at least two periodic points, which contradicting Lemma 6.5. Hence $h^{-1}(x)$ consists of exactly one point. Thus we proved that $x\in\mathrm{Fix}(B^m)$ implies $h^{-1}(x)\in\mathrm{Fix}(g^m)$.

Since $\Omega(g)\subsetneqq\boldsymbol{T}^n$, according to Proposition 6.1(2) we decompose

$$\Omega(g)=\bigcup_{i=1}^{s}\Omega_i$$

where $g_{|\Omega_i}$ is topologically transitive. Then there is $\Omega_i\in\{\Omega_i\}_{i=1}^{s}$ such that $h(\Omega_i)=\boldsymbol{T}^n$, and hence by the above result $N(g^m{}_{|\Omega_i})=N(B^m)$, which means that $N(g^m)>N(B^m)$ for sufficiently large $m>0$, thus contradicting (6.5).

Hereafter let $f: \boldsymbol{T}^n\to\boldsymbol{T}^n$ be a homeomorphism with expansiveness and POTP. By Corollary 6.3 we may assume $f(e)=e$. Then there is a lift $\bar{f}: \boldsymbol{R}^n\to\boldsymbol{R}^n$ of f by π such that $\bar{f}(0)=0$. For $x\in\boldsymbol{R}^n$, let $\overline{W}^s(x)$ and $\overline{W}^u(x)$ be the stable and unstable sets of $\bar{f}$ under the Euclidean metric respectively. As before let

$$\mathcal{F}_{\bar{f}}^{\sigma}=\{\overline{W}^{\sigma}(x):\ x\in\boldsymbol{R}^n\}\qquad(\sigma=s, u).$$

Then by Proposition 3.9, $\mathcal{F}_{\bar{f}}^{\sigma}$ ($\sigma=s, u$) are generalized foliations on $\boldsymbol{R}^n$ and $\mathcal{F}_{\bar{f}}^{s}$ is transverse to $\mathcal{F}_{\bar{f}}^{u}$.

LEMMA 6.7. *For $x, y\in\boldsymbol{R}^n$, $\overline{W}^s(x)\cap\overline{W}^u(y)$ is at most one point.*

PROOF. Assume that $a, b\in\overline{W}^s(x)\cap\overline{W}^u(y)$ and $a\neq b$. Then there is a coordinate domain U around a such that $b\notin \mathrm{cl}(U)$. For $\sigma=s, u$, let $V^{\sigma}=\{z\in\boldsymbol{R}^n : \overline{W}^{\sigma}(z)\cap U\neq\emptyset\}$. Since U is open in $\boldsymbol{R}^n$ and $\mathcal{F}_{\bar{f}}^{\sigma}$ is a generalized foliation, it is easily checked that V^{σ} is open in $\boldsymbol{R}^n$. Since $b\in V^s\cap V^u$ and $b\notin\mathrm{cl}(U)$, we have $V^s\cap V^u\setminus\mathrm{cl}(U)$ is a non-empty open set. By Proposition 6.1(1) and Proposition 6.6 we can find $y'\in V^s\cap V^u\setminus\mathrm{cl}(U)$ such that $\pi(y')\in\mathrm{Per}(f)$. Since U is a coordinate domain, there is $x'\in\overline{W}^s(y')\cap\overline{W}^u(y')\cap U\neq\emptyset$. Choose $k\in\boldsymbol{N}$ such that $f^k(\pi(y'))=\pi(y')$, and put $g=f^k$. Then there is a lift $\bar{g}: \boldsymbol{R}^n\to\boldsymbol{R}^n$ of g such that $\bar{g}(y')=y'$. Since $g=f^k$, clearly $\bar{g}=\beta\circ\bar{f}^k$ for some $\beta\in G(\pi)$. Since g is expansive and has POTP, $\bar{g}$ also is expansive and has POTP (Lemmas 3.1 and 3.3). Let $c>0$ be an expansive constant for $\bar{g}$ and let $\varepsilon=\min\{c/2, \|x'-y'\|/2\}$. Then there is $\delta>0$ such that any δ-pseudo-orbit of $\bar{g}$ is ε-traced. Since $x'\in\overline{W}^s(y')\cap\overline{W}^u(y')$, we have

