J. Math. Soc. Japan
Vol. 41, No. 2, 1989

One-dimensional bi-generalized diffusion processes

Dedicated to Professor Nobuyuki Ikeda on his 60th birthday

By Yukio OGURA

(Received Sept. 14, 1987)

1. Introduction.

The theory of one-dimensional diffusion processes (ODDPs for brief) was extensively developed in 1950's by many mathematicians headed by W. Feller, K. Itô, E. B. Dynkin, H. P. McKean and so on (see the references of [**9**] for the literatures). An ODDP is a strong Markov process with continuous sample paths, and it is determined by the strictly increasing continuous scale function s and the positive speed measure dm on an interval in the real line. The positivity of dm was soon relaxed to nonnegativity, and appeared the notion of generalized diffusion processes (GDPs) or gap processes. A GDP is a strong Markov process with right continuous sample paths, which may jump only to the nearest neighbours in the support of dm, and it is determined by a strictly increasing continuous scale function s and a nonnegative speed measure dm. The set of ODDPs or GDPs forms an effective and beautiful class from both probabilistic and analytic points of view. However, in the recent development of their application, there appeared a one-dimensional Markov process corresponding to the scale function with jumps and the Lebesgue speed measure (see [**7**] and [**12**]). In our introductory lecture [**13**], we tried to define the class of those processes by means of the expression $s^{-1}\circ B(\mathfrak{f}^{-1}(t))$, where B is a Brownian motion and $\mathfrak{f}$ is a random time change function. But it remained to reveal the behavior of the process on the flats of s, when they exist.

In this paper, we first define and construct the one-dimensional Markov process corresponding to a non-decreasing scale function s and a nonnegative speed measure dm, which we call a bi-generalized diffusion process (BGDP). The obtained process neither is strong Markov nor has right continuous sample paths in general anymore. Actually, there are 'chaotic' ponds, where the sample paths are absolutely jumbled, but after identifying each such pond as one point, the sample paths are quite tame; they are right continuous and jump only to the nearest neighbours in the support of dm. This situation is realized by our auxiliary GDP Y given in the following sections, which, I believe, is the same as Ray-Knight process; it is right continuous strong Markov process and the state space is obtained by identifying the 'non-separable' points and splitting the points at which the strong Markov property fails (see [**16**], [**20**], [**24**]).

Our next objective is to give limit theorems for a sequence of BGDPs. Since the sample paths of our process are not right continuous in general, we can expect no general limit theorems of J_1-convergence, so that we give those of finite dimensional distributions. Aside from some additional assumptions, we can almost conclude that the finite dimensional distributions of BGDPs converge to those of a BGDP if the associate scale functions and speed measures converge to the corresponding ones. Our result is a generalization of Golosov's one in [**7**] (including a small correction), which is concerned with the case of Lebesgue speed measure dm (see also [**1**] for the results concerning with multidimensional case).

Actually, our original motivation was to give a unified prospect for the limit theorems for ODDPs in various areas of applications. This is verified at least for three topics, the metastable behavior in statistical physics, asymptotic behavior of a one-dimensional diffusion process in a Wiener medium and discrete approximation of a diffusion process of gene frequency (see [**15**], [**2**] and [**6**] respectively). From this aspect, one would easily recognize that all the above three applications are the same kind of problems, that is the one proposed and extensively studied by A. D. Ventsel and M. L. Freidlin [**23**]. Our study in Section 7 asserts that, as far as the state space is one-dimensional, their problem is reduced to a general convergence theorem.

The moment problem has been one of the most interesting and important subjects in the classical analysis. Further, the class of GDPs includes birth and death processes, and our class of BGDPs contains birth and death processes with more general boundary conditions such as to correspond to non-strong Markov processes. This enables us to study another type of the Stieltjes moment problems than those dealt with by S. Karlin and J. L. McGregor [**11**] (see Section 4 below).

Finally, we note that there naturally arise two open problems of mathematical interest: the characterization of our BGDPs and the behavior of sample paths in the convergence theorems.

The arrangement of this paper is as follows. In the next Section 2, we give our definition of BGDPs and their analytic construction in the exactly standard way. In Section 3, we give a realization of sample paths, and show that the class of our BGDPs includes Ikeda's example, which covers all types of continuous Markov processes violating strong Markov property at a single point. Section 4 is devoted to the study of the Stieltjes moment problem associated to the birth and death processes with the new type of boundary conditions. We give general theorems for the convergence of finite dimensional distributions of a sequence of BGDPs in Sections 5 and 6. In Section 5, we are concerned with the vague convergence, whereas, in Section 6, we proceed to the weak convergence. Section 7 is for examples of application of our limit theorems in Sections 5 and 6 to the three topics in applied mathematics. We add Appendix for the formulas on change of variables and integration by parts in the case where the integrating measures are induced by discontinuous non-decreasing functions.

ACKNOWLEDGEMENT. The author would like to thank Prof. N. Ikeda for his valuable suggestions, and Prof. M. Tomisaki who has read through the original manuscript and pointed out various mistakes.

2. Definition and analytical construction.

Let $\bar{\boldsymbol{R}}=[-\infty, +\infty]$ and $\mathcal{M}$ be the totality of monotone non-decreasing functions φ from $\bar{\boldsymbol{R}}$ into $\bar{\boldsymbol{R}}$. For each $\varphi\in\mathcal{M}$, we set

$$\begin{aligned}
&l_1(\varphi)=\inf\{x\in\bar{\boldsymbol{R}}: \varphi(x)>-\infty\}, \qquad l_2(\varphi)=\sup\{x\in\bar{\boldsymbol{R}}: \varphi(x)<+\infty\},\\
&Q(\varphi)=(l_1(\varphi), l_2(\varphi)), \qquad \bar{Q}(\varphi)=[l_1(\varphi), l_2(\varphi)],\\
&\mathrm{Spt}(\varphi)=\{x\in\boldsymbol{R}: \varphi(x_1)<\varphi(x_2) \ \text{ for every } x_1<x<x_2\},\\
&J(\varphi)=\{x\in Q(\varphi): \Delta_\varphi(x)>0\}\cup\{l_i(\varphi): \lim_{x\to l_i,\, x\in Q(\varphi)}|\varphi(x)|<+\infty \text{ or } |l_i(\varphi)|<+\infty\},
\end{aligned}$$

where $\Delta_\varphi(x)=\varphi(x+)-\varphi(x-)\equiv\lim_{y\downarrow x}\varphi(x)-\lim_{y\uparrow x}\varphi(x)$ (set $\inf\emptyset=+\infty$ and $\sup\emptyset=-\infty$ by convention). The right continuous inverse function of $\varphi\in\mathcal{M}$ is denoted by φ^{-1};

$$\varphi^{-1}(\xi)=\sup\{x\in\bar{\boldsymbol{R}}: \varphi(x)\leqq\xi\}.$$

Then it also belongs to the space $\mathcal{M}_+$, where

$$\mathcal{M}_\pm=\{\varphi\in\mathcal{M}: \varphi(x)=\varphi(x\pm) \ \text{ for all } x\in\boldsymbol{R}\}.$$

Sometimes the same symbol φ is used for the measure induced by φ.

Now fix a pair $(s, m)\in\mathcal{M}\times\mathcal{M}_+$ and set $l_1=l_1(s)\vee l_1(m)$, $l_2=l_2(s)\wedge l_2(m)$, where $a\vee b$ $[a\wedge b]$ stands for the maximum [resp. minimum] of a and b. Throughout, we assume

$$l_1, l_2\notin J(s)\cap J(m), \qquad Q(s)\cap\mathrm{Spt}(m)\neq\emptyset, \qquad Q(m)\cap\mathrm{Spt}(s)\neq\emptyset. \tag{2.1}$$

Put

$$\tilde{m}(\xi)=\begin{cases}\sup\{m(x): s(x)\leqq\xi\}, & \text{if } \{\ \}\neq\emptyset,\\ m(-\infty), & \text{if } \{\ \}=\emptyset,\end{cases} \tag{2.2}$$

which clearly belongs to $\mathcal{M}_+$. With this notation, the last two relations in (2.1) exclude the cases $\tilde{m}=$constant and $\tilde{l}_1=\tilde{l}_2$ respectively, where $\tilde{l}_i=l_i(\tilde{m})$, $i=1, 2$.

Notice that $\tilde{m}=m\circ s^{-1}$ if $s\in\mathcal{M}_-$ or $J(s)\cap J(m)=\emptyset$. We next set $Q=Q(s, m)=(l_1, l_2)$ and put

$$D(Q, s)=\{u: u \text{ is a complex valued function on } Q \text{ such that } u(x)=u(y) \text{ whenever } s(x)=s(y)\},$$

$BD(Q, s)=B(Q)\cap D(Q, s)$, where $B(Q)$ is the space of all bounded measurable functions on Q. Also, we use the convention $\int_{(a,b]}=-\int_{(b,a]}$ whenever $b<a$, and fix an $a\in(l_1, l_2)$ through this section. The space $D(Q, s, m)$ is the set of all functions u which satisfy the relation

$$u(x)=A_1+A_2 s(x)+\int_{(a,x]}(s(x)-s(y))f(y)dm(y), \qquad x\in Q,$$

for some constants A_1, A_2 and some locally bounded f. For each such u, we set

$$u^+(x)=A_2+\int_{(a,x]}f(y)dm(y).$$

Notice that it necessarily holds that $A_1=u(a)-s(a)u^+(a)$ and $A_2=u^+(a)$.

DEFINITION 2.1. The domain $\mathcal{D}(\mathcal{G}_{s,m})$ is the set of all those functions $u\in BD(Q, s)$ which satisfy the following three conditions.

(i) There exist two constants A_1, A_2 and a function $f\in B(Q)$ such that

$$u(x)=A_1+A_2 s(x)+\int_{(a,x]}(s(x)-s(y))f(y)dm(y), \qquad x\in Q. \tag{2.3}$$

(ii) For $i=1, 2$, if

$$(2.4)_i \qquad |l_i(m)-a|\leqq|l_i(s)-a| \quad\text{and}\quad \lim_{x\to l_i,\, x\in Q}|s(x)|<+\infty,$$

then $\lim_{x\to l_i,\, x\in Q}u(x)=0$.

(iii) For $i=1, 2$, if

$$(2.5)_i \qquad |l_i(s)-a|<|l_i(m)-a| \quad\text{and}\quad \lim_{x\to l_i,\, x\in Q}|s(x)|<+\infty,$$

then $\lim_{x\to l_i,\, x\in Q}u^+(x)=0$.

For each $u\in\mathcal{D}(\mathcal{G}_{s,m})$, we denote the function f in (2.3) by $\mathcal{G}_{s,m}u$. Note that a function $u\in\mathcal{D}(\mathcal{G}_{s,m})$ does not uniquely determine $\mathcal{G}_{s,m}u$ in general (this phenomenon already appears in generalized diffusion processes, where $\mathcal{G}_{s,m}u$ is uniquely determined as an element of $L^1_{\mathrm{loc}}(Q, m)$).

In the followings, we add the state l_i to the state space Q in the case where $(2.5)_i$ holds, and denote it by Q again.

Let Y be a generalized diffusion process (GDP for brief) with the natural scale ξ and the speed measure $\tilde{m}$. It is well known that Y is a strong Markov process on the state space $\tilde{Q}=Q(\tilde{m})=(\tilde{l}_1, \tilde{l}_2)$ and it has a right continuous version. We denote the transition density function of Y w.r.t. the speed measure $\tilde{m}$ by $q(t, \xi, \eta)$. Then the corresponding Green function is given by

$$(2.6)\qquad H(\alpha, \xi, \eta)=\Phi(\xi, \eta)+\int_0^{+\infty} e^{-\alpha t}q(t, \xi, \eta)dt\,, \qquad \alpha>0,\ \ \xi, \eta\in\tilde{Q}\,,$$

where the correction function $\Phi(\xi, \eta)$ is defined in (2.17) below (see [**19**; Lemma 1]). Also we set

$$(2.7)\qquad T_t f(x)=\int_Q q(t, s(x), s(y))f(y)dm(y)\,, \qquad f\in B(Q),\ x\in Q\,,$$

$$(2.8)\qquad G_\alpha f(x)=\int_Q H(\alpha, s(x), s(y))f(y)dm(y)\,, \quad f\in B(Q),\ x\in Q\,.$$

It is then clear that $T_t(B(Q))\subset BD(Q, s)$ and $G_\alpha(B(Q))\subset BD(Q, s)$. Further, we have the following

PROPOSITION 2.1. *Let* $(s, m)\in\mathcal{M}\times\mathcal{M}_+$ *satisfy the conditions* (2.1). *Then*

$$(2.9)\qquad 0\leqq T_t f\leqq 1,\quad t>0,\qquad whenever\ 0\leqq f\leqq 1,$$

$$(2.10)\qquad T_t T_s=T_{t+s},\qquad t, s>0,$$

$$(2.11)\qquad 0\leqq G_\alpha f\leqq 1/\alpha,\quad \alpha>0,\qquad whenever\ 0\leqq f\leqq 1,$$

$$(2.12)\qquad G_\alpha-G_\beta+(\alpha-\beta)G_\alpha G_\beta=0,\qquad \alpha, \beta>0.$$

Further, $T_t f(x)$ *is continuous in* $t>0$ *for each* $f\in B(Q)$ *and* $x\in Q$.

The proof of Proposition 2.1 is easy. Indeed, one has only to apply Lemma A.1 carefully and make use of the properties for $q(t, \xi, \eta)$ and $H(\alpha, \xi, \eta)$. The details are omitted (see also Proof of Lemma 2.1 below).

Due to (2.9) and (2.10), there exists a unique Markov process X on Q corresponding to the semigroup T_t. We call it a *bi-generalized diffusion process* (BGDP for brief) corresponding to (s, m).

The next theorem justifies this definition:

THEOREM 2.1. *Let* $(s, m)\in\mathcal{M}\times\mathcal{M}_+$ *satisfy the conditions* (2.1). *Then, for each* $\alpha>0$ *and* $f\in B(Q)$ *with* $\lim_{x\to l_i, x\in Q}f(x)=0$, *the equation*

$$(2.13)\qquad (\alpha 1-\mathcal{G}_{s,m})u=f$$

has a unique solution $u=G_\alpha f$ *in* $\mathcal{D}(\mathcal{G}_{s,m})$. *Further, it holds that*

$$(2.14)\qquad G_\alpha f(x)=\int_0^\infty e^{-\alpha t}T_t f(x)dt,\qquad \alpha>0,\ x\in Q.$$

In order to prove Theorem 2.1, we first review the construction of $q(t, \xi, \eta)$ and $H(\alpha, \xi, \eta)$.

