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Asymptotic behavior of elementary solutions of transient generalized diffusion equations

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1. Introduction.

Let $\mathfrak{G}=(d/dm)(d/dx)$ be a generalized diffusion operator on an interval S and $p(t, x, y)$ the elementary solution of the generalized diffusion equation

$$\partial u(t, x)/\partial t = \mathfrak{G}u(t, x), \qquad t>0,\ x\in S, \tag{1.1}$$

in the sense of McKean [**11**]. We note that $p(t, x, y)dm(y)$ is the transition probability of the generalized diffusion process having $\mathfrak{G}$ as the generator. In this paper we study the asymptotic behavior of $p(t, x, y)$ for large t under the condition that $\mathfrak{G}$ is *transient*, i.e. $\int_0^\infty p(t, x, y)dt<\infty$, and $m(x)$ varies regularly near the end points of S.

In the previous paper [**12**], we discussed the same problem for recurrent $\mathfrak{G}$. The results there verified rigorously long time tails, i.e. $t^{-\gamma}$-decay of moments with $\gamma<1$, for multiplicative stochastic processes in statistical physics. Recently Y. Okabe [**15**] studied the asymptotic behavior of the correlation functions of stationary solutions for Stokes-Boussinesq-Langevin equations in order to observe Alder-Wainwright effect, i.e. $t^{-3/2}$-decay of velocity autocorrelation function for hard sphere. Our results here for transient $\mathfrak{G}$ give an explanation for such long time tails of the type $t^{-\gamma}$ with $\gamma\geqq 1$ from the point of view of one-dimensional generalized diffusion processes.

In [**17**] we obtained a criterion, in terms of m, for the convergence of the integral $\int_1^\infty t^\gamma p(t, x, y)dt$. By using it, we can get a rough asymptotic behavior of $p(t, x, y)$ for large time t. Namely, let $S=(l_1, l_2)$ with $-\infty\leqq l_1<l_2\leqq\infty$ and suppose that one of the following assumptions (A.1), (A.2) and (A.3) is satisfied, where $0<\rho<1$, $L(x)$ is a slowly varying function, and the symbol $a(x)\sim b(x)$ as $x\to\alpha$ stands for $\lim_{x\to\alpha}a(x)/b(x)=1$.

(A.1): $|l_i|<\infty$, $i=1, 2$, there exists the limit $\theta\equiv\lim_{x\to\infty}|m(l_2-1/x)/m(l_1+1/x)|$

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$\in[0, \infty)$, and $m(x)$ satisfies

(1.2) $$|m(l_1+1/x)| \sim x^{1/\rho+1}L(x) \quad \text{as } x\to\infty .$$

(A.2): $l_1>-\infty$, $l_2=\infty$, there exists the limit $\lim_{x\to\infty}|x^2m(x)/m(l_1+1/x)|=0$, and (1.2) is satisfied.

(A.3): $l_1>-\infty$, $l_2=\infty$, there exists the limit $\tau\equiv\lim_{x\to\infty}|x^2m(x)/m(l_1+1/x)|\in(0, \infty]$, and $m(x)$ satisfies

(1.3) $$m(x)\sim x^{1/\rho-1}L(x) \quad \text{as } x\to\infty .$$

Then it follows from [**17**; Theorem 1] that, for $\gamma<\rho$, the integral $\int_1^\infty t^\gamma p(t, x, y)dt$ converges, whence $p(t, x, y)=o(t^{-\gamma-1})$ as $t\to\infty$. Our aim is to give the following explicit asymptotic formula (1.4) in the above case.

(1.4) $$p(t, x, y)\sim c(x, y)t^{-\rho-1}/K(t) \quad \text{as } t\to\infty ,$$
$$\text{uniformly in } x, y\in[a, b],\ a, b\in S ,$$

where $c(x, y)$ is given by (2.3) below and $K(t)$ is a slowly varying function satisfying the asymptotic relation (2.4).

If an extra condition is assumed in the case of (A.3) (see (A.3)′ below), then we obtain a stronger formula

(1.5) $$p(t, x, y)\sim C_2(\rho)(x-l_1)(y-l_1)t^{-\rho-1}/K(t) \quad \text{as } t\to\infty ,$$
$$\text{uniformly in } x, y\in(l_1, a],\ a\in S ,$$

where $C_2(\rho)$ is a positive constant given by (2.2). Further, in this case, it holds that

(1.6) $$T_tf(x)\equiv\int_S p(t, x, y)f(y)dm(y)$$
$$\sim C_2(\rho)(x-l_1)(t^{\rho+1}K(t))^{-1}\int_S f(y)(y-l_1)dm(y) \quad \text{as } t\to\infty ,$$

for every f such that $f(y)(y-l_1)\in L^1(S, m)$. We will also consider the asymptotic behavior of $T_tf(x)$ as $t\to\infty$ for regularly varying functions f such that $f(y)(y-l_1)\notin L^1(S, m)$.

As was shown in [**12**], if $\mathfrak{G}$ is recurrent, there exists the limit $\delta\equiv\lim_{x\to\infty}|m(x)/m(-x)|\in(0, \infty]$, and if $m(x)$ satisfies (1.3), then

(1.7) $$p(t, x, y)\sim D_{\delta\rho}t^{\rho-1}K(t) \quad \text{as } t\to\infty,\quad x, y\in S ,$$

where $D_{\delta\rho}=C_2(\rho)\Gamma(1-\rho)/\Gamma(1+\rho)(1+\delta^{-\rho})$. This shows a sharp contrast with (1.4). In transient case, a particle starting in the interior hits the boundary l_1 or l_2 and disappears up to a finite time, or approaches the boundary l_1 or l_2

ultimately, with positive probability. The difference between (1.4) and (1.7) depends on this fact.

Our results owe to some asymptotic theorem for Krein's correspondence. Krein's one to one correspondence between spectral functions and strings $m(x)$'s plays an important role in the theory of one-dimensional generalized diffusion operators, and many asymptotic theorems for the correspondence have been already obtained ([**4**], [**5**], [**7**], [**8**], [**9**], [**17**] etc.). Most of those are related to the recurrent case. What we need in our case is that for the transient case, which we will also give in this paper.

We will describe our results in § 2. The definition of the elementary solution will be given in § 3. In § 4 we will discuss an asymptotic theorem for Krein's correspondence. Our results will be proved in § 5.

2. Main results.

Let $S=(l_1, l_2)$ be an open interval with $-\infty \leqq l_1 < l_2 \leqq \infty$ and $m(x)$ be a real valued nontrivial right continuous nondecreasing function on it. We may assume that $0 \in S$ and $m(0)=0$ without loss of generality. We denote the induced measure by $dm(x)$. Given a function u on S, we set $u(l_i)=\lim_{x \to l_i, x \in S} u(x)$, $i=1, 2$, and $u^+(x)=\lim_{\varepsilon \downarrow 0}\{u(x+\varepsilon)-u(x)\}/\varepsilon$, if there exist the limits. The integral $\int_{a+}^{b+}$ is always read as $\int_{(a,b]}$ or $-\int_{(b,a]}$ according as $a \leqq b$ or $a>b$. Let $D(\mathfrak{G})$ be the space of all functions $u \in L^2(S, m)$ which have continuous versions u (we use the same symbol) satisfying the following two conditions:

a) There are two complex constants A, B and a function $\mathfrak{G}u \in L^2(S, m)$ such that

$$u(x) = A+Bx+\int_{0+}^{x+}(x-y)\mathfrak{G}u(y)dm(y)\,, \qquad x \in S\,.$$

b) For each $i=1, 2$, if $l_i+m(l_i)$ is finite, then $u(l_i)=0$.

We then define the generalized diffusion operator $\mathfrak{G}$ from $D(\mathfrak{G})$ into $L^2(S, m)$ by $D(\mathfrak{G}) \ni u \mapsto \mathfrak{G}u \in L^2(S, m)$. Due to S. Watanabe's argument ([**18**], see also [**9**]), the above setting includes not only the absorbing boundary condition but also all cases of reflecting or sticky elastic ones for regular boundaries. Indeed, if l_1 is the regular boundary with the boundary condition $\theta_1 u(l_1)-\theta_2 u^+(l_1)+\theta_3 \mathfrak{G}u(l_1)=0$, where $\theta_1+\theta_2+\theta_3=1$, $\theta_i \geqq 0$, $i=1, 3$ and $\theta_2>0$ (such boundary condition is called the sticky elastic one in the case $\theta_1>0$), then we reset $S=(l_1-\theta_2/\theta_1, l_2)$, extend $m(x)$ by setting $m(x)=m(l_1)-\theta_3/\theta_2$ for $l_1-\theta_2/\theta_1<x \leqq l_1$, and take the right continuous modification. Here and hereafter we use the conventions $1/\infty=0$, $\pm a/0=\pm\infty$, $\infty \pm a=\infty$, $-\infty \pm a=-\infty$, $0^{-a}=\infty$ and $\infty^{-a}=0$ for $a>0$.