$$\begin{aligned}\|\bar{g}^m(x')-y'\| &= \|\bar{g}^m(x')-\bar{g}^m(y')\| = \|(\beta\circ\bar{f}^k)^m(x')-(\beta\circ\bar{f}^k)^m(y')\| \\ &= \|\beta_m\circ\bar{f}^{mk}(x')-\beta_m\circ\bar{f}^{mk}(y')\| \quad (\text{for some } \beta_m\in G(\pi)) \\ &= \|\bar{f}^{mk}(x')-\bar{f}^{mk}(y')\| \longrightarrow 0 \quad (\text{as } |m|\to\infty).\end{aligned}$$

Hence there is $m>0$ such that $\|\bar{g}^m(x')-y'\|<\delta$ and $\|\bar{g}^{-m}(x')-y'\|<\delta$, which means

$$y',\ \bar{g}^{-m}(x'),\ \cdots,\ \bar{g}^{-1}(x'),\ x',\ \bar{g}(x'),\ \cdots,\ \bar{g}^{m-1}(x'),\ y'$$

is a $(2m+1)$-periodic δ-pseudo-orbit of $\bar{g}$. Let $z\in\boldsymbol{R}^n$ be a tracing point of this pseudo-orbit. Then by the choice of ε it follows that $z\neq y'$ and $z\in\mathrm{Per}(\bar{g})$, and so $z, y'\in\mathrm{Fix}(\bar{g}^l)$ for some $l\in\boldsymbol{N}$, thus contradicting Lemma 6.5 (since $\bar{g}^l$ is a lift of g^l).

LEMMA 6.8. *For $x, y\in\boldsymbol{R}^n$, $\overline{W}^s(x)\cap\overline{W}^u(y)$ is exactly one point.*

PROOF. Combining Lemma 6.7 with Proposition 6.2 and Corollary 6.4, by the technique of J. Franks [9] the conclusion is obtained.

By Lemma 6.8 we see that $\boldsymbol{R}^n$ is a coordinate domain, and hence the continuous map

$$\bar{\gamma}:\ \boldsymbol{R}^n\times\boldsymbol{R}^n\longrightarrow\boldsymbol{R}^n,$$

which sends (x, y) to $\bar{\gamma}(x, y)\in\overline{W}^s(x)\cap\overline{W}^u(y)$, is obtained. Let us define

$$\bar{\gamma}_0:\ \overline{W}^u(0)\times\overline{W}^s(0)\longrightarrow\boldsymbol{R}^n$$

by $\bar{\gamma}_0=\bar{\gamma}_{|\overline{W}^u(0)\times\overline{W}^s(0)}$. Then $\bar{\gamma}_0$ is a homeomorphism.

Let $0<c\leqq\eta_0$ be as in §3. We recall that c is an expansive constant for $\bar{f}$. Put $\varepsilon_0=c/4$. For $x\in\boldsymbol{R}^n$, let $\overline{W}^s_{\varepsilon_0}(x)$ and $\overline{W}^u_{\varepsilon_0}(x)$ be the local stable and unstable sets of $\bar{f}$ under the Euclidean metric respectively. Then for every $x\in\boldsymbol{R}^n$ a canonical coordinate $\bar{\alpha}_x:\bar{D}^u_x\times\bar{D}^s_x\to N_x$ is constructed as in §3 (Proposition 3.9). Note that $\bar{\alpha}_x=\bar{\gamma}_{|\bar{D}^u_x\times\bar{D}^s_x}$ for all $x\in\boldsymbol{R}^n$.

For $y\in\overline{W}^s(x)$ define

$$d(x,\ y\,;\ \overline{W}^s(x))=\min\{m\geqq 0:\ \bar{f}^m(y)\in\overline{W}^s_{\varepsilon_0}(\bar{f}^m(x))\}.$$

Then by Lemma 3.5 we see easily that $d(x,\ y\,;\ \overline{W}^s(x))<\infty$. Similarly for $y\in\overline{W}^u(x)$ we define

$$d(x,\ y\,;\ \overline{W}^u(x))=\min\{m\geqq 0:\ \bar{f}^{-m}(y)\in\overline{W}^u_{\varepsilon_0}(\bar{f}^{-m}(x))\}<\infty\,.$$

Note that if $y\in\overline{W}^\sigma(x)$ ($\sigma=s,\ u$) then $\beta(y)\in\overline{W}^\sigma(\beta(x))$ for every $\beta\in G(\pi)$ (see Lemma 3.6).

