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# Rings with only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules

Dedicated to Professor Hiroyuki Tachikawa on his 60th birthday

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## 1. Introduction.

The purpose of this paper is to prove the following

THEOREM (1.1). *Let $P=k[\![X_1, X_2, \cdots, X_n]\!]$ be a formal power series ring over an algebraically closed field $k$ of $\operatorname{ch} k\neq 2$. Let $R=P/I$, where $I$ is an ideal of $P$ and suppose that $\dim R=d\geqq 2$. Then the following two conditions are equivalent.*

(1) *$R$ is a regular local ring.*

(2) *$R$ is a Cohen-Macaulay ring that possesses only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules.* (*See Section* 2 *for the notion of maximal Buchsbaum module.*)

*When this is the case, the syzygy modules of the residue class field $k$ of $R$ are the representatives of indecomposable maximal Buchsbaum modules and so there are exactly $d$ non-isomorphic indecomposable maximal Buchsbaum modules over $R$.*

Our contribution in the above theorem is the implication (2)$\Rightarrow$(1). The last assertion and the implication (1)$\Rightarrow$(2) are due to [**6**] (see also [**5**, Theorem 3.2]), where some consequences of the result are discussed too.

We would like to note here that the assumption $\dim R\geqq 2$ in Theorem (1.1) is not superfluous. There actually exist non-regular Cohen-Macaulay local rings $R$ of $\dim R=1$ that possess only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules. The typical example is the ring

$$R=k[\![X, Y]\!]/(X^3+Y^2)$$

($k$, any field), which has exactly 5 indecomposable maximal Buchsbaum modules (cf. (5.3)). So the result of one-dimensional case seems more complicated.

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Our proof of Theorem (1.1) is based on the recent progress [3] and [10] of the theory of Cohen-Macaulay local rings of finite CM-representation type, i.e., Cohen-Macaulay local rings with only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay modules. Let $R$ be as in (1.1). Then the ring $R$ is evidently of finite CM-representation type, if it satisfies the condition (2) of (1.1) (since any Cohen-Macaulay module is by definition Buchsbaum, see (2.1)). According to [1], [3] and [12] such rings $R$ are rather rare and when $R$ is Gorenstein, it is already proved by [3] (see [8, Satz 1.2] too) that $R$ must be a simple hypersurface in the sense of [9]; so the structure of $R$ is completely known.

In Section 2 of this paper we will establish a technical lemma (2.3) which enables us to construct infinitely many non-isomorphic indecomposable maximal Buchsbaum modules, once there is given an indecomposable maximal Cohen-Macaulay module satisfying certain requirements. The lemma also helps us reduce our problem to the case where $R$ is a Gorenstein ring and consequently, to the case where $R$ is a simple hypersurface (Proposition (2.4)). We will prove Theorem (1.1) by paralogism and the proof is divided into two parts, i.e., the case where $\dim R \geqq 3$ and the case where $\dim R=2$. In both cases the theorems on simple hypersurfaces in [3] and [10] are quite helpful to accomplish the proof of Theorem (1.1) by finding indecomposable maximal Cohen-Macaulay $R$-modules which satisfy the requirements of (2.3).

Let us now explain how to organize this paper. The proof of Theorem (1.1) of the case where $\dim R \geqq 3$ (resp. $\dim R=2$) shall be given in Section 3 (resp. Section 4). Section 2 is devoted to preliminaries. The definition of Buchsbaum modules and some basic results on matrix factorizations of maximal Cohen-Macaulay modules over hypersurfaces shall be summarized too. In Section 5 we shall explore the ring $k[\![X, Y]\!]/(X^3+Y^2)$, that is the simplest counterexample to Theorem (1.1) in one-dimensional case.

Throughout this paper let $R$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $\dim R=d$. We denote by $\mathrm{H}^i_{\mathfrak{m}}(\cdot)$ the $i^{\text{th}}$ local cohomology functor of $R$ relative to $\mathfrak{m}$.

## 2. Preliminaries.

We begin with the definition of Buchsbaum modules.

DEFINITION (2.1) ([13]). Let $M$ be a finitely generated $R$-module. Then $M$ is said to be a Buchsbaum module, if the difference

$$\mathrm{I}_R(M) = l_R(M/\mathfrak{q}M) - \mathrm{e}_{\mathfrak{q}}(M)$$

is an invariant of $M$, that is independent of the particular choice of a parameter ideal $\mathfrak{q}$ for $M$ (here $l_R(M/\mathfrak{q}M)$ and $\mathrm{e}_{\mathfrak{q}}(M)$ respectively denote the length of $M/\mathfrak{q}M$ and the multiplicity of $M$ relative to $\mathfrak{q}$).

Consequently, $M$ is a Cohen-Macaulay module if and only if $M$ is a Buchsbaum module of $\mathrm{I}_R(M)=0$. Thus the concept of Cohen-Macaulay module is naturally generalized by that of Buchsbaum module. We say that a Buchsbaum $R$-module $M$ is *maximal*, if $\dim_R M=\dim R$.

The readers may consult the monumental book [**15**] for the general reference on Buchsbaum rings and modules and also for the recent developments of the theory. So let us note here only the next criterion, which we need in the sequel:

LEMMA (2.2) ([**14**, Corollary 1.1]). *Let $M$ be a finitely generated $R$-module with* $\mathrm{depth}_R M=t<\dim_R M=s$. *Assume that* $\mathrm{H}_{\mathfrak{m}}^i(M)=(0)$ *for $i\neq t$, $s$. Then $M$ is a Buchsbaum module if and only if*

$$\mathfrak{m}\cdot \mathrm{H}_{\mathfrak{m}}^t(M)=(0).$$

The following lemma is the key of this paper.

