J. Math. Soc. Japan
Vol. 40, No. 4, 1988

Homogeneous hypersurfaces in Kähler C-spaces with $b_2=1$

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(Received May 28, 1987)

Introduction.

A compact simply connected homogeneous Kähler manifold is called a *Kähler C-space*. Recall that a Kähler C-space Y with $b_2(Y)=1$ can be obtained by a possible pair $(\mathfrak{g}, \alpha_r)$ of a complex simple Lie algebra $\mathfrak{g}$ and a simple root α_r of $\mathfrak{g}$ (cf. Section 1 below). Moreover, since $b_2(Y)=1$, the Picard group of Y is isomorphic to $\boldsymbol{Z}$. We denote its ample generator by $\mathcal{O}_Y(1)$ and $\mathcal{O}_Y(1)^{\otimes a}$ by $\mathcal{O}_Y(a)$, $a\in\boldsymbol{Z}$. For a positive integer d, a member of the linear system $|\mathcal{O}_Y(d)|$ is called a hypersurface of degree d in Y.

We sometimes encounter the phenomena that a certain Kähler C-space can be embedded in another Kähler C-space with $b_2=1$ as a hypersurface. For example, an n-dimensional projective space $\boldsymbol{P}^n$ (resp. a complex quadric Q^n) can be embedded in $\boldsymbol{P}^{n+1}$ as a hypersurface of degree 1 (resp. 2). On the other hand, Kimura [7, II] showed that the cohomology group $H^0(T_X)$ vanishes for a smooth hypersurface X in an irreducible Hermitian symmetric space of compact type if the degree of X is greater than two. This gives us the feeling that the above phenomena can be completely classified. In fact, we show the following:

MAIN THEOREM. *Let Y be a Kähler C-space with $b_2=1$. Then a Kähler C-space X can be embedded as a hypersurface of degree d in Y, if and only if X, Y and d are one of the following (up to isomorphism):*

(1) $X=\boldsymbol{P}^n$, $Y=\boldsymbol{P}^{n+1}$ *and* $d=1$.

(2) $X=Q^n$, $Y=\boldsymbol{P}^{n+1}$ *and* $d=2$.

(3) $X=Q^n$, $Y=Q^{n+1}$ *and* $d=1$.

(4) $X=(C_l, \alpha_2)$, $Y=(A_{2l-1}, \alpha_2)$: *the grassmannian* Grass(2, 2l), *and* $d=1$.

(5) $X=(F_4, \alpha_4)$, $Y=(E_6, \alpha_1)$: *the irreducible Hermitian symmetric space of type* EIII, *and* $d=1$.

In each of the above five cases, it is known that X can be embedded in Y as a hypersurface of the prescribed degree. The first three are standard and the last two examples (4) and (5) are due to Sakane [14] and Kimura [8], respectively. Thus the proof is reduced to showing the converse.

The organization of the present article is as follows: In Section 1, we review the construction of Kähler C-spaces with $b_2=1$. We shall give Table 2 which contains some numerical invariants of them. In Section 2, we study the vanishing of $H^q(\Omega_Y^1(a))$ for each Kähler C-space Y with $b_2(Y)=1$, where Ω_Y^1 is the cotangent bundle of Y and $\Omega_Y^1(a)=\Omega_Y^1\otimes\mathcal{O}_Y(a)$. The complete vanishing theorem for $H^q(\Omega_Y^1(a))$ can be found in (2.5). Finally, in Section 3, we complete the proof of Main Theorem by means of (2.5) and Table 2.

The author wishes to thank Professors Y. Kimura and Y. Sakane for informing him of their excellent results on Kähler C-spaces. He also wishes to thank the referee for his valuable suggestion.

§ 1. Construction of Kähler C-spaces with $b_2=1$.

In this section, we recall the construction of Kähler C-spaces with $b_2=1$ due to Wang [15].

(a) We first review the theory of Lie algebras. A general reference is [6]. Let $\mathfrak{g}$ be a complex simple Lie algebra of rank l and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. An element α in the dual vector space $\mathfrak{h}^*$ of $\mathfrak{h}$ is called a root of $\mathfrak{g}$ (with respect to $\mathfrak{h}$) if there exists a non-zero vector $E_\alpha\in\mathfrak{g}$ such that

$$[H, E_\alpha]=\alpha(H)E_\alpha, \qquad \text{for all} \quad H\in\mathfrak{h}.$$

We denote by Φ the set of all non-zero roots of $\mathfrak{g}$ and put $\mathfrak{g}_\alpha=\boldsymbol{C}E_\alpha$. Then we have a Cartan decomposition

$$\mathfrak{g}=\mathfrak{h}+\sum_{\alpha\in\Phi}\mathfrak{g}_\alpha .$$

Since the Killing form κ is non-degenerate on $\mathfrak{h}\times\mathfrak{h}$, for each $\lambda\in\mathfrak{h}^*$, there exists a unique vector $H_\lambda\in\mathfrak{h}$ satisfying

$$\kappa(H, H_\lambda)=\lambda(H) \qquad \text{for all} \quad H\in\mathfrak{h}.$$

Put $\mathfrak{h}_0=\sum_{\alpha\in\Phi}\boldsymbol{R}H_\alpha$. Then we can define an inner product on the dual vector space $\mathfrak{h}_0^*$ of $\mathfrak{h}_0$ by

$$(\lambda, \mu)=\kappa(H_\lambda, H_\mu) \qquad \text{for} \quad \lambda, \mu\in\mathfrak{h}_0^*.$$

Fix a lexicographic order on $\mathfrak{h}_0^*$ and let Φ^+ (resp. Φ^-) be the set of all positive (resp. negative) roots. Let $\Delta=\{\alpha_1, \cdots, \alpha_l\}$ denote a fundamental root system of $\mathfrak{g}$ consisting of simple roots. The group Λ of weights of $\mathfrak{g}$-modules is the subset of $\mathfrak{h}_0^*$ defined by

$$\Lambda=\{\lambda\in\mathfrak{h}_0^* \mid \langle\lambda, \alpha\rangle:=2(\lambda, \alpha)/(\alpha, \alpha)\in\boldsymbol{Z} \text{ for all } \alpha\in\Phi\}.$$

This is a lattice of $\mathfrak{h}_0^*$ generated by the fundamental weights $\lambda_1, \cdots, \lambda_l$ associated with Δ by $\langle\lambda_i, \alpha_j\rangle=\delta_{ij}$.

(1.1) DEFINITION. A weight $\lambda\in\Lambda$ is called

(1) singular if $(\lambda, \alpha)=0$ for some $\alpha\in\Phi$,

(2) regular with index p if it is not singular and there exist exactly p roots $\alpha\in\Phi^{+}$ with $(\lambda, \alpha)<0$.

Put

$$\Lambda^{+}=\{\lambda\in\Lambda \mid (\lambda, \alpha_i)\geqq 0 \text{ for each } \alpha_i\in\Delta\}.$$

Then this is a fundamental domain for the action of the Weyl group $\mathcal{W}$ of $\mathfrak{g}$ and a weight in Λ^{+} is called a dominant weight.