Fix an $\tilde{a}\in\tilde{Q}$ and let $v_i(\xi, \alpha)$, $i=1, 2$, $\xi\in\tilde{Q}$, $\alpha>0$ be the positive solutions of the integral equation

$$(2.15)\qquad v(\xi)=1+B_2(\xi-\tilde{a})+\alpha\int_{(\tilde{a},\xi]}(\xi-\eta)v(\eta)d\tilde{m}(\eta),\qquad \xi\in\tilde{Q},$$

such that $v_1(\xi, \alpha)$ [$v_2(\xi, \alpha)$] is increasing [resp. decreasing] and satisfies $\lim_{\xi\to\tilde{l}_i,\xi\in\tilde{Q}}v_i(\xi, \alpha)=0$ whenever $|\tilde{l}_i-\tilde{a}|<+\infty$ (see [**17**] and [**19**]). As before, we set, for such a v,

$$D_\xi^+v(\xi)=B_2+\alpha\int_{(\tilde{a},\xi]}v(\eta)d\tilde{m}(\eta),\qquad \xi\in\tilde{Q}.$$

It is then well known that the Wronskian $W(v_1, v_2)(\xi)$ of $v_1(\xi, \alpha)$ and $v_2(\xi, \alpha)$ is constant;

$$W(v_1, v_2)(\xi):=D_\xi^+v_1(\xi, \alpha)v_2(\xi, \alpha)-v_1(\xi, \alpha)D_\xi^+v_2(\xi, \alpha)=1/h(\alpha),\quad \xi\in\tilde{Q}.$$

Now the Green function $H(\alpha, \xi, \eta)$ of the GDP Y is defined by

$$(2.16)\quad H(\alpha, \xi, \eta)=H(\alpha, \eta, \xi)=h(\alpha)v_1(\xi, \alpha)v_2(\eta, \alpha),\qquad \alpha>0,\ \xi\leqq\eta,\ \xi, \eta\in\tilde{Q}.$$

The function $q(t, \xi, \eta)$ is then given by (2.6) with the help of the correction function $\Phi(\xi, \eta)$, which we now define. Let I_k, $k=1, 2, \cdots$ be the disjoint open intervals such that $\tilde{Q}\setminus\mathrm{Spt}(\tilde{m})=\bigcup_{k=1}^{\infty}I_k$ and the end points (if exist) belong to $\mathrm{Spt}(\tilde{m})\cup\{\tilde{l}_1, \tilde{l}_2\}$. For each $\xi, \eta\in\tilde{Q}$ with $\xi\leqq\eta$, we set

$$(2.17)\qquad \Phi(\xi, \eta)=\Phi(\eta, \xi)=\begin{cases}\xi_2-\eta, & -\infty=\xi_1<\xi_2<+\infty,\\ (\xi-\xi_1)(\xi_2-\eta)/(\xi_2-\xi_1), & -\infty<\xi_1<\xi_2<+\infty,\\ \xi-\xi_1, & -\infty<\xi_1<\xi_2=+\infty,\end{cases}$$

if $\xi, \eta\in I_k=(\xi_1, \xi_2)$ for some $I_k\neq\emptyset$, and $=0$ otherwise.

For the later use, we give here three remarks and one convention.

(i) *$\tilde{l}_i=(-1)^i\cdot\infty$ in the case of $(2.5)_i$, and $\tilde{l}_i=\lim_{x\to l_i,x\in Q}s(x)$ otherwise.*

(ii) *$s(Q)\subset(\tilde{l}_1, \tilde{l}_2)$ except for the case where $(2.4)_i$ holds for some $i=1, 2$.*

(iii) *In the case of $\tilde{l}_i\in s(Q)$, the boundary $\tilde{l}_i$ for the* GDP *Y is finite and regular with the absorbing boundary condition. Hence, it holds that* $\lim_{\eta\to\tilde{l}_i,\eta\in\tilde{Q}}q(t, \xi, \eta)=0$.

CONVENTION. *In the case where $\tilde{l}_i$ is finite, we set $q(t, \xi, \eta)=q(t, \eta, \xi)=0$, $H(\alpha, \eta, \eta)=\lim_{\zeta\to\tilde{l}_i,\zeta\in\tilde{Q}}H(\alpha, \zeta, \zeta)$ and $v_j(\eta, \alpha)=\lim_{\zeta\to\tilde{l}_i,\zeta\in\tilde{Q}}v_j(\zeta, \alpha)$, for each $\xi\in\tilde{Q}$, $\eta\in[\tilde{l}_i, (-1)^i\infty)$ and $j=1, 2$, where $[\tilde{l}_1, -\infty)$ is read as $(-\infty, \tilde{l}_1]$ (admitting the possibility that they take the values $\pm\infty$).*

LEMMA 2.1. *Let $u_i(x, \alpha)=v_i(s(x), \alpha)$, $x\in Q$, $i=1, 2$. Then it holds that*

$$(2.18)\qquad u_i(x, \alpha)=u_i(a, \alpha)+u_i^+(a, \alpha)(s(x)-s(a))$$
$$+\alpha\int_{(a,x]}(s(x)-s(y))u_i(y, \alpha)dm(y),\qquad x\in Q.$$

Further, if $(2.4)_i$ holds, then

(2.19) $$\lim_{x \to l_i,\, x \in Q} u_i(x, \alpha) = 0,$$

and, if $(2.5)_i$ *holds, then*

(2.20) $$\lim_{x \to l_i,\, x \in Q} u_i^+(x, \alpha) = 0.$$

PROOF. We will first prove (2.18).

Suppose that $s(x) \neq \tilde{l}_j$, $j=1, 2$. It is then clear from (2.15) and Lemma A.1 that the function $u(x)=u_i(x, \alpha)$ satisfies

$$u(x) = 1+B_2(s(x)-\tilde{a})+\alpha\int_{s^{-1}((\tilde{a}, s(x)])}(s(x)-s(y))u(y)dm(y),$$

where $\int_{s^{-1}((x, y])} = -\int_{s^{-1}((y, x])}$ whenever $y<x$. Noting that $s(x)-s(y)=0$ for all $y \in s^{-1}(\{s(x)\})$, we have

(2.21) $$\int_{s^{-1}((\tilde{a}, s(x)])}(s(x)-s(y))u(y)dm(y)$$
$$= C_1+C_2 s(x)+\int_{(a, x]}(s(x)-s(y))u(y)dm(y),$$

for some constants C_1 and C_2. This proves (2.18) in this case.

Suppose next that $s(x)=\tilde{l}_2$. This can occur only when $(2.4)_2$ holds. On the other hand, we have

$$\lim_{\xi \uparrow \tilde{l}_2}\int_{s^{-1}((\tilde{a}, \xi])}(\xi-s(y))u(y)dm(y)$$
$$= \begin{cases} C_1+C_2 s(x)+\int_{(a, r_2]}(\tilde{l}_2-s(y))u(y)dm(y), & \text{if } s(r_2)<\tilde{l}_2, \\ C_1+C_2 s(x)+\int_{(a, r_2)}(\tilde{l}_2-s(y))u(y)dm(y), & \text{if } s(r_2)=\tilde{l}_2, \end{cases}$$

where $r_2=\lim_{b \uparrow \tilde{l}_2} s^{-1}(b)$. Hence we obtain (2.18) by the same reason as in the above case.

The proof of (2.18) for the case where $s(x)=\tilde{l}_1$ is similar and will be omitted.

We will next prove (2.19) assuming $(2.4)_i$. Note that, in this case, $l_i=l_i(m)$, $\tilde{l}_i=\lim_{x \to l_i,\, x \in Q} s(x)$ and $\tilde{l}_i$ is finite. Hence $v_i(\tilde{l}_i, \alpha)=0$ by our assumption, whence (2.19) follows.

The proof of (2.20) is similar. Indeed, assuming $(2.5)_i$, we have $l_i=l_i(s)$, $\tilde{l}_i=(-1)^i \cdot \infty$ and $l_i \notin J(m)$ by (2.1). Further, setting $\tilde{r}_i=\lim_{x \to l_i,\, x \in Q} s(x)$, one has $\tilde{m}(\xi)=\tilde{m}(\tilde{r}_i)$ for all $\xi \in [\tilde{r}_i, (-1)^i \cdot \infty)$, where $[\tilde{r}_1, -\infty)$ is read as $(-\infty, \tilde{r}_1]$. Hence due to the arguments for GDPs, we have $D_\xi^+ v_i(\xi, \alpha)=0$ for all $\xi \in [\tilde{r}_i, (-1)^i \cdot \infty)$. Thus we obtain (2.20) by making use of Lemma A.1. q. e. d.

COROLLARY 2.2. *The Wronskian* $W(u_1, u_2)(x)$ *of* $u_1(x, \alpha)$ *and* $u_2(x, \alpha)$ *is* *constant;*

(2.22) $$W(u_1, u_2)(x):=u_1^+(x, \alpha)u_2(x, \alpha)-u_1(x, \alpha)u_2^+(x, \alpha)=1/h(\alpha), \quad x\in Q.$$

PROOF. We will prove (2.22) only for $a<x$ and $\tilde{a}<s(x)$. Suppose first that $s(x)<\tilde{l}_2$. It then follows from (2.15) and Lemma A.1 that

$$D_\xi^+ v_i(s(x), \alpha)=D_\xi^+ v_i(\tilde{a}, \alpha)+\alpha\int_{s^{-1}((\tilde{a}, s(x)])} v_i(s(y), \alpha)dm(y).$$

Hence

$$u_i^+(x, \alpha)=D_\xi^+ v_i(s(x), \alpha)-\alpha v_i(s(x), \alpha)m(s^{-1}((\tilde{a}, s(x)])\setminus(-\infty, x]),$$

so that $W(u_1, u_2)(x)=W(v_1, v_2)(s(x))=1/h(\alpha)$.

On the other hand, if $s(x)=\tilde{l}_2$, then, as in the Proof of Lemma 2.1, we have

$$D_\xi^+ v_i(\tilde{l}_2, \alpha)=\begin{cases} u_i^+(r_2, \alpha), & \text{if } s(r_2)<\tilde{l}_2, \\ u_i^+(r_2-, \alpha), & \text{if } s(r_2)=\tilde{l}_2. \end{cases}$$

Hence, we can easily obtain (2.22).

The proof of (2.22) for the case where $x\leqq a$ or $s(x)\leqq\tilde{a}$ is similar and will be omitted. q. e. d.

PROOF OF THEOREM 2.1. We first note that

$$\int_{(a, x]}(s(x)-s(y))f(y)dm(y)=\int_{(a, x]}^{\#}\int_{(a, y]} f(z)dm(z)ds(y)-\Delta_s^+(x)\int_{(a, x]} f(z)dm(z),$$

for $x\in Q$ by Lemma A.2. Hence

(2.23) $$u(x)=u(a)+\int_{(a, x]}^{\#} u^+(y)ds(y)-(\Delta_s^+(x)u^+(x)-\Delta_s^+(a)u^+(a)),$$

for each $u\in D(Q, s, m)$. This also implies that $\Delta_u^{\pm}(x)=\Delta_s^{\pm}(x)u^+(x\pm)$ and, for each function g of bounded variation,

(2.24) $$\int_{(a, x]}^{\#} g(y)u^+(y)ds(y)=\int_{(a, x]}^{\#} g(y)du(y).$$

We will first show (2.13). It follows from (2.8) and (2.16) that

$$\frac{1}{h(\alpha)}\int_{(a, y]}\alpha G_\alpha f(z)dm(z)=\frac{1}{h(\alpha)}\int_{(a, y]} f(w)dm(w)$$
$$+u_2^+(y)g_1(y)-u_2^+(a)g_1(a)+u_1^+(y)g_2(y)-u_1^+(a)g_2(a),$$

where we denote as $u_i(x)=u_i(x, \alpha)$ and

$$g_1(y)=\int_{(l_1, y]} u_1(w)f(w)dm(w), \qquad g_2(y)=\int_{(y, l_2)} u_2(w)f(w)dm(w).$$

Further, by Lemma A.2 and (2.24),

$$\int_{(a,x]}^{\#} g_1(y)u_2^+(y)ds(y)-\Delta_s^+(x)u_2^+(x)g_1(x)$$
$$=u_2(x)g_1(x)-u_2(a)g_1(a)-\int_{(a,x]}u_1(w)u_2(w)f(w)dm(w),$$
$$\int_{(a,x]}^{\#} g_2(y)u_1^+(y)ds(y)-\Delta_s^+(x)u_1^+(x)g_2(x)$$
$$=u_1(x)g_2(x)-u_1(a)g_2(a)+\int_{(a,x]}u_1(w)u_2(w)f(w)dm(w).$$

Hence we have (2.3) with f replaced by $\alpha G_\alpha f-f$.

On the other hand, we know that $\alpha G_\alpha f-f$ belongs to $B(Q)$ by the well known property for GDP and (2.8). Hence it follows that $u\in\mathcal{D}(\mathcal{G}_{s,m})$ and $\mathcal{G}_{s,m}u=\alpha G_\alpha f-f$. The proof of the first assertion is finished.

The uniqueness of the solution of (2.13) in $\mathcal{D}(\mathcal{G}_{s,m})$ is clear by the usual arguments.

For the proof of (2.14), it suffices to show

$$\int_Q \Phi(s(x),\, s(y))f(y)dm(y)=0, \qquad x\in Q. \tag{2.25}$$

But this is clear, since $\mathrm{Spt}(m)\cap Q\subset(s^{-1}(\mathrm{Spt}(\tilde{m}))\cup(J(s)\setminus J(m)))\cap Q$ and $\Phi(\xi,\eta)=0$ for $\eta\in\mathrm{Spt}(\tilde{m})$. q. e. d.

3. Sample paths.

In this section, we give a realization of sample paths for the BGDPs given in Section 2.

Let B be a Brownian motion with $B(0)=0$, and denote the first hitting time for the state ξ and the local time of the process $B+s(x)$ by $\sigma_\xi(B+s(x))$ and $L(u,\xi)=L(u,\xi;B+s(x))$ respectively (with the convention $\sigma_{\pm\infty}(B+s(x))=+\infty$). Let also $\mathfrak{f}(u)=\int_{Q(\tilde{m})}L(u,\xi)d\tilde{m}(\xi)$ and $\mathfrak{f}^{-1}(t)=\sup\{u:\mathfrak{f}(u)\leqq t\}$ $(\sup\emptyset=0)$. Then the GDP Y defined in Section 2 is given by $Y(t)=B(\mathfrak{f}^{-1}(t))+s(x)$, $t<e_\Delta$, where $e_\Delta=\mathfrak{f}(\sigma_{\tilde{l}_1}(B+s(x))\wedge\sigma_{\tilde{l}_2}(B+s(x)))$. Notice that, given an $s\in\mathcal{M}$, the GDP Y is uniquely determined by the value of m on a neighbourhood of $\mathrm{Spt}(s)\cap Q$. To be more precise, let $\mathcal{M}(s,m)=\{\mu\in\mathcal{M}_+:\mu(x)=m(x)+c,\ \mu(x-)=m(x-)+c$ for all $x\in\mathrm{Spt}(s)$, for some constant $c\}$. Then all the (s,μ) with $\mu\in\mathcal{M}(s,m)$ determine the same GDP Y.