LEMMA (2.3). *Suppose that $R$ is a Cohen-Macaulay ring of* $\dim R=d\geqq 2$ *and that $R$ possesses the canonical module $K_R$. Let $L$ be a maximal Cohen-Macaulay $R$-module and let* $\Lambda=\mathrm{End}_R L$ *(resp. $J$) denote the endomorphism ring of $L$ (resp. the Jacobson radical of $\Lambda$). If* $\dim_{R/\mathfrak{m}}\Lambda/J=1$ *and if one of the following conditions*

(a) $\dim_{R/\mathfrak{m}} L/JL\geqq 2$

(b) $\dim_{R/\mathfrak{m}} JL/(J^2L+\mathfrak{m}L)\geqq 2$

*is satisfied, then $R$ has a family $\{M_\lambda\}_{\lambda\in R/\mathfrak{m}}$ of indecomposable maximal Buchsbaum modules such that $M_\lambda\not\cong M_\mu$ for $\lambda\neq\mu$.*

PROOF. Choose elements $f$ and $g$ of $L$ (resp. $JL$), when the condition (a) (resp. (b)) is satisfied, so that the classes $\bar{f}$ and $\bar{g}$ of $f$ and $g$ in $L/JL$ (resp. $JL/(J^2L+\mathfrak{m}L)$) are linearly independent over $R/\mathfrak{m}$. For each $\lambda\in R/\mathfrak{m}$, let $c_\lambda\in R$ be such that $\lambda=c_\lambda \bmod \mathfrak{m}$. We put $h_\lambda=f+c_\lambda\cdot g$ and define

$$M_\lambda=JL+Rh_\lambda \qquad (\text{resp. } M_\lambda=J^2L+\mathfrak{m}L+Rh_\lambda),$$

if (a) (resp. (b)) is the case. Then as $M_\lambda\supset\mathfrak{m}L$, applying the functors $\mathrm{H}_{\mathfrak{m}}^i(\cdot)$ to the exact sequence

$$(\#) \qquad 0\longrightarrow M_\lambda\longrightarrow L\longrightarrow L/M_\lambda\longrightarrow 0$$

we get

$$\mathrm{H}_{\mathfrak{m}}^i(M_\lambda)=\begin{cases} L/M_\lambda & (i=1), \\ \mathrm{H}_{\mathfrak{m}}^d(L) & (i=d), \\ (0) & (i\neq 1,\ d). \end{cases}$$

Hence by (2.2) $M_\lambda$ is a maximal Buchsbaum $R$-module.

Take the $K_R$-dual $[\cdot]^*$ of the sequence (#). Then as $\operatorname{depth}_R K_R = d \geqq 2$, we get $L^* = M_\lambda^*$ whence

$$M_\lambda^{**} = L$$

by [7, Satz 6.1]. This guarantees that $M_\lambda$ is indecomposable, because so is $L$ by the assumption that $\dim_{R/\mathfrak{m}} \Lambda/J = 1$.

Let $\phi: M_\lambda \to M_\mu$ be an isomorphism for some $\lambda, \mu \in R/\mathfrak{m}$. Then as $M_\lambda^{**} = L$ and $M_\mu^{**} = L$, the map $\phi$ extends to an automorphism $\psi$ of $L$. Write $\psi = c + \rho$ with $c \in R$ and $\rho \in J$. Then as $\rho M_\lambda \subset M_\lambda$ by the definition of $M_\lambda$, we get

$$M_\mu = \psi M_\lambda \subset M_\lambda .$$

Hence $h_\mu \in M_\lambda$, i.e., $\bar{f} + \mu\bar{g} \in R/\mathfrak{m}(\bar{f} + \lambda\bar{g})$, which forces $\lambda = \mu$ as required.

PROPOSITION (2.4). *Suppose that $R$ is a Cohen-Macaulay ring of* $\dim R \geqq 2$ *and that $R$ possesses the canonical module $K_R$. If $R$ has only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules and if the field $R/\mathfrak{m}$ is infinite, then the completion $\hat{R}$ of $R$ is a hypersurface.*

PROOF. If $R$ were not a Gorenstein ring, then by (2.3) we can construct from $L = K_R$ infinitely many non-isomorphic indecomposable maximal Buchsbaum $R$-modules, because $\operatorname{Hom}_R(K_R, K_R) = R$ and because $\dim_{R/\mathfrak{m}} K_R/\mathfrak{m}K_R \geqq 2$ by [7, Satz 6.10]. Hence $R$ has to be Gorenstein. Since $R$ is of finite CM-representation type, by [8, Satz 1.2] $\hat{R}$ is even a hypersurface.

PROPOSITION (2.5). *Let $R$ be a normal ring of* $\dim R = 2$ *and suppose that the field $R/\mathfrak{m}$ is infinite. If $R$ has only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules, then $R$ is a* UFD.

PROOF. Assume that $R$ is not a UFD and take a non-principal prime ideal $\mathfrak{p}$ of $R$ so that $\dim R_\mathfrak{p} = 1$. Then $\operatorname{End}_R \mathfrak{p} = R$ and $\dim_{R/\mathfrak{m}} \mathfrak{p}/\mathfrak{m}\mathfrak{p} \geqq 2$. Since $\mathfrak{p} \cong \operatorname{Hom}_R(\operatorname{Hom}_R(\mathfrak{p}, R), R)$, by the proof of (2.3) (replacing the $K_R$-dual by the $R$-dual) we can construct from the $R$-module $L = \mathfrak{p}$ a family $\{M_\lambda\}_{\lambda \in R/\mathfrak{m}}$ of non-isomorphic indecomposable maximal Buchsbaum $R$-modules — this is a contradiction.

The rest of this section is devoted to a brief survey on matrix factorizations.

Let $P_0$ be a regular ring with maximal ideal $\mathfrak{n}_0$ and let $0 \neq f \in \mathfrak{n}_0$. Then a matrix factorization of $f$ is a pair

$$(F \xrightarrow{\phi} G,\ G \xrightarrow{\psi} F)$$

of homomorphisms of finitely generated free $P_0$-modules such that

$$\phi \circ \psi = f \cdot 1_G \quad \text{and} \quad \psi \circ \phi = f \cdot 1_F .$$

A morphism between two matrix factorizations $(F\xrightarrow{\phi}G,\ G\xrightarrow{\psi}F)$ and $(F'\xrightarrow{\phi'}G',\ G'\xrightarrow{\psi'}F')$ is a pair

$$(F\xrightarrow{\alpha}F',\ G\xrightarrow{\beta}G')$$

of homomorphisms which make the following square

$$\begin{array}{ccc} F & \xrightarrow{\phi} & G \\ \alpha\downarrow & & \downarrow\beta \\ F' & \xrightarrow{\phi'} & G' \end{array}$$

commutative. The matrix factorizations of $f$ form an additive category, which we denote by $\mathrm{MF}(f)$.

