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On the boundary limits of Green potentials of functions

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1. Introduction.

In the half space $D=\{x=(x_1, \cdots, x_n); x_n>0\}$, $n\geqq 2$, let $G(\cdot, \cdot)$ be the Green function in D, that is,

$$G(x, y)=\begin{cases} |x-y|^{2-n}-|\bar{x}-y|^{2-n} & \text{if } n>2, \\ \log(|\bar{x}-y|/|x-y|) & \text{if } n=2, \end{cases}$$

where $\bar{x}=(x_1, \cdots, x_{n-1}, -x_n)$ for $x=(x_1, \cdots, x_{n-1}, x_n)$. For a nonnegative measurable function f on D, we define

$$Gf(x)=\int_D G(x, y)f(y)dy.$$

Then it is noted (see e.g. [2; Lemma 2]) that $Gf\not\equiv\infty$ if and only if

(1) $$\int_D (1+|y|)^{-n} y_n f(y)dy<\infty.$$

In this paper we study the existence of nontangential limits of Gf with f satisfying (1) and the additional condition:

(2) $$\int_D y_n^{\alpha} f(y)^{n/2}\omega(f(y))dy<\infty,$$

where $\omega(t)$ is a positive nondecreasing function on R^1. In case $n\geqq 3$, ω is assumed to satisfy the following conditions:

(ω1) There exists a positive constant A such that $\omega(2r)\leqq A\omega(r)$ for any $r>0$.

(ω2) $\int_1^{\infty}\omega(t)^{-1/(n/2-1)}t^{-1}dt<\infty$.

(ω3) $\lim_{r\to\infty}\omega(r)^{-1/(n/2-1)}\int_r^{\infty}\omega(t)^{-1/(n/2-1)}t^{-1}dt=\infty$.

As typical examples of ω, we give

$$\omega(t)=[\log(2+t)]^{\delta},\ [\log(2+t)]^{n/2-1}[\log(2+(\log(2+t)))]^{\delta},\ \cdots,$$

where $\delta>n/2-1$.

We say that a function u on D has a nontangential limit l at $\xi\in\partial D$ if $u(x)$

tends to l as x tends to ξ along any cone $\Gamma(\xi, a)=\{x=(x', x_n)\in R^{n-1}\times R^1; |(x', 0)-\xi|<ax_n\}$. To evaluate the size of the set of all points at which u fails to have a nontangential limit, we use the Hausdorff measures. For a positive nondecreasing function h on an interval $(0, A_h)$, $A_h>0$, we denote by H_h the Hausdorff measure with the measure function h; if $h(r)=r^\alpha$, $\alpha>0$, then we shall write H_α for H_h. Our aim in this paper is to give generalizations of results of Widman [**6**], and, in fact, our main result is as follows:

THEOREM 1. *Let $n\geqq 3$, $0<\alpha\leqq n-1$ and f be a nonnegative measurable function on D satisfying* (1) *and* (2). *Then there exists $E\subset\partial D$ such that $H_h(E)=0$ and Gf has nontangential limit zero at any $\xi\in\partial D-E$, where $h(r)=r^\alpha\omega^*(r^{-1})$ with $\omega^*(r)=\left(\int_r^\infty \omega(t)^{-1/(n/2-1)}t^{-1}dt\right)^{-n/2+1}$.*

In the case $\alpha=n-1$, this theorem gives an improvement of Widman [**6**; Theorem 6.7], where he proved that $H_{n-1}(E)=0$. As will be shown later, Theorem 1 is best possible as to the size of the exceptional sets.

If ω fails to satisfy condition (ω2), then we are concerned with the existence of weak sense limits such as they were discussed in the author's papers [**2**], [**3**], [**4**]. As to the existence of fine limits of Green potentials, in the final section we shall add one result, which is an extension of the result of [**4**] to the case $p>1$.

In case $n=2$, letting $\omega(r)=\log(2+r)$, we aim to generalize the results of Tolsted [**5**].

2. Proof of Theorem 1.

We first note by condition (ω1) that $\omega^*(r)\leqq A^*\omega(r)$ and $\omega^*(2r)\leqq A^*\omega^*(r)$ for $r>0$ with a positive constant A^*. Further, in view of (ω3), we can show that $r^{-\delta}\omega^*(r)$ is nonincreasing on an interval (A_δ, ∞) for any $\delta>0$. Thus, H_h with $h(r)=r^\alpha\omega^*(r^{-1})$ is well defined.

For a proof of Theorem 1, we need several lemmas.