Now the elementary solution $p(t, x, y)$ of the generalized diffusion equation (1.1) is defined following McKean [**11**] (see §3 below for details). $\mathfrak{G}$ is called *recurrent* or *transient* if $\int_0^\infty p(t, x, y)dt=\infty$ or $<\infty$ for any $x, y\in S$, respectively. It will be seen in the next section that $\mathfrak{G}$ is transient if and only if $|l_i|<\infty$ for $i=1$ or 2.

For $0<\rho<1$ and $x, y\in S$, let

$$C_1(\rho)=\{\rho(1+\rho)\}^\rho/\Gamma(\rho)\,, \tag{2.1}$$

$$C_2(\rho)=\{\rho(1-\rho)\}^\rho/\Gamma(\rho)\,, \tag{2.2}$$

$$c(x, y)=\begin{cases} C_1(\rho)(l_2-l_1)^{-2}\{\theta^\rho(x-l_1)(y-l_1)+(l_2-x)(l_2-y)\}, & \text{if (A.1) holds,} \\ C_1(\rho)\,, & \text{if (A.2) holds,} \\ \tau^{-\rho}C_1(\rho)+C_2(\rho)(x-l_1)(y-l_1)\,, & \text{if (A.3) holds.} \end{cases} \tag{2.3}$$

Besides the assumptions (A.1), (A.2) and (A.3), we are also concerned with the following assumptions.

(A.2)$'$: All conditions in (A.2) are fulfilled and $m(x)\sim x^\beta L_1(x)$ as $x\to\infty$ for some $\beta>0$ and some slowly varying function $L_1(x)$.

(A.3)$'$: $l_1+\int_{l_1}^0(y-l_1)^{-1/2}m(y)dy>-\infty$, $l_2=\infty$ and (1.3) is satisfied.

The condition $l_1+\int_{l_1}^0 m(y)dy>-\infty$ implies that the boundary l_1 is either the regular boundary with the absorbing boundary condition or the sticky elastic one, or the exit boundary in the sense of Feller [**1**] (see also [**3**; §4.6]). Further note that (A.3)$'$ implies (A.3) with $\tau=\infty$.

Given a $\rho\in(0, 1)$ and an $L(x)$ in the condition (1.2) or (1.3), let $K(x)$ be another slowly varying function such that

$$\lim_{x\to\infty} K(x)^{1/\rho}L(x^\rho K(x))=\lim_{x\to\infty} L(x)^\rho K(x^{1/\rho}L(x))=1\,. \tag{2.4}$$

Throughout this paper, slowly varying functions L, K etc. are real valued, positive, locally bounded functions defined on $\boldsymbol{R}$ such that $\lim_{x\to\infty}L(cx)/L(x)=1$ for any $c>0$.

THEOREM 1. *If one of the assumptions* (A.1), (A.2) *and* (A.3) *is satisfied, then* (1.4) *holds.*

THEOREM 2. *If* (A.2)$'$ *or* (A.3) *is satisfied, then*

$$\limsup_{t\to\infty} t^{\rho+1}K(t)\sup_{a\leq y<\infty} p(t, x, y)/y<\infty\,, \qquad x, a\in S\,. \tag{2.5}$$

THEOREM 3. *Assume* (A.3)$'$. *Then* (1.5) *as well as the following* (2.6) *holds.*

(2.6) $$\limsup_{t\to\infty} t^{\rho+1}K(t)\sup_{y\in S} p(t, x, y)/(y-l_1) < \infty, \qquad x\in S.$$

Further (1.6) *is valid for every* f *such that* $f(y)(y-l_1)\in L^1(S, m)$.

Finally we consider the asymptotic behavior of $T_t f(x)$ for f satisfying

(2.7) $$f(y)(y-l_1) \in L^1((l_1, a), m)\setminus L^1(S, m), \qquad l_1<a<\infty,$$

(2.8) $$f(x)\sim x^{\gamma}L_f(x) \qquad \text{as } x\to\infty,$$

where L_f is a slowly varying function. Note that (2.7) implies $\gamma\geqq -1/\rho$. By means of [**14**; Corollary 1], we get

(2.9) $$T_t f(x)\sim (x-l_1)t^{\rho(\gamma-1)}\kappa(t) \qquad \text{as } t\to\infty, \quad x\in S,$$

provided (A.3)′ and $\gamma\geqq 1-1/\rho$ are satisfied, where $\kappa(t)$ is a slowly varying function given by

(2.10) $$\kappa(t) = \{\rho(1-\rho)\}^{\rho(1-\gamma)}\Gamma(\rho\gamma+1)\Gamma(\rho+1)^{-1}L_f(t^{\rho}K(t))K(t)^{\gamma-1}.$$

We show that (2.9) also holds for $-1/\rho\leqq\gamma<1-1/\rho$.

THEOREM 4. *Suppose* (2.7), (2.8) *and* (A.3)′. *Then* (2.9) *is valid for* $-1/\rho<\gamma<1-1/\rho$. *Further* (2.9) *is still valid for* $\gamma=-1/\rho$ *if* $\kappa(t)$ *is replaced by the following one.*

(2.11) $$\kappa(t) = \rho^{\rho-1}(1-\rho)^{\rho+1}\Gamma(\rho)^{-1}K(t)^{-1}\int_1^{t^{\rho}K(t)} y^{-1}L_f(y)L(y)dy.$$

3. Preliminaries.

We define the elementary solution $p(t, x, y)$ of the generalized diffusion equation (1.1) following [**3**], [**11**] and [**19**]. Let S and $m(x)$ be those mentioned at the beginning of §2. For each $i=1, 2$, $\lambda\in \boldsymbol{C}$, let $\varphi_i(x, \lambda)$ be the solution of the integral equation

(3.1) $$\varphi_i(x, \lambda) = 2-i+(i-1)x+\lambda\int_{0+}^{x+}(x-y)\varphi_i(y, \lambda)dm(y), \qquad x\in S.$$

Then, for each $\alpha>0$ and $i=1, 2$, there exists the limit

(3.2) $$h_i(\alpha) = (-1)^i\lim_{x\to l_i,\, x\in S}\varphi_2(x, \alpha)/\varphi_1(x, \alpha).$$

We set

$$1/h(\alpha) = 1/h_1(\alpha)+1/h_2(\alpha), \quad h_{11}(\alpha) = h(\alpha),$$
$$h_{22}(\alpha) = -(h_1(\alpha)+h_2(\alpha))^{-1}, \quad h_{12}(\alpha) = h_{21}(\alpha) = -h(\alpha)/h_2(\alpha).$$

The functions $h_{ij}(\alpha)$, $i, j=1, 2$, can be analytically continued to $\boldsymbol{C}\setminus(-\infty, 0]$. The spectral measures σ_{ij}, $i, j=1, 2$, are defined by

$$\sigma_{ij}([\lambda_1, \lambda_2]) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi}\int_{\lambda_1}^{\lambda_2} \mathcal{I}m\, h_{ij}(-\lambda-\sqrt{-1}\varepsilon)d\lambda ,$$

for all continuity points $\lambda_1<\lambda_2$. The matrix valued measure $[\sigma_{ij}]_{i,j=1,2}$ is symmetric nonnegative definite. Now the elementary solution of the generalized diffusion equation (1.1) is given by

$$p(t, x, y) = \sum_{i,j=1,2}\int_{0-}^{\infty} e^{-\lambda t}\varphi_i(x, -\lambda)\varphi_j(y, -\lambda)\sigma_{ij}(d\lambda), \quad t>0, \ x, y\in S. \tag{3.3}$$

In particular, if $l_1+\int_{l_1}^{0} m(y)dy>-\infty$, then (3.3) is reduced to

$$p(t, x, y) = \int_{0+}^{\infty} e^{-\lambda t}\psi(x, -\lambda)\psi(y, -\lambda)\sigma(d\lambda) , \qquad t>0, \quad x, y\in S , \tag{3.4}$$

where $\psi(x, \lambda)$ is given by

$$\psi(x, \lambda) = -\varphi_2(l_1, \lambda)\varphi_1(x, \lambda)+\varphi_1(l_1, \lambda)\varphi_2(x, \lambda) , \tag{3.5}$$

and σ is a Borel measure on $(0, \infty)$ satisfying the following relations.

$$\begin{aligned} &\varphi_2^2(l_1, -\lambda)\sigma(d\lambda) = \sigma_{11}(d\lambda) , \\ &-\varphi_1(l_1, -\lambda)\varphi_2(l_1, -\lambda)\sigma(d\lambda) = \sigma_{12}(d\lambda) = \sigma_{21}(d\lambda) , \\ &\varphi_1^2(l_1, -\lambda)\sigma(d\lambda) = \sigma_{22}(d\lambda) . \end{aligned} \tag{3.6}$$