For each $X=(F\xrightarrow{\phi}G,\ G\xrightarrow{\psi}F)\in\mathrm{MF}(f)$, we define

$$\operatorname{cok} X := \operatorname{Coker}\phi.$$

Then one can easily check that $\operatorname{cok} X$ is a maximal Cohen-Macaulay $R_0$-module (here $R_0:=P_0/fP_0$) and the operation cok is an additive functor from $\mathrm{MF}(f)$ to the category $\mathrm{MCM}(R_0)$ of maximal Cohen-Macaulay $R_0$-modules. We say that a matrix factorization $X$ of $f$ is projective (resp. trivial), if $\operatorname{cok} X$ is free (resp. $\operatorname{cok} X=(0)$).

The next result due to D. Eisenbud is fundamental.

PROPOSITION (2.6) ([**4**, Chapter 6]). *The functor* cok *induces an equivalence between the category* $\mathrm{MF}(f)/J$ *and the category* $\mathrm{MCM}(R_0)$, *where* $J$ *denotes the ideal in* $\mathrm{MF}(f)$ (*in the sense of* [**11**, 2.2]) *generated by the morphisms that factor through the trivial matrix factorization*

$$(P_0\xrightarrow{1}P_0,\ P_0\xrightarrow{f}P_0).$$

Now suppose that $P_0=k[\![X_1, X_2, \cdots, X_n]\!]$ is a formal power series ring over an algebraically closed field $k$ of $\operatorname{ch} k\neq 2$. Let $P=P_0[\![Y, Z]\!]$ be a formal power series ring over $P_0$. We consider the ring $R:=P/(f+Y^2+Z^2)P$ and the category $\mathrm{MF}(f+Y^2+Z^2)$ of matrix factorizations of $f+Y^2+Z^2$. The purpose is to compare $\mathrm{MCM}(R)$ and $\mathrm{MCM}(R_0)$. First let us write

$$u=Y+iZ \quad\text{and}\quad v=Y-iZ$$

(so that $uv=Y^2+Z^2$) and introduce an additive functor

$$\mathrm{H}:\mathrm{MF}(f)\longrightarrow\mathrm{MF}(f+Y^2+Z^2)$$

that associates to $(\phi, \psi)\in\mathrm{MF}(f)$ the matrix factorization

$$\left(\begin{pmatrix} u & \psi \\ \phi & -v \end{pmatrix}, \begin{pmatrix} v & \psi \\ \phi & -u \end{pmatrix}\right)$$

of $f+Y^2+Z^2$, and to a morphism $(\alpha, \beta)$ in $\mathrm{MF}(f)$ the morphism

$$\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\right)$$

in $\mathrm{MF}(f+Y^2+Z^2)$. Let $\underline{\mathrm{MCM}}(R_0)$ (resp. $\underline{\mathrm{MCM}}(R)$) be the quotient of the category $\mathrm{MCM}(R_0)$ (resp. $\mathrm{MCM}(R)$) by the ideal which is generated by the morphisms that factor through free $R_0$-modules (resp. free $R$-modules). By (2.6) $\underline{\mathrm{MCM}}(R_0)$ and $\underline{\mathrm{MCM}}(R)$ are respectively the quotients of $\mathrm{MF}(f)$ and $\mathrm{MF}(f+Y^2+Z^2)$ by the ideals of morphisms factoring through projective matrix factorizations, too. So H induces a functor from $\underline{\mathrm{MCM}}(R_0)$ to $\underline{\mathrm{MCM}}(R)$ and Knörrer's periodicity theorem can be stated as follows:

THEOREM (2.7) ([**10**, Theorem 3.1]). H *induces an equivalence between the categories* $\underline{\mathrm{MCM}}(R_0)$ *and* $\underline{\mathrm{MCM}}(R)$.

## 3. Proof of Theorem (1.1) in the case where $\dim R \geqq 3$.

Let $P=k[\![X_1, X_2, \cdots, X_n, Y, Z]\!]$ be a formal power series ring over an algebraically closed field $k$ of $\mathrm{ch}\, k \neq 2$. We put

$$P_0 = k[\![X_1, X_2, \cdots, X_n]\!] \quad \text{and} \quad \mathfrak{n}_0 = (X_1, X_2, \cdots, X_n)P_0.$$

Let $0 \neq f \in \mathfrak{n}_0^2$ and define

$$R_0 := P_0/fP_0 \quad \text{and} \quad R := P/(f+Y^2+Z^2)P.$$

Then we have the following

THEOREM (3.1). *Let $L$ be an indecomposable maximal Cohen-Macaulay $R$-module such that $L \not\cong R$. Then*

$$\dim_k \Lambda/J = 1 \quad \textit{and} \quad \dim_k L/JL \geqq 2,$$

*where $\Lambda = \mathrm{End}_R L$ (resp. $J$) denotes the endomorphism ring of $L$ (resp. the Jacobson radical of $\Lambda$).*

To prove Theorem (3.1) we need three more functors T, $\rho$ and rest. First, for each $X=(F_1 \to F_2, F_2 \to F_1) \in \mathrm{MF}(f)$, let

$$\mathrm{T}X = (F_2 \xrightarrow{\psi} F_1, F_1 \xrightarrow{\phi} F_2).$$

Then

$$\mathrm{T} : \mathrm{MF}(f) \longrightarrow \mathrm{MF}(f)$$

is an involutive functor and $\mathrm{cok}\,\mathrm{T}X$ is the first syzygy module of $\mathrm{cok}\,X$. Let

$$\rho : \mathrm{MF}(f+Y^2+Z^2) \longrightarrow \mathrm{MF}(f)$$

be the functor which sends a matrix factorization

$$(F_1 \xrightarrow{\Phi} F_2,\ F_2 \xrightarrow{\Psi} F_1)$$

of $f+Y^2+Z^2$ to the matrix factorization of $f$ defined by the induced maps between $F_1/(Y, Z)F_1$ and $F_2/(Y, Z)F_2$. On the level of maximal Cohen-Macaulay modules we have a functor

$$\begin{aligned} \mathrm{rest} : \mathrm{MCM}(R) &\longrightarrow \mathrm{MCM}(R_0), \\ M &\longmapsto M/(Y, Z)M. \end{aligned}$$

The following identification is easily checked.

(3.2) $$\rho\circ\mathrm{H} = \mathrm{id}\oplus\mathrm{T} \quad \textit{and} \quad \mathrm{rest}\circ\mathrm{cok} = \mathrm{cok}\circ\rho.$$

In the proof of (3.1) we need the following remark. It is almost obvious and we omit the proof.