LEMMA 1. *For a nonnegative function g in $L^1(D)$, set $E=\left\{\xi\in\partial D; \limsup_{r\downarrow 0} k(r)^{-1}\int_{B(\xi,r)\cap D} g(y)dy>0\right\}$, where k is a positive nondecreasing function on an interval $(0, A_k)$, $A_k>0$, such that $k(2r)\leqq Mk(r)$ whenever $0<2r<A_k$, with a positive constant M. Then $H_k(E)=0$.*

PROOF. Letting $E_a=\left\{\xi\in\partial D; \limsup_{r\downarrow 0} k(r)^{-1}\int_{B(\xi,r)\cap D} g(y)dy>a\right\}$, $a>0$, we shall prove that $H_k(E_a)=0$. For this we have only to prove that $H_k(K)=0$ for any compact subset of E_a, since E_a is seen to be a Borel subset of ∂D. Let ε,

$0<\varepsilon<10A_h$, and K be a compact subset of E_a. By the definition of E_a, for each $\xi \in K$ there exists $r(\xi)<\varepsilon$ such that $\int_{B(\xi, r(\xi)) \cap D} g(y)dy > ak(r(\xi))$. Now we can find a finite family $\{B(\xi_j, r(\xi_j))\}$ of $\{B(\xi, r(\xi))\}$ such that $\{B(\xi_j, r(\xi_j))\}$ is mutually disjoint and $\bigcup_j B(\xi_j, 5r(\xi_j)) \supset K$. Then we note that

$$\int_{\{y \in D; y_n < \varepsilon\}} g(y)dy \geqq \sum_j \int_{B(\xi_j, r(\xi_j)) \cap D} g(y)dy \geqq \sum_j ak(r(\xi_j)) \geqq M'a \sum_j k(5r(\xi_j))$$

with a positive constant M'. Letting $\varepsilon \to 0$, we establish $H_k(K)=0$. Thus the proof of Lemma 1 is completed.

LEMMA 2. *Let $n \geqq 3$, $0<\alpha \leqq n-1$ and f be a nonnegative measurable function on D satisfying* (2). *If we set $F=\left\{\xi \in \partial D;\ \limsup_{r \downarrow 0} r^{1-n} \int_{B(\xi, r) \cap D} y_n f(y)dy > 0\right\}$, then $H_h(F)=0$ with $h(r)=r^\alpha \omega^*(r^{-1})$.*

PROOF. For simplicity, we set $p=n/2$ and $p'=p/(p-1)$. By Hölder's inequality we have

$$\begin{aligned} & r^{1-n} \int_{\{y \in B(\xi, r) \cap D; f(y) > 1/y_n\}} y_n f(y)dy \\ & \leqq r^{1-n} \left(\int_{B(\xi, r) \cap D} y_n^\alpha f(y)^p \omega(f(y))dy \right)^{1/p} \left(\int_{B(\xi, r) \cap D} y_n^{p'(1-\alpha/p)} \omega(1/y_n)^{-p'/p} dy \right)^{1/p'} \\ & \leqq M_1 \left(r^{-\alpha} \omega^*(r^{-1})^{-1} \int_{B(\xi, r) \cap D} y_n^\alpha f(y)^p \omega(f(y))dy \right)^{1/p} \end{aligned}$$

with a positive constant M_1 independent of r. On the other hand we easily find a positive constant M_2 such that

$$r^{1-n} \int_{\{y \in B(\xi, r) \cap D; f(y) < 1/y_n\}} y_n f(y)dy \leqq M_2 r$$

for any $r>0$. Now we can apply Lemma 1 to prove that $H_h(F)=0$. Thus the lemma is established.

LEMMA 3. *For a nonnegative measurable function f on D satisfying* (1), *we set*

$$u_1(x) = \int_{D - B(x, x_n/2)} G(x, y) f(y) dy.$$

Then $\lim_{x \to \xi, x \in \Gamma(\xi, a)} u_1(x)=0$ for any $a>0$ if and only if $\xi \in \partial D - F$, where F is defined as in Lemma 2.

PROOF. We shall prove the lemma in the case $n \geqq 3$, because the case $n=2$ can be proved similarly. In case $n \geqq 3$, we note easily that $G(x, y) \leqq M_1 x_n y_n |x-y|^{2-n}(|x-y|^2 + x_n^2)^{-1}$ for any x and y in D, where M_1 is a positive constant. Let $\xi \in \partial D - F$ and $\varepsilon>0$. Then we have

$$\begin{aligned}
&\limsup_{x\to\xi,\, x\in\Gamma(\xi,a)} u_1(x) \\
&\leqq M_1 \limsup_{x\to\xi,\, x\in\Gamma(\xi,a)} \int_{B(\xi,\varepsilon)\cap D} x_n(|\xi-y|+x_n)^{-n} y_n f(y)dy \\
&\leqq M_1 \limsup_{x\to\xi,\, x\in\Gamma(\xi,a)} \left\{ x_n \int_0^\varepsilon \left(\int_{B(\xi,r)\cap D} y_n f(y)dy \right) d(-(r+x_n)^{-n}) \right. \\
&\qquad\qquad \left. + x_n(\varepsilon+x_n)^{-n} \int_{B(\xi,\varepsilon)\cap D} y_n f(y)dy \right\} \\
&\leqq M_2 \sup_{r\leqq\varepsilon} r^{1-n} \int_{B(\xi,r)\cap D} y_n f(y)dy,
\end{aligned}$$

where M_2 is a positive constant independent of x and ε. Since $\xi\in\partial D-F$, the right hand side tends to zero as $\varepsilon\downarrow 0$, and hence the "if" part follows.