(b) Fix a simple root α_r $(1\leqq r\leqq l)$ and define

$$\Phi_1=\{\alpha\in\Phi \mid n_r(\alpha)=0\},$$
$$\Phi(\mathfrak{n}^{+})=\{\alpha\in\Phi^{+} \mid n_r(\alpha)>0\},$$
$$\Phi(\mathfrak{u})=\Phi_1\cup\Phi(\mathfrak{n}^{+}),$$

where we denote by $n_r(\alpha)$ the coefficient of α_r when we express α as

$$\alpha=\sum_{i=1}^{l} n_i(\alpha)\alpha_i, \qquad n_i(\alpha)\in \boldsymbol{Z}.$$

Using these, we define Lie subalgebras of $\mathfrak{g}$ as follows:

$$\mathfrak{g}_1=\mathfrak{h}+\sum_{\alpha\in\Phi_1}\mathfrak{g}_\alpha,$$
$$\mathfrak{n}^{+}=\sum_{\alpha\in\Phi(\mathfrak{n}^{+})}\mathfrak{g}_\alpha,$$
$$\mathfrak{u}=\mathfrak{h}+\sum_{\alpha\in\Phi(\mathfrak{u})}\mathfrak{g}_\alpha.$$

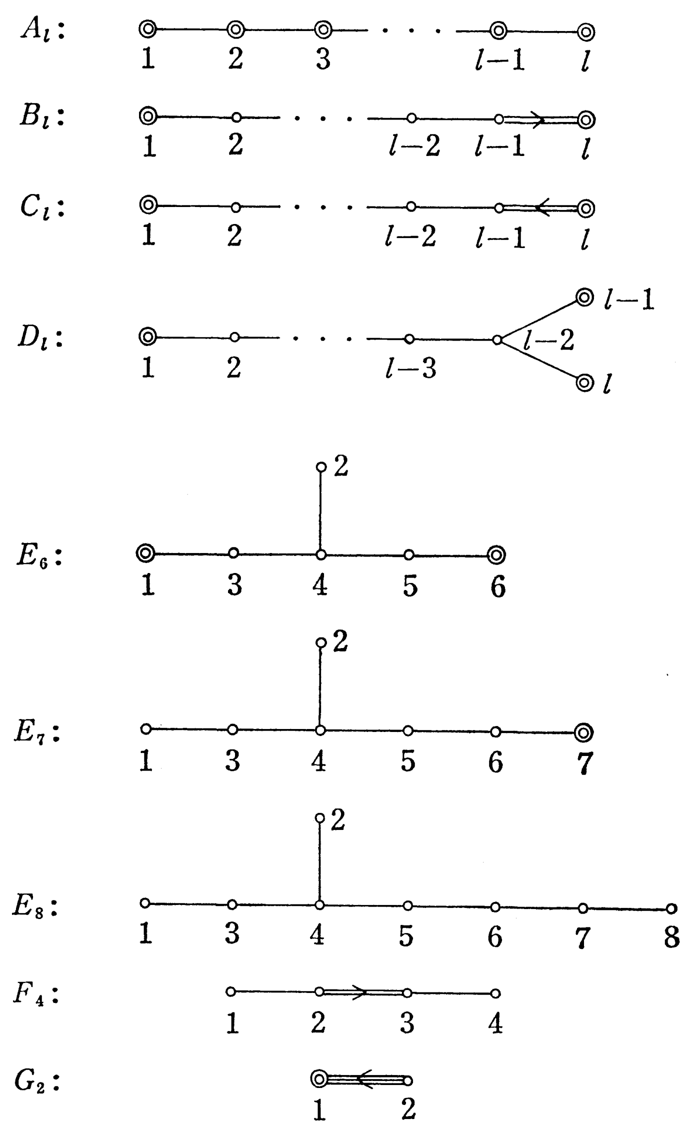
Then $\mathfrak{g}_1$ (resp. $\mathfrak{n}^{+}$) is a reductive (resp. nilpotent) subalgebra of $\mathfrak{g}$ and $\mathfrak{u}=\mathfrak{g}_1+\mathfrak{n}^{+}$ is parabolic. Let G be the connected, simply connected complex Lie group with $\mathrm{Lie}\, G=\mathfrak{g}$ and U the connected complex Lie subgroup of G with $\mathrm{Lie}\, U=\mathfrak{u}$. Take a compact real form $\mathfrak{g}_{\boldsymbol{R}}$ of $\mathfrak{g}$ such that $\mathfrak{g}_{\boldsymbol{R}}\cap\mathfrak{h}=\sqrt{-1}\mathfrak{h}_0$. Let $G_{\boldsymbol{R}}$ be the connected Lie subgroup of G such that $\mathrm{Lie}\, G_{\boldsymbol{R}}=\mathfrak{g}_{\boldsymbol{R}}$ and put $K=G_{\boldsymbol{R}}\cap U$. Note that the injection of $G_{\boldsymbol{R}}$ into G induces a homeomorphism of a compact homogeneous manifold $Y=G_{\boldsymbol{R}}/K$ onto a simply connected complex homogeneous manifold G/U. Under this homeomorphism, Y becomes a complex manifold on which $G_{\boldsymbol{R}}$ (and also G) acts transitively as a group of holomorphic transformations. Moreover, we have the following (see [2], p. 507):

(1.2) LEMMA. $H^2(Y, \boldsymbol{Z})\cong \boldsymbol{Z}\lambda_r$.

Thus we have constructed a Kähler C-space Y with $b_2(Y)=1$ from each complex simple Lie algebra $\mathfrak{g}$ and each simple root α_r of $\mathfrak{g}$. Conversely, any Kähler C-space with $b_2=1$ can be obtained in the way just mentioned (cf. [15]). For this reason, we express the manifold Y obtained above by the pair $(\mathfrak{g}, \alpha_r)$. Since this notation depends on the choice of the numbering of the simple roots, we fix it as in Table 1. Note in particular that (A_n, α_1) denotes the complex projective space $\boldsymbol{P}^n$. Moreover, the complex hyperquadric Q^n is written as (B_l, α_1) (where $n=2l-1$) or (D_l, α_1) (where $n=2l-2$), according as n is odd or even, respectively.

There are some important examples of Kähler C-spaces with $b_2=1$, the *irreducible Hermitian symmetric spaces of compact type*. In Table 1, the notation "$\circledcirc r$" means that the Kähler C-space corresponding to α_r is an irreducible Hermitian symmetric space of compact type.

Table 1.



(c) Put

$$\Lambda_1^+ = \{\lambda \in \Lambda \mid (\lambda, \alpha_i) \geqq 0 \text{ for each } \alpha_i, \quad i \neq r\}.$$

For a weight $\lambda \in \Lambda_1^+$, we denote by $(\rho_{-\lambda}^1, V_{-\lambda})$ the irreducible representation of $\mathfrak{g}_1$ with lowest weight $-\lambda$ ([**11**, 6.3]). Let $(\tilde{\rho}, V)$ be a finite dimensional holomorphic representation of U. Then this defines the holomorphic vector bundle $\mathcal{V}_{\tilde{\rho}}$ over Y associated to the principal bundle $G \to Y = G/U$ by $\tilde{\rho}$. We call such $\mathcal{V}_{\tilde{\rho}}$ a homogeneous vector bundle over Y. Assume that $\tilde{\rho}$ is irreducible and denote by ρ the differential of $\tilde{\rho}$ so that (ρ, V) is an irreducible representation of $\mathfrak{u}$. Since ρ is trivial on $\mathfrak{n}^+$, we see that, on $\mathfrak{g}_1$, (ρ, V) is equivalent to $(\rho_{-\lambda}^1, V_{-\lambda})$ for some $\lambda \in \Lambda_1^+$. Conversely, if we take an irreducible representation $(\rho_{-\lambda}^1, V_{-\lambda})$ of $\mathfrak{g}_1$ with lowest weight $-\lambda$, $\lambda \in \Lambda_1^+$, then we can extend it to an irreducible representation $(\rho_{-\lambda}, V_{-\lambda})$ of $\mathfrak{u}$ by making it trivial on $\mathfrak{n}^+$ and find a representation $(\tilde{\rho}_{-\lambda}, V_{-\lambda})$ of U such that the differential of $\tilde{\rho}_{-\lambda}$ is $\rho_{-\lambda}$. In this case, we denote the vector bundle $\mathcal{V}_{\tilde{\rho}_{-\lambda}}$ by $\mathcal{V}_{-\lambda}$ and call it the homogeneous vector bundle induced by an irreducible representation of U with the lowest weight $-\lambda$.