We also define the Green function $G(\alpha, x, y)$ of (1.1) by

$$G(\alpha, x, y) = G(\alpha, y, x) = h(\alpha)u_1(x, \alpha)u_2(y, \alpha), \quad \alpha>0, \ x\leqq y, \ x, y\in S, \tag{3.7}$$

where $u_i(x, \alpha)=\varphi_1(x, \alpha)+(-1)^{i+1}\varphi_2(x, \alpha)/h_i(\alpha)$, $i=1, 2$, $\alpha>0$, $x\in S$. $u_1(x, \alpha)$ [resp. $u_2(x, \alpha)$] is positive and nondecreasing [resp. nonincreasing] in $x\in S$ (see [8]). Denote the Laplace transform of $p(t, x, y)$ by $\tilde{G}(\alpha, x, y)$:

$$\tilde{G}(\alpha, x, y) = \int_0^{\infty} e^{-\alpha t}p(t, x, y)dt , \qquad \alpha>0, \quad x, y\in S . \tag{3.8}$$

We should notice that $G(\alpha, x, y)$ is not necessarily identical with $\tilde{G}(\alpha, x, y)$. Indeed, we need a correction function $\Phi(x, y)$ to combine $G(\alpha, x, y)$ and $\tilde{G}(\alpha, x, y)$. Let I_k, $k=1, 2, \cdots$ be the disjoint open intervals such that $S\setminus\mathrm{Supp}(dm)=\bigcup_{k=1}^{\infty} I_k$ and the end points (if exist) belong to $\mathrm{Supp}(dm)\cup\{l_1, l_2\}$. For each $x, y\in S$ with $x\leqq y$, we set

$$\Phi(x, y) = \Phi(y, x) = \begin{cases} (x-x_1)(x_2-y)/(x_2-x_1), & -\infty<x_1<x_2<\infty, \\ x-x_1 , & -\infty<x_1<x_2=\infty, \\ x_2-y , & -\infty=x_1<x_2<\infty, \end{cases}$$

if $x, y\in \bar{I}_k\equiv[x_1, x_2]$ for some $I_k\neq\emptyset$, and $=0$ otherwise. Then it holds that

(3.9) $$G(\alpha, x, y)=\tilde{G}(\alpha, x, y)+\Phi(x, y), \qquad \alpha>0, \quad x, y\in S$$

(see [12; Lemma 1]).

For each $a\in S$, let

$$\Psi_a(y)=\begin{cases}(x_2-y)/(x_2-a), & x_2<\infty,\\ 1, & x_2=\infty,\end{cases}$$

if $a, y\in \bar{I}_k\equiv[x_1, x_2]$, $a\neq x_2$ for some $I_k\neq\emptyset$, and $=0$ otherwise. Then there exists a nonnegative function $q_a(t, y)$, $t>0$, $y\in(a, l_2)$ such that

(3.10) $$u_2(y, \alpha)/u_2(a, \alpha)=\int_0^\infty e^{-\alpha t}q_a(t, y)dt+\Psi_a(y), \qquad y\in(a, l_2), \quad \alpha>0.$$

Hence, by means of (3.7) and (3.8),

(3.11) $$p(t, x, y)=\int_0^t p(t-s, x, a)q_a(s, y)ds+\Phi(x, a)q_a(t, y)+p(t, x, a)\Psi_a(y),$$
$$t>0, \ l_1<x\leqq a<y<l_2,$$

(see [12; §3] for details).

We next observe some estimates of $\varphi_i(x, \lambda)$. Due to [6; (2.27)], for $i=1, 2$, $x\in S$, $\lambda\in \boldsymbol{C}$,

(3.12) $$|\varphi_1(x, \lambda)|\leqq \exp\sqrt{2|\lambda x m(x)|},$$

(3.13) $$|\varphi_2(x, \lambda)|\leqq |x|\exp\sqrt{2|\lambda x m(x)|},$$

(3.14) $$|\varphi_1(x, \lambda)-1|\leqq |\lambda x m(x)|\exp\sqrt{2|\lambda x m(x)|},$$

(3.15) $$|\varphi_2(x, \lambda)-x|\leqq |\lambda x^2 m(x)|\exp\sqrt{2|\lambda x m(x)|}.$$

It is also easy to see that, if $l_1+\int_{l_1}^0 m(y)dy>-\infty$, then $\phi(x, \lambda)$ defined by (3.5) satisfies

(3.16) $$|\phi(x, \lambda)|\leqq |x-l_1|e^{|\lambda|M(x)},$$

(3.17) $$|\phi(x, \lambda)-(x-l_1)|\leqq |\lambda|(x-l_1)M(x)e^{|\lambda|M(x)},$$

for $x\in S$, $\lambda\in\boldsymbol{C}$, where $M(x)=\int_{l_1}^x (m(x)-m(y))dy$.

Finally we note that $\mathfrak{G}$ is transient if and only if $l_1>-\infty$ or $l_2<\infty$. Indeed, since $\varphi_1(x, \lambda)\varphi_2^+(x, \lambda)-\varphi_1^+(x, \lambda)\varphi_2(x, \lambda)=1$, $x\in S$, $\lambda\in\boldsymbol{C}$, we get by (3.2)

$$h_i(\alpha)=\left|\int_0^{l_i}\varphi_1(x, \alpha)^{-2}dx\right|, \qquad i=1, 2, \quad \alpha>0,$$

from which $\lim_{\alpha\downarrow 0}h_i(\alpha)=|l_i|$, $i=1, 2$. By using this and (3.7), (3.8), (3.9), we see that $\mathfrak{G}$ is transient if and only if $\lim_{\alpha\downarrow 0}h(\alpha)<\infty$, which is equivalent to $l_1>-\infty$ or $l_2<\infty$.

4. Asymptotic theorem for Krein's correspondence.

In this section we give some asymptotic theorems for Krein's correspondence. The arguments of Krein's correspondence are due to [6] and [9]. Let $m(x)$ be a nonnegative right continuous nondecreasing function on $[0, \infty]$ such that $m(x)\not\equiv\infty$ and $m(\infty)=\infty$. We denote the totality of such m by $\mathcal{M}$. For $m\in\mathcal{M}$, we always set $m(0-)=0$ and consider the solution $\varphi_i(x, \lambda)$ of the following integral equation

$$(4.1)\qquad \varphi_i(x, \lambda)=2-i+(i-1)x+\lambda\int_{0-}^{x+}(x-y)\varphi_i(y, \lambda)dm(y), \qquad 0\leqq x<l,$$

where $i=1, 2$, $\lambda\in\boldsymbol{C}$, and $l=\sup\{x: m(x)<\infty\}$. Set

$$h(\alpha)=\lim_{x\uparrow l}\varphi_2(x, \alpha)/\varphi_1(x, \alpha)=\int_0^l\varphi_1(x, \alpha)^{-2}dx, \qquad \alpha>0.$$

h is called the *characteristic function* of m and the correspondence $m\in\mathcal{M}\rightarrow h$ is called *Krein's correspondence*. Let $\mathcal{H}$ be the class of functions h on $(0, \infty)$ such that

$$h(\alpha)=c+\int_{0-}^{\infty}(\alpha+\lambda)^{-1}\sigma(d\lambda), \qquad \alpha>0.$$

for some $c\geqq 0$ and some nonnegative Borel measure σ on $[0, \infty)$ satisfying $\int_{[0,\infty)}(1+\lambda)^{-1}\sigma(d\lambda)<\infty$. It is well known that Krein's correspondence $m\in\mathcal{M}\rightarrow h$ is a one to one map from $\mathcal{M}$ onto $\mathcal{H}$ (see [6], e.g.). From now on we denote Krein's correspondence by $m\in\mathcal{M}\leftrightarrow h\in\mathcal{H}$. It is easy to see that

$$c=\inf\{x>0: m(x)>0\},$$

$$l=\lim_{\alpha\downarrow 0}h(\alpha)=c+\int_{0-}^{\infty}\lambda^{-1}\sigma(d\lambda).$$

In the following, $0<\rho<1$, and $L(x)$ and $K(x)$ are slowly varying functions satisfying (2.4).

Theorem 4.1 (Kasahara [7]). *Let $m\in\mathcal{M}\leftrightarrow h\in\mathcal{H}$ and $l=\infty$. Then the following* (4.2) *and* (4.3) *are equivalent each other.*

$$(4.2)\qquad m(x)\sim x^{1/\rho-1}L(x) \qquad \text{as } x\rightarrow\infty.$$

$$(4.3)\qquad h(\alpha)\sim\{\rho/\Gamma(1-\rho)C_2(\rho)\}\alpha^{-\rho}K(1/\alpha) \qquad \text{as } \alpha\downarrow 0.$$

Now we will establish a version of Theorem 4.1 corresponding to the case where $l<\infty$. The proof for the sufficiency of the first assertion (i) of the following theorem is due to Y. Okabe.