For each $\xi\in\bar{\boldsymbol{R}}$, denote $Q_\xi=s^{-1}(\{\xi\})$ and let $\mathcal{B}(Q_\xi)$ be the topological Borel field on Q_ξ. For all $\xi\in J(s^{-1})$ with $0<m(Q_\xi)<+\infty$, we define a stationary process (X_ξ,P) on Q_ξ such that

$$P(X_\xi(t)\in E)=m(E)/m(Q_\xi)\,,\qquad E\in\mathcal{B}(Q_\xi),\ t\geqq 0,$$

and that the system $\{X_\xi(t): t\geqq 0\}$ is independent of each other, i.e., $X_\xi(t_1)$, $X_\xi(t_2)$, $\cdots$, $X_\xi(t_n)$ are independent for all $t_1<t_2<\cdots<t_n$. For $\xi\in J(s^{-1})$ with $m(Q_\xi)=0$ or $+\infty$, we define $X_\xi(t)\equiv s^{-1}(\xi)$ if $\xi\in\tilde{Q}$, and $\equiv l_i$ if $\xi=\tilde{l}_i$. Let now $\{B, X_\xi: \xi\in J(s^{-1})\}$ be a system of independent processes on the probability space $(\Omega, \mathcal{F}, P)$ such that X_ξ is a stationary process with the same law as that of the above X_ξ (we use the same symbol). Also, we set $\mathfrak{Z}=\{t\in[0, e_\Delta): Y(t)\in J(s^{-1})\}$. Then the sample paths of our BGDP X are realized by the formula

$$X(t; x)=\begin{cases} s^{-1}(Y(t)), & \text{if } t\notin\mathfrak{Z},\ t<e_\Delta, \\ X_\xi(t), & \text{if } t\in\mathfrak{Z} \text{ and } Y(t)=\xi. \end{cases}$$

Indeed, we have the following

THEOREM 3.1. *Let* $(s, m)\in\mathcal{M}\times\mathcal{M}_+$ *satisfy the conditions* (2.1). *Then, the process* $(X(t; x), e_\Delta, P)$ *defined above corresponds to the semigroup* T_t, *i.e., for any* $0<t_1<t_2<\cdots<t_N$ *and* $f_1, f_2, \cdots, f_N\in B(Q)$, *it holds that*

$$(3.1)\qquad \begin{aligned} &E[f_1(X(t_1; x))f_2(X(t_2; x))\cdots f_N(X(t_N; x)): t_N<e_\Delta] \\ &\qquad = T_{\tau_1}(f_1T_{\tau_2}(\cdots(f_{\tau_{N-1}}T_{\tau_N}f_N)\cdots))(x), \qquad x\in Q, \end{aligned}$$

where $\tau_1=t_1$ *and* $\tau_k=t_k-t_{k-1}$ *for* $2\leqq k\leqq N$.

The proof is straightforward. Indeed, we have only divide the expectation according as that $X(t_k)$, $k=1, 2, \cdots, N$ belong to $\mathfrak{Z}$ or not, and make use of Lemma A.1 and Corollary A.1. The details are omitted.

The assertion 2) of the following Corollary is a slight generalization of that in [**12**].

COROLLARY 3.1. *Let the assumption of Theorem* 3.1 *be satisfied.*

1) *If* s *is strictly increasing in* $x\in Q(s)$, *then* $X(t; x)$, $t\in[0, e_\Delta)$, *is right continuous and has left limit.*

2) *If* s *and* m *are strictly increasing in* $x\in Q(s)$, *then* $X(t; x)$ *is continuous in* $t\in[0, e_\Delta)$.

PROOF. 1) Assuming that s is strictly increasing in $x\in Q(s)$, we have $s(Q)\subset\tilde{Q}$ and e_Δ is the first leaving time of $Y(t)$ from $\tilde{Q}$. Further, it holds that $J(s^{-1})\cap\tilde{Q}=\emptyset$, so that $\mathfrak{Z}=\emptyset$ a.s. Since s^{-1} is continuous on $\tilde{Q}$ and $Y(t)$, $t\in[0, e_\Delta)$ is right continuous and has left limit, so is and does $X(t; x)=s^{-1}(Y(t))$.

2) In this case, $\tilde{m}$ is strictly increasing in $\xi\in\tilde{Q}\cap\mathrm{Spt}(s^{-1})$. Noticing the relation $s^{-1}(\xi)=x$, $\xi\in[s(x-), s(x+)]$ for this case, we see that $X(t; x)=s^{-1}(Y(t))$ is continuous. q.e.d.

The following example shows that the assertions in Corollary 3.1 fail without the assumptions.

EXAMPLE 3.1. Let $s(x)=x$ for $x\leqq 0$, $=0$ for $0<x\leqq 1$, $=x-1$ for $1\leqq x$, and $m(x)=2x$ for all $x\in\bar{\boldsymbol{R}}$. It then follows that $\tilde{m}(\xi)=2\xi$, for $\xi<0$, $=2(1+\xi)$, for $\xi\geqq 0$, and so $Y(t)=B(\mathfrak{f}^{-1}(t))+s(x)$, where $\mathfrak{f}(u)=2L(u, 0)+\int_{\boldsymbol{R}\setminus\{0\}} L(u, \xi)2d\xi=2L(u, 0)+u$. Further, $J(s^{-1})=\{0\}$, $e_{\Delta}=+\infty$, and $\mathfrak{Z}=\{t\geqq 0: Y(t)=0\}$. Since $\mathfrak{f}$ is a homeomorphism on $\boldsymbol{R}$, we then have $\mathfrak{Z}=\mathfrak{f}(\mathfrak{Z}(B))$, where $\mathfrak{Z}(B)=\{t\geqq 0: B(t)+s(x)=0\}$. Similarly, letting $\mathfrak{Z}_{\pm}=\{t\geqq 0: \pm Y(t)>0\}$ and $\mathfrak{Z}_{\pm}(B)=\{t\geqq 0: \pm(B(t)+s(x))>0\}$, we also have $\mathfrak{Z}_{\pm}=\mathfrak{f}(\mathfrak{Z}_{\pm}(B))$.

On the other hand, it is well known that, for each $\varepsilon>0$,

$$\begin{aligned}&\#(\mathfrak{Z}_{+}(B)\cap(\sigma_0(B+s(x)), \sigma_0(B+s(x))+\varepsilon))\\ &\quad = \#(\mathfrak{Z}_{-}(B)\cap(\sigma_0(B+s(x)), \sigma_0(B+s(x))+\varepsilon)) = +\infty, \quad \text{a. s.}\end{aligned}$$

Let σ_{ξ} be the first hitting time of $Y(t)$ for the state ξ. Then $\sigma_0=\mathfrak{f}(\sigma_0(B+s(x)))$, and it follows that

$$\#(\mathfrak{Z}_{+}\cap(\sigma_0, \sigma_0+\varepsilon)) = \#(\mathfrak{Z}_{-}\cap(\sigma_0, \sigma_0+\varepsilon)) = +\infty, \quad \text{a. s.}$$

Noting that $X(t; x)\geqq 1$, for $t\in\mathfrak{Z}_{+}$, and $X(t; x)<0$, for $t\in\mathfrak{Z}_{-}$, we see that the variation of $X(t; x)$ on $(\sigma_0, \sigma_0+\varepsilon)$ is infinite with probability 1. Thus it neither is right continuous nor has left limit.

We next show that Ikeda's example given in [**9**; § 5.8] is already a typical example of our BGDPs. It covers all kinds of motions of Markov processes with local property, which behaves by the law of one-dimensional Brownian motion off the origin and violates the strong Markov property at the origin.

EXAMPLE 3.2 (Ikeda's example). Let $s(x)=x$ for $x<0$, $=p$ for $x=0$, $=x+1$ for $0<x$, and $m(x)=2x$ for $x<0$, $=2x+q$ for $x\geqq 0$, where p and q are constants such that $0\leqq p\leqq 1$ and $q>0$. It then follows that $\tilde{m}(\xi)=2\xi$ for $\xi<0$, $=0$ for $0\leqq\xi<p$, $=q$ for $p\leqq\xi<1$, $=2(\xi-1)+q$ for $\xi\geqq 1$. Hence, $Y(t)=B(\mathfrak{f}^{-1}(t))+s(x)$, where $\mathfrak{f}(u)=2\int_{\boldsymbol{R}\setminus(0,1)} L(u, \xi)d\xi+qL(u, p)$, and $X(t; x)=s^{-1}\circ Y(t)$. Notice that $Y(t)$ is a GDP on $(\boldsymbol{R}\setminus(0, 1))\cup\{p\}$ and it is continuous at t for which $t\in\boldsymbol{R}\setminus[0, 1]$. Further the operation s^{-1} identifies the points 0, p and 1, so that the sample path $X(t; x)$ is continuous (see Corollary 3.1). The proof of violating strong Markov property is very similar to that in [**9**; § 5.8]. Indeed, one can easily check that the value $E\exp(-\sigma_{0+}(X))$ is different from 0 and 1, if the process starts at $x=0$, where $\sigma_x(X)$ is the first hitting time of the process X for the state x and $\sigma_{0+}(X)=\lim_{x\downarrow 0}\sigma_x(X)$.

The relation (2.3) in this case is reduced to

$$u(x) = u(0)+u^{+}(0)(x-p)-(x-p)q\mathcal{G}u(0)-\int_x^0(x-y)\mathcal{G}u(y)2dy, \quad x<0,$$

$$u(x)=u(0)+u^{+}(0)(x+1-p)+\int_0^x (x-y)\mathcal{G}u(y)2dy, \quad x>0,$$

where we choose $a=0$ and denote $\mathcal{G}=\mathcal{G}_{s,m}$. Hence

$$\begin{aligned}\mathcal{D}(\mathcal{G}) = \{u : u \text{ is } C^2 \text{ on } \boldsymbol{R}\setminus\{0\}, &\text{ has the limits } u(0\pm),\ u'(0\pm) \text{ and}\\ &\text{satisfies } (1-p)u'(0+)=u(0+)-u(0),\ pu'(0-)=u(0)-u(0-)\},\\ \mathcal{G}u(x) = d^2u(x)/2dx^2, &\quad \text{for } x\neq 0,\\ \mathcal{G}u(0) = q^{-1}\{u'(0+)-u'(0-)\}.&\end{aligned}$$

The behavior of sample paths is as follows. Suppose first that $p(1-p)\neq 0$. Then, the generator at $x=0$ is reduced to

$$\mathcal{G}u(0)=\{qp(1-p)\}^{-1}\{pu(0+)+(1-p)u(0-)-u(0)\}.$$

Thus the sample path starting at $x\neq 0$ behaves as a Brownian motion (reflected at 0) until the time $\mathfrak{f}(\sigma_p(B+s(x)))$. From that time it stays at 0 for exponential random time with parameter $1/qp(1-p)$, and after the stay it behaves as a reflecting barrier Brownian motion on $[0,+\infty)$ or as that on $(-\infty,0]$ starting at 0 with probabilities p and $1-p$ respectively for a random time $\mathfrak{f}(\sigma_p(B+1))$ and $\mathfrak{f}(\sigma_p(B))$ respectively. It then stays for exponential random time again and repeats the above procedure.

Suppose next that $p=0$. Then, the generator at 0 is reduced to $\mathcal{G}u(0)=q^{-1}\{u(0+)-u'(0-)-u(0)\}$. Thus a sample path starting at $x>0$ behaves as a Brownian motion (reflected at 0) until the time $\mathfrak{f}(\sigma_0(B+x+1))$. From that time it behaves as $B(\mathfrak{f}^{-1}(\cdot))$ for a random time $\mathfrak{f}(\sigma_1(B))$ (notice that $B(\mathfrak{f}^{-1}(\cdot))$ restricted on $(-\infty,-\varepsilon]$ behaves as a Brownian motion restricted on $(-\infty,-\varepsilon]$ for each $\varepsilon>0$). It then behaves again as a Brownian motion (on $[0,+\infty)$ reflected at 0) starting at 0 until the time $\mathfrak{f}(\sigma_0(B+1))$ and so on.

The behavior for the case of $p=1$ is just the symmetry.

We finally note that the precise example given in [**9**; §5.8] is obtained by putting $p=0$ and reforming m so that $m(x)=0$ for $x<0$.

4. BGDP and Stieltjes moment problem.

In their paper [**11**], Karlin and McGregor revealed the close relation between birth and death processes (B&D processes for brief) and the classical Stieltjes moment problems. Since B&D processes are regarded as GDPs as Feller pointed out in [**5**], moment problems come upon our stage, and our generalization to BGDPs makes it possible for us to generalize the results in [**11**]. Actually, N. Ikeda already done this by the truncation method in his private note of about 30 years ago.

Let X be a B&D process with the transition matrix

$$A=\begin{bmatrix} -\beta_0 & \beta_0 & 0 & \cdots\cdots & & \\ \delta_1 & -(\delta_1+\beta_1) & \beta_1 & 0 & \cdots\cdots & \\ 0 & \delta_2 & -(\delta_2+\beta_2) & \beta_2 & 0 & \cdots \\ \cdots\cdots & & & & & \end{bmatrix}$$

where $\beta_0, \beta_1, \cdots$ and $\delta_1, \delta_2, \cdots$ are positive constants (the boundary condition at $+\infty$, if necessary, will be given later). Let also

$$\mu_0=1, \quad \mu_n=\beta_0\beta_1\cdots\beta_{n-1}/\delta_1\delta_2\cdots\delta_n, \qquad \text{for } n\geqq 1,$$
$$\rho_0=1/\beta_0, \quad \rho_n=\delta_1\delta_2\cdots\delta_n/\beta_0\beta_1\cdots\beta_n, \qquad \text{for } n\geqq 1.$$

As is noted in [**11**], there corresponds the following moment problem. Given the recurrence relations

$$-\lambda Q_0(\lambda)=-\beta_0 Q_0(\lambda)+\beta_0 Q_1(\lambda),$$
$$-\lambda Q_n(\lambda)=\delta_n Q_{n-1}(\lambda)-(\delta_n+\beta_n)Q_n(\lambda)+\beta_n Q_{n+1}(\lambda), \qquad \text{for } n\geqq 1,$$

together with the normalizing condition $Q_0(\lambda)\equiv 1$, we have a unique solution $\{Q_n\}_{n=0}^{\infty}$. The solution Q_n is a polynomial in λ of order n, and the Stieltjes moment problem for our case is to find a nonnegative measure Ψ on $\mathcal{B}([0, +\infty))$ such that

$$\int_{[0,+\infty)} Q_i(\lambda)Q_j(\lambda)\Psi(d\lambda)=\delta_{ij}/\mu_i, \qquad i, j=0, 1, 2, \cdots,$$

where δ_{ij} is Kronecker's delta.