Let Φ_1^+ be the set of all positive roots in Φ_1. We define a subset $\mathcal{W}^1$ of $\mathcal{W}$ by

$$\mathcal{W}^1 = \{\sigma \in \mathcal{W} \mid \sigma^{-1}(\Phi_1^+) \subset \Phi^+\}.$$

If we denote by $\# A$ the cardinality of a set A, then the index $n(\sigma)$ of $\sigma \in \mathcal{W}$ is defined by

$$n(\sigma) = \#(\sigma(\Phi^+) \cap \Phi^-).$$

Put $\delta = \sum_{i=1}^{l} \lambda_i = (1/2) \sum_{\alpha \in \Phi^+} \alpha$. Then the generalized Borel-Weil theorem [**11**, p. 371], originally obtained by Bott [**5**, p. 228], can be stated as follows.

(1.3) THEOREM. *Let $\lambda \in \Lambda_1^+$ and $\mathcal{V}_{-\lambda}$ be the homogeneous vector bundle on a Kähler C-space $Y = G/U$ induced by an irreducible representation of U with lowest weight $-\lambda$.*

(1) *If $\lambda + \delta$ is singular, then*

$$H^i(Y, \mathcal{V}_{-\lambda}) = 0 \quad \text{for all } i.$$

(2) *If $\lambda + \delta$ is regular with index p, then $\lambda + \delta$ can be expressed uniquely as $\lambda + \delta = \sigma(\mu + \delta)$, where $\mu \in \Lambda^+$ and $\sigma \in \mathcal{W}^1$ with $n(\sigma) = p$. Moreover,*

$$H^i(Y, \mathcal{V}_{-\lambda}) = 0 \quad \text{for } i \neq p,$$

and $H^p(Y, \mathcal{V}_{-\lambda})$ is the irreducible G-module with lowest weight $-\mu$.

The following can be found in [**7**, Lemmas 1 and 2].

(1.4) LEMMA. *Let $\lambda \in \Lambda_1^+$.*

Table 2.

$\mathfrak{g}$	r	dim	k	$h^0(\mathcal{O}(1))$
A_l $(l\geqq 1)$	$1\leqq r\leqq l+1-r$	$r(l+1-r)$	$l+1$	$\binom{l+1}{r}$
B_l $(l\geqq 2)$	$1\leqq r\leqq l-1$	$2r(l-r)+r(r+1)/2$	$2l-r$	$\binom{2l+1}{r}$
C_l $(l\geqq 3)$	$2\leqq r\leqq l$	$2r(l-r)+r(r+1)/2$	$2l-r+1$	$\binom{2l}{r}-\binom{2l}{r-2}$
D_l $(l\geqq 3)$	$1\leqq r\leqq l-2$	$2r(l-r)+r(r-1)/2$	$2l-r-1$	$\binom{2l}{r}$
	l	$l(l-1)/2$	$2l-2$	2^{l-1}
E_6	1	16	12	27
	2	21	11	78
	3	25	9	351
	4	29	7	2925
E_7	1	33	17	133
	2	42	14	912
	3	47	11	8645
	4	53	8	365750
	5	50	10	27664
	6	42	13	1539
	7	27	18	56
E_8	1	78	23	3875
	2	92	17	147250
	3	98	13	6696000
	4	106	9	6899079264
	5	104	11	146325270
	6	97	14	2450240
	7	83	19	30380
	8	57	29	248
F_4	1	15	8	52
	2	20	5	1274
	3	20	7	273
	4	15	11	26
G_2	2	5	3	14

(1) *If* $(\lambda+\delta, \beta)\neq 0$ *for all* $\beta\in\Phi(\mathfrak{n}^+)$, *then* $\lambda+\delta$ *is regular.*
(2) *Assume that there exist* $\mu\in\Lambda^+$ *and* $\sigma\in W^1$ *such that* $\lambda+\delta=\sigma(\mu+\delta)$. *Then*

$$n(\sigma)=\#\{\beta\in\Phi(\mathfrak{n}^+)\mid (\lambda+\delta, \beta)<0\}.$$

Let $\mathcal{O}_Y(1)$ be the homogeneous vector bundle on $Y=G/U=(\mathfrak{g}, \alpha_r)$ induced by the irreducible representation of U with the lowest weight $-\lambda_r$. Then this is known to be the ample generator of $\mathrm{Pic}(Y)$ ([4] and [**13**, § 4]). Since the weight $\lambda_r+\delta$ is itself a dominant weight, we see from (1.3) that $H^0(\mathcal{O}_Y(1))$ is the irreducible G-module with lowest weight $-\lambda_r$. Note that there exists a positive integer $k=k(Y)$ such that $K_Y=\mathcal{O}_Y(-k)$ since Y is known to be rational (cf. [**3**, Satz I]). We can compute $h^0(\mathcal{O}_Y(1))$ and $k(Y)$ by the formula (1.5) and (1.2) in [**10**], respectively.

We close this section by listing in Table 2 all the Kähler C-spaces with $b_2=1$ (up to isomorphism) together with some of their numerical invariants. See also Remark (1.6) and Table 1 in [**10**].

(1.5) Remark. Though the value of $h^0(\mathcal{O}_Y(1))-1$ is given in [**13**, Table 1], there are some mistakes for (E_8, α_r).

§ 2. Vanishing theorem for $H^q(\Omega_Y^1(a))$.

Let Y be a Kähler C-space with $b_2(Y)=1$ in Table 2. In this section, we determine when $H^q(\Omega_Y^1(a))$ does not vanish.

(a) It is known that the cotangent bundle Ω_Y^1 is induced by the representation $(\mathrm{Ad}(U), \mathfrak{n}^+)$. Since, in general, $\mathfrak{n}^+$ is not even a completely reducible U-module, we have defined in [**10**, § 3] a descending filtration on $\mathfrak{n}^+$:

$$0\subset\cdots\subset F^i(\mathfrak{n}^+)\subset F^{i-1}(\mathfrak{n}^+)\subset\cdots\subset F^1(\mathfrak{n}^+)=\mathfrak{n}^+,$$

such that the linear subspaces $F^i(\mathfrak{n}^+)=\sum_{n_r(\alpha)\geq i}\mathfrak{g}_\alpha$ are also invariant by $\mathrm{Ad}(U)$ and the graduations $G^i(\mathfrak{n}^+)=F^i(\mathfrak{n}^+)/F^{i+1}(\mathfrak{n}^+)$ are completely reducible U-modules. Further, we gave the lowest weights of $G^i(\mathfrak{n}^+)$ in Table 3 in [**10**]. Thus we can examine the vanishing of $H^q(\Omega_Y^1(a))$ by means of (1.3) and the following spectral sequence:

$$E_1^{i,q-i}=H^q(G^i\Omega_Y^1(a))\Rightarrow H^q(\Omega_Y^1(a)), \tag{2.1}$$

where we denote by $G^i\Omega_Y^1$ the homogeneous vector bundle induced by the U-module $G^i(\mathfrak{n}^+)$. Here we recall some facts about $G^i(\mathfrak{n}^+)$ which follows from the definition and [**10**, Table 3].