Theorem 4.2. *Let $m\in\mathcal{M}\leftrightarrow h\in\mathcal{H}$ and $l<\infty$.* (i) *The integral $I\equiv$*

$\int_0^l dx\int_0^x m(y)dy$ *converges if and only if there exists the limit* $J\equiv\lim_{\alpha\downarrow 0}(l-h(\alpha))/\alpha<\infty$. *Then* $2I=J$. (ii) *The following* (4.4) *and* (4.5) *are equivalent each other.*

$$m(l-1/x)\sim x^{1/\rho+1}L(x) \quad \text{as } x\to\infty. \tag{4.4}$$

$$l-h(\alpha)\sim \rho^{-1}\Gamma(1-\rho)C_1(\rho)\alpha^{\rho}/K(1/\alpha) \quad \text{as } \alpha\downarrow 0. \tag{4.5}$$

PROOF. (i) First note that, by means of (4.1),

$$\lim_{\alpha\downarrow 0}(\varphi_1(x,\alpha)-1)/\alpha=\int_{0-}^{x+}(x-y)dm(y)=\int_0^x m(y)dy.$$

Assume $I<\infty$. Since, for each $\alpha>0$, $\varphi_1(x,\alpha)$ is nondecreasing and $\varphi_1(x,\alpha)\geqq 1$, $x\geqq 0$, we get by (4.1)

$$\begin{aligned}(\varphi_1^2(x,\alpha)-1)/\alpha\varphi_1^2(x,\alpha)&\leqq 2(\varphi_1(x,\alpha)-1)/\alpha\varphi_1(x,\alpha)\\ &=2\int_{0-}^{x+}(x-y)\varphi_1(y,\alpha)dm(y)/\varphi_1(x,\alpha)\\ &\leqq 2\int_{0-}^{x+}(x-y)dm(y)=2\int_0^x m(y)dy.\end{aligned}$$

By the assumption the last term is integrable on $[0, l)$. Therefore it follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned}J&=\lim_{\alpha\downarrow 0}(l-h(\alpha))/\alpha=\int_0^l\lim_{\alpha\downarrow 0}\{(\varphi_1^2(x,\alpha)-1)/\alpha\varphi_1^2(x,\alpha)\}dx\\ &=2\int_0^l dx\int_0^x m(y)dy=2I<\infty.\end{aligned}$$

Conversely, if $J<\infty$, then in view of Fatou's lemma,

$$\begin{aligned}\infty>J&=\lim_{\alpha\downarrow 0}(l-h(\alpha))/\alpha\\ &\geqq\int_0^l\liminf_{\alpha\downarrow 0}\{(\varphi_1^2(x,\alpha)-1)/\alpha\varphi_1^2(x,\alpha)\}dx\\ &=2\int_0^l dx\int_0^x m(y)dy=2I.\end{aligned}$$

Applying the first half of the proof, we have $J=2I$.

(ii) Let $m^{-1}(x)$ be the right continuous inverse function of $x\mapsto m(x)$. Further, let $\mu(x)=l^{-2}\int_0^x\{l-m^{-1}(y)\}^2dy$ and $\mu^{-1}(x)$ be the inverse function of $x\mapsto\mu(x)$. Then it holds that

$$\mu\circ m(l-1/x)=l^{-2}\int_{(1/l,\,x]}y^{-2}dm(l-1/y), \qquad 1/l\leqq x<\infty. \tag{4.6}$$

In view of [**6**; (12.5)] or [**9**; (1.10)], $m^{-1}(x)\in\mathcal{M}\leftrightarrow 1/\alpha h(\alpha)\in\mathcal{H}$. By virtue of [**10**; (2)] or [**9**; (1.11)], we get

$$(4.7)\qquad m_0(x)\equiv lm^{-1}\circ\mu^{-1}(x)/\{l-m^{-1}\circ\mu^{-1}(x)\}\in\mathcal{M}$$
$$\longleftrightarrow h_0(\alpha)\equiv(l-h(\alpha))/l\alpha h(\alpha)\in\mathcal{H}\,.$$

Noting (4.6), we now find that (4.4) is equivalent to

$$\mu\circ m(l-1/x)\sim l^{-2}(1+\rho)(1-\rho)^{-1}x^{(1-\rho)/\rho}L(x)\qquad\text{as }x\to\infty\,.$$

Further, this is equivalent to

$$(4.8)\qquad m_0(x)\sim c_1x^{\rho/(1-\rho)}K_1(x)\qquad\text{as }x\to\infty\,,$$

where $c_1=l^{2/(1-\rho)}\{(1-\rho)/(1+\rho)\}^{\rho/(1-\rho)}$ and $K_1(x)$ is a slowly varying function such that

$$(4.9)\qquad \lim_{x\to\infty}K_1(x)^{(1-\rho)/\rho}L(x^{\rho/(1-\rho)}K_1(x))$$
$$=\lim_{x\to\infty}L(x)^{\rho/(1-\rho)}K_1(x^{(1-\rho)/\rho}L(x))=1\,.$$

(4.7) and Theorem 4.1 assure that (4.8) is equivalent to

$$(4.10)\qquad h_0(\alpha)\sim c_2\alpha^{\rho-1}K_2(1/\alpha)\qquad\text{as }\alpha\downarrow 0\,,$$

where $c_2=l^{-2}\{\rho(1+\rho)\}^{\rho}\Gamma(1-\rho)/\Gamma(1+\rho)$ and $K_2(x)$ is a slowly varying function satisfying

$$(4.11)\qquad \lim_{x\to\infty}K_1(x)^{1-\rho}K_2(x^{1/(1-\rho)}K_1(x))$$
$$=\lim_{x\to\infty}K_2(x)^{1/(1-\rho)}K_1(x^{1-\rho}K_2(x))=1\,.$$

Incidentally, if two slowly varying functions $r_1(x)$ and $r_2(x)$ satisfy the asymptotic relation

$$\lim_{x\to\infty}r_1(x)^{1/\gamma}r_2(x^{\gamma}r_1(x))=\lim_{x\to\infty}r_2(x)^{\gamma}r_1(x^{1/\gamma}r_2(x))=1$$

for some $\gamma>0$, then

$$\lim_{x\to\infty}r_1(x^{1/\gamma})r_2(x)^{\gamma}=1$$

(see [**16**; (1.38) and Lemma 1.10]). Therefore (2.4), (4.9) and (4.11) imply that

$$\lim_{x\to\infty}K(x)K_2(x)=1\,.$$

Thus (4.10) is equivalent to (4.5). q. e. d.

5. Proof of Theorems.

Throughout this section, we assume that one of the assumptions (A.1), (A.2) and (A.3) is satisfied. Let us recall $h_i(\alpha)$, $i=1, 2$, defined by (3.2). First we note

LEMMA 5.1. *The assumptions* (A.1), (A.2) *and* (A.3) *imply the following* (5.1) *and* (5.3), (5.1) *and* (5.4) *with* $\tau=0$, *and* (5.2) *and* (5.4), *respectively.*

$$-l_1-h_1(\alpha)\sim\rho^{-1}\Gamma(1-\rho)C_1(\rho)\alpha^{\rho}/K(1/\alpha) \qquad as\ \alpha\downarrow 0. \tag{5.1}$$

$$h_2(\alpha)\sim\{\rho/\Gamma(1-\rho)C_2(\rho)\}\alpha^{-\rho}K(1/\alpha) \qquad as\ \alpha\downarrow 0. \tag{5.2}$$

$$\lim_{\alpha\downarrow 0}(l_2-h_2(\alpha))/(-l_1-h_1(\alpha))=\theta^{\rho}. \tag{5.3}$$

$$\lim_{\alpha\downarrow 0}(-l_1-h_1(\alpha))h_2(\alpha)=\tau^{-\rho}C_1(\rho)/C_2(\rho). \tag{5.4}$$

PROOF. Put $m_1(x)=-m(-x)$, $0\leqq x<-l_1$, $=\infty$, $-l_1\leqq x\leqq\infty$, and $m_2(x)=m(x)$, $0\leqq x<l_2$, $=\infty$, $l_2\leqq x\leqq\infty$. Taking the right continuous modification of m_1, we notice $m_i\in\mathcal{M}\leftrightarrow h_i\in\mathcal{H}$, $i=1, 2$. Therefore it is obvious that (1.2) [resp. (1.3)] implies (5.1) [resp. (5.2)] by means of Theorem 4.2 [resp. Theorem 4.1]. Let $U_i(x)$ and $V_i(x)$ be the inverse functions of $x\mapsto\int_0^x m_i(y)dy$ and $x\mapsto x\int_0^{|l_i|-1/x}dy\int_0^y m_i(z)dz$, respectively, $i=1, 2$. Then

$$\lim_{x\to\infty}V_1(x)/V_2(x)=0, \qquad \text{if } \theta=0 \text{ in (A.1)},$$

$$\lim_{x\to\infty}V_1(x)/U_2(x)=\begin{cases}0, & \text{if (A.2) holds,}\\ \infty, & \text{if } \tau=\infty \text{ in (A.3).}\end{cases}$$

By using [**9**; Theorem 2.3] and [**17**; Proposition 4.1], we have positive constants c_i, $i=1, 2$ such that

$$c_1\leqq h_2(\alpha)/U_2(1/\alpha)\leqq c_2,\quad \alpha>0, \qquad \text{if } l_2=\infty,$$

$$c_1\leqq\{|l_j|-h_j(\alpha)\}V_j(1/\alpha)\leqq c_2,\quad \alpha>0, \qquad \text{if } |l_j|<\infty,$$

$j=1, 2$. Consequently, (A.1) with $\theta=0$, (A.2), and (A.3) with $\tau=\infty$ imply (5.3) with $\theta=0$, (5.4) with $\tau=0$, and (5.4) with $\tau=\infty$, respectively.