We assume $\sum_{i=0}^{\infty}\rho_i<+\infty$ and $m_\infty:=\sum_{i=0}^{\infty}\mu_i<+\infty$. This assumption is equivalent to that the Stieltjes moment problem has more than one solutions Ψ ([**11**]). Further, in this case, we have the limits $Q_\infty(\lambda)=\lim_{n\to\infty}Q_n(\lambda)$ and $H_\infty(\lambda)=\lim_{n\to\infty}H_n(\lambda)$, where

$$H_0(\lambda)\equiv 0 \quad \text{and} \quad H_n(\lambda)=(Q_{n+1}(\lambda)-Q_n(\lambda))/\rho_n \qquad \text{for } n\geqq 1$$

(*ibid.*). In order to specify Ψ uniquely, we introduce an additional condition that the support of Ψ is included in the set of solutions of the equation

$$(a-b\lambda)Q_\infty(\lambda)+(1-c\lambda)H_\infty(\lambda)=0, \tag{4.1}$$

where a, b and c are nonnegative constants satisfying $b-ac>0$. Soon later, it will be seen that all the solutions of (4.1) are nonnegative. Notice that the relation (4.1) corresponds to the (limit of) quasiorthogonal polynomials in the truncation method utilized in [**11**].

THEOREM 4.1. *The Stieltjes moment problem for* $\{Q_n\}_{n=0}^{\infty}$ *and* $\{\mu_n\}_{n=0}^{\infty}$ *has a unique solution* Ψ *which is supported on the set of solutions of* (4.1). *Further, the system of the functions* $\{Q_n(\lambda)\}_{n=0}^{\infty}\cup\{Q_\infty(\lambda)/(1-c\lambda)\}$ *is a complete orthogonal basis in the space* $L^2([0, +\infty); \Psi)$.

REMARK 4.1. We will show in the following that the class of the problems with this condition corresponds to that of B&D processes with local property, but I can not clarify its significance in the theory of moment problems. The class of the problems treated by Karlin and McGregor [11] is that for the case $c=0$, and correspond to strong Markov B&D processes. N. Ikeda treated the class for the general case $c\geqq 0$ (and $a=0$) long ago.

Assume first that $a>0$ and $c>0$, and let $p=b^2/(b-ac)$, $q=c/b$, $r=ab/(b-ac)$. Following [5], let also

$$x_0>0, \quad x_n=x_0+\rho_0+\rho_1+\cdots+\rho_{n-1} \quad \text{for } n\geqq 1, \quad x_\infty=\lim_{n\to\infty} x_n,$$

$s(x)=x$ for $x<x_\infty$, $=x_\infty+q$ for $x=x_\infty$, $=x_\infty+q+(1/r)$ for $x_\infty<x$, $m(x)=\sum_{x_i\leq x}\mu_i$ for $x<x_\infty$, $=m_\infty+p+x$ for $x_\infty\leqq x$. It then follows that the generator $\mathcal{G}=\mathcal{G}_{s,m}$ of the corresponding BGDP satisfies

$$\begin{aligned}
\mathcal{G}u(x_0)&=-\beta_0u(x_0)+\beta_0u(x_1),\\
\mathcal{G}u(x_n)&=\delta_nu(x_{n-1})-(\delta_n+\beta_n)u(x_n)+\beta_nu(x_{n+1}), \qquad \text{for } n\geqq 1,\\
p\mathcal{G}u(x_\infty)&=\{u(x_\infty-)-u(x_\infty)\}/q-ru(x_\infty)
\end{aligned}$$

(see [5] and the arguments in Example 3.2). Thus our BGDP X is a B&D process with the transition matrix A violating the strong Markov property at x_∞. Notice that the last formula amounts to setting boundary condition at x_∞. The transformed speed measure function $\tilde{m}$ is given by $\tilde{m}(\xi)=m(\xi)$ for $\xi<x_\infty$, $=m_\infty$ for $x_\infty\leqq\xi<x_\infty+q$, $=m_\infty+p$ for $x_\infty+q\leqq\xi<x_\infty+q+(1/r)$, $=+\infty$ for $\xi\geqq x_\infty+q+(1/r)$. In view of the shape of $\tilde{m}$, the GDP Y corresponds the one on the interval $(0, x_\infty+q+(1/r))$ with the reflecting boundary condition at 0 and the absorbing boundary condition at $x_\infty+q+(1/r)$. Hence, due to the general theory for GDPs, we have the eigenfunction expansion

$$q(t,\xi,\eta)=\sum_{\nu=0}^{\infty}\exp\{-\lambda_\nu t\}\psi(\xi;-\lambda_\nu)\psi(\eta;-\lambda_\nu)\sigma_\nu, \qquad \xi,\eta\in\boldsymbol{R},\ t>0,$$

where $\psi(\xi;\alpha)$ is a solution of the equation

$$\psi(\xi)=1+\alpha\int_{(0,\xi]}(\xi-\eta)\psi(\eta)d\tilde{m}(\eta), \qquad \xi\in\boldsymbol{R}, \tag{4.2}$$

$\sigma_\nu=\left\{\int_{\boldsymbol{R}}|\psi(\xi;-\lambda_\nu)|^2d\tilde{m}(\xi)\right\}^{-1}$ and $0\leqq\lambda_0<\lambda_1\leqq\lambda_2\leqq\cdots$ are the solution of the equation

$$-\lambda p\psi(x_\infty+q;-\lambda)+\{\psi(x_\infty+q;-\lambda)-\psi(x_\infty;-\lambda)\}/q+r\psi(x_\infty+q;-\lambda)=0. \tag{4.3}$$

The transition density $p(t,x,y)$ and the Green function $G(\alpha,x,y)$ of the B&D process X is now given by

$$p(t,x,y)=\sum_{\nu=0}^{\infty}\exp\{-\lambda_\nu t\}\varphi(x;-\lambda_\nu)\varphi(y;-\lambda_\nu)\sigma_\nu, \qquad x,y\in(-\infty,x_\infty),\ t>0,$$

$$G(\alpha, x, y) = \sum_{\nu=0}^{\infty}(\lambda_\nu+\alpha)^{-1}\varphi(x\,;\,-\lambda_\nu)\varphi(y\,;\,-\lambda_\nu)\sigma_\nu+\Phi(s(x),\, s(y)),$$
$$x,\ y\in(-\infty,\ x_\infty),\ \alpha>0,$$

where $\varphi(x\,;\alpha)=\phi(s(x)\,;\alpha)$, especially, $\varphi(x_n\,;-\lambda)=\phi(x_n\,;-\lambda)$ for $n=0,1,2,\cdots$ and $\varphi(x_\infty\,;-\lambda)=\phi(x_\infty+q\,;-\lambda)$. Due to the general theory for GDPs, we also have

$$G(\alpha, 0, 0) = x_0+\sum_{\nu=0}^{\infty}\frac{\sigma_\nu}{\lambda_\nu+\alpha}=x_0+\sum_{n=1}^{\infty}\frac{\rho_{n-1}}{\varphi(x_{n-1}\,;\,\alpha)\varphi(x_n\,;\,\alpha)}$$
$$+\frac{q(1+q(p\alpha+r))}{\varphi(x_\infty-\,;\,\alpha)\{\varphi(x_\infty\,;\,\alpha)(1+q(p\alpha+r))-\varphi(x_\infty-\,;\,\alpha)\}},\quad \alpha>0. \tag{4.4}$$

We finally note that all the above arguments are valid also for the case of $c=0$ with the convention of $\{u(x_\infty)-u(x_\infty-)\}/q=u^+(x_\infty-)$ and $\{\phi(x_\infty+q\,;-\lambda)-\phi(x_\infty\,;-\lambda)\}/q=D_\xi^+\phi(x_\infty-\,;-\lambda)$.

PROOF OF THEOREM 4.1. Assume first that $a>0$. It then follows from (4.2) and the definition of φ that

$$\varphi(x_0\,;\,-\lambda)=1,\qquad -\lambda\varphi(x_0\,;\,-\lambda)=-\beta_0\varphi(x_0\,;\,-\lambda)+\beta_0\varphi(x_1\,;\,-\lambda),$$
$$-\lambda\varphi(x_n\,;\,-\lambda)=\delta_n\varphi(x_{n-1}\,;\,-\lambda)-(\delta_n+\beta_n)\varphi(x_n\,;\,-\lambda)+\beta_n\varphi(x_{n+1}\,;\,-\lambda)$$
$$\text{for } n\geqq 1.$$

Further, (4.3) is transformed to

$$-\lambda p\varphi(x_\infty\,;\,-\lambda)+\varphi^+(x_\infty-\,;\,-\lambda)+r\varphi(x_\infty\,;\,-\lambda)=0. \tag{4.3$'$}$$

Thus it follows that $Q_n(\lambda)=\varphi(x_n\,;-\lambda)$, $n=0,1,2,\cdots$, $Q_\infty(\lambda)=\varphi(x_\infty-\,;-\lambda)$, $H_\infty(\lambda)=\varphi^+(x_\infty-\,;-\lambda)$, and $\varphi(x_\infty\,;-\lambda)=Q_\infty(\lambda)+qH_\infty(\lambda)$ by (4.2). Hence the relation (4.3)$'$ is reduced to (4.1). On the other hand, (4.3)$'$ in turn is rewritten as $\varphi(x_\infty\,;-\lambda)=Q_\infty(\lambda)(b-ac)/b(1-c\lambda)$. The solution measure Ψ of the moment problem is now given by $\Psi(E)=\sum_{\nu=0}^{\infty}\sigma_\nu 1_E(\lambda_\nu)$, $E\in\mathcal{B}(\boldsymbol{R})$, where $\sigma_\nu=\{\sum_{n=0}^{\infty}|Q_n(\lambda_\nu)|^2\mu_n+p\,|Q_\infty(\lambda_\nu)(b-ac)/b(1-c\lambda_\nu)|^2\}^{-1}$. It is also easy to see that the system of the functions $\{Q_n(\lambda)\}_{n=0}^{\infty}\cup\{Q_\infty(\lambda)/(1-c\lambda)\}$ is a complete orthogonal basis in the space $L^2([0,+\infty)\,;\Psi)$. Finally, (4.4) is rewritten as

$$\sum_{\nu=0}^{\infty}\frac{\sigma_\nu}{\lambda_\nu+\alpha}=\sum_{n=1}^{\infty}\frac{\rho_{n-1}}{Q_{n-1}(-\alpha)Q_n(-\alpha)}$$
$$+\frac{1+c\alpha}{Q_\infty(-\alpha)\{(a+b\alpha)Q_\infty(-\alpha)+(1+c\alpha)H_\infty(-\alpha)\}}, \tag{4.4$'$}$$

for all $\alpha>0$, which proves the uniqueness of Ψ.

The result for the case of $a=0$ is obtained from the above by letting $a\downarrow 0$.

q. e. d.

5. Limit theorem for a sequence of BGDPs I.

In this section, we will give a vague convergence theorem of finite dimensional distributions for a sequence of BGDPs. The space of the test functions in this section is $C_0(Q)$, the space of all continuous functions on Q with compact support. In the following, we let $(s_n, m_n) \in \mathcal{M}\times\mathcal{M}_+$, $n=1, 2, \cdots$, $(s, m) \in \mathcal{M}\times\mathcal{M}_+$ and denote the corresponding semigroups by $T_t^{(n)}$. The associate measures and the transition densities are denoted by $\tilde{m}_n$ and $q_n(t, \xi, \eta)$ respectively. Further, we assume that $a \in Q\setminus(J(s)\cup J(m))$ and $\tilde{a} \in \tilde{Q}\setminus J(\tilde{m})$.

THEOREM 5.1. *Assume that* $J(s)\cap J(m)\cap Q=\emptyset$,

$$(5.1)\qquad \lim_{n\to\infty} s_n(x) = s(x), \quad x\in \boldsymbol{R}\setminus J(s), \qquad \lim_{n\to\infty} m_n(x) = m(x), \quad x \in \boldsymbol{R}\setminus J(m).$$

Then, for every $t_1, t_2, \cdots, t_N>0$ *and* $f_1, f_2, \cdots, f_N \in C_0(Q)$, *it holds that*

$$(5.2)\qquad \lim_{n\to\infty} T_{t_1}^{(n)}(f_1 T_{t_2}^{(n)}(\cdots(f_{N-1}T_{t_N}^{(n)}f_N)\cdots))(x) = T_{t_1}(f_1 T_{t_2}(\cdots(f_{N-1}T_{t_N}f_N)\cdots))(x),$$

for all $x\in Q$ *with* $\lim_{n\to\infty} s_n(x)=s(x)$.

Notice that the condition (5.1) implies $\overline{\lim}_{n\to\infty}\tilde{l}_1^{(n)}\leqq \tilde{l}_1$ and $\underline{\lim}_{n\to\infty}\tilde{l}_2^{(n)}\geqq\tilde{l}_2$, where $\tilde{l}_i^{(n)}=l_i(\tilde{m}_n)$. Actually, we have more;

LEMMA 5.1. *The assumptions of Theorem* 5.1 *imply*

$$(5.3)\qquad \lim_{n\to\infty}\tilde{m}_n(\xi) = \tilde{m}(\xi), \qquad \xi\in\boldsymbol{R}\setminus J(\tilde{m}).$$

PROOF. We can first show that

$$\lim_{n\to\infty} s_n^{-1}(\xi) = s^{-1}(\xi), \qquad \xi \in Q(s^{-1})\setminus J(s^{-1}),$$

and then the formula (5.3) for $\xi\in Q(s^{-1})\setminus J(s^{-1})$ with $s^{-1}(\xi)\in Q(m)\setminus J(m)$. This with the monotone non-decreasing property of $\tilde{m}_n$ and $\tilde{m}$ proves (5.3). The details are omitted. q. e. d.

PROPOSITION 5.1. *Suppose that* (5.3) *holds. Then the transition density* $q_n(t, \xi, \eta)$ *converges to* $q(t, \xi, \eta)$ *uniformly in the wide sense in* $(t, \xi, \eta)\in(0, +\infty)\times\tilde{Q}\times\tilde{Q}$ *as* $n\to\infty$.