(2.2) Fact. *Let* $Y=(\mathfrak{g}, \alpha_r)$ *be a Kähler C-space with* $b_2(Y)=1$ *in Table* 2. *Then any weight* α *of* $G^i(\mathfrak{n}^+)$ *is an element in* $\Phi(\mathfrak{n}^+)$ *satisfying* $n_r(\alpha)=i$. *We* *further have:*

(1) *For each i, the U-module $G^i(\mathfrak{n}^+)$ is irreducible and the lowest weight is the minimal one among the roots $\alpha \in \Phi(\mathfrak{n}^+)$ such that $n_r(\alpha)=i$. In particular,*

(i) *the lowest weight of $G^1(\mathfrak{n}^+)$ is α_r, and*

(ii) *if $Y=(B_l, \alpha_2)$, (D_l, α_2), (E_6, α_2), (E_7, α_1), (E_8, α_8), (F_4, α_1) or (G_2, α_2), then the lowest weight of $G^2(\mathfrak{n}^+)$ is λ_r and $G^i(\mathfrak{n}^+)=0$ for $i \geqq 3$.*

(2) *For an irreducible Hermitian symmetric space of compact type, we have $G^1(\mathfrak{n}^+)=\mathfrak{n}^+$. Thus the cotangent bundle is the homogeneous vector bundle induced by the irreducible representation of U with lowest weight α_r.*

For convenience, we list in Table 3 the lowest weights of $G^i(\mathfrak{n}^+)$, $i \geqq 2$, for Y which does not appear in (2.2).

(b) First, we recall the following:

(2.3) LEMMA. *Let Y be a Kähler C-space and denote by T_Y the tangent bundle of Y.*

(1) *Assume that* $\dim Y \geqq 3$. *Then for an ample divisor D of Y, the group $H^0(T_Y(-D))$ does not vanish if and only if Y is a projective space and D is a hyperplane.*

(2) *The group $H^1(T_Y)$ vanishes.*

(3) *$H^q(\Omega_Y^p)$ does not vanish if and only if $p=q$.*

PROOF. (1) is a special case of Theorem 8 in [**12**]. (2) and (3) can be found in [**5**], Theorem VII and Lemma 14.2, respectively. Q.E.D.

(2.4) LEMMA. *Let Y be an n-dimensional Kähler C-space with $b_2(Y)=1$, $n \geqq 3$. Then the group $H^q(\Omega_Y^1(a))$ vanishes in the following cases:*

(1) *$a<0$ and $q \leqq n-2$.*

(2) *$a>0$ and $q \geqq 1$.*

PROOF. (1) follows from the vanishing theorem of Kodaira-Nakano. (2) can be found in [**1**, p. 66]. Q.E.D.

For the proof of the following (2.5), we need some explicit calculations. Since they are quite elementary, we only give data to carry them out in Appendix and leave the detail to the reader.

(2.5) THEOREM. *Let Y be an n-dimensional Kähler C-space with $b_2(Y)=1$ in Table 2. Then the group $H^q(\Omega_Y^1(a))$ vanishes except for the following cases:*

(1) *$q=0$ and*

$$a \geqq \begin{cases} 2 & \text{if } Y=(C_l, \alpha_r),\ (F_4, \alpha_4) \text{ or a symmetric space,} \\ 1 & \text{otherwise.} \end{cases}$$

(2) *$q=1$ and $a=0$.*

(3) *$q=n-1$ and*

either

(i) *$Y=Q^n$ and $a=2-n$,*

or

(ii) *$Y=(C_l, \alpha_2)$ or (F_4, α_4), and $a=1-k(Y)$.*

(4) *$q=n$ and*

$$a \leqq \begin{cases} 1-k(Y)=-n & \text{if } Y=\boldsymbol{P}^n, \\ -k(Y) & \text{otherwise.} \end{cases}$$

PROOF. If $q=1$ (resp. $q=n$), then, by (2.3) and (2.4), only the situation (2) (resp. (4)) above is possible. Thus, by (2.3) (3) and (2.4), we only have to consider the cases: (i) $q=0$ and $a>0$, and (ii) $q=n-1$ and $a<0$. Let $-\lambda$ be the lowest weight of $G^i(\mathfrak{n}^+)$. First, consider the case (i). Note that the coefficient of λ_r in $-\lambda$ (i.e., the integer $\langle -\lambda, \alpha_r \rangle$) is either 1 or 2. By an argument in

Table 3.

$\mathfrak{g}$	r	lowest weights of $G^i(\mathfrak{n}^+)$
B_l	$3 \leqq r \leqq l-1$	(G^2) $\lambda_r-\lambda_{r-2}$
D_l	$3 \leqq r \leqq l-2$	
C_l	$2 \leqq r \leqq l-1$	(G^2) $2\lambda_r-2\lambda_{r-1}$
E_6	3	(G^2) $\lambda_3-\lambda_6$
	4	(G^2) $\lambda_4-\lambda_1-\lambda_6$, (G^3) $\lambda_4-\lambda_2$
E_7	2	(G^2) $\lambda_2-\lambda_7$
	3	(G^2) $\lambda_3-\lambda_6$, (G^3) $\lambda_3-\lambda_1$
	4	(G^2) $\lambda_4-\lambda_1-\lambda_6$, (G^3) $\lambda_4-\lambda_2-\lambda_7$, (G^4) $\lambda_4-\lambda_3$
	5	(G^2) $\lambda_5-\lambda_1-\lambda_7$, (G^3) $\lambda_5-\lambda_2$
	6	(G^2) $\lambda_6-\lambda_1$
E_8	1	(G^2) $\lambda_1-\lambda_8$
	2	(G^2) $\lambda_2-\lambda_7$, (G^3) $\lambda_2-\lambda_1$
	3	(G^2) $\lambda_3-\lambda_6$, (G^3) $\lambda_3-\lambda_1-\lambda_8$, (G^4) $\lambda_3-\lambda_2$
	4	$\begin{cases} (G^2)\ \lambda_4-\lambda_1-\lambda_6,\ (G^3)\ \lambda_4-\lambda_2-\lambda_7,\ (G^4)\ \lambda_4-\lambda_3-\lambda_8, \\ (G^5)\ \lambda_4-\lambda_1-\lambda_2,\ (G^6)\ \lambda_4-\lambda_5 \end{cases}$
	5	(G^2) $\lambda_5-\lambda_1-\lambda_7$, (G^3) $\lambda_5-\lambda_2-\lambda_8$, (G^4) $\lambda_5-\lambda_3$, (G^5) $\lambda_5-\lambda_6$
	6	(G^2) $\lambda_6-\lambda_1-\lambda_8$, (G^3) $\lambda_6-\lambda_2$, (G^4) $\lambda_6-\lambda_7$
	7	(G^2) $\lambda_7-\lambda_1$, (G^3) $\lambda_7-\lambda_8$
F_4	2	(G^2) $\lambda_2-2\lambda_4$, (G^3) $\lambda_2-\lambda_1$
	3	(G^2) $2\lambda_3-\lambda_1-2\lambda_4$, (G^3) $\lambda_3-\lambda_4$, (G^4) $2\lambda_3-\lambda_2$
	4	(G^2) $2\lambda_4-\lambda_1$