On the other hand it follows that

$$u_1(x) \geqq \int_{B(\xi,x_n/2)\cap D} G(x,y)f(y)dy \geqq M_3 x_n^{1-n} \int_{B(\xi,x_n/2)} y_n f(y)dy,$$

with a positive constant M_3 independent of x. Hence if $u_1(x)$ tends to zero as x tends to ξ along $\Gamma(\xi,a)$ for some $a>0$, then we see readily that $\xi\in\partial D-F$. Thus the "only if" part of the lemma follows, and the lemma is established.

LEMMA 4. *If $n\geqq 3$ and g is a nonnegative measurable function on R^n, then*

$$\begin{aligned}
&\int_{\{y;g(y)\geqq a\}} |x-y|^{2-n} g(y)dy \\
&\leqq M\left(\int g(y)^{n/2}\omega(g(y))dy\right)^{2/n} \left(\int_a^\infty \omega(t)^{-1/(n/2-1)} t^{-1}dt\right)^{1-2/n}
\end{aligned}$$

for $a>0$, where M is a positive constant independent of g, x and a.

PROOF. Define $G_j=\{y\in D;\, 2^{j-1}a\leqq g(y)<2^j a\}$ for each positive integer j, and take $r_j\geqq 0$ such that $|G_j|=|B(0,r_j)|$, where $|E|$ denotes the Lebesgue measure of a set $E\subset R^n$. Then we note that

$$\begin{aligned}
&\int_{\{y;g(y)\geqq a\}} |x-y|^{2-n} g(y)dy = \sum_{j=1}^\infty \int_{G_j} |x-y|^{2-n} g(y)dy \\
&\leqq \sum_{j=1}^\infty 2^j a \int_{G_j} |x-y|^{2-n}dy \leqq \sum_{j=1}^\infty 2^j a \int_{B(x,r_j)} |x-y|^{2-n}dy \\
&= M_1 \sum_{j=1}^\infty 2^j a |G_j|^{2/n} \\
&\leqq M_2\left(\sum_{j=1}^\infty (2^{j-1}a)^{n/2}\omega(2^{j-1}a)|G_j|\right)^{2/n} \left(\sum_{j=1}^\infty \omega(2^j a)^{-1/(n/2-1)}\right)^{1-2/n} \\
&\leqq M_3\left(\int g(y)^{n/2}\omega(g(y))dy\right)^{2/n} \left(\int_a^\infty \omega(t)^{-1/(n/2-1)} t^{-1}dt\right)^{1-2/n},
\end{aligned}$$

where M_1, M_2 and M_3 are positive constants independent of g, x and a.

We are now ready to prove Theorem 1.

PROOF OF THEOREM 1. Suppose f is a nonnegative measurable function on D satisfying (1) and (2), and define F as in Lemma 2. Then, in view of Lemmas 1 and 2, it follows that $H_h(F)=0$ with $h(t)=t^\alpha\omega^*(t^{-1})$. Write $Gf=u_1+u_2$, where u_1 is defined as in Lemma 3 and $u_2(x)=\int_{B(x,x_n/2)}G(x,y)f(y)dy$. If $\xi\in\partial D-F$, then Lemma 3 implies that u_1 has nontangential limit zero at ξ. On the other hand, since $u_2(x)\leqq\int_{B(x,x_n/2)}|x-y|^{2-n}f(y)dy$, it follows from Lemma 4 that

$$\begin{aligned}u_2(x) &\leqq x_n^{-1}\int_{B(x,x_n/2)}|x-y|^{2-n}dy\\ &\quad+M_1\Big(\int_{B(x,x_n/2)}f(y)^{n/2}\omega(f(y))dy\Big)^{2/n}\Big(\int_{x_n^{-1}}^{\infty}\omega(t)^{-1/(n/2-1)}t^{-1}dt\Big)^{1-2/n}\\ &\leqq M_2x_n+M_2\Big(\omega^*(x_n^{-1})^{-1}\int_{B(x,x_n/2)}f(y)^{n/2}\omega(f(y))dy\Big)^{2/n},\end{aligned}$$

where M_1 and M_2 are positive constants independent of x. Hence we derive

$$u_2(x)\leqq M_2x_n+M_3\Big(h(x_n)^{-1}\int_{B(x,x_n/2)}y_n^\alpha f(y)^{n/2}\omega(f(y))dy\Big)^{2/n}$$

with a positive constant M_3. By Lemma 1 we see that the right hand side has nontangential limit zero at $\xi\in\partial D-F'$, where $H_h(F')=0$. Therefore Gf has nontangential limit 0 at $\xi\in\partial D-F\cup F'$ and $H_h(F\cup F')=0$. Thus the theorem is established.

3. Further results concerning nontangential limits.

We begin with giving a similar result in the two dimensional case.

THEOREM 2. *Let $n=2$ and $0<\alpha\leqq 1$. If f is a nonnegative measurable function on D satisfying* (1) *and*

$$\int_D y_2^\alpha f(y)[\log(2+f(y))]dy<\infty. \tag{3}$$

Then Gf has nontangential limit zero at $\xi\in\partial D$ except for those in a set E such that $H_\alpha(E)=0$.