Assume $0<\theta<\infty$ in (A.1) or $0<\tau<\infty$ in (A.3). Let $i=2$, $\delta=\theta$ in case (A.1), and $i=1$, $\delta=\tau^{-1}$ in case (A.3). Then

$$m_i(|l_i|-1/x)\sim\delta x^{1/\rho+1}L(x) \qquad \text{as } x\to\infty.$$

Noting (2.4) with $\delta L(x)$ in place of $L(x)$, we get by Theorem 4.2,

$$|l_i|-h_i(\alpha)\sim\delta^{\rho}\rho^{-1}\Gamma(1-\rho)C_1(\rho)\alpha^{\rho}/K(1/\alpha) \qquad \text{as } \alpha\downarrow 0.$$

Hence (5.3) [resp. (5.4)] is also valid for $0<\theta<\infty$ [resp. $0<\tau<\infty$] in case (A.1) [resp. (A.3)]. q. e. d.

We turn to the asymptotic estimate of the Green function $G(\alpha, x, y)$ as $\alpha\downarrow 0$.

LEMMA 5.2. *It holds that as* $\alpha\downarrow 0$,

$$G(0+, x, y)-G(\alpha, x, y) \sim \rho^{-1}\Gamma(1-\rho)c(x, y)\alpha^{\rho}/K(1/\alpha), \quad x, y \in S. \tag{5.5}$$

PROOF. Let $\alpha>0$ and $x \leqq y$, $x, y \in S$. Then by means of (3.7),

$$G(0+, x, y)-G(\alpha, x, y)=\sum_{i=1}^{3} I_i(\alpha, x, y), \quad \text{if } l_2<\infty,$$

$$G(0+, x, y)-G(\alpha, x, y)=\sum_{i=1}^{3} J_i(\alpha, x, y), \quad \text{if } l_2=\infty,$$

where

$$\begin{aligned}
I_1(\alpha, x, y) &= (h_1(\alpha)+h_2(\alpha))^{-1}(\{-h_1(\alpha)(\varphi_1(x, \alpha)-1)-\varphi_2(x, \alpha)+x\} \\
&\quad \times\{h_2(\alpha)\varphi_1(y, \alpha)-\varphi_2(y, \alpha)\} \\
&\quad -(h_1(\alpha)+x)\{h_2(\alpha)(\varphi_1(y, \alpha)-1)-\varphi_2(y, \alpha)+y\}), \\
I_2(\alpha, x, y) &= (l_2-l_1)^{-1}(h_1(\alpha)+h_2(\alpha))^{-1}(-l_1-h_1(\alpha)) \\
&\quad \times\{(l_2-l_1)(h_2(\alpha)-y)-(x-l_1)(l_2-y)\}, \\
I_3(\alpha, x, y) &= (l_2-l_1)^{-1}(h_1(\alpha)+h_2(\alpha))^{-1}(l_2-h_2(\alpha))(x-l_1)(y-l_1), \\
J_1(\alpha, x, y) &= h_2(\alpha)(h_1(\alpha)+h_2(\alpha))^{-1}(-h_1(\alpha)(\varphi_1(x, \alpha)-1)\{\varphi_1(y, \alpha)-\varphi_2(y, \alpha)/h_2(\alpha)\} \\
&\quad -(\varphi_2(x, \alpha)-x)\{1-\varphi_2(y, \alpha)/h_2(\alpha)\}-(h_1(\alpha)+\varphi_2(x, \alpha))(\varphi_1(y, \alpha)-1)), \\
J_2(\alpha, x, y) &= h_2(\alpha)(h_1(\alpha)+h_2(\alpha))^{-1}(-l_1-h_1(\alpha))\{1-\varphi_2(y, \alpha)/h_2(\alpha)\}, \\
J_3(\alpha, x, y) &= (h_1(\alpha)+h_2(\alpha))^{-1}(x-l_1)(\varphi_2(y, \alpha)+h_1(\alpha)).
\end{aligned}$$

We put

$$\xi(x)=\int_0^x m(y)dy, \qquad \eta(x)=\int_{0+}^{x+}(x-y)ydm(y).$$

Note that by (3.1)

$$\lim_{\alpha\downarrow 0}\{\varphi_1(x, \alpha)-1\}/\alpha=\xi(x), \qquad \lim_{\alpha\downarrow 0}\{\varphi_2(x, \alpha)-x\}/\alpha=\eta(x).$$

Since $\lim_{\alpha\downarrow 0}h_i(\alpha)=|l_i|$, $i=1, 2$, it holds that as $\alpha\downarrow 0$,

$$\begin{aligned}
I_1(\alpha, x, y) &\sim \alpha(l_2-l_1)^{-1}(\{l_1\xi(x)-\eta(x)\}(l_2-y)+\{l_2\xi(y)-\eta(y)\}(l_1-x)), \\
I_2(\alpha, x, y) &\sim (-l_1-h_1(\alpha))(l_2-x)(l_2-y)/(l_2-l_1)^2, \\
I_3(\alpha, x, y) &\sim (l_2-h_2(\alpha))(x-l_1)(y-l_1)/(l_2-l_1)^2, \\
J_1(\alpha, x, y) &\sim \alpha\{l_1\xi(x)-\eta(x)+\xi(y)(l_1-x)\}, \\
J_2(\alpha, x, y) &\sim -l_1-h_1(\alpha), \\
J_3(\alpha, x, y) &\sim h_2(\alpha)^{-1}(x-l_1)(y-l_1).
\end{aligned}$$

Combining these asymptotic estimates with those in Lemma 5.1, we have (5.5). The proof for the case $x>y$ is just the same as above. q.e.d.

The idea of the following proof is due to S. Kotani.

PROPOSITION 5.3.

$$\lim_{t\to\infty} t^{\rho+1}K(t)p(t, x, y) = c(x, y), \qquad x, y \in S. \tag{5.6}$$

PROOF. (The present proof which is simpler than the original one is due to S. Kotani.) We note that the matrix $P(t, x, y)=\begin{pmatrix} p(t, x, x) & p(t, x, y) \\ p(t, y, x) & p(t, y, y) \end{pmatrix}$ is nonnegative definite and the derivative $\partial P(t, x, y)/\partial t$ is nonpositive definite for all $t>0$ and $x, y\in S$. Further, putting $g(\alpha, x, y)=G(0+, x, y)-G(\alpha, x, y)$, we get

$$\alpha\int_0^\infty e^{-\alpha t}dt\int_t^\infty P(t, x, y)ds = \begin{pmatrix} g(\alpha, x, x) & g(\alpha, x, y) \\ g(\alpha, y, x) & g(\alpha, y, y) \end{pmatrix}, \qquad \alpha>0, \quad x, y\in S.$$

It is easy to see that Hardy-Littlewood-Karamata theorem ([**16**; Theorem 2.3], e.g.) is available for matrix valued functions. Therefore (5.6) follows from Lemma 5.2 immediately. q. e. d.

Next we will study the asymptotic behaviors of spectral measures. By means of (3.3) and Proposition 5.3,

$$p(t, 0, 0) = \int_{0-}^\infty e^{-\lambda t}\sigma_{11}(d\lambda) \sim c(0, 0)t^{-\rho-1}/K(t) \qquad \text{as } t\to\infty. \tag{5.7}$$

Accordingly, by Hardy-Littlewood-Karamata theorem,

$$\sigma_{11}([0, \lambda]) \sim (c(0, 0)/\Gamma(\rho+2))\lambda^{\rho+1}/K(1/\lambda) \qquad \text{as } \lambda\downarrow 0. \tag{5.8}$$

We also get

LEMMA 5.4. *It holds that as* $t\to\infty$,

$$\int_{0-}^\infty e^{-\lambda t}\varphi_1(x, -\lambda)\varphi_1(y, -\lambda)\sigma_{11}(d\lambda) \sim c(0, 0)t^{-\rho-1}/K(t), \tag{5.9}$$

uniformly in $x, y\in[a, b]$, $a, b\in S$.