LEMMA (3.3). *Let $M$ and $N$ be maximal Cohen-Macaulay $R_0$-modules such that $M$ is indecomposable and $M \not\cong R_0$. Let $\alpha \in \mathrm{Hom}_{R_0}(M, N)$ and assume that $\alpha$ factors through a free $R_0$-module. Then*

$$\alpha(M) \subset \mathfrak{m}_0 N,$$

*where $\mathfrak{m}_0$ denotes the maximal ideal of $R_0$.*

PROOF OF THEOREM (3.1). By virtue of (2.6) and [**10**, (3.6)] we may choose an indecomposable matrix factorization $X$ of $f$ so that $L = \mathrm{cok}\,\mathrm{H}(X)$. Let $M_1 = \mathrm{cok}\,X$ and $M_2 = \mathrm{cok}\,\mathrm{T}X$. Then by (3.2) we have

$$L/(Y, Z)L = M_1 \oplus M_2.$$

Let $\varepsilon : L \to M_1 \oplus M_2$ denote the canonical epimorphism. To see $\dim_k L/JL \geqq 2$, it is enough to show

$$\varepsilon(JL) \subset J_1 M_1 \oplus J_2 M_2,$$

where $J_i$ denotes the Jacobson radical of $\mathrm{End}_{R_0} M_i$.

Consider the following commutative diagram

$$\begin{array}{ccccc} \mathrm{End}\,X & \xrightarrow{\mathrm{H}} & \mathrm{End}\,\mathrm{H}(X) & \xrightarrow{\rho} & \mathrm{End}(X\oplus\mathrm{T}X) \\ & & \mathrm{cok}\downarrow & & \downarrow\mathrm{cok} \\ & & \mathrm{End}_R L & \xrightarrow{\mathrm{rest}} & \mathrm{End}_{R_0}(M_1\oplus M_2) \end{array}$$

of algebras, where all the homomorphisms are induced by the corresponding functors (hence the vertical homomorphisms are surjective, see (2.6)). Let $\phi \in J$

and write $\phi=\mathrm{cok}\psi$ with $\psi\in\mathrm{End}\,\mathrm{H}(X)$. Then as $\mathrm{End}\,\mathrm{H}(X)$ is local by [**10**, (3.6)], $\psi$ is in the Jacobson radical of $\mathrm{End}\,\mathrm{H}(X)$. So by (2.7) we can write that

$$\psi=\mathrm{H}(\xi)+\eta$$

with $\xi$ in the Jacobson radical of $\mathrm{End}\,X$ and $\eta\in\mathrm{End}\,\mathrm{H}(X)$ which factors through a projective matrix factorization of $f+Y^2+Z^2$. Hence

$$\begin{aligned}\mathrm{rest}\,\phi &= \mathrm{cok}(\rho(\mathrm{H}(\xi)))+\mathrm{cok}(\rho(\eta))\\ &= \mathrm{cok}(\xi\oplus\mathrm{T}\xi)+\mathrm{cok}(\rho(\eta)),\end{aligned}$$

as $\rho\circ\mathrm{H}=\mathrm{id}\oplus\mathrm{T}$ by (3.2).

Note that $\mathrm{T}\xi$ is in the Jacobson radical of $\mathrm{End}\,\mathrm{T}X$ (since $\xi$ is in the Jacobson radical of $\mathrm{End}\,X$) and we get $\mathrm{cok}\,\xi\in J_1$ and $\mathrm{cok}\,\mathrm{T}\xi\in J_2$. Hence the image of the endomorphism $\mathrm{cok}(\xi\oplus\mathrm{T}\xi)=\mathrm{cok}\,\xi\oplus\mathrm{cok}\,\mathrm{T}\xi$ of $M_1\oplus M_2$ is contained in $J_1M_1\oplus J_2M_2$. Write that

$$\mathrm{cok}(\rho(\eta))=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$$

with $a_{ij}\in\mathrm{Hom}_{R_0}(M_j, M_i)$ and recall that the morphism $\eta$ factors through a projective matrix factorization of $f+Y^2+Z^2$. Then we find that each $a_{ij}$ factors through a free $R_0$-module, as so does $\mathrm{cok}(\rho(\eta))=\mathrm{rest}(\mathrm{cok}\,\eta)$. Hence by (3.3) we get $\mathrm{Im}\,a_{ij}\subset\mathfrak{m}_0M_i$ and therefore

$$\mathrm{Im}(\mathrm{cok}(\rho(\eta)))\subset J_1M_1\oplus J_2M_2\,.$$

Thus for any $\phi\in J$, $\mathrm{Im}(\mathrm{rest}\,\phi)\subset J_1M_1\oplus J_2M_2$ which guarantees

$$\varepsilon(JL)\subset J_1M_1\oplus J_2M_2$$

as required. This completes the proof of Theorem (3.1).

We are now in position to prove Theorem (1.1) in the higher dimensional case. Let $R$ be as in (1.1) and assume that $R$ satisfies the condition (2) of (1.1). Then by (2.4) $R$ must be a hypersurface and so we may assume the ideal $I$ is principal, say $I=gP$. Let us suppose that $R$ is non-regular. Then as $R$ has finite CM-representation type, by virtue of [**3**, Theorem A] the possible normal form of the $g$'s is completely classified. In particular if $\dim R=d\geqq 3$, we must have

$$gP=(f+Y^2+Z^2)P$$

for some regular system $X_1, X_2, \cdots, X_{d-1}, Y, Z$ of parameters of $P$ and for some non-zero element $f$ of $\mathfrak{n}_0^2$ where $\mathfrak{n}_0$ is the maximal ideal of $k[\![X_1, X_2, \cdots, X_{d-1}]\!]$. However in this situation, Lemma (2.3) and Theorem (3.1) claim that $R$ cannot satisfy the condition (2) in (1.1)——hence $\dim R=2$.