In case $\alpha=1$, this theorem was proved by Tolsted [5].

PROOF OF THEOREM 2. We write $Gf=u_1+u_2$ as in the proof of Theorem 1. By Lemmas 1 and 3 we see that u_1 has nontangential limit zero at $\xi\in\partial D-E_1$, where E_1 is a subset of ∂D such that $H_\alpha(E_1)=0$. If we note the following result instead of Lemma 4, then we can show that u_2 has nontangential limit zero at $\xi\in\partial D$ except those in a set E_2 satisfying $H_\alpha(E_2)=0$.

LEMMA 5. *If f is a nonnegative measurable function on D, then* $\int_{\{y;f(y)\geqq 1\}}G(x, y)f(y)dy\leqq M\eta\log(1/\eta)$, *whenever* $\eta\equiv\int_D f(y)\log(2+f(y))dy<e^{-1}$, *where M is a positive constant independent of x and f.*

PROOF. For each positive integer j, set $F_j=\{y\in D;\, 2^{j-1}\leqq f(y)<2^j\}$. Then we have

$$\int_{\{y;f(y)\geqq 1\}}G(x, y)f(y)dy \leqq \sum_{j=1}^{\infty}2^j\int_{F_j}\log(1+4x_ny_n|x-y|^{-1}|\bar{x}-y|^{-1})dy$$

$$\leqq \sum_{j=1}^{\infty}2^j\int_{B(x,r_j)}\log(1+4|x-y|^{-1})dy \leqq M_1\sum_{j=1}^{\infty}2^jr_j^2\log(2+4r_j^{-1}),$$

where $x_n<1$, $|F_j|=|B(0, r_j)|$ and M_1 is a positive constant. Let I' be the set of all positive integer j such that $r_j\leqq\eta^j$ $(<e^{-j})$, and note

$$\sum_{j\in I'}2^jr_j^2\log(2+4r_j^{-1}) \leqq \sum_{j\in I'}2^j\eta^j\log(2+4\eta^{-j})$$

$$\leqq \sum_{j\in I'}2^jj\eta^j\log(2+4\eta^{-1}) \leqq M_2\eta\log(1/\eta)$$

with a positive constant M_2. On the other hand, letting I'' be the set of all positive integers j such that $j\notin I'$, we obtain

$$\sum_{j\in I''}2^jr_j^2\log(2+4r_j^{-1}) \leqq \sum_{j\in I''}2^jr_j^2\log(2+4\eta^{-j})$$

$$\leqq \sum_{j\in I''}2^jjr_j^2\log(2+4\eta^{-1}) \leqq M_3\eta\log(2+4\eta^{-1})$$

with a positive constant M_3. Thus the lemma is proved.

THEOREM 3. *Let $n\geqq 3$ and f be a nonnegative measurable function on D satisfying* (1) *and* (2) *with $\alpha=0$. Then* $\lim_{x_n\to 0,\, x\in D}(1+|x|)^{-n}Gf(x)=0$.

PROOF. Let $\varepsilon>0$. Then we can find a positive number M_1 depending on ε such that $G(x, y)\leqq M_1x_ny_n(1+|x|)^n(1+|y|)^{-n}$ whenever $y_n>\varepsilon$ and $0<x_n<\varepsilon/2$. By (1) we can apply Lebesgue's dominated convergence theorem to obtain $\lim_{x_n\downarrow 0}(1+|x|)^{-n}\int_{\{y\in D;y_n>\varepsilon\}}G(x, y)f(y)dy=0$. On the other hand, in view of Lemma 4, we establish

$$\int_{\{y\in D;y_n<\varepsilon\}}G(x, y)f(y)dy \leqq \int_{\{y\in D;\, y_n<\varepsilon\}}G(x, y)dy+\int_{\{y\in D;y_n<\varepsilon,\, f(y)\geqq 1\}}|x-y|^{2-n}f(y)dy$$

$$\leqq M_2x_n\varepsilon+M_2\left(\int_{\{y\in D;y_n<\varepsilon\}}f(y)^{n/2}\omega(f(y))dy\right)^{2/n}$$

with a positive constant M_2, which tends to zero with ε uniformly on the set

$\{x \in D;\ x_n < 1\}$. Thus the theorem is obtained.

In the same manner we can prove the following result.

THEOREM 4. *Let $n=2$ and f be a nonnegative measurable function on D satisfying* (1) *and* (3) *with $\alpha=0$. Then* $\lim_{x_2 \downarrow 0} (1+|x|)^{-2} Gf(x)=0$.

4. The existence of nontangential limits of $x_n^\beta Gf(x)$, $\beta>0$.

In this section we deal with the Green potentials of functions satisfying condition (2) with $\alpha > n-1$.

THEOREM 5. *Let $n \geqq 3$, $0<\beta<n-1$ and f be a nonnegative measurable function on D satisfying* (1) *and*

$$\int_D y_n^{n-1-\beta} [y_n^\beta f(y)]^{n/2} \omega(f(y)) dy < \infty . \tag{4}$$

Then $x_n^\beta Gf(x)$ has nontangential limit zero at any $\xi \in \partial D - E$, where $H_{n-1-\beta}(E)=0$.