LEMMA 6.9. *For $\beta\in G(\pi)$ and $y\in\overline{W}^\sigma(x)$ ($\sigma=s,\ u$)*

$$d(x,\ y\,;\ \overline{W}^\sigma(x))=d(\beta(x),\ \beta(y)\,;\ \overline{W}^\sigma(\beta(x)))\,.$$

PROOF. For $y\in\overline{W}^s(x)$ and $\beta\in G(\pi)$, let $m=d(\beta(x),\ \beta(y)\,;\ \overline{W}^s(\beta(x)))$. By the definition $\bar{f}^m(\beta(y))\in W^s_{\varepsilon_0}(\bar{f}^m\circ\beta(x))$. Remark that $\bar{f}^m\circ\beta=\beta_m\circ\bar{f}^m$ for some $\beta_m\in G(\pi)$. Then $\beta_m\circ\bar{f}^m(y)\in\overline{W}^s_{\varepsilon_0}(\beta_m\circ\bar{f}^m(x))$. By Lemma 3.2 it is easily checked that $\overline{W}^s_{\varepsilon_0}(\beta_m\circ\bar{f}^m(x))=\beta_m\overline{W}^s_{\varepsilon_0}(\bar{f}^m(x))$, and hence $\bar{f}^m(y)\in\overline{W}^s_{\varepsilon_0}(\bar{f}^m(x))$, which implies $d(x,\ y\,;\ \overline{W}^s(x))\leqq m$. In the same way we have $d(\beta(x),\ \beta(y)\,;\ \overline{W}^s(\beta(x)))\leqq d(x,\ y\,;\ \overline{W}^s(x))$, and therefore the conclusion for $\sigma=s$ is obtained. The analogous result for $\sigma=u$ is obtain in the same manner.

For $m\geqq 0$, define

$$B^\sigma(m)=\{x\in\overline{W}^\sigma(0):\ \ d(x,\,0\,;\ \overline{W}^\sigma(0))\leqq m\}\qquad(\sigma=s,\ u)\,.$$

Since $\bar{f}(0)=0$, by the definition we see that

$$B^s(m)=\bigcup_{i=0}^m\bar{f}^{-i}\overline{W}^s_{\varepsilon_0}(0)\,,\qquad B^u(m)=\bigcup_{i=0}^m\bar{f}^{i}\overline{W}^u_{\varepsilon_0}(0)\,,$$

and hence $B^\sigma(m)$ is compact in $\overline{W}^\sigma(0)$. Since $\bar{\gamma}_0$ is a homeomorphism, $B(m)=\bar{\gamma}_0(B^u(m)\times B^s(m))$ is compact in $\boldsymbol{R}^n$. For $m\geqq 0$, let $U^s(m)=\bar{f}^{-m}\bar{D}^s_0$ and $U^u(m)=\bar{f}^m\bar{D}^u_0$. Since $\bar{D}^\sigma_0$ is open in $\overline{W}^\sigma(0)$, it follows that $U^\sigma(m)$ also is open in $\overline{W}^\sigma(0)$. Hence $U(m)=\bar{\gamma}_0(U^u(m)\times U^s(m))$ is open in $\boldsymbol{R}^n$. Note that $U(m)\subset B(m)$ for all $m\geqq 0$. By Lemma 3.5 we have

$$\boldsymbol{R}^n=\bigcup_{m\geqq 0}B(m)=\bigcup_{m\geqq 0}U(m)\,. \tag{6.6}$$

LEMMA 6.10. *For* $m\geqq 0$ *and* $\sigma=s, u$

$$\{d(x, y; \overline{W}^{\sigma}(x)): \ x, y\in B(m), y\in\overline{W}^{\sigma}(x)\}$$

is bounded in $\boldsymbol{N}$.