PROOF. Fix $a, b\in S$ and put $c_1=\sup_{x\in[a,b]}(2|xm(x)|)^{1/2}$. Then, by means of (3.12) and (3.14),

$$\begin{aligned}
&\sup_{a\le x, y\le b}|\varphi_1(x, \lambda)\varphi_1(y, \lambda)-1| \\
&\le \sup_{a\le x, y\le b}|\varphi_1(x, \lambda)-1|\,|\varphi_1(y, \lambda)| + \sup_{a\le x, y\le b}|\varphi_1(y, \lambda)-1| \\
&\le c_1^2|\lambda|\exp\{2c_1\sqrt{|\lambda|}\}.
\end{aligned}$$

Therefore, by virtue of (5.7) and (5.8),

$$\sup_{a\leq x,\, y\leq b}\left|t^{\rho+1}K(t)\int_{0-}^{\infty}e^{-\lambda t}\varphi_1(x, -\lambda)\varphi_1(y, -\lambda)\sigma_{11}(d\lambda)-c(0, 0)\right|$$
$$\leqq c_1^2 t^{\rho+1}K(t)\int_{0-}^{\infty}\lambda\exp\{-\lambda t+2c_1\sqrt{\lambda}\}\sigma_{11}(d\lambda)+\left|t^{\rho+1}K(t)\int_{0-}^{\infty}e^{-\lambda t}\sigma_{11}(d\lambda)-c(0, 0)\right|$$
$$\longrightarrow 0 \quad \text{as } t\to\infty .$$

Thus (5.9) follows. q.e.d.

Next we put

$$A=\begin{cases} C_1(\rho)(\theta^{\rho}+1)(l_2-l_1)^{-2}, & \text{in case (A.1)},\\ 0, & \text{in case (A.2)},\\ C_2(\rho), & \text{in case (A.3)}.\end{cases}$$

LEMMA 5.5.

(5.10) $$\lim_{\lambda\downarrow 0}\lambda^{-\rho-1}K(1/\lambda)\sigma_{22}((0, \lambda])=A/\Gamma(\rho+2).$$

(5.11) $$\lim_{t\to\infty}t^{\rho+1}K(t)\int_{0+}^{\infty}e^{-\lambda t}\sigma_{22}(d\lambda)=A.$$

(5.12) $$\lim_{t\to\infty}t^{\rho+1}K(t)\int_{0+}^{\infty}e^{-\lambda t}\varphi_2(x, -\lambda)\varphi_2(y, -\lambda)\sigma_{22}(d\lambda)=Axy,$$

uniformly in $x, y\in[a, b]$, $a, b\in S$.

PROOF. Let $\alpha>0$. Then, by using [**17**; (5.7)],

$$\int_{0+}^{\infty}(\lambda(\alpha+\lambda))^{-1}\sigma_{22}(d\lambda)=\alpha^{-1}(\{h_1(\alpha)+h_2(\alpha)\}^{-1}-(l_2-l_1)^{-1}).$$

First we consider the case (A.1). By means of (5.1) and (5.3),

$$\int_{0+}^{\infty}(\lambda(\alpha+\lambda))^{-1}\sigma_{22}(d\lambda)=\{-l_1-h_1(\alpha)+l_2-h_2(\alpha)\}/(h_1(\alpha)+h_2(\alpha))(l_2-l_1)$$
$$\sim\rho^{-1}\Gamma(1-\rho)C_1(\rho)(\theta^{\rho}+1)(l_2-l_1)^{-2}\alpha^{\rho-1}/K(1/\alpha) \quad \text{as } \alpha\downarrow 0.$$

In view of Tauberian theorem on Stieltjes transform ([**16**; Theorem 2.5], e.g.),

$$\int_{0+}^{\lambda+}u^{-1}\sigma_{22}(du)\sim\{\rho\Gamma(\rho+1)\}^{-1}C_1(\rho)(\theta^{\rho}+1)(l_2-l_1)^{-2}\lambda^{\rho}/K(1/\lambda) \quad \text{as } \lambda\downarrow 0.$$

from which (5.10) follows.

Next assume (A.2). Then,

$$\sigma_{22}((0, \lambda])\leqq 2\lambda^2\int_{0+}^{\lambda+}(u(\lambda+u))^{-1}\sigma_{22}(du)\leqq 2\lambda(h_1(\lambda)+h_2(\lambda))^{-1}\leqq 2\lambda/h_2(\lambda).$$

By virtue of (5.1) and (5.4) with $\tau=0$,

$$\lambda^{-\rho-1}K(1/\lambda)\sigma_{22}((0, \lambda])\leqq 2\lambda^{-\rho}K(1/\lambda)/h_2(\lambda)\longrightarrow 0 \quad \text{as } \lambda\downarrow 0.$$

Thus (5.10) is valid.

Suppose (A.3). Then by (5.2) and by the fact $h_1(0+)=-l_1$,

$$\int_{0+}^{\infty}(\lambda(\alpha+\lambda))^{-1}\sigma_{22}(d\lambda)=\alpha^{-1}(h_1(\alpha)+h_2(\alpha))^{-1}$$
$$\sim \rho^{-1}\Gamma(1-\rho)C_2(\rho)\alpha^{\rho-1}/K(1/\alpha) \qquad \text{as } \alpha\downarrow 0.$$

By using Tauberian theorem on Stieltjes transform again, we get (5.10).

(5.11) is obvious by (5.10) and Abelian theorem.

(5.12) follows in the same way as in Lemma 5.4 by using (3.13) and (3.15). q. e. d.

PROOF OF THEOREM 1. In order to derive (1.4) from (5.6), we observe the asymptotic behavior of σ_{12}. We use an idea in [13] to do it.

First we note that $\sigma_{ij}(\{0\})=0$, $i, j=1, 2$, because of (5.6). The measure σ_{12} is not necessarily nonnegative but of bounded variation. Therefore σ_{12} is expressed as $\sigma_{12}=\sigma_1-\sigma_2$, where σ_i, $i=1, 2$ are nonnegative Borel measures on $[0, \infty)$ and $\operatorname{Supp}\sigma_1\cap\operatorname{Supp}\sigma_2=\emptyset$.

For any fixed sequence $\{t_n\}$ such that $t_n\to\infty$, we set $\xi_i^{(n)}(\lambda)=t_n^{\rho+1}K(t_n)\sigma_i((0, \lambda/t_n])$, $\xi_{ij}^{(n)}(\lambda)=t_n^{\rho+1}K(t_n)\sigma_{ij}((0, \lambda/t_n])$, $i, j=1, 2$, $\lambda>0$. Since $\sigma_{12}^2(E)\leqq\sigma_{11}(E)\sigma_{22}(E)\leqq(\sigma_{11}(E)+\sigma_{22}(E))^2/2$ for Borel sets $E\subset[0, \infty)$, we find that $\xi_i^{(n)}(\lambda)\leqq\xi_{11}^{(n)}(\lambda)+\xi_{22}^{(n)}(\lambda)$, $\lambda\geqq 0$, as well as

$$\int_{0+}^{\infty}e^{-\lambda s}d\xi_i^{(n)}(\lambda)\leqq\sum_{j=1,2}\int_{0+}^{\infty}e^{-\lambda s}d\xi_{jj}^{(n)}(\lambda), \qquad s>0, \quad i=1, 2. \tag{5.13}$$

In view of (5.8) and (5.10), we can choose a subsequence $\{n_k\}$ and nondecreasing functions $\xi_i^*(\lambda)$, $i=1, 2$, on $[0, \infty)$ such that $\xi_i^{(n_k)}(\lambda)\to\xi_i^*(\lambda)$ for every continuity point λ of ξ_i^*. We will show that

$$\xi_{12}^*(\lambda)\equiv\xi_1^*(\lambda)-\xi_2^*(\lambda)=B\Gamma(\rho+2)^{-1}\lambda^{\rho+1}, \qquad \lambda>0, \tag{5.14}$$

where

$$B=\begin{cases} C_1(\rho)(-\theta^{\rho}l_1-l_2)(l_2-l_1)^{-2}, & \text{in case (A.1)},\\ 0, & \text{in case (A.2)},\\ -C_2(\rho)l_1, & \text{in case (A.3)}.\end{cases}$$

By virtue of (5.7), (5.11) and (5.13), the sequence $\left\{\int_{0+}^{\infty}e^{-\lambda s}d\xi_i^{(n_k)}(\lambda)\right\}_{k=1}^{\infty}$ is bounded. In view of [2; Ch. 13, Theorem 2a],

$$\int_{0+}^{\infty}e^{-\lambda s}d\xi_i^{(n_k)}(\lambda)\longrightarrow\int_{0+}^{\infty}e^{-\lambda s}d\xi_i^*(\lambda), \qquad s>0, \quad i=1, 2.$$

Combining this with (3.12)-(3.15), by the same argument as in Lemma 5.4, we see that

$$\int_{0+}^{\infty}e^{-\lambda s}\varphi_1(x, -\lambda/t_{n_k})\varphi_2(y, -\lambda/t_{n_k})d\xi_{12}^{(n_k)}(\lambda)\longrightarrow y\int_{0+}^{\infty}e^{-\lambda s}d\xi_{12}^*(\lambda), \tag{5.15}$$

uniformly in $x, y\in[a, b]$, $a, b\in S$, $s>0$.