[**10**; §3], we can show $H^0(G^i\Omega_Y^1(b))=0$ for $b\leqq 1$ (resp. $b\leqq 0$) if $\langle -\lambda, \alpha_r\rangle=2$ (resp. 1). Thus (1) follows from (2.1), (2.2) and Table 3. Next, consider the case (ii). A direct calculation shows that $H^{n-1}(G^i\Omega_Y^1(a))$ does not vanish in the following cases:

(A) $Y=Q^n$: $i=1$ and $a=2-n$,
$Y=(C_l, \alpha_2)$ or (F_4, α_4): $i=1$ and $a=1-k(Y)$,

(B) $Y=(B_l, \alpha_2), (D_l, \alpha_2), (E_6, \alpha_2), (E_7, \alpha_1), (E_8, \alpha_8), (F_4, \alpha_1)$ or (G_2, α_2):
$i=1$ and $a=1-k(Y)$,

(C) $Y=(F_4, \alpha_3)$: $i=2$ and $a=-k(Y)$.

If Y is one in (A), then $G^j(\mathfrak{n}^+)=0$ for $j\geqq 3$ and we can show $H^*(G^2\Omega_Y^1(a))=0$ for a in (A). Thus $H^{n-1}(\Omega_Y^1(a))$ does not vanish. If Y is one in (B), then we see that $H^n(G^2\Omega_Y^1(1-k))$ does not vanish. In fact, since the lowest weight of $G^2(\mathfrak{n}^+)$ is λ_r (cf. (2.2)), $\lambda_r-(1-k(Y))\lambda_r=k(Y)\lambda_r$ is nothing but the lowest weight corresponding to the canonical bundle K_Y. Thus we get $H^n(G^2\Omega_Y^1(1-k))\cong H^n(\Omega_Y^n)\cong \boldsymbol{C}$. Now, (2.1) gives the following exact sequence:

$$\begin{aligned} 0 \longrightarrow H^{n-1}(\Omega_Y^1(1-k)) &\longrightarrow H^{n-1}(G^1\Omega_Y^1(1-k)) \\ &\longrightarrow H^n(G^2\Omega_Y^1(1-k)) \longrightarrow H^n(\Omega_Y^1(1-k)) \longrightarrow 0. \end{aligned}$$

Since we have $H^n(\Omega_Y^1(1-k))=0$ by (2.3)(1), the above sequence shows the equality,

$$(*) \qquad h^{n-1}(\Omega_Y^1(1-k)) = h^{n-1}(G^1\Omega_Y^1(1-k))-h^n(G^2\Omega_Y^1(1-k)) = h^{n-1}(G^1\Omega_Y^1(1-k))-1,$$

where we set $h^{n-1}(\Omega_Y^1(1-k)):=\dim H^{n-1}(\Omega_Y^1(1-k))$, etc. We show $h^{n-1}(G^1\Omega_Y^1(1-k))=1$. A direct calculation shows $\langle\delta, \lambda_r\rangle=k(Y)$ and $\langle\alpha_r, \lambda_r\rangle=1$. Let σ_1 (resp. σ_2) denote the reflection with respect to the hyperplane orthogonal to α_r (resp. λ_r). Then we have

$$\sigma_2\sigma_1(\delta)=\sigma_2(\delta-\alpha_r)=\delta-\alpha_r-\langle\delta-\alpha_r, \lambda_r\rangle\lambda_r=-\alpha_r+(1-k)\lambda_r+\delta.$$

Since the lowest weight of $G^1(\mathfrak{n}^+)$ is α_r, this and (1.3) imply that $h^{n-1}(G^1\Omega_Y^1(1-k))=\deg(0)=1$. Thus, by $(*)$, we get $H^{n-1}(\Omega_Y^1(1-k))=0$ for Y in (B). Finally, consider the case $Y=(F_4, \alpha_3)$. By (C), the group $H^{n-1}(\Omega_Y^1(-k))$ may not vanish. However, we see $H^{n-1}(\Omega_Y^1(-k))\cong H^1(T_Y)^*$ by Serre's duality theorem. Thus (2.3)(2) implies that $H^{n-1}(\Omega_Y^1(-k))$ vanishes. Q.E.D.

(2.6) REMARKS. (1) Kimura [**7**] showed (2.5) in the case where Y is an irreducible Hermitian symmetric spaces of compact type. He also tried to extend it to an arbitrary Kähler C-spaces with $b_2=1$ in [**9**]. Unfortunately, his result seems incomplete since he ignored (2.3)(1).

(2) It can be shown that $h^{n-1}(\Omega_Y^1(1-k))=1$ for $Y=(C_l, \alpha_2)$ or (F_4, α_4), and $h^{n-1}(\Omega_Y^1(2-n))=1$ for $Y=Q^n$.

(2.7) COROLLARY. *Let Y be a Kähler C-space with $b_2(Y)=1$ which is not a projective space. Then the following hold for any positive integer d:*

(1) $H^0(T_Y(-d))=0$,

(2) $H^1(T_Y(-d))=0$ *except for the cases:*

(i) $Y=Q^n$ *and* $d=2$,

(ii) $Y=(C_l, \alpha_2)$ *or* (F_4, α_4), *and* $d=1$.

§ 3. Proof of Main Theorem.

In this section, we complete the proof of Main Theorem. Let Y be a Kähler C-space with $b_2(Y)=1$ and X a non-singular hypersurface of degree d in Y. We first study cohomological properties of X. In doing so, we assume that Y is neither a projective space nor a complex hyperquadric. Thus, in particular, we get $\dim Y \geqq 5$ by Table 2. Since X is an ample divisor of Y, it follows from Lefschetz's hyperplane-section theorem that $\mathrm{Pic}(X)\cong\mathrm{Pic}(Y)\cong\boldsymbol{Z}$. To begin with, we observe the following easy fact:

(3.1) LEMMA. *Let X and Y be as above. Then*

(1) $\dim Y=\dim X+1$,

(2) $h^0(\mathcal{O}_Y(1))=h^0(\mathcal{O}_X(1))+\begin{cases} 0 & \text{for } d\geqq 2, \\ 1 & \text{for } d=1, \end{cases}$

(3) $k(Y)=k(X)+d$,

where $\mathcal{O}_X(1)$ denotes the restriction of $\mathcal{O}_Y(1)$ to X and $k(X)$ is the integer defined by $K_X=\mathcal{O}_X(-k(X))$.

PROOF. (1) is clear and (3) follows from the adjunction formula. To see (2), consider the following exact sequence:

$$0 \longrightarrow \mathcal{O}_Y(1-d) \longrightarrow \mathcal{O}_Y(1) \longrightarrow \mathcal{O}_X(1) \longrightarrow 0.$$

Since it follows from [7, Theorem 6] that the group $H^1(\mathcal{O}_Y(a))$ vanishes for any integer a, the derived cohomology exact sequence shows the equality $h^0(\mathcal{O}_Y(1)) =h^0(\mathcal{O}_X(1))+h^0(\mathcal{O}_Y(1-d))$. This implies (2). Q. E. D.

The following is the heart of our argument.

(3.2) LEMMA. *Let X and Y be as above. Then the group $H^1(T_X(-d))$ does not vanish.*

PROOF. Consider the following exact sequences:

(1) $0 \longrightarrow T_X \longrightarrow T_Y|_X \longrightarrow N_{X/Y} \longrightarrow 0$,

(2) $0 \longrightarrow T_Y(-d) \longrightarrow T_Y \longrightarrow T_Y|_X \longrightarrow 0$.