PROOF. Since $B(m)$ is compact and $U(m)$ is open, by (6.6) for $m\geqq 0$ there is $k\geqq 0$ such that $B(m)\subset U(k)$. Remark that $\mathrm{cl}(U(k))$ is compact (since $U(k)\subset B(k)$). Since each $\overline{N}_x$ is open in $\boldsymbol{R}^n$, we can find a finite set $\{x_i\}_{i=1}^{l}\subset\mathrm{cl}(U(k))$ such that $\mathrm{cl}(U(k))\subset\bigcup_{i=1}^{l}\overline{N}_{x_i}$. To obtain the conclusion for $\sigma=s$, let $D(x)=\overline{W}^s(x)\cap\mathrm{cl}(U(k))$ for $x\in B(m)$. Then we claim that $D(x)$ is connected. Indeed, since $\bar{\gamma}_0$ is a homeomorphism, $\mathrm{cl}(U(k))=\bar{\gamma}_0(\mathrm{cl}(U^u(k))\times\mathrm{cl}(U^s(k)))$. Since $x\in\mathrm{cl}(U(k))$, there are $x_1\in\mathrm{cl}(U^u(k))$ and $x_2\in\mathrm{cl}(U^s(k))$ such that $\bar{\gamma}_0(x_1, x_2)=x$. By the definition of $\bar{\gamma}_0$, $\overline{W}^s(x)=\bar{\gamma}_0(\{x_1\}\times\overline{W}^s(0))$, and hence

$$\begin{aligned} D(x) &= \bar{\gamma}_0(\{x_1\}\times\overline{W}^s(0))\cap\bar{\gamma}_0(\mathrm{cl}(U^u(k))\times\mathrm{cl}(U^s(k))) \\ &= \bar{\gamma}_0(\{x_1\}\times\mathrm{cl}(U^k(k))) \end{aligned}$$

which implies $D(x)$ is connected.

Let $K=\{i\in\{1, \cdots, l\}: D(x)\cap\overline{N}_{x_i}\neq\emptyset\}$. Then $D(x)\subset\bigcup_{i\in K}\overline{N}_{x_i}$. Choose $y_i\in D(x)\cap\overline{N}_{x_i}$ for $i\in K$. Then it is checked that $D(x)\cap\overline{N}_{x_i}\subset\overline{W}^s_{2\varepsilon_0}(y_i)$. Indeed, recall

$$\overline{N}_{x_i}=\bigcup_{z\in\overline{D}^u_{x_i}}\overline{D}^s_{x_i, z} \qquad \text{(disjoint union)}$$

where $\overline{D}^s_{x_i, z}=\overline{N}_{x_i}\cap\overline{W}^s_{\varepsilon_0}(z)$. Since $\boldsymbol{R}^n$ is a coordinate domain we see easily that $D(x)\cap\overline{N}_{x_i}\subset\overline{D}^s_{x_i, z}$ for some $z\in\overline{D}^u_{x_i}$, and so $y_i\in\overline{D}^s_{x_i, z}$. Therefore $D(x)\cap\overline{N}_{x_i}\subset\overline{W}^s_{2\varepsilon_0}(y_i)$.

By the above result, $D(x)\subset\bigcup_{i\in K}\overline{W}^s_{2\varepsilon_0}(y_i)$. Since $D(x)$ is connected, for $i, j\in K$ there is $\{i_1, \cdots, i_s\}\subset K$ such that $i_1=i$, $i_s=j$ and $\overline{W}^s_{2\varepsilon_0}(y_{i_n})\cap\overline{W}^s_{2\varepsilon_0}(y_{i_{n+1}})\neq\emptyset$ for $n=1, 2, \cdots, s-1$. By Lemma 3.4 we can find $L\in\boldsymbol{N}$ such that $\bar{f}^L\overline{W}^s_{2\varepsilon_0}(z)\subset\overline{W}^s_{\varepsilon_0/2l}(\bar{f}^L(z))$. Then $\bar{f}^L(D(x))\subset\overline{W}^s_{\varepsilon_0/2}(\bar{f}^L(x))$ and therefore $d(x, y; \overline{W}^s(x))\leqq L$ if $y\in\overline{W}^s(x)\cap B(m)\subset D(x)$. Since L is independent of $x\in B(m)$, the conclusion for $\sigma=s$ is obtained. The analogous result for $\sigma=u$ is obtained in the same way.