PROOF. Let f be as in the theorem and consider $E_1 = \left\{\xi \in \partial D;\ \limsup_{r \downarrow 0} r^{\beta+1-n} \int_{B(\xi, r) \cap D} y_n f(y) dy > 0\right\}$. Since (1) holds, we find, with the aid of Lemma 1, that $H_{n-1-\beta}(E_1)=0$. Write $Gf=u_1+u_2$ as in the proof of Theorem 1. For $\varepsilon>0$, we set $F(\varepsilon)=\sup\left\{r^{\beta+1-n} \int_{B(\xi, r) \cap D} y_n f(y) dy;\ 0<r \leqq \varepsilon\right\}$, where $\xi \in \partial D$. Then we note that

$$\begin{aligned} &\limsup_{x_n \to 0,\, x \in \Gamma(\xi, a)} x_n^\beta u_1(x) \\ &\leqq M_1 \limsup_{x_n \to 0,\, x \in \Gamma(\xi, a)} x_n^{\beta+1} \int_{B(\xi, \varepsilon) \cap D} (|\xi-y|+x_n)^{-n} y_n f(y) dy \leqq M_2 F(\varepsilon) \end{aligned}$$

with positive constants M_1 and M_2, which implies that the left hand side is equal to zero as long as $\xi \in \partial D - E_1$. On the other hand, we derive from Lemma 4

$$\begin{aligned} x_n^\beta u_2(x) &\leqq x_n^\beta \int_{B(x, x_n/2)} |x-y|^{2-n} f(y) dy \\ &\leqq M_3 x_n^{\beta+2} + M_3 x_n^\beta \left(\int_{B(x, x_n/2)} f(y)^{n/2} \omega(f(y)) dy\right)^{2/n} \\ &\leqq M_3 x_n^{\beta+2} + M_4 \left(x_n^{\beta+1-n} \int_{B(x, x_n/2)} y_n^{n-1-\beta} [y_n^\beta f(y)]^{n/2} \omega(f(y)) dy\right)^{2/n} \end{aligned}$$

with positive constants M_3 and M_4. Consequently, Lemma 1 implies that $x_n^\beta u_2(x)$ has nontangential limit zero at any $\xi \in \partial D - E_2$, where $H_{n-1-\beta}(E_2)=0$. Thus $E=E_1 \cup E_2$ satisfies the required conditions in the theorem.

In the same manner we can prove the following result.

THEOREM 6. *Let $n=2$, $0<\beta<1$ and f be a nonnegative measurable function on D satisfying* (1) *and* (3) *with $\alpha=1$. Then $x_2^{\beta}Gf(x)$ has nontangential limit zero at any $\xi\in\partial D-E$, where $H_{1-\beta}(E)=0$.*

Finally we note the following results, which can be proved in the same way as the above theorems.

THEOREM 7. *Let $n\geqq 3$ and f be a nonnegative measurable function on D satisfying* (1) *and* (4) *with $\beta=n-1$. Then $x_n^{n-1}(1+|x|)^{-n}Gf(x)$ has limit zero as x tends to the boundary ∂D.*

THEOREM 8. *Let $n=2$ and f be a nonnegative measurable function on D satisfying* (1) *and* (3) *with $\alpha=1$. Then $x_2(1+|x|)^{-2}Gf(x)$ has limit zero as x tends to ∂D.*

5. Best possibility as to the size of the exceptional sets.

We here prove that Theorem 1 is best possible as to the size of the exceptional set if we assume further that

(ω4) $\omega(r^2)\leqq A'\omega(r)$ whenever $r>1$,

where A' is a positive constant independent of r.

PROPOSITION 1. *For a compact set $K\subset\partial D$ such that $H_h(K)=0$ there exists a nonnegative measurable function f on D satisfying* (1) *and* (2) *such that Gf does not have nontangential limit zero at any $\xi\in K$.*