We will be concerned with the proof of Proposition 5.1 for a while. We first supply a little more facts on GDPs than those in Section 2. Let $\psi_i(\xi, \alpha)$, $i=1, 2$, $\xi\in\tilde{Q}$, $\alpha\in\boldsymbol{C}$ be the solutions of the integral equations

$$\psi_1(\xi, \alpha) = 1+\alpha\int_{(\tilde{a}, \xi]}(\xi-\eta)\psi_1(\eta, \alpha)d\tilde{m}(\eta),$$

$$\psi_2(\xi, \alpha) = \xi-\tilde{a}+\alpha\int_{(\tilde{a}, \xi]}(\xi-\eta)\psi_2(\eta, \alpha)d\tilde{m}(\eta).$$

Then,

$$(5.4)\qquad \begin{aligned} |\psi_1(\xi, \alpha)| &\leqq \cosh\{(2|\alpha(\xi-\tilde{a})(\tilde{m}(\xi)-\tilde{m}(\tilde{a}))|)^{1/2}\}, \\ |\psi_2(\xi, \alpha)| &\leqq |\xi-\tilde{a}|\cosh\{(2|\alpha(\xi-\tilde{a})(\tilde{m}(\xi)-\tilde{m}(\tilde{a}))|)^{1/2}\}. \end{aligned}$$

and, for each $\alpha>0$, there exist the limits

$$h_1(\alpha) = -\lim_{\xi\downarrow l_1}\psi_2(\xi,\alpha)/\psi_1(\xi,\alpha), \qquad h_2(\alpha) = \lim_{\xi\uparrow l_2}\psi_2(\xi,\alpha)/\psi_1(\xi,\alpha). \tag{5.5}$$

In the above, we use the usual convention $1/+\infty=0$, $(\pm A)/0=\pm\infty$, $+\infty\pm A=+\infty$ and $-\infty\pm A=-\infty$ for a positive A. Actually, $h_i(\alpha)$, $i=1, 2$ are analytically continued to the domain $\boldsymbol{C}\setminus(-\infty, 0]$ and (5.5) is valid there. Define the functions $h(\alpha)$, $h_{ij}(\alpha)$, $i, j=1, 2$, $\alpha\in\boldsymbol{C}\setminus(-\infty, 0]$, by

$$\begin{aligned} 1/h(\alpha) &= 1/h_1(\alpha)+1/h_2(\alpha), \\ h_{11}(\alpha) &= h(\alpha), \qquad h_{22}(\alpha) = -(h_1(\alpha)+h_2(\alpha))^{-1}, \\ h_{12}(\alpha) &= h_{21}(\alpha) = -h(\alpha)/h_2(\alpha). \end{aligned}$$

Then the α-harmonic functions $v_i(x, \alpha)$ in (2.15) are given by

$$v_i(\xi,\alpha) = \psi_1(\xi,\alpha)+(-1)^{i+1}\psi_2(\xi,\alpha)/h_i(\alpha), \qquad i=1, 2.$$

The functions h_{ij}, $i, j=1, 2$ are also analytic in $\alpha\in\boldsymbol{C}\setminus(-\infty, 0]$ and we define the spectral measures $\sigma_{ij}(d\lambda)$, $i, j=1, 2$ by

$$\sigma_{ij}([\lambda_1, \lambda_2]) = \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2}\mathcal{I}m\, h_{ij}(-\lambda-\sqrt{-1}\,\varepsilon)d\lambda,$$

for all continuity points $\lambda_1<\lambda_2$ of σ_{ij}. The matrix valued measure $[\sigma_{ij}]_{i,j=1,2}$ is symmetric and nonnegative definite, and the transition density $q(t, \xi, \eta)$ and the Green function $H(\alpha, \xi, \eta)$ are given by the relations

$$q(t,\xi,\eta) = \sum_{i,j=1,2}\int_{[0,\infty)} e^{-\lambda t}\psi_i(\xi,-\lambda)\psi_j(\eta,-\lambda)\sigma_{ij}(d\lambda), \qquad t>0,\ \ \xi, \eta\in\tilde{Q}, \tag{5.6}$$

$$H(\alpha,\xi,\eta) = \Phi(\xi,\eta)+\sum_{i,j=1,2}\int_{[0,\infty)}(\lambda+\alpha)^{-1}\psi_i(\xi,-\lambda)\psi_j(\eta,-\lambda)\sigma_{ij}(d\lambda), \qquad \xi, \eta\in\tilde{Q}. \tag{5.7}$$

We then have from formulas (5.6) and (5.7) together with [**19**; Lemma 2] that

$$\begin{aligned} q(t,\xi,\eta) &\leqq t^{-1}H(1/t,\xi,\xi)^{1/2}H(1/t,\eta,\eta)^{1/2} \\ &\leqq t^{-1}(h(1/t)+|\xi-\tilde{a}|)^{1/2}(h(1/t)+|\eta-\tilde{a}|)^{1/2}, \qquad t>0,\ \ \xi, \eta\in\tilde{Q}. \end{aligned} \tag{5.8}$$

The corresponding items for $(s_n, m_n)\in\mathcal{M}\times\mathcal{M}_+$ are denoted as $H_n(\alpha, \xi, \eta)$, $\sigma_{ij}^{(n)}$ and so on.

We owe to S. Kotani the following proof of the assertion 5) of the next lemma, which is simpler than our original one.

LEMMA 5.2. *Suppose that* (5.3) *holds. Then, the following assertions hold.*

1) *For each* $i=1, 2$, *it holds that* $\lim_{n\to\infty}\psi_i^{(n)}(\xi,\alpha)=\psi_i(\xi,\alpha)$ *uniformly in the wide sense in* $\xi\in\tilde{Q}$ *and* $\alpha\in\boldsymbol{C}$.

2) *For each* $\alpha\in\boldsymbol{C}\setminus(-\infty, 0]$, $\lim_{n\to\infty}h_i^{(n)}(\alpha)=h_i(\alpha)$, $i=1, 2$.

3) *For each* $\alpha\in\boldsymbol{C}\setminus(-\infty, 0]$ *and* $i=1, 2$, *it holds that* $\lim_{n\to\infty}v_i^{(n)}(\xi,\alpha)=v_i(\xi,\alpha)$

uniformly in the wide sense in $\xi \in \tilde{Q}$.

4) *For each* $\alpha \in \boldsymbol{C} \setminus (-\infty, 0]$, *it holds that* $\lim_{n\to\infty} H_n(\alpha, \xi, \eta) = H(\alpha, \xi, \eta)$ *uniformly in the wide sense* $\xi, \eta \in \tilde{Q}$.

5) *For each* $t>0$ *and* $\xi, \eta \in \tilde{Q}$, $\lim_{n\to\infty} q_n(t, \xi, \eta) = q(t, \xi, \eta)$.

PROOF. The assertions through 1) to 4) are well known. To see 5), we note that, by virtue of (5.6), the matrix $Q(t, \xi, \eta) = \begin{pmatrix} q(t, \xi, \xi) & q(t, \xi, \eta) \\ q(t, \eta, \xi) & q(t, \eta, \eta) \end{pmatrix}$ is nonnegative definite and its derivative $\partial Q(t, \xi, \eta)/\partial t$ is nonpositive definite for all $t>0$ and $\xi, \eta \in \tilde{Q}$. Hence, the assertion 4) together with the continuity theorem of Laplace transformation for matrix valued functions proves 5). q. e. d.

LEMMA 5.3. *Suppose that* (5.3) *holds. Then, it holds that*

$$\lim_{n\to\infty} \int_{[0,\infty)} f(\lambda) \sigma_{ij}^{(n)}(d\lambda) = \int_{[0,\infty)} f(\lambda) \sigma_{ij}(d\lambda), \tag{5.9}$$

for all $f \in C([0, +\infty))$ *with* $\sup_{\lambda \in [0,+\infty)} |f(\lambda)|(1+\lambda^3) < +\infty$.

PROOF. We first note that

$$\begin{aligned} h(\alpha) &= h(+\infty) + \int_{[0,+\infty)} (\lambda+\alpha)^{-1} \sigma_{11}(d\lambda), \\ \{\alpha(h_1(\alpha)+h_2(\alpha))\}^{-1} &= \{\alpha(h_1(0+)+h_2(0+))\}^{-1} + \int_{(0,+\infty)} \{\lambda(\lambda+\alpha)\}^{-1} \sigma_{22}(d\lambda), \end{aligned} \tag{5.10}$$

for all $\alpha \in \boldsymbol{C} \setminus (-\infty, 0]$, and that $h_1(0+)+h_2(0+) = \tilde{l}_2 - \tilde{l}_1$ (see [**10**; pp. 13-14 and p. 18]). It then follows that $\sigma_{11}([0, \lambda]) \leqq h(1)(1+\lambda)$ and $\int_{(0,\lambda]} \lambda^{-1} \sigma_{22}(d\lambda) \leqq \{h_1(1) + h_2(1)\}^{-1}(1+\lambda)$. Further, it holds that $|\sigma_{12}|(E) = |\sigma_{21}|(E) \leqq (\sigma_{11}(E) + \sigma_{22}(E))/2$, $E \in \mathscr{B}([0, +\infty))$. Hence, for each subsequence of the sequence $\{(\sigma_{ij}^{(n)})_{i,j=1}^2\}_{n=1}^\infty$ of matrix valued signed measures, we can find its subsequence (denoted by the same symbol $\{(\sigma_{ij}^{(n)})_{i,j=1}^2\}_{n=1}^\infty$) and a matrix $(\sigma_{ij}^*)_{i,j=1}^2$ of signed measures such that $\lim_{n\to\infty} \sigma_{ij}^{(n)} = \sigma_{ij}^*$, $i, j = 1, 2$, vaguely.

On the other hand, (5.10) also implies the inequalities $\int_{[K,+\infty)} (1+\lambda^3)^{-1} \sigma_{11}(d\lambda) \leqq h(1)(1+K)/(1+K^3)$ and $\int_{[K,+\infty)} (1+\lambda^3)^{-1} \sigma_{22}(d\lambda) \leqq \{h_1(1) + h_2(1)\}^{-1}(1+K)/(1+K^2)$ for $K \geqq 1$. Hence, putting

$$\begin{aligned} \mathfrak{e}_n(\xi, \eta, d\lambda) &= \sum_{i,j=1}^2 \phi_i^{(n)}(\xi, -\lambda) \phi_i^{(n)}(\eta, -\lambda) \sigma_{ij}^{(n)}(d\lambda), \\ \mathfrak{e}^*(\xi, \eta, d\lambda) &= \sum_{i,j=1}^2 \phi_i(\xi, -\lambda) \phi_i(\eta, -\lambda) \sigma_{ij}^*(d\lambda), \\ \mathfrak{e}(\xi, \eta, d\lambda) &= \sum_{i,j=1}^2 \phi_i(\xi, -\lambda) \phi_i(\eta, -\lambda) \sigma_{ij}(d\lambda), \end{aligned}$$

we have

$$\lim_{n\to\infty} \int_{[0,+\infty)} e^{-\lambda t} \mathfrak{e}_n(\xi, \eta, d\lambda) = \int_{[0,+\infty)} e^{-\lambda t} \mathfrak{e}^*(\xi, \eta, d\lambda),$$

with the aid of (5.4) and Lemma 5.2. Combining this with (5.6) and Lemma 5.2, we obtain

$$\int_{[0,+\infty)} e^{-\lambda t}\mathrm{e}^*(\xi, \eta, d\lambda) = \int_{[0,+\infty)} e^{-\lambda t}\mathrm{e}(\xi, \eta, d\lambda), \qquad t>0, \ \ \xi, \eta \in \tilde{Q}.$$

This implies $\sigma_{ij}^*=\sigma_{ij}$, proving the assertion of Lemma. q. e. d.

The assertion of Proposition 5.1 is now clear from Lemma 5.3 together with (5.4) and (5.6).

Due to Convention just before Lemma 2.1, we also have

LEMMA 5.4. *Suppose that* (5.3) *holds and that* $\tilde{l}_i$ *is finite. Then, for each* $t>0$ *and compact set* $K\subset\tilde{Q}$ *it holds that*

$$(5.11) \qquad \lim_{\delta\downarrow 0}\varlimsup_{n\to\infty}\sup_{\xi\in K,\,\eta\in[\tilde{l}_i-\delta,\,\tilde{l}_i+\delta]} q_n(t, \xi, \eta) = 0.$$

PROOF. We first note that the boundary $\tilde{l}_i$ is not entrance by our conventions for GDP. Hence it holds that $v_i(\tilde{l}_i, \alpha)=0$.

Let $i=2$ for simplicity and $\alpha>0$. Due to the inequality $v_2(\xi_1, \alpha)-v_2(\xi_2, \alpha) \leqq(-D_\xi^+ v_2(\xi_1, \alpha))(\xi_2-\xi_1)$, $\tilde{l}_1<\xi_1\leqq\xi_2\leqq\tilde{l}_2$, we then have $H(\alpha, \xi, \xi)\leqq|\tilde{l}_2-\xi|$. Take an $\varepsilon>0$ and a $\xi_0\in(\tilde{l}_2-\varepsilon, \tilde{l}_2)$. It then follows from Lemma 5.2 that $\varlimsup_{n\to\infty}H_n(\alpha, \xi_0, \xi_0)\leqq\varepsilon$, and from the relation $|H_n(\alpha, \xi, \xi)-H_n(\alpha, \eta, \eta)|\leqq|\xi-\eta|$ that $\varlimsup_{n\to\infty}\sup_{\eta\in[\tilde{l}_2-\varepsilon,\tilde{l}_2+\varepsilon]}H_n(\alpha, \eta, \eta)\leqq 3\varepsilon$. This implies $\lim_{\delta\downarrow 0}\varlimsup_{n\to\infty}\sup_{\eta\in[\tilde{l}_2-\delta,\tilde{l}_2+\delta]} H_n(\alpha, \eta, \eta)=0$. We now have (5.11) from (5.8). q. e. d.

PROOF OF THEOREM 5.1. We will only show that, for each $t>0$ and $f\in C_0(Q)$,

$$(5.12) \qquad \lim_{n\to\infty} T_t^{(n)}f(x) = T_t f(x) \qquad \text{for all} \quad x\in Q \text{ with } \lim_{n\to\infty} s_n(x)=s(x).$$

Case 1. If $f\in C_0(Q)$ and $s(\mathrm{Spt}(f))\subset\tilde{Q}$, then (5.12) is a direct consequence of Proposition 5.1 and our assumption $J(s)\cap J(m)\cap Q=\emptyset$.

Case 2. Suppose that $\tilde{l}_i\in s(\mathrm{Spt}(f))$. As is noted in Section 2, this can occur only when $(2.4)_i$ holds, and $\tilde{l}_i$ is finite and $\varliminf_{n\to\infty}|\tilde{l}_i^{(n)}-\tilde{a}|\geqq|\tilde{l}_i-\tilde{a}|$. Hence we obtain (5.12) from Lemma 5.4 and the argument in Case 1. q. e. d.

6. Limit theorem for a sequence of BGDPs II.

Theorem 5.1 in the previous section does not assure the convergence of (finite dimensional) distribution functions, since we have assumed $f_k\in C_0(Q)$. In this section, we will discuss on this subject. For simplicity we only deal with the intervals (b, l_2) for $b\in(l_1, l_2)$ and denote the law of $(X(\cdot, x), P)$ by $(X(\cdot), P_x)$. As before, we fix an $a\in(l_1, l_2)\setminus(J(s)\cup J(m))$.