PROOF OF THEOREM. Let $A: \boldsymbol{T}^n\to\boldsymbol{T}^n$ be the group automorphism homotopic to f. Then by Proposition 6.2 A is hyperbolic. Since $f(e)=e$, by Corollary 6.4 there is a continuous map $h: \boldsymbol{T}^n\to\boldsymbol{T}^n$ homotopic to the identity map id such that $h(e)=e$ and $A\circ h=h\circ f$. Therefore it is enough to prove that h is a homeomorphism. To do this, we take lifts $\overline{A}$ and $\bar{h}$ by π such $\overline{A}\circ\bar{h}=\bar{h}\circ\bar{f}$ holds. Since h is homotopic to id, there is $M_0>0$ such that for all $x\in\boldsymbol{R}^n$

$$\|\bar{h}(x)-x\|\leqq M_0. \tag{6.7}$$

To show that $\bar{h}$ is injective, assuming that there are $x, y\in \boldsymbol{R}^n$ such that $\bar{h}(x)=\bar{h}(y)$, put $z=\bar{\gamma}(x, y)$. Then we have

$$\|\bar{A}^i\circ\bar{h}(z)-\bar{A}^i\circ\bar{h}(y)\| = \begin{cases} \|\bar{h}\circ\bar{f}^i(z)-\bar{h}\circ\bar{f}^i(x)\| & (i\geqq 0) \\ \|\bar{h}\circ\bar{f}^i(z)-\bar{h}\circ\bar{f}^i(y)\| & (i\leqq 0). \end{cases}$$

Since $z\in\overline{W}^s(x)\cap\overline{W}^u(y)$, by (6.7) we can find $M_1>0$ such that $\|\bar{A}^i\circ\bar{h}(z)-\bar{A}^i\circ\bar{h}(y)\|\leqq M_1$ for all $i\in\boldsymbol{Z}$. Since $\bar{A}$ is hyperbolic, M_1 is an expansive constant for $\bar{A}$ and hence $\bar{h}(z)=\bar{h}(y)$. Now we claim that there is $M_2\in\boldsymbol{N}$ such that if $z\in\overline{W}^u(y)$ and $\bar{h}(z)=\bar{h}(y)$, then $d(y, z; \overline{W}^u(y))\leqq M_2$. Indeed, let $K\subset\boldsymbol{R}^n$ is a compact covering domain for π. Since $\bar{h}$ is proper by (6.7), there is $m\geqq 0$ such that $\bar{h}^{-1}(K)\subset B(m)$. Take $\beta\in G(\pi)$ such that $\beta(\bar{h}(z))\in K$. Then we have

$$\bar{h}(\beta(z)) = \beta\circ\bar{h}(z) = \beta\circ\bar{h}(y) = \bar{h}(\beta(y)) \in K$$

and hence $\beta(z), \beta(y)\in B(m)$. By Lemma 6.10 there is

$$M_2 = \max\{d(x, y; \overline{W}^u(x)):\ x, y\in B(m),\ y\in\overline{W}^u(x)\},$$

and so $d(\beta(y), \beta(z); \overline{W}^u(\beta(y)))\leqq M_2$. By Lemma 6.9 we have $d(y, z; \overline{W}^u(y))\leqq M_2$, and therefore the claim holds.

If $y\neq z$, then by expansiveness there is $l\in\boldsymbol{N}$ such that $d(\bar{f}^l(y), \bar{f}^l(z); \overline{W}^u(\bar{f}^l(y)))>M_2$, which contradicts the above claim since

$$\bar{h}\circ\bar{f}^l(y) = \bar{A}^l\circ\bar{h}(y) = \bar{A}^l\circ\bar{h}(z) = \bar{h}\circ\bar{f}^l(z).$$

Therefore $y=z$. In the same way, we have $x=z$, and so $\bar{h}$ is injective. We know by Brouwer's Theorem that h is a covering map. Since h is homotopic to id, h is a homeomorphism. The proof of our theorem is completed.

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Koichi HIRAIDE
Institute of Mathematics
University of Tsukuba
Tsukuba-shi, Ibaraki 305
Japan