When $\dim R=2$, the possible normal form of the $g$'s in $P=k[\![X, Y, Z]\!]$ is

$f+Z^2$ where $f$ is one of the following type ([**3**, Theorem A] and [**9**]):

$(\mathrm{A}_n)$ $X^2+Y^{n+1}$ $(n\geqq 1)$

$(\mathrm{D}_n)$ $X^{n-1}+XY^2$ $(n\geqq 4)$

$(\mathrm{E}_6)$ $X^3+Y^4$ $(\mathrm{ch}\, k\neq 3)$

$X^3+Y^4,\ X^3+X^2Y^2+Y^4$ $(\mathrm{ch}\, k=3)$

$(\mathrm{E}_7)$ $X^3+XY^3$ $(\mathrm{ch}\, k\neq 3)$

$X^3+XY^3,\ X^3+X^2Y^2+XY^3$ $(\mathrm{ch}\, k=3)$

$(\mathrm{E}_8)$ $X^3+Y^5$ $(\mathrm{ch}\, k\neq 3, 5)$

$X^3+Y^5,\ X^3+X^2Y^3+Y^5,\ X^3+X^2Y^2+Y^5$ $(\mathrm{ch}\, k=3)$

$X^3+Y^5,\ X^3+XY^4+Y^5$ $(\mathrm{ch}\, k=5)$.

In the above list, one can easily check that any $P/(f+Z^2)$ is a normal ring but except $(\mathrm{E}_8)$ it cannot be a UFD. Consequently, by (2.5) our ring $R$ must be of type $(\mathrm{E}_8)$.

In the next section we will prove that the ring $R$ still fails to be of type $(\mathrm{E}_8)$.

## 4. Proof of Theorem (1.1) in the case where $\dim R=2$.

Let $P=k[\![X, Y, Z]\!]$ be a formal power series ring over an algebraically closed field $k$ of $\mathrm{ch}\, k\neq 2$. Let

$$F = X^3+Y^2G+Y^5+Z^2,$$

where $G$ is either 0 or one of the following:

$$X^2Y,\ X^2 \quad (\mathrm{ch}\, k=3),$$
$$XY^2 \quad (\mathrm{ch}\, k=5).$$

We put $R:=P/FP$. Recall that $R$ is a normal ring. Let $x$, $y$, $z$ and $g$ respectively denote $X$, $Y$, $Z$ and $G$ mod $FP$. We denote by $\mathfrak{m}$ the maximal ideal $(x, y, z)R$ of $R$.

Let $L$ be the $R$-submodule of $R^2$ generated by

$$f_1=\begin{pmatrix} x^2 \\ z \end{pmatrix},\quad f_2=\begin{pmatrix} -y \\ 0 \end{pmatrix},\quad f_3=\begin{pmatrix} -z \\ x \end{pmatrix} \quad \text{and} \quad f_4=\begin{pmatrix} 0 \\ y \end{pmatrix}.$$

Let $\Lambda=\mathrm{End}_R L$ (resp. $J$) denote the endomorphism ring of $L$ (resp. the Jacobson radical of $\Lambda$).

With the above notation the purpose of this section is to check the following assertions (4.1), which will complete the proof of Theorem (1.1) as is noted in the end of Section 3.

THEOREM (4.1). (1) *$L$ is an indecomposable maximal Cohen-Macaulay $R$-module.*

(2) $\dim_k L/JL = 1$.

(3) $\dim_k JL/(J^2L+\mathfrak{m}L) \geqq 2$.

We divide the proof of (4.1) into several steps. First of all let $L'$ denote the second syzygy module of $R/\mathfrak{m}$:

$$0 \longrightarrow L' \longrightarrow R^3 \xrightarrow[(x\ y\ z)]{} R \longrightarrow R/\mathfrak{m} \longrightarrow 0.$$

Then $L'$ is a maximal Cohen-Macaulay $R$-module of rank 2. The next assertion is directly checked.

(4.2) *$L'$ is generated by*

$$\begin{pmatrix} x^2 \\ y^4+yg \\ z \end{pmatrix}, \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}, \begin{pmatrix} -z \\ 0 \\ x \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}.$$

As $\operatorname{rank}_R L=2$ and as $L$ is a homomorphic image of $L'$ (via the homomorphism $\phi: R^3 \to R^2$ defined by $\phi(\begin{pmatrix} a \\ b \\ c \end{pmatrix})=\begin{pmatrix} a \\ c \end{pmatrix}$), we have $L' \cong L$ and hence $L$ is a maximal Cohen-Macaulay $R$-module. By (4.2) we check that the following sequence

$$(\#) \qquad R^4 \xrightarrow[\begin{pmatrix} z & 0 & x & y \\ 0 & z & -(y^4+yg) & x^2 \\ x^2 & -y & -z & 0 \\ y^4+yg & x & 0 & -z \end{pmatrix}]{} R^4 \xrightarrow[\begin{pmatrix} x^2 & -y & -z & 0 \\ z & 0 & x & y \end{pmatrix}]{} R^2$$

is exact. So the matrix factorization of $F=X^3+Y^2G+Y^5+Z^2$ corresponding to $L$ is

$$\left(\begin{pmatrix} Z & 0 & X & Y \\ 0 & Z & -(Y^4+YG) & X^2 \\ X^2 & -Y & -Z & 0 \\ Y^4+YG & X & 0 & -Z \end{pmatrix}, \begin{pmatrix} Z & 0 & X & Y \\ 0 & Z & -(Y^4+YG) & X^2 \\ X^2 & -Y & -Z & 0 \\ Y^4+YG & X & 0 & -Z \end{pmatrix}\right).$$

By (4.2) we similarly have the following

PROPOSITION (4.3). *The sequence*

$$0 \longrightarrow R \xrightarrow{q} L \xrightarrow{p} R \longrightarrow R/\mathfrak{m} \longrightarrow 0$$

*is exact, where* $q(1)=f_2$ *and* $p(\begin{pmatrix} a \\ b \end{pmatrix})=b$.

To see that $L$ is indecomposable we consider the ring $S:=k[\![X, Y]\!]/(X^3+Y^2G+Y^5)$. The ideal $(x^2, y)S$ is the first syzygy module of the ideal $(x, y)S$, and $(x^2, y)S\not\cong(x, y)S$; actually the matrix factorization of $X^3+Y^2G+Y^5$ corresponding to $(x, y)S$ is

$$\left(\begin{pmatrix} X^2 & -Y \\ Y^4+YG & X \end{pmatrix}, \begin{pmatrix} X & Y \\ -(Y^4+YG) & X^2 \end{pmatrix}\right).$$

On the other hand the matrix factorization of $X^3+Y^2G+Y^5+Z^2$ corresponding to $L$ has the form

$$\left(\begin{pmatrix} Z & 0 & X & Y \\ 0 & Z & -(Y^4+YG) & X^2 \\ X^2 & -Y & -Z & 0 \\ Y^4+YG & X & 0 & -Z \end{pmatrix}, \begin{pmatrix} Z & 0 & X & Y \\ 0 & Z & -(Y^4+YG) & X^2 \\ X^2 & -Y & -Z & 0 \\ Y^4+YG & X & 0 & -Z \end{pmatrix}\right),$$

whence by [**10**, (2.7)] we readily get that $L$ is indecomposable.