THEOREM 6.1. *In addition to the assumptions of Theorem* 5.1, *assume that*

(6.1) $\int_{(a_0, l_2)}(m(x)-m(a_0))ds(x)=+\infty$ *for some* $a_0 \in Q$ *or* $l_2(s)<l_2(m)$.

Then it holds that

(6.2) $$\lim_{n\to\infty} P_x^{(n)}(X(t_1)>a_1, \cdots, X(t_N)>a_N, t_N<e_\Delta) = P_x(X(t_1)>a_1, \cdots, X(t_N)>a_N, t_N<e_\Delta),$$

for every $0<t_1<t_2<\cdots<t_N$, $a_1, a_2, \cdots, a_N \in Q \setminus J(m)$ *and* $x\in Q$ *with* $\lim_{n\to\infty} s_n(x)=s(x)$.

Before proceeding to the proof of Theorem 6.1, we prepare two formulas and introduce a dual process. It is well known that, for a fixed $c\in[\tilde{l}_1, \tilde{l}_2]$, there exists a continuous nonnegative function $q_c(t, \xi)$ in $(0, +\infty)\times(c, \tilde{l}_2)$ such that

$$v_2(\xi, \alpha)/v_2(c, \alpha)=\Psi_c(\xi)+\int_0^{+\infty} e^{-\alpha t} q_c(t, \xi)dt, \qquad \alpha>0, \ \xi\in(c, \tilde{l}_2),$$

where $\Psi_c(\xi)$ is the correction function given in [**19**; (3.20)]. We then have

(6.3) $$q(t, \xi, \eta)=\int_0^t q(t-\tau, \xi, c)q_c(\tau, \eta)d\tau+q(t, \xi, c)\Psi_c(\eta)+\Phi(\xi, c)q_c(t, \eta), \qquad \xi<c<\eta$$

(see [**19**; (3.21)]). The function $q_c(t, \xi)$ in $(0, +\infty)\times(\tilde{l}_1, c)$ is also defined in the similar way with the function $v_2(\xi, \alpha)$ replaced by $v_1(\xi, \alpha)$, and it satisfies the corresponding formula to (6.3). Let next $h_{1,c}^{\tau}(\alpha)=-\psi_2(c, \alpha)/\psi_1(c, \alpha)$, $h_{2,c}^{\tau}(\alpha)=h_2(\alpha)$ and the corresponding items defined in Sections 2 and 5 by $v_{i,c}(\xi, \alpha)$, $q_c(t, \xi, \eta)$, $H_c(\alpha, \xi, \eta)$, where the base point is taken as $\tilde{a}\in(c, \tilde{l}_2)\setminus J(\tilde{m})$. We then have

$$H(\alpha, \xi, \eta)=H_c(\alpha, \xi, \eta)+h(\alpha)v_1(c, \alpha)v_2(\xi, \alpha)v_2(\eta, \alpha)/v_2(c, \alpha), \qquad \alpha>0, \ \xi, \eta\in(c, \tilde{l}_2),$$

which implies

(6.4) $$q(t, \xi, \eta)=q_c(t, \xi, \eta)+\int_0^t q(t-\tau, \xi, c)q_c(\tau, \eta)d\tau+q(t, \xi, c)\Psi_c(\eta) +\Phi(\xi, c)q_c(t, \eta), \qquad \xi, \eta\in(c, \tilde{l}_2).$$

The corresponding formula for $\xi, \eta\in(\tilde{l}_1, c)$ is similar.

Let next $\tilde{m}_c(\xi)=\tilde{m}(\xi)$ for $\xi>c$ and $=-\infty$ for $\xi\leqq c$, and $q_c^*(t, \xi, \eta)$ be the transition density function for the GDP corresponding to the speed measure $d\xi$ and the scale function $\tilde{m}_c$. Then it is easy to see that

(6.5) $$-D_\xi^+ v_{1,c}(\xi, \alpha)D_\xi^+ v_2(\eta, \alpha)/\alpha D_\xi^+ v_{1,c}(c, \alpha)v_2(c, \alpha) =\int_0^\infty e^{-\alpha t} q_c^*(t, \xi, \eta)dt=\int_0^\infty e^{-\alpha t} q_c^*(t, \eta, \xi)dt, \qquad \alpha>0, \ c<\xi\leqq\eta<\tilde{l}_2.$$

PROOF OF THEOREM 6.1. We will only show that, for each $t>0$, $a_1\in Q\setminus J(m)$ and $x\in Q$ with $\lim_{n\to\infty}s_n(x)=s(x)$, $\lim_{n\to\infty}P_x^{(n)}(X(t)>a_1,\ t<e_\Delta)=P_x(X(t)>a_1,\ t<e_\Delta)$. By Theorem 5.1, it is enough to show that

$$\lim_{b\uparrow l_2}\overline{\lim_{n\to\infty}}P_x^{(n)}(X(t)>b,\ t<e_\Delta)=0. \tag{6.6}$$

Notice first that $\underline{\lim}_{n\to\infty}l_2(s_n)\geqq l_2$ by the assumptions. Also, for simplicity, set $u_i(y,\alpha)=\lim_{z\uparrow l_2}u_i(z,\alpha)$, $u_i^+(y,\alpha)=\lim_{z\uparrow l_2}u_i^+(z,\alpha)$ for all $y\in[l_2,+\infty]$ and $i=1, 2$ (admitting the possibility that they take values $\pm\infty$). It then follows that $u_2^+(l_2,\alpha)=0$, since either $(2.5)_2$ holds or the boundary l_2 is non-exit in Feller's classification (cf. [9; §4.6]). Hence we have $\lim_{b\uparrow l_2}\overline{\lim}_{n\to\infty}|u_2^{(n)+}(b,\alpha)|=0$, by the same reason as for Lemma 5.2. This with the relation

$$u_2^+(l_2,\alpha)-u_2^+(b,\alpha)=\alpha\int_{(b,l_2)}u_2(y,\alpha)dm(y),\qquad b<l_2, \tag{6.7}$$

implies

$$\lim_{b\uparrow l_2}\overline{\lim_{n\to\infty}}\int_{(b,l_2^{(n)})}v_2^{(n)}(s_n(y),\alpha)dm_n(y)=0. \tag{6.8}$$

Suppose first that $\tilde{m}((\xi,\tilde{l}_2))>0$ for each $\xi\in\tilde{Q}$. Then, due to Lemma 5.1, we can find a $c\in(s(x),\tilde{l}_2)$ and an $n_1\in\boldsymbol{N}$ such that the set $(s_n(x),c)\cap\mathrm{Spt}(\tilde{m}_n)$ contains more than two points and $c<\tilde{l}_2^{(n)}$ for all $n\geqq n_1$. On the other hand, (6.3) implies

$$T_t1_{(b,l_2)}(x)=\int_0^t q(t-\tau,s(x),c)\int_{(b,l_2)}q_c(\tau,s(y))dm(y)d\tau,\qquad c<s(b)\leqq\tilde{l}_2.$$

Combining now the arguments in [18; p. 538] with [19; Lemma 4], we obtain $\sup_{n\geqq n_2,0\leqq\tau\leqq t}q_n(\tau,s_n(x),c)<+\infty$ for some $n_2\in\boldsymbol{N}$. This with (6.8) proves (6.6).

Suppose next that $\tilde{m}((\xi_0,\tilde{l}_2))=0$ for some $\xi_0\in\tilde{Q}$. In this case, we have $\tilde{l}_2=+\infty$ by (6.1). Hence, we can find a $c>\xi_0$ such that $s_n(x)<c<\tilde{l}_2^{(n)}$ for all sufficiently large n. Let now $A_n=s_n^{-1}((-\infty,c])$ and make the decomposition

$$\begin{aligned}T_t^{(n)}1_{(b,l_2^{(n)})}(x)&=\int_{(b,l_2^{(n)})\cap A_n}q_n(t,s_n(x),s_n(y))dm_n(y)\\&\quad+\int_{(b,l_2^{(n)})\setminus A_n}q_n(t,s_n(x),s_n(y))dm_n(y)=:\mathrm{I}+\mathrm{II}.\end{aligned}$$

In view of (5.8), we have $\lim_{b\uparrow l_2}\overline{\lim}_{n\to\infty}\mathrm{I}=0$. On the other hand, (6.3) implies

$$\begin{aligned}\mathrm{II}&=\int_0^{t/2}q_n(t-\tau,s_n(x),c)\int_{(b,l_2^{(n)})\setminus A_n}q_{n,c}(\tau,s_n(y))dm_n(y)d\tau\\&\quad+\int_{t/2}^t q_n(t-\tau,s_n(x),c)\int_{(b,l_2^{(n)})\setminus A_n}q_{n,c}(\tau,s_n(y))dm_n(y)d\tau\\&\quad+\Phi(s_n(x),c)\int_{(b,l_2^{(n)})\setminus A_n}q_{n,c}(t,s_n(y))dm_n(y)=:\mathrm{III}+\mathrm{IV}+\mathrm{V}.\end{aligned}$$

Noting that $\overline{\lim}_{n\to\infty}\sup_{t/2\leqq\tau\leqq t}q_n(\tau,s_n(x),c)<+\infty$, we obtain $\lim_{b\uparrow l_2}\overline{\lim}_{n\to\infty}\mathrm{III}=0$ by the same way as in the above. Further, it is not hard to see

$$\int_{(b,l_2^{(n)})\setminus A_n} q_{n,c}(\tau, s_n(y))dm_n(y) \leqq \int_{(c,\tilde{l}_2^{(n)})} q_{n,c}(\tau, \xi)d\tilde{m}_n(\xi)$$
$$\leqq q_{n,c}^*(\tau, c, c) \leqq -D_\xi^+ v_2^{(n)}(c, 1/\tau)/v_2^{(n)}(c, 1/\tau),$$

which implies

$$\lim_{b\uparrow l_2} \overline{\lim_{n\to\infty}} \sup_{t/2\leqq\tau\leqq t} \int_{(b,l_2^{(n)})\setminus A_n} q_{n,c}(\tau, s_n(y))dm_n(y) = 0. \tag{6.9}$$

But $\overline{\lim}_{n\to\infty}\left(\int_0^{t/2} q_n(\tau, s_n(x), c)d\tau + \Phi(s_n(x), c)\right) < +\infty$. Thus, we obtain $\lim_{b\uparrow l_2} \overline{\lim}_{n\to\infty}(\mathrm{IV}+\mathrm{V})=0$, proving (6.6) for this case. q. e. d.

In the case where (6.1) fails, we need some additional conditions to get (6.2). Indeed, we have the following

COROLLARY 6.1. *Suppose that* (6.1) *fails but* $\lim_{n\to\infty} l_2^{(n)} = l_2$, $l_2(m_n) \leqq l_2(s_n)$ *and*

$$\lim_{n\to\infty} \int_{(a,l_2^{(n)})} (m_n(x)-m_n(a))ds_n(x) = \int_{(a,l_2)} (m(x)-m(a))ds(x). \tag{6.10}$$

Assume further that $\lim_{n\to\infty} s_n(l_2^{(n)}-) = s(l_2-) < +\infty$. *Then we obtain* (6.2).

PROOF. Note first that, under the assumptions, $l_2(m) \leqq l_2(s)$ and

$$\begin{aligned} u_1(x_1, \alpha)+(s(x_2)-s(x_1))u_1^+(x_1, \alpha) &\leqq u_1(x_2, \alpha) \\ &\leqq \{u_1(x_1, \alpha)+(s(x_2)-s(x_1))u_1^+(x_1, \alpha)\}\exp\left\{\alpha\int_{(x_1,x_2]}(m(y)-m(x_1))ds(y)\right\}, \\ &\qquad a \leqq x_1 < x_2 < l_2,\ \alpha > 0. \end{aligned} \tag{6.11}$$

We then obtain $\overline{\lim}_{n\to\infty} u_1^{(n)}(l_2^{(n)}, \alpha) < +\infty$ and $\lim_{b\uparrow l_2} \overline{\lim}_{n\to\infty}\{u_1^{(n)}(l_2^{(n)}, \alpha)-u_1^{(n)}(b, \alpha)\}=0$. Hence, using the formula $u_1^+(b, \alpha)u_2(b, \alpha) \leqq -u_2^+(b, \alpha)(u_1(l_2, \alpha)-u_1(b, \alpha))$, we have $\lim_{b\uparrow l_2} \overline{\lim}_{n\to\infty} u_1^{(n)+}(b, \alpha)u_2^{(n)}(b, \alpha)=0$. This with Corollary 2.2 implies $\lim_{b\uparrow l_2} \overline{\lim}_{n\to\infty}\{u_2^{(n)+}(l_2^{(n)}, \alpha)-u_2^{(n)+}(b, \alpha)\}=0$. We thus obtain (6.8) in this case too.

Suppose first that $s(x) < \tilde{l}_2$. Then we can repeat the argument in the proof of Theorem 6.1 to obtain (6.6), exploiting the decomposition

$$\begin{aligned} T_t^{(n)} 1_{(b,l_2^{(n)})}(x) &= \int_0^{t/2} q_{n,c}(t-\tau, s_n(x)) \int_{(b,l_2^{(n)})} q_n(t, c, s_n(y))dm_n(y)d\tau \\ &+ \int_{t/2}^t q_{n,c}(t-\tau, s_n(x)) \int_{(b,l_2^{(n)})} q_n(t, c, s_n(y))dm_n(y)d\tau \\ &+ \Psi_c(s_n(x)) \int_{(b,l_2^{(n)})} q_n(t, c, s_n(y))dm_n(y) \end{aligned}$$

and making use of the estimate

$$T_t^{(n)} 1_{(b,l_2^{(n)})}(x) \leqq T_t^{(n)} 1(x) \leqq \{1-v_1^{(n)}(s_n(x), \alpha)/v_1^{(n)}(\tilde{l}_2, \alpha)\}/(1-e^{-\alpha t}).$$

The details are omitted.

Suppose next that $s(x)=\tilde{l}_2$. In this case, we have $\lim_{n\to\infty}s_n(x)=\tilde{l}_2$ and (6.6) by the above method. q. e. d.

REMARK 6.1. Under the assumptions of Theorem 5.1, the condition $\lim_{n\to\infty}s_n(l_2^{(n)}-)=s(l_2-)<+\infty$ is automatically satisfied provided $l_2(s)<l_2(m)$. Thus that condition is necessary only when $l_2(s)=l_2(m)=+\infty$.