PROOF. First take a mutually disjoint finite family $\{B(x_{j,1}, r_{j,1})\}$ of balls such that $x_{j,1}\in\partial D$, $\bigcup_j B(x_{j,1}, 5r_{j,1})\supset K$ and $\sum_j h(r_{j,1})<1$, and define $f_1(y)=a_{j,1}|z_{j,1}-y|^{-2}\omega(|z_{j,1}-y|^{-1})^{-1/(n/2-1)}$ for $y\in B(z_{j,1}, r_{j,1})$, where $z_{j,1}=x_{j,1}+(0, 2r_{j,1})$ and $a_{j,1}=\omega^*(r_{j,1}^{-1})^{1/(n/2-1)}$; set $f_1(y)=0$ otherwise. Letting $\varepsilon_1=\min_j r_{j,1}$, we take a mutually disjoint finite family $\{B(x_{j,2}, r_{j,2})\}$ of balls such that $x_{j,2}\in\partial D$, $r_{j,2}<\varepsilon_1/4$, $\sum_j h(r_{j,2})<2^{-1}$ and $\bigcup_j B(x_{j,2}, 5r_{j,2})\supset K$. As above, we define $f_2(y)=a_{j,2}|z_{j,2}-y|^{-2}\omega(|z_{j,2}-y|^{-1})^{-1/(n/2-1)}$ for $y\in B(z_{j,2}, r_{j,2})$, where $z_{j,2}=x_{j,2}+(0, 2r_{j,2})$ and $a_{j,2}=\omega^*(r_{j,2}^{-1})^{1/(n/2-1)}$; define $f_2(y)=0$ otherwise. In the same manner, for each positive integer m we can find a mutually disjoint finite family $\{B(x_{j,m}, r_{j,m})\}$ and a function f_m such that $x_{j,m}\in\partial D$, $\sum_j h(r_{j,m})<2^{-m+1}$, $\bigcup_j B(x_{j,m}, 5r_{j,m})\supset K$ and $f_m(y)=a_{j,m}|z_{j,m}-y|^{-2}\omega(|z_{j,m}-y|^{-1})^{-1/(n/2-1)}$ for $y\in B(z_{j,m}, r_{j,m})$, where $z_{j,m}=x_{j,m}+(0, 2r_{j,m})$, $a_{j,m}=\omega^*(r_{j,m}^{-1})^{1/(n/2-1)}$ and $r_{j,m}<\varepsilon_{m-1}/4$ with $\varepsilon_{m-1}=\min_j r_{j,m-1}$; we set $f_m(y)=0$ outside $\bigcup_j B(z_{j,m}, r_{j,m})$ as above. Then, since $f_m(y)\leqq M_1|z_{j,m}-y|^{-2}$ on $B(z_{j,m}, r_{j,m})$ with a positive constant M_1, we note by the aid of condition (ω4)

$$\int_D y_n^\alpha f_m(y)^{n/2}\omega(f_m(y))dy$$
$$\leqq M_2 \sum_j a_{j,m}^{n/2}\int_{B(z_{j,m},r_{j,m})} y_n^\alpha |z_{j,m}-y|^{-n}\omega(|z_{j,m}-y|^{-1})^{1-(n/2)/(n/2-1)}dy$$
$$\leqq M_3 \sum_j a_{j,m}^{n/2} r_{j,m}^\alpha \int_0^{r_{j,m}} \omega(t^{-1})^{-1/(n/2-1)}t^{-1}dt$$
$$= M_3 \sum_j h(r_{j,m}) < M_3 2^{-m+1},$$
$$\int_D y_n f_m(y)dy \leqq M_4 \sum_j a_{j,m}\int_{B(z_{j,m},r_{j,m})} y_n |z_{j,m}-y|^{-2}\omega(|z_{j,m}-y|^{-1})^{-1/(n/2-1)}dy$$
$$\leqq M_5 \sum_j a_{j,m} r_{j,m}^{n-1}\int_0^{r_{j,m}} \omega(t^{-1})^{-1/(n/2-1)}t^{-1}dt$$
$$= M_5 \sum_j r_{j,m}^{n-1} \leqq M_6 \sum_j h(r_{j,m}) \leqq M_6 2^{-m+1}$$

and

$$Gf_m(z_{j,m}) \geqq M_7\int_{B(z_{j,m},r_{j,m})} |z_{j,m}-y|^{2-n} f_m(y)dy$$
$$\geqq M_8 a_{j,m}\int_0^{r_{j,m}} \omega(t^{-1})^{-1/(n/2-1)}t^{-1}dt = M_8,$$

where $M_2 \sim M_8$ are positive constants independent of j and m. Consequently, since $\{B(z_{j,m}, r_{j,m})\}$ is mutually disjoint, $f=\sum_{m=1}^\infty f_m$ satisfies conditions (1) and (2). Moreover, if $\xi \in K$, then for each m there exists $j(m)$ such that $\xi \in B(x_{j(m),m}, 5r_{j(m),m})$, so that $z_{j(m),m} \in \Gamma(\xi, 5)$. This implies that $\limsup_{x\to\xi, x\in\Gamma(\xi,5)} Gf(x) \geqq M_8 > 0$ and hence Gf does not have nontangential limit zero at ξ.

PROPOSITION 2. *Let ω be a positive nondecreasing function on R^1 such that ω satisfies condition (ω1), $r^{-1}\omega(r)$ is nonincreasing on $[1, \infty)$ and ω does not satisfy condition (ω2). Then for a sequence $\{x_j\} \subset D$ which is everywhere dense in D, there exists a nonnegative measurable function f on D satisfying* (1) *and* (2) (*with* $\alpha=0$) *such that* $\inf_j Gf(x_j)>0$, *so that Gf does not have nontangential limit zero at any* $\xi \in \partial D$.