Let us show $\dim_k L/JL=1$. As $\dim_k \Lambda/J=1$, it is enough to see that $L=\Lambda f_2$. Take the $R$-dual $[\cdot]^*$ of the exact sequence

$$0 \longrightarrow R \xrightarrow{q} L \longrightarrow \mathfrak{m} \longrightarrow 0$$

given in (4.3). Then since $R$ is Gorenstein, the resulting exact sequence has the following form

$$0 \longrightarrow R \longrightarrow L^* \xrightarrow{q^*} R^* \longrightarrow R/\mathfrak{m} \longrightarrow 0.$$

As $L^*$ is again a maximal Cohen-Macaulay $R$-module, this sequence is fundamental in the sense of M. Auslander [2]. Let $v\in L$ and let $\phi: R\to L$ denote the homomorphism defined by $\phi(1)=v$. Then as $L$ is indecomposable and $L\not\cong R$, the induced homomorphism $\phi^*: L^*\to R^*$ is not surjective and so by [**2**, Proposition 6.1] there is a homomorphism $\psi^*: L^*\to L^*$ such that $\phi^*=q^*\circ\psi^*$. Hence $\phi=\psi\circ q$ for some $\psi\in\Lambda$ and thus $L=\Lambda f_2$.

To show that $\dim_k JL/(J^2L+\mathfrak{m}L)\geqq 2$ we consider the ring $T:=R/yR$ $(=k[\![X, Z]\!]/(X^3+Z^2))$. Let $\bar{T}$ denote the normalization of $T$ and put $t=-z/x$. Then

$$\bar{T}=k[\![t]\!], \quad x=-t^2 \quad \text{and} \quad z=t^3.$$

Let $\bar{L}=L/yL$ and recall that any indecomposable maximal Cohen-Macaulay $T$-module is isomorphic to $T$ or $\bar{T}$ ([8, Satz 1.6]). Then we have that

$$\bar{L}=\bar{T}\oplus\bar{T},$$

as $\mathrm{rank}_T \bar{L}=2$ and as $\bar{L}$ is minimally generated by the four elements $\{\bar{f}_i\}_{1\leq i\leq 4}$ (here $\bar{\cdot}$ denotes the reduction $\mathrm{mod}\, yL$).

LEMMA (4.4). *$\bar{f}_2$ and $\bar{f}_3$ form a $\bar{T}$-free basis of $\bar{L}$.*

PROOF. As $xf_1=(y^4+yg)f_2+zf_3$ and $zf_4=yf_1+x^2f_2$, we get $\bar{f}_1=-t\bar{f}_3$ and $\bar{f}_4=t\bar{f}_2$. Hence the $\bar{T}$-module $\bar{L}$ is generated by $\bar{f}_2$ and $\bar{f}_3$.

Since $\mathrm{End}_T\bar{L}=\mathrm{End}_{\bar{T}}\bar{L}$, we shall identify $\mathrm{End}_T\bar{L}$ with $\Gamma=M_2(\bar{T})$ (the matrix algebra) via the $\bar{T}$-free basis $\bar{f}_2$ and $\bar{f}_3$. Let $\bar{\Lambda}=\Lambda/y\Lambda$. Then $\bar{\Lambda}$ may be canonically considered to be a subalgebra of $\mathrm{End}_T\bar{L}$ and we have a homomorphism

$$\phi : \Lambda \longrightarrow \mathrm{End}_T\bar{L}=\Gamma$$

of $R$-algebras. Thus via $\phi$ we may write each element of $\Lambda$ as a $2\times2$ matrix with entries in $\bar{T}$. For example since

$$\begin{pmatrix} x^2 & -y & -z & 0 \\ z & 0 & x & y \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -(y^3+g) & 0 \\ 0 & 1 & 0 & 0 \\ y^3+g & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} z & 0 & x & y \\ 0 & z & -(y^4+yg) & x^2 \\ x^2 & -y & -z & 0 \\ y^4+yg & x & 0 & -z \end{pmatrix}=0,$$

the middle matrix induces an element $\rho$ of $J$ (see the exact sequence (#) above). As $\rho(f_2)=f_3$ and $\rho(f_3)=-(y^3+g)f_2$,

$$\phi(\rho)=\begin{pmatrix} 0 & h \\ 1 & 0 \end{pmatrix}$$

(here $h$ denotes $-g \bmod yR$). Later, in the proof of (4.6), this endomorphism $\rho$ will play a key role.

LEMMA (4.5). *$\phi(J^2)\subset t\Gamma$.*

PROOF. Let $I$ be a maximal left ideal of $\Gamma$ and put $V=\Gamma/I$. Then $I\supset t\Gamma$, as $t\Gamma$ is the Jacobson radical of $\Gamma$ and so $V$ is a simple $\Gamma/t\Gamma$-module. Because $\Gamma/t\Gamma=M_2(k)$, we get $\dim_k V=2$ and therefore the $\Lambda$-module $V$ has a composition series of length at most 2. Hence $J^2V=(0)$ and we have $\phi(J^2)\subset t\Gamma$.

PROPOSITION (4.6). *Let $\xi\in J$ and write*

$$\phi(\xi)=\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

*Then $a$, $d\in t\bar{T}$ and $b\in t^2\bar{T}$.*

PROOF. As $\rho\in J$, $\phi(\rho\xi)=\begin{pmatrix} 0 & h \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in t\Gamma$ by (4.5). Hence $a$, $b\in t\bar{T}$. Considering $\phi(\xi\rho)$, we get $d\in t\bar{T}$ too. Notice that $\det\phi(\eta)=\det\eta \bmod yR$ for any $\eta\in\Lambda$. Then we see that $ad-bc\in t^2\bar{T}$, since $ad-bc\in t\bar{T}\cap T=t^2\bar{T}$. Therefore if $c\in t\bar{T}$, we readily get $b\in t^2\bar{T}$. When $c\in t\bar{T}$, considering $\phi(\rho+\xi)=\begin{pmatrix} a & h+b \\ 1+c & d \end{pmatrix}$ instead, we get $h+b\in t^2\bar{T}$ and so $b\in t^2\bar{T}$ (as $h\in t^2\bar{T}$) also in this case.