Note that $\sigma_{12}=\sigma_{21}$ and hence $\xi_{12}^{(n)}(\lambda)=\xi_{21}^{(n)}(\lambda)$. By means of Proposition 5.3, (5.9), (5.12) and (5.15),

$$c(x, y)s^{-\rho-1} = (c(0, 0)+Axy)s^{-\rho-1}+(x+y)\int_{0+}^{\infty} e^{-\lambda s}d\xi_{12}^{*}(\lambda), \qquad x, y\in S, \quad s>0.$$

By a simple calculation, $c(x, y)-c(0, 0)-Axy=B(x+y)$, from which

$$Bs^{-\rho-1} = \int_{0+}^{\infty} e^{-\lambda s}d\xi_{12}^{*}(\lambda), \qquad s>0.$$

This means (5.14).

ξ_{12}^{*} is independent of choice of $\{t_n\}$ and by a standard argument, we get

$$\lim_{t\to\infty} t^{\rho+1}K(t)\int_{0+}^{\infty} e^{-\lambda t}\varphi_1(x, -\lambda)\varphi_2(y, -\lambda)\sigma_{12}(d\lambda) = By,$$

uniformly in $x, y\in[a, b]$, $a, b\in S$.

This coupled with (5.9) and (5.12) gives us the assertion of the theorem. q.e.d.

PROOF OF THEOREM 2. Note that the assumption of Theorem 1 is satisfied. Further note that, for each $x\in S$, there are $c_1>0$ and $t_1>0$ such that, for $t\geqq t_1$ and $y\geqq x$,

$$p(t, x, y) \leqq p(t/2, x, x)+\{c_1(y-x)/tk(t)\}\Big(\int_0^{t/2} p(s, x, x)ds+\Phi(x, x)\Big),$$

where $k(t)$ is the inverse function of $t\mapsto tm(t)$, $t>0$ (see [**14**; Lemma 4.2]). Therefore, by means of (3.8) and (3.9),

$$\begin{aligned} p(t, x, y)/y &\leqq p(t/2, x, x)/a+\{c_1(y-x)/ytk(t)\}\Big(\int_0^{t/2} p(s, x, x)ds+\Phi(x, x)\Big) \\ &\leqq p(t/2, x, x)/a+(c_1/tk(t))G(0+, x, x), \end{aligned}$$

for $t\geqq t_1$ and $y\geqq\max\{x, a\}$. By virtue of (1.4),

$$\lim_{t\to\infty} t^{\rho+1}K(t)p(t/2, x, x) < \infty.$$

Since $\limsup_{t\to\infty} h_2(1/t)/k(t)<\infty$ (see [**9**; Theorem 2.3]), we have by Lemma 5.1

$$\limsup_{t\to\infty} t^{\rho}K(t)/k(t) < \infty.$$

Thus (2.5) follows. q.e.d.

PROOF OF THEOREM 3. We notice that (A.3) with $\tau=\infty$ holds. (1.5) and (2.5) imply (2.6). (1.6) is deduced from (1.5), (2.6) and Lebesgue's dominated convergence theorem. Hence it is enough to show (1.5).

We use the expression (3.4). Noting the relation (3.6) and (5.8), we see

$$\sigma((0, \lambda]) = \int_{0+}^{\lambda+}\varphi_2(l_1, -\lambda)^{-2}\sigma_{11}(d\lambda) \sim (C_2(\rho)/\Gamma(\rho+2))\lambda^{\rho+1}/K(1/\lambda) \qquad \text{as } \lambda\downarrow 0.$$

By Abelian theorem

$$\int_{0+}^{\infty} e^{-\lambda t}\sigma(d\lambda) \sim C_2(\rho)t^{-\rho-1}/K(t) \qquad \text{as } t\to\infty .$$

Fix an $a\in S$ arbitrarily. Then by virtue of (3.16) and (3.17),

$$\begin{aligned}
&\sup_{l_1<x,\, y\leq a} |\phi(x, \lambda)\phi(y, \lambda)/(x-l_1)(y-l_1)-1| \\
&\leqq \sup_{l_1<x,\, y\leq a} |\phi(x, \lambda)-(x-l_1)|\,|\phi(y, \lambda)|/(x-l_1)(y-l_1) \\
&\quad + \sup_{l_1<x,\, y\leq a} |\phi(y, \lambda)-(y-l_1)|/(y-l_1) \\
&\leqq 2M(a)|\lambda| e^{2M(a)|\lambda|}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sup_{l_1<x,\, y\leq a} |t^{\rho+1}K(t)p(t, x, y)/(x-l_1)(y-l_1)-C_2(\rho)| \\
&\leqq 2M(a)t^{\rho+1}K(t)\int_{0+}^{\infty}\lambda\exp\{-\lambda(t-2M(a))\}\sigma(d\lambda)+\left|t^{\rho+1}K(t)\int_{0+}^{\infty}e^{-\lambda t}\sigma(d\lambda)-C_2(\rho)\right| \\
&\longrightarrow 0 \qquad \text{as } t\to\infty ,
\end{aligned}$$

which implies (1.5). q. e. d.

We turn to the proof of Theorem 4. In the following, we assume all the conditions in Theorem 4. Let $k(x)$ be the inverse function of $x\mapsto xm(x)$, $x\geqq 0$, as in the proof of Theorem 2. Then it holds that

(5.16) $$k(x) \sim x^{\rho}K(x) \qquad \text{as } x\to\infty ,$$

(5.17) $$f(k(x))m(k(x)) \sim x^{\rho\gamma-\rho+1}L_f(k(x))K(x)^{\gamma-1} \qquad \text{as } x\to\infty .$$

For any fixed $a\in S$, we set

$$F(x) = \int_{(a, x]} yf(y)dm(y) , \qquad x>a .$$

Due to [**14**; Lemma 5.1],

$$\lim_{x\to\infty} F(x)/xf(x)m(x) = (1-\rho)/(1+\rho\gamma) , \qquad \text{if } \gamma>-1/\rho ,$$

$$\lim_{x\to\infty} F(x)/\int_1^x y^{-1}L_f(y)L(y)dy = 1/\rho-1 , \qquad \text{if } \gamma=-1/\rho .$$

Therefore, by means of (5.16) and (5.17),

(5.18) $$\lim_{t\to\infty} t^{-1}k(t)^{-1}F(k(t))/t^{\rho(\gamma-1)}\kappa(t) = \{\rho(1-\rho)\}^{\rho\gamma-\rho+1}\Gamma(\rho)/\Gamma(\rho\gamma+2) .$$

Since $\lim_{x\to\infty}F(x)=\infty$ by the assumption of the theorem, we also have, by (5.16) and (5.17),

(5.19) $$\lim_{t\to\infty} (t^{\rho+1}K(t))^{-1}/t^{\rho(\gamma-1)}\kappa(t) = 0 .$$

For any $c>0$ with $m(c)>0$, let $m^{(c)}(x)=m(cx)/m(c)$, $l_1/c<x<\infty$. Then the condition (1.3) implies

$$\lim_{c\to\infty} m^{(c)}(x) = m^*(x) \equiv x^{1/\rho-1}, \qquad x\geqq 0.$$

We denote $q_a(t, y)$ in (3.10) by $q_a^{(c)}(t, y)$ and $q_a^*(t, y)$ if m is replaced by $m^{(c)}$ and m^*, respectively. Then it holds that

$$q_0(t, y) = q_0^{(c)}(t/cm(c), y/c)/cm(c), \qquad t, y>0, \tag{5.20}$$

$$q_0^*(t, y) = C_2(\rho)yt^{-\rho-1}\exp\{-\rho(1-\rho)y^{1/\rho}t^{-1}\}, \quad t, y>0. \tag{5.21}$$

Now we give

PROOF OF THEOREM 4. It suffices to show (2.9) for $x=0$. The argument for the case $-1/\rho<\gamma<1-1/\rho$ is similar to that in [**14**]. So we omit the proof for that case. Suppose $\gamma=-1/\rho$, whence $f\in L^1((a, \infty), m)$. Fix a sufficiently large $a\in\operatorname{Supp}(dm)$ so that f is positive on (a, ∞). Put

$$\begin{aligned} T_t f(0) &= \int_{l_1+}^{a+} p(t, 0, y)f(y)dm(y) + \int_{a+}^{\infty} p(t, 0, y)f(y)dm(y) \\ &\equiv T_1(t)+T_2(t). \end{aligned}$$