PROOF. For each positive integer j, take r_j and s_j such that $1>r_j>2s_j>0$, and define

$$f_j(y) = \begin{cases} a_j|x_j-y|^{-2}\omega(|x_j-y|^{-1})^{-1/(n/2-1)} & \text{on } B(x_j, r_j)-B(x_j, s_j), \\ 0 & \text{elsewhere,} \end{cases}$$

where $a_j=\left(\int_{s_j}^{r_j}\omega(t^{-1})^{-1/(n/2-1)}t^{-1}dt\right)^{-1}$. Then

$$\int_D f_j(y)^{n/2}\omega(f_j(y))dy$$
$$\leqq M_1 a_j^{n/2}\int_{B(x_j,r_j)-B(x_j,s_j)} |x_j-y|^{-n}\omega(|x_j-y|^{-1})^{1-(n/2)/(n/2-1)}dy$$
$$= M_2 a_j^{n/2-1}.$$

On the other hand, if r_j is chosen so that $B(x_j, 2r_j)\subset D$, then

$$Gf_j(x_j) \geqq M_3\int_{B(x_j, r_j)-B(x_j, s_j)} |x_j-y|^{2-n}f_j(y)dy \geqq M_4.$$

Now we choose $\{r_j\}$, $\{s_j\}$ so that $B(x_j, 2r_j)\subset D$, $\sum_{j=1}^{\infty} j^{n/2}A^j a_j^{n/2-1}<\infty$ and $\max_{k\leqq j} f_k(y)\leqq f_{j+1}(y)$ on $B(x_{j+1}, r_{j+1})$. Then it is not difficult to see that $f=\sum_{j=1}^{\infty} f_j$ satisfies the required conditions.

6. Fine boundary limits.

If f is a nonnegative measurable function on D satisfying (1) and $\int_D y_n^{\alpha} f(y)^{n/2}dy<\infty$ with $0\leqq\alpha<n-1$, then Gf may fail to have nontangential limit zero at any $\xi\in\partial D$ as seen in Proposition 2, but Gf is shown to have a weak sense limit at many boundary points. For example, in view of [2], Gf has fine nontangential limit zero at any $\xi\in\partial D-E$, where $H_{\alpha}(E)=0$. In this section we investigate a global behavior of Gf near the boundary. More precisely, we aim to find a function $A(x)$ such that $A(x)Gf(x)$ tends to zero as x tends to ∂D along a set $F\subset D$ whose complement is thin near ∂D in a certain sense.

For a set $E\subset D$ and an open set $G\subset R^n$, we define $C_{2,p}(E\,;G)=\inf\int_G f(y)^p dy$, where the infimum is taken over all nonnegative measurable functions f on G such that $\int_G |x-y|^{2-n}f(y)dy\geqq 1$ for every $x\in E$.

We now give the following result.

THEOREM 9. *Let $1<p\leqq n/2$, $p-n<\alpha<2p-1$ and f be a nonnegative measurable function on D such that $\int_D y_n^{\alpha} f(y)^p dy<\infty$. Then there exists a set $E\subset D$ having the following properties.*

(i) $\lim_{x_n\downarrow 0,\, x\in D-E} x_n^{(n-2p+\alpha)/p}Gf(x)=0.$

(ii) $\sum_{j=j_0}^{\infty} 2^{j(n-2p)}C_{2,p}(E_j\cap G_1\,;G_2)<\infty$ *for any open sets G_1 and G_2 for which there exists $r>0$ such that $B(x, r)\subset G_2$ whenever $x\in G_1$, where $E_j=\{x\in E\,;\ 2^{-j}\leqq x_n<2^{-j+1}\}$ and j_0 is a positive integer which may depend on G_1 and G_2.*

PROOF. Write $Gf=u_1+u_2$ as in the proof of Theorem 1. In this proof, $M_1, M_2, \cdots$ will denote positive constants. First we shall prove

$$\int_{D-B(x, x_n/2)} [|x-y|^{2-n}(|x-y|+x_n)^{-2}y_n^{1-\alpha/p}]^{p'}dy \leqq M_1 x_n^{-p'(n-p+\alpha)/p},$$

where $1/p+1/p'=1$. If $1-\alpha/p\leqq 0$, then

$$\int_{D-B(x,x_n/2)}[|x-y|^{2-n}(|x-y|+x_n)^{-2}y_n^{1-\alpha/p}]^{p'}dy$$
$$\leqq \int_{\{y\in D-B(x,x_n/2);y_n>x_n/2\}}[|x-y|^{2-n}(|x-y|+x_n)^{-2}|x_n-y_n|^{1-\alpha/p}]^{p'}dy$$
$$+\int_{\{y\in D-B(z,x_n/2);y_n\leqq x_n/2\}}[|z-y|^{2-n}(|z-y|+x_n)^{-2}y_n^{1-\alpha/p}]^{p'}dy$$
$$+M_2x_n^{-np'}\int_{D\cap B(z,x_n/2)}y_n^{(1-\alpha/p)p'}dy\leqq M_3x_n^{p'[-n+1-\alpha/p]+n},$$

where $z=(x',0)$ with $x=(x',x_n)$. If $1-\alpha/p>0$, then

$$\int_{D-B(x,x_n/2)}[|x-y|^{2-n}(|x-y|+x_n)^{-2}y_n^{1-\alpha/p}]^{p'}dy$$
$$\leqq \int_{D-B(x,x_n/2)}[|x-y|^{2-n}(|x-y|+x_n)^{-2}(|x_n-y_n|+x_n)^{1-\alpha/p}]^{p'}dy$$
$$\leqq M_4x_n^{(1-\alpha/p)p'}x_n^{-np'+n}$$
$$+M_4\int_{D-B(x,x_n/2)}[|x-y|^{2-n}(|x-y|+x_n)^{-2}|x_n-y_n|^{1-\alpha/p}]^{p'}dy$$
$$\leqq M_5x_n^{-p'(n-p+\alpha)/p}.$$