Note that the normal bundle $N_{X/Y}$ is isomorphic to $\mathcal{O}_X(d)$. Tensoring the above sequences with $\mathcal{O}_Y(-d)$, we get

$$0 \longrightarrow T_X(-d) \longrightarrow T_Y(-d)|_X \longrightarrow \mathcal{O}_X \longrightarrow 0$$

and

$$0 \longrightarrow T_Y(-2d) \longrightarrow T_Y(-d) \longrightarrow T_Y(-d)|_X \longrightarrow 0.$$

From these, we derive the cohomology diagram of exact sequences:

$$\begin{array}{ccccccc} & & H^0(T_Y(-d)) & & & & \\ & & \downarrow & & & & \\ H^0(T_X(-d)) & \longrightarrow & H^0(T_Y(-d)|_X) & \longrightarrow & H^0(\mathcal{O}_X) & \longrightarrow & H^1(T_X(-d)) \\ & & \downarrow & & & & \\ & & H^1(T_Y(-2d)). & & & & \end{array}$$

It follows from (2.7) that $H^0(T_Y(-d))=H^1(T_Y(-2d))=0$, since we have assumed that Y is neither a projective space nor a complex quadric. Thus $H^0(T_Y(-d)|_X)$ vanishes and in particular we have an injection of $H^0(\mathcal{O}_X)$ into $H^1(T_X(-d))$. Hence $H^1(T_X(-d))$ does not vanish. Q. E. D.

The following is due to the referee.

(3.3) LEMMA. *Let X be (C_l, α_r), (F_4, α_4) or an irreducible Hermitian symmetric space of compact type. Suppose that X can be embedded in another Kähler C-space Y with $b_2(Y)=1$ as a non-singular hypersurface of degree d. Then $H^0(\Omega_Y^1(1))=0$.*

PROOF. Put $n=\dim X$. Tensoring the exact sequence (1) in the proof of (3.2) with $\mathcal{O}_X(-k(X)-1)$, we get

$$0 \longrightarrow T_X(-k(X)-1) \longrightarrow T_Y(d-k(Y)-1)|_X \longrightarrow \mathcal{O}_X(d-k(X)-1) \longrightarrow 0,$$

because we have $k(Y)=k(X)+d$ by (3.1) (3). From this, we derive the cohomology exact sequence

$$H^n(T_X(-k(X)-1)) \longrightarrow H^n(T_Y(d-k(Y)-1)|_X) \longrightarrow H^n(\mathcal{O}_X(d-k(X)-1)) \longrightarrow 0.$$

By Serre's duality theorem and (2.5), we have $H^n(T_X(-k(X)-1)) \cong H^0(\Omega_X^1(1))^*=0$ and thus $H^n(T_Y(d-k(Y)-1)|_X) \cong H^n(\mathcal{O}_X(d-k(X)-1))$. Since $H^n(\mathcal{O}_X(d-k(X)-1)) \cong H^0(\mathcal{O}_X(1-d))^*$, we get

$$H^n(T_Y(d-k(Y)-1)|_X) \cong \begin{cases} 0 & \text{if } d \geqq 2, \\ \boldsymbol{C} & \text{if } d=1. \end{cases}$$

Similarly, tensoring the exact sequence (2) in the proof of (3.2) with $\mathcal{O}_Y(d-k(Y)-1)$, we get

$$0 \longrightarrow T_Y(-k(Y)-1) \longrightarrow T_Y(d-k(Y)-1) \longrightarrow T_Y(d-k(Y)-1)|_X \longrightarrow 0.$$

From this, we derive the cohomology exact sequence

$$\cdots \longrightarrow H^n(T_Y(-k(Y)-1)) \longrightarrow H^n(T_Y(d-k(Y)-1)) \longrightarrow H^n(T_Y(d-k(Y)-1)|_X)$$
$$\longrightarrow H^{n+1}(T_Y(-k(Y)-1)) \longrightarrow H^{n+1}(T_Y(d-k(Y)-1)) \longrightarrow 0.$$

Then, as above, we get $H^n(T_Y(-k(Y)-1))\cong H^1(\Omega_Y^1(1))^*=0$ and

$$H^n(T_Y(d-k(Y)-1)) \cong H^1(\Omega_Y^1(1-d))^* \cong \begin{cases} 0 & \text{if } d\geqq 2, \\ \boldsymbol{C} & \text{if } d=1. \end{cases}$$

Thus we get $H^n(T_Y(d-k(Y)-1))\cong H^n(T_Y(d-k(Y)-1)|_X)$ and $H^{n+1}(T_Y(-k(Y)-1))$ $\cong H^{n+1}(T_Y(d-k(Y)-1))$. By Serre's duality theorem and (2.5), this last isomorphism shows $H^0(\Omega_Y^1(1))\cong H^0(\Omega_Y^1(1-d))=0$. Q. E. D.

PROOF OF MAIN THEOREM: If Y is $\boldsymbol{P}^{n+1}$ or Q^{n+1}, then the result is well-known. Thus we assume that Y is neither a projective space nor a complex quadric. Further, it is shown by Kimura [8] and Sakane [**14**, Theorem 1] that (F_4, α_4) (resp. (C_l, α_2)) can be embedded in (E_6, α_1) (resp. (A_{2l-1}, α_2)) as a hypersurface of degree 1. Thus, it suffices to show the "only if" part of Main Theorem. Let X be a non-singular hypersurface of degree d in Y and assume that X is homogeneous. Then the fact $\mathrm{Pic}(X)\cong\boldsymbol{Z}$ implies that X is a Kähler C-space with $b_2(X)=1$. Thus, by (2.7) and (3.2), we see that X and d must be one of the following:

(a) $X=Q^n$ and $d=2$,
(b) $X=(C_l, \alpha_2)$, $l\geqq 3$, and $d=1$,
(c) $X=(F_4, \alpha_4)$ and $d=1$.

Then, it follows from (3.3) and (2.5) (1) that Y is (C_l, α_r), (F_4, α_4) or a symmetric space. If we use Table 2 to find Y satisfying (1), (2) and (3) in (3.1), then a simple calculation shows that the only possible Y is $\boldsymbol{P}^{n+1}$ for (a), (A_{2l-1}, α_2) for (b), and (E_6, α_1) for (c), respectively. Q. E. D.

§ 4. Appendix.

We shall describe $\Phi(\mathfrak{n}^+)$ and give the value $\langle\delta, \alpha\rangle$ for each $\alpha\in\Phi(\mathfrak{n}^+)$. A construction of Φ for each $\mathfrak{g}$ can be found in [**6**, § 12].

First, we consider the case where $\mathfrak{g}$ is a classical Lie algebra. We denote by $\varepsilon_1, \cdots, \varepsilon_n$ the orthonormal basis for $\boldsymbol{R}^n$ with respect to the usual inner product $(\cdot, \cdot)$.