Now we are ready to check the assertion (3) in (4.1). We put

$$W=\mathfrak{n}^2\bar{f}_2+\mathfrak{n}\bar{f}_3,$$

where $\mathfrak{n}=t\bar{T}$. Let $\xi\in J^2$ and write

$$\phi(\xi)=\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then by (4.6) we get $c\in\mathfrak{n}$, $a$, $d\in\mathfrak{n}^2$ and $b\in\mathfrak{n}^3$, from which it follows that $\phi(\xi)(\bar{f}_i)\in W$ for $i=2$, 3. Hence $\operatorname{Im}\phi(\xi)\subset W$ for any $\xi\in J^2$. Let $M=J^2L+\mathfrak{m}L$ and let $\varepsilon: L\to\bar{L}=L/yL$ denote the canonical epimorphism. Then as $\mathfrak{m}\bar{T}=\mathfrak{n}^2$ and $\operatorname{Im}\phi(\xi)\subset W$ for $\xi\in J^2$, we get that $\varepsilon(M)\subset W$ and therefore an epimorphism

$$L/(J^2L+\mathfrak{m}L)\longrightarrow\bar{L}/W.$$

Hence $\dim_k L/(J^2L+\mathfrak{m}L)\geqq\dim_k\bar{L}/W=3$, by which we have

$$\dim_k JL/(J^2L+\mathfrak{m}L)\geqq 2$$

since $\dim_k L/JL=1$ by the assertion (2). This completes the proof of both Theorems (4.1) and (1.1).

### 5. An example of one-dimensional case.

As is noted in Section 1, the assumption that $\dim R\geqq 2$ in Theorem (1.1) cannot be omitted. When $\dim R=1$, maximal Buchsbaum $R$-modules $M$ are characterized by the condition that

$$\dim_R M=1 \quad\text{and}\quad \mathfrak{m}\cdot\mathrm{H}^0_{\mathfrak{m}}(M)=(0)$$

(see (2.2)). This condition (is not too much weak but) seems not quite strong. Nevertheless, in some sense surprisingly, there exist such Cohen-Macaulay local rings $R$ of $\dim R=1$ that are non-regular but possess only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules. In what follows we will explore the typical example $R=k[\![t^2, t^3]\!]$.

Now let $k$ be any field and $S=k[\![t]\!]$ a formal power series ring. We put $R:=k[\![t^2, t^3]\!]$. Then $R$ and $S$ are the only indecomposable maximal Cohen-Macaulay $R$-modules and the $R$-module $S$ has a resolution of the following form

$$\cdots\longrightarrow R^2\xrightarrow[\begin{pmatrix} t^3 & t^4 \\ -t^2 & -t^3 \end{pmatrix}]{} R^2\xrightarrow[\begin{pmatrix} t^3 & t^4 \\ -t^2 & -t^3 \end{pmatrix}]{} R^2\xrightarrow{\varepsilon} S\longrightarrow 0,$$

where $\varepsilon(\begin{pmatrix} a \\ b \end{pmatrix})=a+bt$. Therefore we have an embedding $\sigma: S\to R^2$ which sends 1 (resp. $t$) to $\begin{pmatrix} t^3 \\ -t^2 \end{pmatrix}$ (resp. $\begin{pmatrix} t^4 \\ -t^3 \end{pmatrix}$) and makes the diagram

$$\begin{array}{ccccc} S & \xrightarrow{\sigma} & R^2 & \xrightarrow{\varepsilon} & S \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle -t} \\ S & \xrightarrow{\sigma} & R^2 & \xrightarrow{\varepsilon} & S \end{array}$$

commutative, where $\rho=\begin{pmatrix} 0 & -t^2 \\ -1 & 0 \end{pmatrix}$. Let $A=(a_{ij})$ and $B=(b_{ij})$ be $m\times m$ matrices with entries in $k$ and let

$$\phi : (R^2)^m \longrightarrow (R^2)^m$$

denote the homomorphism defined by

$$\phi((x_i)) = \left(\sum_{j=1}^{m} a_{ij}x_j + \sum_{j=1}^{m} b_{ij}\rho(x_j)\right).$$

Then we clearly have the following

(5.1) *The diagram*

$$\begin{array}{ccccc} S^m & \xrightarrow{\sigma^m} & (R^2)^m & \xrightarrow{\varepsilon^m} & S^m \\ \downarrow{\scriptstyle A+tB} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle A-tB} \\ S^m & \xrightarrow[\sigma^m]{} & (R^2)^m & \xrightarrow[\varepsilon^m]{} & S^m \end{array}$$

*is commutative (here $\sigma^m$ and $\varepsilon^m$ respectively denote the direct sum of $m$ copies of $\sigma$ and $\varepsilon$).*

Let $\mathfrak{m}$ $(=t^2S)$ denote the maximal ideal of $R$ and let $N$ be an $R$-submodule of $S$ containing $\mathfrak{m}$. We put $M:=R^2/\sigma(N)$. Then

PROPOSITION (5.2). *$M$ is an indecomposable maximal Buchsbaum $R$-module with $\mathrm{H}^0_{\mathfrak{m}}(M)=S/N$.*

PROOF. Considering the exact sequence

$$0 \longrightarrow S/N \longrightarrow M \longrightarrow S \longrightarrow 0,$$

we get $\mathrm{H}^0_{\mathfrak{m}}(M)=S/N$, as $\mathfrak{m}\cdot(S/N)=(0)$ and as $S$ is Cohen-Macaulay. So $M$ is a maximal Buchsbaum $R$-module. Assume that $M=M_1\oplus M_2$ for some non-zero submodules $M_1$ and $M_2$. Then $M_i$'s are cyclic, since $M$ is generated by two elements. If $\dim_R M_i=1$ for $i=1, 2$, the isomorphisms $S\cong M/\mathrm{H}^0_{\mathfrak{m}}(M)\cong M_1/\mathrm{H}^0_{\mathfrak{m}}(M_1)\oplus M_2/\mathrm{H}^0_{\mathfrak{m}}(M_2)$ claim that $S$ is decomposable. Hence $\dim_R M_i$ must be 0 for some $i$, say $i=2$. Then $M_2\subset \mathrm{H}^0_{\mathfrak{m}}(M)$ and so $S$ is a homomorphic image of $M_1$ —— this is impossible, because $M_1$ is cyclic while $S$ is not. Thus we see $M$ is indecomposable.