Hence we obtain from Hölder's inequality

$$u_1(x)\leqq M_6x_n\int_{\{y\in D-B(x,x_n/2);y_n>\delta\}}|x-y|^{2-n}|\bar{x}-y|^{-2}y_nf(y)dy$$
$$+M_6x_n\int_{\{y\in D-B(x,x_n/2);y_n\leqq\delta\}}|x-y|^{2-n}|\bar{x}-y|^{-2}y_nf(y)dy$$
$$\leqq M_7x_n\delta^{-(n-p+\alpha)/p}\left(\int_D y_n^\alpha f(y)^pdy\right)^{1/p}+M_7x_n^{-(n-2p+\alpha)/p}\left(\int_{\{y\in D;y_n\leqq\delta\}}y_n^\alpha f(y)^pdy\right)^{1/p}$$

whenever $\delta>4x_n$. Consequently,

$$\limsup_{x_n\downarrow 0}x_n^{(n-2p+\alpha)/p}u_1(x)\leqq M_7\left(\int_{\{y\in D;y_n\leqq\delta\}}y_n^\alpha f(y)^pdy\right)^{1/p},$$

which implies that the left hand side is equal to zero.

Put $D_j=\{y=(y',y_n);2^{-j-1}<y_n<2^{-j+2}\}$ for each positive integer j. Since $\sum_{j=1}^\infty\int_{D_j}y_n^\alpha f(y)^pdy<\infty$, we can find a sequence $\{a_j\}$ of positive integers such that $\lim_{j\to\infty}a_j=\infty$ and $\sum_{j=1}^\infty a_j\int_{D_j}y_n^\alpha f(y)^pdy<\infty$. Now we define the sets

$$E_j=\left\{x\in D;\ 2^{-j}\leqq x_n<2^{-j+1},\ \int_{B(x,x_n/2)}|x-y|^{2-n}f(y)dy>a_j^{-1/p}2^{j(n-2p+\alpha)/p}\right\}$$

and $E=\bigcup_{j=1}^\infty E_j$. Let G_1 and G_2 be open sets for which there exists $r>0$ such that $B(x,r)\subset G_2$ whenever $x\in G_1$. If $2^{-j}\leqq 2^{-j_0}<r$, then $B(x,x_n/2)\subset D_j\cap G_2$ for $x\in E_j\cap G_1$. Hence we obtain by the definition of capacity $C_{2,p}$

$$C_{2,p}(E_j\cap G_1;G_2)\leqq a_j2^{-j(n-2p+\alpha)}\int_{D_j}f(y)^pdy\leqq M_8a_j2^{-j(n-2p)}\int_{D_j}y_n^\alpha f(y)^pdy,$$

so that,

$$\sum_{j=j_0}^{\infty} 2^{j(n-2p)} C_{2,p}(E_j \cap G_1; G_2) < \infty.$$

Moreover, since $u_2(x) \leqq \int_{B(x, x_n/2)} |x-y|^{2-n} f(y) dy$, we see that

$$\limsup_{x_n \downarrow 0, x \in D-E} x_n^{(n-2p+\alpha)/p} u_2(x) \leqq M_9 \limsup_{j \to \infty} a_j^{-1/p} = 0.$$

Thus Theorem 9 is proved.

COROLLARY. *If* $0 \leqq \alpha < n-1$, $n \geqq 3$ *and* f *is a nonnegative measurable function on* D *satisfying* (2), *then* $\lim_{x_n \downarrow 0} x_n^{\alpha/(n/2)} Gf(x) = 0$.

REMARK. Following Aikawa [1], we say that a set E satisfying (ii) of Theorem 9 is $C_{2,p}$-thin on ∂D.

Finally we collect some results corresponding to the case $p=1$. Let $0 \leqq \alpha \leqq 1$. Then:

(i) *If* f *is a nonnegative measurable function on* D *such that* $\int_D y_n^{\alpha} f(y) dy < \infty$, *then* Gf *has minimally semi-fine nontangential limit zero at* $\xi \in \partial D - E$, *where* $H_{\alpha}(E) = 0$ (*cf.* [3]).

(ii) *If* f *is as above, then* $x_n^{n-2+\alpha} Gf(x)$ *tends to zero as* x *tends to* ∂D *along* $D - F$, *where* F *is thin on* ∂D (*cf.* [4]).

(iii) *In case* $n=2$, *if* f *is a nonnegative measurable function on* D *such that* $\int_D y_2^{\alpha} f(y)[\log(2+f(y))] dy < \infty$, *then* $x_2^{\alpha} Gf(x)$ *has limit zero as* x *tends to* ∂D.

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