The condition (6.10) is almost necessary for (6.2). Indeed, in view of the inequality

$$u_1(x_1, \alpha)\left\{1+\alpha\int_{(x_1, x_2]}^{\#}(m(y)-m(x_1))ds(y)-\alpha(s(x_2+)-s(x_2))(m(x_2)-m(x_1))\right\}$$
$$\leqq u_1(x_2, \alpha), \qquad a\leqq x_1<x_2<l_2,$$

and (6.11), we can find a sequence $(s_n, m_n)\in\mathcal{M}\times\mathcal{M}_+$, $n=1, 2, 3, \cdots$ and $(s, m)\in\mathcal{M}\times\mathcal{M}_+$ (violating the condition (6.10)) such that $\int_{(a, l_2)}(m(x)-m(a))ds(x)<+\infty$ and $\lim_{n\to\infty}u_1^{(n)}(l_2^{(n)}, \alpha)>u_1(l_2, \alpha)$. It then follows from the formulas in the proof of Corollary 6.1 and Lemma 5.2 that $\lim_{n\to\infty}u_2^{(n)+}(l_2^{(n)}, \alpha)<u_2^+(l_2, \alpha)$. This with the relation (6.7) and Lemma 5.2 causes the violation of (6.8). Thus the formula (6.2) fails in this case.

The above argument poses a doubt on the assertion of the convergence in $L_1(\boldsymbol{R}^N)$ of the density functions in [7; Theorem 2]. In order to avoid those uncomfortable conditions, we introduce the stopped processes (actually, it is not hard to believe that A. O. Golosov imagined the stopped processes by his modification of $s(x)$). Let $T_t^{\cdot}f$ be the semigroup defined by

$$\text{(6.12)}\quad \int_0^{+\infty}e^{-\alpha t}T_t^{\cdot}f(x)dt=\int_Q H(\alpha, s(x), s(y))f(y)dm(y)$$
$$+f(l_1)v_2(s(x), \alpha)/\alpha v_2(\tilde{l}_1, \alpha)+f(l_2)v_1(s(x), \alpha)/\alpha v_1(\tilde{l}_2, \alpha),$$

for $x\in\bar{Q}$, $t>0$ and $f\in B(\bar{Q})$. Then we have a unique Markov process $(X^{\cdot}(t), P_x)$ on $\bar{Q}$ corresponding to $T_t^{\cdot}$. To realize the sample paths in the same fashion as in Section 3, we let $\mathfrak{f}^{\cdot}(u)=\int_R L(u, \xi)d\tilde{m}(\xi)$ and define $X^{\cdot}(t\,;x)$ by the same way as in Section 3 using $\mathfrak{f}^{\cdot}$ in place of $\mathfrak{f}$ (with the convention $s^{-1}(\tilde{l}_i)=l_i$). Denoting $(X^{\cdot}(\cdot\,;x), P)$ by $(X^{\cdot}(\cdot), P_x)$, we obtain the desired process.

THEOREM 6.2. *Under the assumptions of Theorem* 5.1 *and that* $\tilde{m}(\tilde{Q})>0$, *it holds that*

$$\text{(6.13)}\quad \lim_{n\to\infty}P_x^{(n)}(X^{\cdot}(t_1)>a_1, \cdots, X^{\cdot}(t_N)>a_N)=P_x(X^{\cdot}(t_1)>a_1, \cdots, X^{\cdot}(t_N)>a_N),$$

for every $0<t_1<t_2<\cdots<t_N$, $a_1, a_2, \cdots, a_N\in Q\setminus J(m)$ *and* $x\in Q$ *with* $\lim_{n\to\infty}s_n(x)=s(x)$.

PROOF. We will also show that $\lim_{n\to\infty}P_z^{(n)}(X^{\cdot}(t)>a_1)=P_x(X^{\cdot}(t)>a_1)$ for each $t>0$, $a_1\in Q\setminus J(m)$ and $x\in Q$ with $\lim_{n\to\infty}s_n(x)=s(x)$. Further, we may assume that $l_2(m)\leqq l_2(s)$ and $\int_{(a,l_2)}(m(x)-m(a))ds(x)<+\infty$. Hence it holds that $u_2(l_2, \alpha)=0$, whence $\lim_{b\uparrow l_2}\overline{\lim}_{n\to\infty}u_2^{(n)}(b, \alpha)=0$. We first assume that $s(a_1)<\tilde{l}_2$. It then follows from (6.3), (6.4) and (6.12) that

$$
\begin{aligned}
(6.14)\quad P_x^{(n)}(X^{\cdot}(t)>a_1) &= \int_{(a_1,b)}q_{n,s_n(b)}(t, s_n(x), s_n(y))dm_n(y)\\
&\quad+\int_0^t\int_{(a_1,l_2^{(n)})}q_n(t-\tau, s_n(b), s_n(y))dm_n(y)q_{n,s_n(b)}(\tau, s_n(x))d\tau\\
&\quad+\int_{(a_1,l_2^{(n)})}q_n(t, s_n(b), s_n(y))dm_n(y)\Psi_{n,s_n(b)}(\tau, s_n(x))\\
&\quad+\left\{\int_0^t q_{n,\tilde{l}_2^{(n)}}(\tau, s_n(x))d\tau+\Psi_{n,\tilde{l}_2^{(n)}}(s_n(x))\right\}\\
&=:\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV},
\end{aligned}
$$

with the convention $\Psi_c(c)=1$ and $q_c(t, \xi, c)=q_c(t, c)=0$ for $t>0$. Further, it is clear that

$$
\begin{aligned}
\mathrm{II}+\mathrm{III}+\mathrm{IV} &= \int_0^t\{1-P_b^{(n)}(X^{\cdot}(t-\tau)\leqq a_1)\}q_{n,s_n(b)}(\tau, s_n(x))d\tau\\
&\quad+(1-P_b^{(n)}(X^{\cdot}(t)\leqq a_1))\Psi_{n,s_n(b)}(s_n(x)), \qquad b\in(x, l_2).
\end{aligned}
$$

Since $\sup_{0<\tau\leqq t}P_b^{(n)}(X^{\cdot}(\tau)\leqq a_1)\leqq eu_2^{(n)}(b, 1/t)/u_2^{(n)}(a_1, 1/t)$, we first have $\lim_{b\uparrow l_2}\overline{\lim}_{n\to\infty}\sup_{0<\tau\leqq t}P_b^{(n)}(X^{\cdot}(t)\leqq a_1)=0$. Further, due to Corollary 6.1, $\lim_{n\to\infty}\mathrm{I}=\int_{(a_1,b)}q_{s(b)}(t, s(x), s(y))dm(y)$ and $\lim_{n\to\infty}\left\{\int_0^t q_{n,s_n(b)}(\tau, s_n(x))d\tau+\Psi_{n,s_n(b)}(s_n(x))\right\}=\int_0^t q_{s(b)}(\tau, s(x))d\tau+\Psi_{s(b)}(s(x))$ for $b\in(x, l_2)\setminus(J(s)\cup(J(m))$. Thus we obtain the desired assertion.

In the case of $s(a_1)=\tilde{l}_2$, we have $\tilde{l}_1<s(a_1)$. Hence by all the above arguments, we have $\lim_{n\to\infty}P_x^{(n)}(X^{\cdot}(t)\leqq a_1)=P_x(X^{\cdot}(t)\leqq a_1)$. This proves the desired assertion. q. e. d.

REMARK 6.2. The assertion of Theorem 6.2 is also valid for $\tilde{m}(\tilde{Q})=0$ under the assumptions of Theorem 6.1 or of Corollary 6.1.

7. Application of limit theorems.

In this section, we give three examples which are direct applications of Theorems 6.1 and 6.2 in the previous section.

EXAMPLE 7.1 (Metastable behavior of [**14**]). Let $G(x)$ be a C^1 function on $\boldsymbol{R}$ such that, for some $M_1<S<M_2$, G is strictly decreasing on $(-\infty, M_1]\cup[S, M_2]$, strictly increasing on $[M_1, S]\cup[M_2, +\infty)$, $G(M_1)>G(M_2)$ and that $\lim_{|x|\to\infty}G(x)=+\infty$. Let also $X^{\varepsilon}(t)$ be the solution of the stochastic differential equation

(7.1) $$dX_t = \varepsilon^{1/2}dW_t - G'(X_t)dt\,, \qquad X_0 = x_0\,,$$

where W is a standard Wiener process on $\boldsymbol{R}$ and ε is a positive constant. For each $a, b\in\boldsymbol{R}$, we set $I_\pm^\varepsilon(a, b)=\int_a^b \exp\{\pm 2G(y)/\varepsilon\}dy$. Fix now a $\delta\in(0, (S-M_1)\wedge(M_2-S))$ and consider the scaled process $X_\varepsilon(t)=X^\varepsilon(\lambda_\varepsilon t)$ with $\lambda_\varepsilon=2I_-^\varepsilon(M_1-\delta, M_1+\delta)I_+^\varepsilon(S-\delta, S+\delta)/\varepsilon$. Then the generator of the process X_ε is given by

(7.2) $$\mathcal{G}_\varepsilon = \theta_\varepsilon\{d^2/dx^2-(2G'(x)/\varepsilon)d/dx\},$$

where $\theta_\varepsilon=I_-^\varepsilon(M_1-\delta, M_1+\delta)I_+^\varepsilon(S-\delta, S+\delta)$. We denote $S_1=\min\{x: G(x)=G(S)\}$, $S_2=\max\{x: G(x)=G(S)\}$ and $M_3=\min\{x\neq M_1: G(x)=G(M_1)\}$, and treat the process separately according as the regions of its starting position x_0. The height $H=G(S)-G(M_1)$ plays an important role in the following.

Case 1. Suppose that $x_0\leqq S_1$. Take, in this case, an $x_1<x_0$ such that $G(x_1)-G(x_0)<H$ and let $x_2=\min\{x: G(x_1)-G(x)=H\}$, $x_3=\max\{x: G(x)=G(x_1)\}$. It is then clear that $x_1<x_0<x_2<M_1<S<M_3<M_2<x_3$. We now define the associate pair $(s_\varepsilon, m_\varepsilon)\in\mathcal{M}\times\mathcal{M}_+$ by $s_\varepsilon(x)=I_+^\varepsilon(x_0, x)\exp\{-2G(x_1)/\varepsilon\}$ and $m_\varepsilon(x)=I_-^\varepsilon(x_0, x)\exp\{2G(x_1)/\varepsilon\}/\theta_\varepsilon$. Then, by a simple computation, we obtain (5.1) with $s(x)=-\infty$ for $x<x_1$, $=0$ for $x_1\leqq x\leqq x_3$, $=+\infty$ for $x_3<x$, and $m(x)=0$ for $x<x_2$, $=+\infty$ for $x\geqq x_2$. Hence, by virtue of Theorem 6.1, we have

(7.3) $$\lim_{\varepsilon\downarrow 0} P_{x_0}(X_\varepsilon(t_1)<a_1, \cdots, X_\varepsilon(t_N)<a_N) = 0\,,$$

for every $0<t_1<t_2<\cdots<t_N$ and $a_1, a_2, \cdots, a_N\in(x_1, x_2)$. Thus, as $\varepsilon\downarrow 0$, the process $X_\varepsilon(t)$ starting at x_0 hits a_2 very soon and it scarcely comes back to the interval $(-\infty, a_1)$ for every $x_0<a_1<a_2<x_2$.

Case 2. Suppose that $S_1<x_0<M_3$. In this case, we define the associate pair $(s_\varepsilon, m_\varepsilon)\in\mathcal{M}\times\mathcal{M}_+$ by $s_\varepsilon(x)=I_+^\varepsilon(M_1, x)/I_+^\varepsilon(S-\delta, S+\delta)$ and $m_\varepsilon(x)=I_-^\varepsilon(S_1, x)/I_-^\varepsilon(M_1-\delta, M_1+\delta)$. Then we have (5.1) with $s(x)=-\infty$ for $x<S_1$, $=0$ for $S_1\leqq x\leqq S$, $=1$ for $S<x\leqq S_2$, $=+\infty$ for $S_2<x$, and $m(x)=0$ for $x<M_1$, $=1$ for $M_1\leqq x<M_3$, $=+\infty$ for $x\geqq M_3$. Hence, by virtue of Theorem 6.1, we see that, if $S_1<x_0<S$, then

(7.4) $$\lim_{\varepsilon\downarrow 0} P_{x_0}(X_\varepsilon(t_1)<a_1, \cdots, X_\varepsilon(t_N)<a_N) = \exp\{-t_N\}\prod_{k=1}^{N} 1_{(-\infty, a_k]}(M_1)$$

for every $0<t_1<t_2<\cdots<t_N$ and $a_1, a_2, \cdots, a_N\in(S_1, M_3)\setminus\{M_1\}$ and, if $S<x_0<M_3$, then (7.3) holds for all $0\leqq t_1<t_2<\cdots<t_N$ and $a_1, a_2, \cdots, a_N\in(S_1, M_3)$. Thus, as $\varepsilon\downarrow 0$, the process $X_\varepsilon(t)$ starting at $x_0\in(S_1, S)$ hits M_1 very soon, where it stays for an exponential holding time and then it goes to a_2 and it scarcely comes back to the interval $(-\infty, a_1)$ for every $S<a_1<a_2<M_3$. Further, if it starts at $x_0\in(S, M_3)$ then it hits a_2 very soon and scarcely comes back.

Case 3. Suppose that $M_3\leqq x_0\leqq M_2$. If $G(x_0)-G(M_2)\geqq H$, then taking an

$x_1 \in (S, x_0)$ such that $G(x_1)-G(x_0)<H$ and setting $x_2=\min\{x: G(x_1)-G(x)=H\}$, we can reduce the argument to that in Case 1.

Thus we assume that $G(x_0)-G(M_2)<H$. Set in this case $x_2=\max\{x: G(x)-G(M_2)=H\}$, $x_1=\max\{x\neq x_2: G(x)-G(M_2)=H\}$. It is then clear that $S<x_1<x_0\leqq M_2<x_2$. Let now $s_\varepsilon(x)=I_+^\varepsilon(x_0, x)I_-^\varepsilon(M_2-\delta, M_2+\delta)/\theta_\varepsilon$ and $m_\varepsilon(x)=I_-^\varepsilon(x_0, x)/I_-^\varepsilon(M_2-\delta, M_2+\delta)$. Then we have (5.1) with $s(x)=-\infty$ for $x<x_1$, $=0$ for $x_1\leqq x\leqq x_2$, $=+\infty$ for $x_2<x$, and $m(x)=0$ for $x<M_2$, $=1$ for $x\geqq M_2$. Hence, by virtue of Theorem 6.1, we see that

$$\lim_{\varepsilon\downarrow 0} P_{x_0}(X_\varepsilon(t_1)<a_1, \cdots, X_\varepsilon(t_N)<a_N)=\prod_{k=1}^N 1_{(-\infty, a_k]}(M_2) \tag{7.5}$$

for every $0<t_1<t_2<\cdots<t_N$ and $a_1, a_2, \cdots, a_N \in (x_1, x_2)\setminus\{M_2\}$. Thus, as $\varepsilon\downarrow 0$, the process $X_\varepsilon(t)$ starting at $x_0\in(x_1, x_2)$ hits M_2 very soon, where it stays forever with very high probability.