(1) $Y=(A_l, \alpha_r)$, $2\leqq r\leqq l+1-r$, $l\geqq 4$:
$\Delta=\{\alpha_i=\varepsilon_i-\varepsilon_{i+1};\ 1\leqq i\leqq l\}$,
$\Phi^+=\{\varepsilon_i-\varepsilon_j;\ 1\leqq i<j\leqq l+1\}$,

$$\lambda_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{l+1}\sum_{j=1}^{l+1}\varepsilon_j, \quad 1 \leqq i \leqq l,$$
$$2\delta = l\varepsilon_1 + \cdots + (l-2i+2)\varepsilon_i + \cdots - l\varepsilon_{l+1},$$
$$\Phi(\mathfrak{n}^+) = \{\varepsilon_i - \varepsilon_j;\ \ 1 \leqq i \leqq r < j \leqq l+1\},$$
$$\langle \delta, \varepsilon_i - \varepsilon_j \rangle = j - i.$$

(2) $Y = (B_l, \alpha_r)$, $1 \leqq r \leqq l-1$, $l \geqq 2$:

$$\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1};\ \ 1 \leqq i \leqq l-1,\ \alpha_l = \varepsilon_l\},$$
$$\Phi^+ = \{\varepsilon_i \pm \varepsilon_j;\ \ 1 \leqq i < j \leqq l,\ \varepsilon_i;\ 1 \leqq i \leqq l\},$$
$$\begin{cases} \lambda_i = \varepsilon_1 + \cdots + \varepsilon_i, & 1 \leqq i \leqq l-1, \\ \lambda_l = (\varepsilon_1 + \cdots + \varepsilon_l)/2. \end{cases}$$
$$2\delta = (2l-1)\varepsilon_1 + \cdots + (2l-2i+1)\varepsilon_i + \cdots + \varepsilon_l,$$
$$\Phi(\mathfrak{n}^+) = \{\varepsilon_i \pm \varepsilon_j;\ \ 1 \leqq i \leqq r < j \leqq l,\ \ \varepsilon_i;\ \ 1 \leqq i \leqq r,\ \ \varepsilon_i + \varepsilon_j;\ \ 1 \leqq i < j \leqq r\},$$
$$\langle \delta, \varepsilon_i - \varepsilon_j \rangle = j - i, \quad \langle \delta, \varepsilon_i + \varepsilon_j \rangle = 2l + 1 - (i+j),$$
$$\langle \delta, \varepsilon_i \rangle = 2l - 2i + 1.$$

(3) $Y = (C_l, \alpha_r)$, $2 \leqq r \leqq l$, $l \geqq 3$:

$$\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1};\ \ 1 \leqq i \leqq l-1,\ \alpha_l = 2\varepsilon_l\},$$
$$\Phi^+ = \{\varepsilon_i \pm \varepsilon_j;\ \ 1 \leqq i < j \leqq l,\ 2\varepsilon_i;\ 1 \leqq i \leqq l\},$$
$$\lambda_i = \varepsilon_1 + \cdots + \varepsilon_i, \quad 1 \leqq i \leqq l,$$
$$\delta = l\varepsilon_1 + \cdots + (l-i+1)\varepsilon_i + \cdots + \varepsilon_l,$$
$$\Phi(\mathfrak{n}^+) = \{\varepsilon_i \pm \varepsilon_j;\ \ 1 \leqq i \leqq r < j \leqq l,\ \ 2\varepsilon_i;\ \ 1 \leqq i \leqq r,\ \ \varepsilon_i + \varepsilon_j;\ \ 1 \leqq i < j \leqq r\},$$
$$\langle \delta, \varepsilon_i - \varepsilon_j \rangle = j - i,\ \langle \delta, \varepsilon_i + \varepsilon_j \rangle = 2l + 2 - (i+j),$$
$$\langle \delta, 2\varepsilon_i \rangle = l - i - 1.$$

(4) $Y = (D_l, \alpha_r)$, $1 \leqq r \leqq l-2$ or $r = l$, $l \geqq 3$:

$$\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1};\ \ 1 \leqq i \leqq l-1,\ \alpha_l = \varepsilon_{l-1} + \varepsilon_l\},$$
$$\Phi^+ = \{\varepsilon_i \pm \varepsilon_j;\ \ 1 \leqq i < j \leqq l\},$$
$$\begin{cases} \lambda_i = \varepsilon_1 + \cdots + \varepsilon_i, & 1 \leqq i \leqq l-2, \\ \lambda_{l-1} = (\varepsilon_1 + \cdots + \varepsilon_{l-1} - \varepsilon_l)/2, \\ \lambda_l = (\varepsilon_1 + \cdots + \varepsilon_l)/2, \end{cases}$$
$$\delta = (l-1)\varepsilon_1 + \cdots + (l-i)\varepsilon_i + \cdots + \varepsilon_{l-1},$$
$$\Phi(\mathfrak{n}^+) = \{\varepsilon_i \pm \varepsilon_j;\ \ 1 \leqq i \leqq r < j \leqq l,\ \ \varepsilon_i + \varepsilon_j;\ \ 1 \leqq i < j \leqq r\},$$
$$\langle \delta, \varepsilon_i - \varepsilon_j \rangle = j - i, \quad \langle \delta, \varepsilon_i + \varepsilon_j \rangle = 2l - (i+j).$$

In case $\mathfrak{g}$ is an exceptional Lie algebra, we give the table of the positive roots classified according to the value $\langle \delta, \alpha \rangle$. We abbreviate a positive root $\alpha = \sum n_i \alpha_i$ as $(n_1, \cdots, n_l)$ in the following. Moreover, if $\mathfrak{g} = F_4$ or G_2, then two distinct root lengths can occur. We denote by $(n_1, \cdots, n_l)_L$ a long root and by $(n_1, \cdots, n_l)_S$ a short root.

(5) E_6, E_7 and E_8. The positive roots of E_8 can be found in Table A.1. The positive roots of E_6 (resp. E_7) can be identified with those of E_8 satisfying $n_7 = n_8 = 0$ (resp. $n_8 = 0$). In this case, $\langle \lambda_i, \alpha \rangle = n_i$ for $\alpha = (n_1, \cdots, n_8)$.

Table A. 1.