Since $\int_{(l_1, a]} |f(y)|(y-l_1)dm(y)<\infty$, Theorem 3 asserts that

$$T_1(t) \sim C_2(\rho)(-l_1)(t^{\rho+1}K(t))^{-1}\int_{l_1+}^{a+} f(y)(y-l_1)dm(y) \qquad \text{as } t\to\infty.$$

By this and (5.19),

$$\lim_{t\to\infty} T_1(t)/t^{\rho(\gamma-1)}\kappa(t) = 0.$$

In order to get (2.9), we only have to show

$$T_2(t) \sim (-l_1)t^{\rho(\gamma-1)}\kappa(t) \qquad \text{as } t\to\infty. \tag{5.22}$$

We decompose $T_2(t)$ for each $u\in(0, 1)$ and $\eta>0$ as

$$\begin{aligned} T_2(t) &= \Big(\int_{\substack{a<y<\infty\\ 0<s<ut}} + \int_{\substack{a<y\leqq\eta k(t)\\ ut\leqq s<t}} + \int_{\substack{\eta k(t)<y<\infty\\ ut\leqq s<t}}\Big) p(t-s, 0, 0)q_0(s, y)f(y)dm(y)ds \\ &\quad + \Phi(0, 0)\Big(\int_{a<y\leqq\eta k(t)} + \int_{\eta k(t)<y<\infty}\Big) f(y)q_0(t, y)dm(y) \\ &\equiv H_1(t, u)+H_2(t, \eta, u)+H_3(t, \eta, u)+H_4(t, \eta)+H_5(t, \eta). \end{aligned}$$

Since $p(t, 0, 0)$ is nonincreasing in t,

$$H_1(t, u) \leqq p(t(1-u), 0, 0)\int_{a+}^{\infty} f(y)dm(y)\int_0^{ut} q_0(s, y)ds.$$

Combining this with (1.4), [**14**; (3.16)] and (5.19), we get

(5.23) $$\lim_{t\to\infty} H_1(t, u)/t^{\rho(\gamma-1)}\kappa(t) = 0\,.$$

By using (5.20),

$$H_2(t, \eta, u) = \int_u^1 p(t(1-s), 0, 0)ds\int_{a+}^{\eta k(t)+} f(y)q_0^{(k(t))}(s, y/k(t))dm(y)\,.$$

In view of [**14**; (4.7)], (5.21), (3.8), and (3.9), for any $\varepsilon\in(0, 1)$, there is $t_1>0$ such that

$$\begin{aligned}&(1-\varepsilon)C_2(\rho)\exp\{-\rho(1-\rho)\eta^{1/\rho}u^{-1}\}k(t)^{-1}F(\eta k(t))t^{-1}(-l_1-\Phi(0, 0))\\ &\leqq H_2(t, \eta, u)\\ &\leqq (1+\varepsilon)C_2(\rho)u^{-\rho-1}k(t)^{-1}F(\eta k(t))t^{-1}(-l_1-\Phi(0, 0))\,, \qquad t\geqq t_1\,.\end{aligned}$$

Since $\lim_{c\to\infty}F(cx)/F(c)=1$ as in [**14**; Lemma 5.1], we see that

$$\begin{aligned}&\lim_{u\uparrow 1}\lim_{\eta\downarrow 0}\liminf_{t\to\infty} H_2(t, \eta, u)/t^{-1}k(t)^{-1}F(k(t))\\ &\geqq (1-\varepsilon)C_2(\rho)(-l_1-\Phi(0, 0))\,,\\ &\lim_{u\uparrow 1}\lim_{\eta\downarrow 0}\limsup_{t\to\infty} H_2(t, \eta, u)/t^{-1}k(t)^{-1}F(k(t))\\ &\leqq (1+\varepsilon)C_2(\rho)(-l_1-\Phi(0, 0))\,.\end{aligned}$$

In the same way as above,

$$\begin{aligned}(1-\varepsilon)C_2(\rho)\Phi(0, 0) &\leqq \lim_{\eta\downarrow 0}\liminf_{t\to\infty} H_4(t, \eta)/t^{-1}k(t)^{-1}F(k(t))\\ &\leqq \lim_{\eta\downarrow 0}\limsup_{t\to\infty} H_4(t, \eta)/t^{-1}k(t)^{-1}F(k(t))\\ &\leqq (1+\varepsilon)C_2(\rho)\Phi(0, 0)\,.\end{aligned}$$

Hence, by means of (5.18),

(5.24) $$\begin{aligned}(1-\varepsilon)(-l_1) &\leqq \lim_{u\uparrow 1}\lim_{\eta\downarrow 0}\liminf_{t\to\infty}\{H_2(t, \eta, u)+H_4(t, \eta)\}/t^{\rho(\gamma-1)}\kappa(t)\\ &\leqq \lim_{u\uparrow 1}\lim_{\eta\downarrow 0}\limsup_{t\to\infty}\{H_2(t, \eta, u)+H_4(t, \eta)\}/t^{\rho(\gamma-1)}\kappa(t)\\ &\leqq (1+\varepsilon)(-l_1)\,.\end{aligned}$$

Notice that we can repeat the argument for [**14**; (4.18)] in this case, too. Therefore by (3.8) and (3.9),

$$\begin{aligned}&\limsup_{t\to\infty} H_3(t, \eta, u)/t^{-1}f(k(t))m(k(t))\\ &\leqq (-l_1-\Phi(0, 0))\max_{u\leqq s\leqq 1}\int_\eta^\infty y^\gamma q_0^*(s, y)dm^*(y) < \infty.\end{aligned}$$

Since $\lim_{x\to\infty}F(x)/xf(x)m(x)=\infty$ (see [**14**; Lemma 5.1]), we obtain by (5.18),

$$\lim_{t\to\infty} H_3(t, \eta, u)/t^{\rho(\gamma-1)}\kappa(t) = 0\,.$$

By the same argument as above,

$$\lim_{t\to\infty} H_5(t, \eta, u)/t^{\rho(\gamma-1)}\kappa(t) = 0.$$

Combining these estimates with (5.23) and (5.24), we arrive at (5.22). q. e. d.

EXAMPLE 5.6 (Elastic Brownian motion). Let us consider the differential operator $\mathcal{L}=d^2/2dx^2$ on $(0, \infty)$. We impose the boundary condition $\theta_1 u(0)-\theta_2 u^+(0)+\theta_3 \mathcal{L}u(0)=0$, where $\theta_1+\theta_2+\theta_3=1$, $\theta_i\geqq 0$, $i=1, 3$, $\theta_2>0$, or $\theta_1-1=\theta_2=\theta_3=0$. Set $l_1=-\theta_2/\theta_1$ and $m(x)=2x$, $x\geqq 0$, $=-\theta_3/\theta_2$, $l_1<x<0$. As was noted in §2, $\mathcal{L}$ reduces to $\mathfrak{G}=(d/dm)(d/dx)$ on (l_1, ∞). The diffusion process having $\mathfrak{G}$ as the generator is called the elastic Brownian motion in the case that $0<\theta_1<1$ and $\theta_3=0$. If $\theta_1=0$, then $\mathfrak{G}$ is recurrent and the asymptotic behaviors of $p(t, x, y)$ and $T_t f(x)$ were observed in [**12**] and [**14**]. So we only consider the case $0<\theta_1\leqq 1$. Suppose that $(y-l_1)f(y)\in L^1((l_1, a), m)$, $l_1<a<\infty$, and $f(x)\sim x^\gamma$ as $x\to\infty$. Since the assumption (A.3)′ is satisfied, it follows from Theorems 3 and 4 that, for any a, $x\in S$,

$$\lim_{t\to\infty} \sup_{l_1<y,z\leq a} |p(t, y, z)/(y-l_1)(z-l_1)-(2\pi)^{-1/2}t^{-3/2}| = 0,$$

$$\limsup_{t\to\infty} t^{3/2} \sup_{y>l_1} p(t, x, y)/(y-l_1) < \infty,$$

$$T_t f(x) \sim \begin{cases} C_0(2\pi)^{-1/2}(x-l_1)t^{-3/2}, & \gamma<-2, \\ (2\pi)^{-1/2}(x-l_1)t^{-3/2}\log t, & \gamma=-2, \\ 2^{(\gamma+1)/2}\pi^{-1/2}\Gamma(\gamma/2+1)(x-l_1)t^{(\gamma-1)/2}, & \gamma>-2, \end{cases}$$

as $t\to\infty$, where $C_0=2\int_0^\infty f(y)(y-l_1)dy$ if $\theta_1=1$, $=2\int_0^\infty f(y)(y-l_1)dy+f(0)\theta_3/\theta_2$ if $0<\theta_1<1$.

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