The case $x_0>M_2$ can be treated in the exactly same way.

Finally, we note that, since the process X_ε is a usual diffusion process, one can easily glue together the processes in the above cases as is done in [**14**]. The limit process is as follows. If it starts at $x_0\in(-\infty, S)$, then it hits M_1 instantaneously, where it stays for an exponential holding time and then it jumps to the trap state M_2. If it starts at $x_0\in(S, +\infty)$, then it hits the trap state M_2 instantaneously. Suppose finally it starts at S and the limit $p=\lim_{\varepsilon\downarrow 0} I_+^\varepsilon(S-\delta, S)/I_+^\varepsilon(S-\delta, S+\delta)$ exits. Then it jumps to M_1 and to the trap state M_2 instantaneously with probabilities $1-p$ and p respectively (note that, for the limit process in Case 2, $s(S)=p$ and the density function $q(t, \xi, \eta)$ of the GDP Y is linear in $\xi\in(s(S-), s(S+))$).

EXAMPLE 7.2 (Diffusion process in Wiener medium of [2]). After scaling the Wiener medium, one can reduce the study of the diffusion process in a Wiener medium of [2] to that of the solution $X^\gamma(t)$ of the stochastic differential equation

$$dX_t=dB_t-(\gamma/2)W'(X_t)dt, \qquad X_0=x_0, \tag{7.6}$$

where B and W are standard Wiener processes on $\boldsymbol{R}$ which are independent of each other, γ is a positive constant which will be let to go to $+\infty$ later, and W' is the derivative of W symbolically understood (see [2]). Thus the problem is reduced to the same type of that in Example 7.1, and we can deal with it by our method (note that, in Example 7.1, we do not need the derivative of G, since all the effective functionals of G for the argument are $I_\pm^\varepsilon(a, b)$ and the values of G itself). It is well known that $\overline{\lim}_{x\to\pm\infty}W(x)=\overline{\lim}_{x\to\pm\infty}(-W(x))=+\infty$ and, for each $0<a<b$, the maximum $\max_{a\leq x\leq b}W(x)$ and the minimum $\min_{a\leq x\leq b}W(x)$ are attained by single points S and M respectively with probability 1. Further, for each $x_0\in\boldsymbol{R}$, we have $x_0\neq S, M$ with probability 1. Thus,

we may use all the above properties (see [2]).

Let now $x^+=\inf\{y>x: W(y)=W(x)\}$, $x^-=\sup\{y<x: W(y)=W(x)\}$, with the conventions $\inf\emptyset=+\infty$, $\sup\emptyset=-\infty$, and let $BT_1=\{x:$ there exist y^+ and y^- such that $y^\pm\in(x, x^\pm)$ and $W(y^\pm)-W(x)\geqq 1\}$, where $(a, b)=(a\wedge b, a\vee b)$. We further set $b_1^+(x_0)=\min BT_1\cap[x_0, +\infty)$, $b_1^-(x_0)=\max BT_1\cap(-\infty, x_0)$ and let $\max\{W(x): x\in[b_1^-(x_0), b_1^+(x_0)]\}=W(S)$ for a unique $S\in[b_1^-(x_0), b_1^+(x_0)]\setminus\{x_0\}$. Finally, we define $b_1=b_1(x_0)$ by $b_1=b_1^-(x_0)$ if $b_1^-(x_0)<x_0<S$, $=b_1^+(x_0)$ if $S<x_0<b_1^+(x_0)$. Further, due to the symmetry of the argument, we may assume $S<x_0<b_1^+(x_0)$, that is $b_1=b_1^+(x_0)$. As in Example 7.1, we set $I_\pm^\gamma(a, b)=\int_a^b \exp\{\pm\gamma W(y)\}dy$ for each $a, b\in\boldsymbol{R}$, and $X_\gamma(t)=X^\gamma(te^\gamma)$, $t\geqq 0$.

Case 1. Suppose first that $W(b_1)+1<W(S_1)$, where $\max_{x\in[x_0, b_1]}W(x)=W(S_1)$, $S_1\in[x_0, b_1]$. If $W(x_0)\geqq W(S_1)$, then we can employ the argument in Example 7.1 to obtain

$$\lim_{\gamma\to\infty}P_{x_0}(X_\gamma(t_1)<a_1, \cdots, X_\gamma(t_N)<a_N)=0, \tag{7.7}$$

for every $0<t_1<t_2<\cdots<t_N$ and $a_1, a_2, \cdots, a_N\in(x_0, x_1)$, where $x_1=\min\{x>x_0: W(x)=W(x_0)-1\}$.

Thus we assume $W(x_0)<W(S_1)$. In this case, let $x_1=\sup\{x<S_1: W(x)<W(S_1)-1\}$, $x_2=\sup\{x<S_1: W(x)>W(S_1)\}$, $x_3=\inf\{x>S_1: W(x)<W(S_1)-1\}$, and $x_4=\inf\{x>S_1: W(x)>W(S_1)\}$. It is then clear that $x_1<x_2<x_0<S_1<x_3<b_1<x_4$ and $W([x_2, x_3])\subset[W(S_1)-1, W(S_1)]$. We assume that $W(x)<W(S_1)$ for all $x\in(x_2, S_1)$ and $W(x)>W(x_3)$ for all $x\in(x_1, x_3)$ (it is easy to see that the following arguments work well without these assumptions with a slight modification, or one can even take a version of W, which satisfy these assumptions). Take now a $\delta\in(0, (S_1-x_2)\wedge(b_1-S_1))$ and set $s_\gamma(x)=I_+^\gamma(x_0, x)/I_+^\gamma(S_1-\delta, S_1+\delta)$, $m_\gamma(x)=2I_-^\gamma(x_0, x)I_+^\gamma(S_1-\delta, S_1+\delta)/e^\gamma$. Then, we have (5.1) with $s(x)=-\infty$ for $x\leqq x_2$, $=0$ for $x_2<x\leqq S_1$, $=1$ for $S_1<x\leqq x_4$, $=+\infty$ for $x>x_4$, and $m(x)=-\infty$ for $x<x_1$, $=0$ for $x_1\leqq x<x_3$, $=+\infty$ for $x\geqq x_3$. Hence, by virtue of Theorem 6.1, we obtain (7.7) for every $0<t_1<t_2<\cdots<t_N$ and $a_1, a_2, \cdots, a_N\in(x_0, x_3)$.

Case 2. Suppose next that $W(b_1)+1\notin W((x_0, b_1))$. In this case, setting $x_1=\max\{x<b_1: W(x)=W(b_1)+1\}$, $x_2=\min\{x>b_1: W(x)=W(b_1)+1\}$, we have $x_1<x_0<x_2$. Take then a $\delta\in(0, (b_1-x_1)\wedge(x_2-b_1))$ and set $s_\gamma(x)=I_+^\gamma(x_0, x)I_-^\gamma(b_1-\delta, b_1+\delta)/e^\gamma$ and $m_\gamma(x)=2I_-^\gamma(x_0, x)/I_-^\gamma(b_1-\delta, b_1+\delta)$. Then, in the exactly same way as in Example 7.1, we have

$$\lim_{\gamma\to\infty}P_{x_0}(X_\gamma(t_1)<a_1, \cdots, X_\gamma(t_N)<a_N)=\prod_{k=1}^N 1_{(-\infty, a_k]}(b_1) \tag{7.8}$$

for every $0<t_1<t_2<\cdots<t_N$ and $a_1, a_2, \cdots, a_N\in(x_1, x_2)\setminus\{b_1\}$.

Finally, we note that, by gluing together with the processes in Cases 1 and 2, one sees that the limit process of X_γ starting at an x_0 hits the trap state

$b_1(x_0)$ instantaneously, and this gives enough information for the results in [**2**].

REMARK 7.1. It is announced in a couple of symposiums that K. Kawazu, Y. Tamura and H. Tanaka extended the results in [2] to those in the self-similar or even asymptotically self-similar random media and obtained further properties (see [**20**] and also its References for the literatures on this subject). Notice that, in their models, it fails the assumption that every local minimum and local maximum of the media function are attained by single points. But, as far as the first primitive convergence theorem of the above type, one can easily apply the above method to those models.

EXAMPLE 7.3 (Gene frequency model in [**6**]). Let us consider the diffusion process X^ε on (0, 1) with the generator

$$\mathcal{G}_\varepsilon = \{x(1-x)/2\}\, d^2/dx^2 + \{\alpha_\varepsilon x(1-x)(1-2x)+\theta_\varepsilon(1-2x)\}\, d/dx, \tag{7.9}$$

where α_ε and θ_ε are positive constants corresponding to the selection rate and the mutation rate respectively. The boundaries 0 and 1 are both entrance and non-exit if $\theta_\varepsilon \geqq 1/2$, and regular if $0<\theta_\varepsilon<1/2$. In the latter case, the reflection boundary conditions are set implicitly. Further, in the following, we let $\alpha_\varepsilon \to +\infty$ and $\theta_\varepsilon \to 0$ so that $\lim_{\varepsilon\downarrow 0}\{\log(1/\theta_\varepsilon)\}/\alpha_\varepsilon=0$. Thus we may assume $\theta_\varepsilon \in (0, 1/2)$, and set

$$I_-^\varepsilon(a, b) = \int_a^b \{y(1-y)\}^{-2\theta_\varepsilon} e^{-2\alpha_\varepsilon y(1-y)} dy,$$

$$I_+^\varepsilon(a, b) = 2\int_a^b \{y(1-y)\}^{-1+2\theta_\varepsilon} e^{2\alpha_\varepsilon y(1-y)} dy,$$

for each $a, b \in [0, 1]$. Define then the associate pair $(s^\varepsilon, m^\varepsilon) \in \mathcal{M}\times\mathcal{M}_+$ by $s^\varepsilon(x)=0$ for $x<0$, $=I_-^\varepsilon(0, x)$ for $0\leqq x\leqq 1$, $=I_-^\varepsilon(0, 1)$ for $x>1$, and $m^\varepsilon(x)=0$ for $x<0$, $=I_+^\varepsilon(0, x)$ for $0\leqq x\leqq 1$, $=I_+^\varepsilon(0, 1)$ for $x>1$.

Set now $X_\varepsilon(t)=X^\varepsilon(\lambda_\varepsilon t)$ with $\lambda_\varepsilon=I_-^\varepsilon(0, 1)I_+^\varepsilon(0, 1)$ and $s_\varepsilon(x)=s^\varepsilon(x)/I_-^\varepsilon(0, 1)$, $m_\varepsilon(x)=m^\varepsilon(x)/I_+^\varepsilon(0, 1)$. Then we have (5.1) with $s(x)=0$ for $x\leqq 0$, $=1/2$ for $0<x<1$, $=1$ for $x\geqq 1$ and $m(x)=0$ for $x<1/2$, $=1$ for $x\geqq 1/2$. Hence, by virtue of Theorem 6.2, we obtain

$$\lim_{\varepsilon\downarrow 0} P_x(X_\varepsilon(t_1)<a_1, \cdots, X_\varepsilon(t_N)<a_N) = \prod_{k=1}^N 1_{(-\infty, a_k]}(1/2), \quad x\in[0, 1], \tag{7.10}$$

for every $0<t_1<t_2<\cdots<t_N$ and $a_1, a_2, \cdots, a_N \in (0, 1)\setminus\{1/2\}$. Notice further that $I_-^\varepsilon(0, 1)\sim\exp\{(2\theta_\varepsilon-1)\log\alpha_\varepsilon\}$ and $I_+^\varepsilon(0, 1)\sim 4\sqrt{2\pi}\exp\{\alpha_\varepsilon/2-(\log\alpha_\varepsilon)/2\}$ so that $\lambda_\varepsilon\sim 4\sqrt{2\pi}\exp\{\alpha_\varepsilon/2+(2\theta_\varepsilon-3)(\log\alpha_\varepsilon)/2\}$ as $\varepsilon\downarrow 0$. We also notice that we are preparing a systematic study on this subject in [8].

A. Appendix.

In this section, we will summarize some formulas on change of variables and integration by parts for the integration with respect to the measures induced by discontinuous non-decreasing functions. We continue to exploit the notation in Section 2. Especially, $\tilde{m}$ is that given in Section 2 and $\tilde{Q}=Q(\tilde{m})$. Although we do not assume that our functions s and f are either right continuous or left continuous, the obtained formulas are just natural extensions of those in [3]. We thus omit the proof.

LEMMA A.1. *Suppose that v is a function in $L^1(\tilde{Q}, \tilde{m})$. Then, for each b_1, $b_2 \in \tilde{Q}$ such that $b_1 < b_2$, it holds that*

$$\int_{s^{-1}((b_1, b_2])} v(s(x))dm(x) = \int_{(b_1, b_2]} v(\xi)d\tilde{m}(\xi). \tag{A.1}$$

COROLLARY A.1. *Let v and f be bounded Borel functions in $\tilde{Q}$. Then, for each b_1, $b_2 \in \tilde{Q}$ such that $b_1 < b_2$, it holds that*

$$\int_{(s^{-1}((b_1, b_2]\setminus J(s^{-1}))} v(s(x))f(s^{-1}\circ s(x))dm(x) = \int_{(b_1, b_2]\setminus J(s^{-1})} v(\xi)f(s^{-1}(\xi))d\tilde{m}(\xi). \tag{A.2}$$

Let f and g be two functions of bounded variation on $\boldsymbol{R}$. For each interval I, we define an integral

$$\int_I^{\#} g(x)df(x) = \int_{I\setminus J(f)} g(x)df(x) + \sum_{x\in J(f)\cap I} \{g(x+)\Delta_f^+(x) + g(x-)\Delta_f^-(x)\},$$

where $\Delta_f^+(x) = f(x+) - f(x)$ and $\Delta_f^-(x) = f(x) - f(x-)$. Notice that $\int_I^{\#} g(x)df(x) = \int_I g(x)df(x)$ if $J(f)\cap J(g) = \emptyset$, but $\int_{\boldsymbol{R}}^{\#} 1_I(x)g(x)df(x) \neq \int_I^{\#} g(x)df(x)$ in general.

LEMMA A.2 (Integration by parts). *Suppose that f and g are of bounded variation on $[a, b]$, and that g is right continuous. Then it holds that*

$$\int_{(a, b]} f(x)dg(x) = (fg)(b+) - (fg)(a+) - \int_{(a, b]}^{\#} g(x)df(x). \tag{A.3}$$

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Yukio OGURA
Department of Mathematics
Faculty of Science and Engineering
Saga University
Saga 840
Japan