$\langle\delta, \alpha\rangle$	α
1	(1, 0, 0, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0, 0, 0), (0, 0, 0, 1, 0, 0, 0, 0), (0, 0, 0, 0, 1, 0, 0, 0), (0, 0, 0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 0, 0, 1, 0), (0, 0, 0, 0, 0, 0, 0, 1)
2	(1, 0, 1, 0, 0, 0, 0, 0), (0, 1, 0, 1, 0, 0, 0, 0), (0, 0, 1, 1, 0, 0, 0, 0), (0, 0, 0, 1, 1, 0, 0, 0), (0, 0, 0, 0, 1, 1, 0, 0), (0, 0, 0, 0, 0, 1, 1, 0), (0, 0, 0, 0, 0, 0, 1, 1)
3	(1, 0, 1, 1, 0, 0, 0, 0), (0, 1, 0, 1, 1, 0, 0, 0), (0, 1, 1, 1, 0, 0, 0, 0), (0, 0, 1, 1, 1, 0, 0, 0), (0, 0, 0, 1, 1, 1, 0, 0), (0, 0, 0, 0, 1, 1, 1, 0), (0, 0, 0, 0, 0, 1, 1, 1)
4	(1, 0, 1, 1, 1, 0, 0, 0), (0, 1, 0, 1, 1, 1, 0, 0), (1, 1, 1, 1, 0, 0, 0, 0), (0, 1, 1, 1, 1, 0, 0, 0), (0, 0, 1, 1, 1, 1, 0, 0), (0, 0, 0, 1, 1, 1, 1, 0), (0, 0, 0, 0, 1, 1, 1, 1)
5	(1, 0, 1, 1, 1, 1, 0, 0), (0, 1, 0, 1, 1, 1, 1, 0), (1, 1, 1, 1, 1, 0, 0, 0), (0, 1, 1, 1, 1, 1, 0, 0), (0, 0, 1, 1, 1, 1, 1, 0), (0, 0, 0, 1, 1, 1, 1, 1), (0, 1, 1, 2, 1, 0, 0, 0)
6	(1, 0, 1, 1, 1, 1, 1, 0), (0, 1, 0, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 0, 0), (0, 1, 1, 1, 1, 1, 1, 0), (0, 0, 1, 1, 1, 1, 1, 1), (1, 1, 1, 2, 1, 0, 0, 0), (0, 1, 1, 2, 1, 1, 0, 0)
7	(1, 0, 1, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 0), (0, 1, 1, 1, 1, 1, 1, 1), (1, 1, 1, 2, 1, 1, 0, 0), (0, 1, 1, 2, 1, 1, 1, 0), (1, 1, 2, 2, 1, 0, 0, 0), (0, 1, 1, 2, 2, 1, 0, 0)
8	(1, 1, 1, 1, 1, 1, 1, 1), (1, 1, 1, 2, 1, 1, 1, 0), (0, 1, 1, 2, 1, 1, 1, 1), (1, 1, 1, 2, 2, 1, 0, 0), (0, 1, 1, 2, 2, 1, 1, 0), (1, 1, 2, 2, 1, 1, 0, 0)
9	(1, 1, 1, 2, 1, 1, 1, 1), (1, 1, 1, 2, 2, 1, 1, 0), (0, 1, 1, 2, 2, 1, 1, 1), (1, 1, 2, 2, 1, 1, 1, 0), (1, 1, 2, 2, 2, 1, 0, 0), (0, 1, 1, 2, 2, 2, 1, 0)
10	(1, 1, 1, 2, 2, 1, 1, 1), (1, 1, 2, 2, 1, 1, 1, 1), (1, 1, 2, 2, 2, 1, 1, 0), (0, 1, 1, 2, 2, 2, 1, 1), (1, 1, 1, 2, 2, 2, 1, 0), (1, 1, 2, 3, 2, 1, 0, 0)
11	(1, 1, 1, 2, 2, 2, 1, 1), (1, 1, 2, 2, 2, 1, 1, 1), (1, 1, 2, 2, 2, 2, 1, 0), (0, 1, 1, 2, 2, 2, 2, 1), (1, 1, 2, 3, 2, 1, 1, 0), (1, 2, 2, 3, 2, 1, 0, 0)
12	(1, 1, 1, 2, 2, 2, 2, 1), (1, 1, 2, 2, 2, 2, 1, 1), (1, 1, 2, 3, 2, 1, 1, 1), (1, 2, 2, 3, 2, 1, 1, 0), (1, 1, 2, 3, 2, 2, 1, 0)
13	(1, 1, 2, 2, 2, 2, 2, 1), (1, 2, 2, 3, 2, 1, 1, 1), (1, 1, 2, 3, 2, 2, 1, 1), (1, 1, 2, 3, 3, 2, 1, 0), (1, 2, 2, 3, 2, 2, 1, 0)
14	(1, 1, 2, 3, 2, 2, 2, 1), (1, 2, 2, 3, 2, 2, 1, 1), (1, 1, 2, 3, 3, 2, 1, 1), (1, 2, 2, 3, 3, 2, 1, 0)
15	(1, 2, 2, 3, 2, 2, 2, 1), (1, 1, 2, 3, 3, 2, 2, 1), (1, 2, 2, 3, 3, 2, 1, 1), (1, 2, 2, 4, 3, 2, 1, 0)
16	(1, 2, 2, 3, 3, 2, 2, 1), (1, 1, 2, 3, 3, 3, 2, 1), (1, 2, 2, 4, 3, 2, 1, 1), (1, 2, 3, 4, 3, 2, 1, 0)
17	(1, 2, 2, 3, 3, 3, 2, 1), (1, 2, 2, 4, 3, 2, 2, 1), (1, 2, 3, 4, 3, 2, 1, 1), (2, 2, 3, 4, 3, 2, 1, 0)
18	(1, 2, 2, 4, 3, 3, 2, 1), (1, 2, 3, 4, 3, 2, 2, 1), (2, 2, 3, 4, 3, 2, 1, 1)
19	(1, 2, 3, 4, 3, 3, 2, 1), (1, 2, 2, 4, 4, 3, 2, 1), (2, 2, 3, 4, 3, 2, 2, 1)
20	(1, 2, 3, 4, 4, 3, 2, 1), (2, 2, 3, 4, 3, 3, 2, 1)
21	(1, 2, 3, 5, 4, 3, 2, 1), (2, 2, 3, 4, 4, 3, 2, 1)
22	(1, 3, 3, 5, 4, 3, 2, 1), (2, 2, 3, 5, 4, 3, 2, 1)
23	(2, 2, 4, 5, 4, 3, 2, 1), (2, 3, 3, 5, 4, 3, 2, 1)
24	(2, 3, 4, 5, 4, 3, 2, 1)
25	(2, 3, 4, 6, 4, 3, 2, 1)
26	(2, 3, 4, 6, 5, 3, 2, 1)
27	(2, 3, 4, 6, 5, 4, 2, 1)
28	(2, 3, 4, 6, 5, 4, 3, 1)
29	(2, 3, 4. 6, 5, 4, 3, 2)

(6) F_4. The positive roots can be found in Table A.2. In this case, for $\alpha=(n_1, n_2, n_3, n_4)$, we have

$$\langle\lambda_i, \alpha\rangle = \begin{cases} 2n_i & \text{if } \alpha \text{ is shorter than } \alpha_i, \\ n_i & \text{if } \alpha \text{ has the same length as } \alpha_i, \\ n_i/2 & \text{if } \alpha \text{ is longer than } \alpha_i. \end{cases}$$

Table A. 2.

$\langle\delta, \alpha\rangle$	α
1	$(1,0,0,0)_L$, $(0,1,0,0)_L$, $(0,0,1,0)_S$, $(0,0,0,1)_S$
2	$(1,1,0,0)_L$, $(0,1,2,0)_L$, $(0,0,1,1)_S$
3	$(1,1,2,0)_L$, $(0,1,2,2)_L$, $(0,1,1,0)_S$
4	$(1,2,2,0)_L$, $(1,1,2,2)_L$, $(0,1,1,1)_S$
5	$(1,2,2,2)_L$, $(1,1,1,0)_S$, $(0,1,2,1)_S$
6	$(1,2,4,2)_L$, $(1,1,1,1)_S$
7	$(1,3,4,2)_L$, $(1,1,2,1)_S$
8	$(2,3,4,2)_L$
9	$(1,2,2,1)_S$
10	$(1,2,3,1)_S$
11	$(1,2,3,2)_S$

(7) G_2. The positive roots can be found in Table A.3. In this case, we have, for $\alpha=(n_1, n_2)$,

$$\langle\lambda_i, \alpha\rangle = \begin{cases} 3n_i & \text{if } \alpha \text{ is shorter than } \alpha_i, \\ n_i & \text{if } \alpha \text{ has the same length as } \alpha_i, \\ n_i/3 & \text{if } \alpha \text{ is longer than } \alpha_i. \end{cases}$$

Table A. 3.

$\langle\delta, \alpha\rangle$	α
1	$(1,0)_S$, $(0,1)_L$
2	$(3,1)_L$
3	$(3,2)_L$
4	$(1,1)_S$
5	$(2,1)_S$

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