We define

$$M_1 := R^2/\sigma(\mathfrak{m}),$$
$$M_2 := R^2/\sigma(R),$$
$$M_3 := R^2/\sigma(tS).$$

By (5.2) $M_i$'s are indecomposable maximal Buchsbaum $R$-modules and $M_1 \not\cong M_i$ for $i=2, 3$, because

$$\dim_k \mathrm{H}^0_{\mathfrak{m}}(M_i) = \begin{cases} 2 & (i=1), \\ 1 & (i=2, 3). \end{cases}$$

$M_2$ is of projective dimension 1 but $M_3$ is not; so $M_2 \not\cong M_3$.

The goal of this section is the following

THEOREM (5.3). *$M_1$, $M_2$, $M_3$, $S$ and $R$ are the indecomposable maximal Buchsbaum $R$-modules.*

To prove this theorem we need one more lemma (5.4), the proof of which is routine (use the induction on the size of matrices $C$) and will be omitted.

LEMMA (5.4). *Let $C$ be an $m\times n$ matrix with entries in $S/t^2S$. Then there exist an invertible $m\times m$ matrix $P$ with entries in $S/t^2S$ and an invertible $n\times n$ matrix $Q$ with entries in $k$ such that $PCQ$ has the following form*

$$\left(\begin{array}{c|c|c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 0 \\ \hline 0 \end{matrix} & \begin{matrix} \begin{matrix} t & & \\ & \ddots & \\ & & t \end{matrix} \\ \hline 0 \end{matrix} & \begin{matrix} 0 \\ \hline 0 \end{matrix} \\ \hline 0 & \begin{matrix} t & & \\ & \ddots & \\ & & t \end{matrix} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \mod t^2S\,.$$

PROOF OF THEOREM (5.3). Let $M$ be an indecomposable maximal Buchsbaum $R$-module such that $M \not\cong R$. Let $V=\mathrm{H}^0_{\mathfrak{m}}(M)$. Then $\mathfrak{m}V=(0)$ by (2.2).

CLAIM. *$V \subset \mathfrak{m}M$ and $M/V \cong S^m$ for some $m \geqq 1$.*

For let $W=V\cap \mathfrak{m}M$ and write $V=W\oplus W'$. Then $W'\cap\mathfrak{m}M=(0)$ and we have an embedding $W'\to M\to M/\mathfrak{m}M$, which naturally splits. Hence $W'=(0)$ as $M$ is indecomposable and thus $V\subset\mathfrak{m}M$. Since $M/V$ is Cohen-Macaulay, the second assertion is clear.

By the above claim we get a commutative diagram

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & V \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d, "i"] & (R^2)^m \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^m \arrow[r, "\sigma^m"'] & (R^2)^m \arrow[r, "\varepsilon^m"'] & S^m \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}
$$

with exact rows and columns. Here we consider $N$ to be an $R$-submodule of $S^m$ and the homomorphism $i: N \to S^m$ to be the inclusion map. Hence $\mathfrak{m}\cdot S^m \subset N$, as $V \cong S^m/N$. Let $\tau: S^m \to S^m/\mathfrak{m}S^m = (S/t^2S)^m$ denote the canonical epimorphism. We put $U = \tau(N)$ and $n = \dim_k U$. If $n=0$, then $N = \mathfrak{m}\cdot S^m$ and so $M = (R^2/\sigma(\mathfrak{m}))^m$. Consequently, we get $m=1$ and $M = M_1$.

Now suppose $n \geqq 1$ and let $v_1, v_2, \cdots, v_n$ be a $k$-basis of $U$. Let us apply (5.4) to the $m \times n$ matrix $C = (v_1, v_2, \cdots, v_n)$. Then Lemma (5.4) asserts that by some automorphism $P$ of $(S/t^2S)^m$, $U$ is mapped onto the $k$-subspace $U'$ which is spanned by the columns of an $m \times n$ matrix of the following form:

$$
(\#) \qquad \left(\begin{array}{c:c:c}
\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 0 \\ \hdashline 0 \end{matrix} & \begin{matrix} \begin{matrix} t & & \\ & \ddots & \\ & & t \end{matrix} \\ \hdashline 0 \end{matrix} \\
\hdashline
0 & \begin{matrix} t & & \\ & \ddots & \\ & & t \end{matrix} & 0 \\
\hdashline
0 & 0 & 0
\end{array}\right) \mod t^2S .
$$

Let $L$ be the $R$-submodule of $S^m$ generated by the columns of the above matrix (#) and put $N' = \mathfrak{m}\cdot S^m + L$. Then clearly $U' = \tau(N')$.

We write $P = A + tB \mod t^2S$ with $m \times m$ matrices $A$ and $B$ with entries in $k$. Then since the following diagram

$$
\begin{tikzcd}
S^m \arrow[r, "\tau"] \arrow[d, "A+tB"'] & (S/t^2S)^m \arrow[d, "P"] \\
S^m \arrow[r, "\tau"] & (S/t^2S)^m
\end{tikzcd}
$$

is commutative and since $U' = \tau(N')$, we get that $N' = (A+tB)N$.

Let us now recall the diagram in (5.1):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^m & \xrightarrow{\sigma^m} & (R^2)^m & \xrightarrow{\varepsilon^m} & S^m & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle A+tB} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle A-tB} & & \\ 0 & \longrightarrow & S^m & \xrightarrow[\sigma^m]{} & (R^2)^m & \xrightarrow[\varepsilon^m]{} & S^m & \longrightarrow & 0. \end{array}$$

Then as the rows of this diagram are exact and as both the matrices $A+tB$ and $A-tB$ are invertible, the middle $\phi$ has to be an isomorphism whence, via $\phi$, we find that

$$M=(R^2)^m/\sigma^m(N)\cong(R^2)^m/\sigma^m(N').$$

Consequently we may assume that $N=N'$. The condition that $M$ is indecomposable now causes a very tight restriction on the form of the matrix (#) above. We readily see that $m=1$ and the matrix (#) must be one of

$$(1\ t),\quad (1)\quad \text{and}\quad (t).$$

Thus $M=R^2/\sigma(S)$ $(=S)$, $M=R^2/\sigma(R)$ $(=M_2)$, or $M=R^2/\sigma(tS)$ $(=M_3)$ as claimed. This completes the proof of Theorem (5.3).

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