Periodicity of the asymptotic curves
on flat tori in S^3

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1. Introduction.

It seems to be interesting to develop a method of construction of flat surfaces in the unit 3-sphere S^3, since the problem of the classification of flat tori in S^3 remains open (see Yau [6]).

There is a method which is due to Sasaki [4] and Spivak [5, pp. 139-163]. To explain the method we recall the notion of asymptotic curves. A curve c on a flat surface M in S^3 is called an *asymptotic curve* if $\sigma(\dot{c}, \dot{c})=0$, where σ denotes the second fundamental form on M. It is well-known that for each point $x \in M$, there are exactly two asymptotic curves on M through the point x. They proved that one of the two curves has torsion $\tau=1$ and the other has torsion $\tau=-1$, and if M is complete and connected, then M is determined by the two curves (see Lemma 2.2). Moreover they obtained a method of construction of flat surfaces in S^3 which says that if a_1 and a_2 are curves in S^3 with torsions $\tau_1=1$ and $\tau_2=-1$, respectively and these curves satisfy some suitable conditions, then there exists a flat surface M in S^3 such that a_1 and a_2 are asymptotic curves on M (see Lemma 4.1). Infinitely many complete flat surfaces in S^3 are constructed by this method. However it is not easy to give a criterion for these surfaces to be compact. So it seems to be difficult to apply this method to the problem of the classification of flat tori in S^3.

In this paper modifying the method of Sasaki and Spivak, we establish a new method and give a criterion for the surfaces constructed by the method to be compact. To explain our method we introduce the notion of admissible pairs. A pair $\Gamma=(\gamma_1, \gamma_2)$ of regular curves γ_1 and γ_2 on the unit 2-sphere S^2 is called an *admissible pair* if the geodesic curvature of γ_1 is greater than that of γ_2 and some additional conditions are satisfied (for details, see Section 4). For each admissible pair Γ, using the Hopf fibration $p: S^3 \rightarrow S^2$, we construct a flat surface M_Γ in S^3 (Theorem 4.2). Conversely we show that if M is a complete connected flat surface in S^3 with bounded mean curvature, then there exists an admissible pair Γ such that M is congruent to M_Γ (Theorem 4.3). Moreover

we prove that for each admissible pair $\Gamma=(\gamma_1, \gamma_2)$, the surface M_Γ is compact iff Γ is *periodic*, that is, γ_1 and γ_2 are periodic (Theorem 5.1). Consequently we see that the problem of the classification of flat tori in S^3 completely reduces to that of periodic admissible pairs. These results are applied to prove the following theorems.

THEOREM A. *If M is a flat torus isometrically immersed in S^3, then all the asymptotic curves on M are periodic.*

THEOREM B. *There exists a flat torus M isometrically embedded in S^3 such that M contains no great circle in S^3.*

The outline of this paper is as follows. In Section 2 we explain a Lie group structure on S^3 and give some basic facts on flat surfaces in S^3. In Section 3 we consider the Hopf fibration $p: S^3 \to S^2$ and discuss the behavior of asymptotic curves on Hopf cylinders. In Section 4 we introduce the notion of admissible pairs and establish a method of construction of flat surfaces in S^3. In Section 5 we give a criterion for the surfaces constructed by our method to be compact. In Section 6 we prove Theorem A. In Section 7 we prove Theorem B.

Throughout this paper we assume that all manifolds and maps are differentiable of class C^∞.

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2. Preliminaries.

Let $SU(2)$ be the group of all 2×2 unitary matrices with determinant 1. Its Lie algebra $\mathfrak{su}(2)$ consists of all 2×2 skew Hermitian matrices of trace 0. The adjoint representation Ad of $SU(2)$ is given by

$$\mathrm{Ad}(a)x = a \cdot x \cdot a^{-1},$$

where $a \in SU(2)$ and $x \in \mathfrak{su}(2)$. We define a positive definite inner product $\langle , \rangle$ on $\mathfrak{su}(2)$ by

$$\langle x, y \rangle = -\frac{1}{2} \mathrm{trace}(x \cdot y).$$

The inner product is invariant under the adjoint action of $SU(2)$. We set

$$e_1 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$

Then $\{e_1, e_2, e_3\}$ is an orthonormal basis of $\mathfrak{su}(2)$. Note that

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = 2e_1, \quad [e_3, e_1] = 2e_2,$$

where $[,]$ denotes the Lie bracket on $\mathfrak{su}(2)$. For each e_i, we define a left invariant vector field E_i on $SU(2)$ by

$$E_i(a) = \frac{d}{dt} \{ a \cdot \exp(te_i) \} |_{t=0}.$$

Then $\{E_1, E_2, E_3\}$ is a frame field on $SU(2)$. We define a Riemannian metric $\langle , \rangle$ on $SU(2)$ by $\langle E_i, E_j \rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Then $SU(2)$ is a Riemannian manifold isometric to the unit 3-sphere S^3. Hereafter we identify S^3 with $SU(2)$.

Let L_a (resp. R_a) denote the left (resp. right) translation of S^3 by $a \in S^3$. Then L_a and R_a are isometries of S^3. We denote by D the Riemannian connection on S^3 with respect to the Riemannian metric $\langle , \rangle$. Since the metric is bi-invariant, we obtain

$$(2.1) \qquad D_{E_i} E_j = \frac{1}{2} [E_i, E_j].$$

We choose an orientation of S^3 such that $\{E_1, E_2, E_3\}$ is a positive frame field. A vector product $\times$ on each tangent space of S^3 is defined by the metric and the orientation in the usual way. Then we obtain

$$(2.2) \qquad E_i \times E_j = \frac{1}{2} [E_i, E_j].$$

Let $c : \mathbf{R} \to S^3$ be a curve in S^3 and let $\dot{c}$ be the tangent vector field of c. A vector field v along the curve c is called left (resp. right) invariant along c if the following relation (2.3) (resp. (2.4)) holds for all $t \in \mathbf{R}$.

$$(2.3) \qquad v(t) = \{ L_{c(t)c(0)^{-1}} \}_* v(0),$$

$$(2.4) \qquad v(t) = \{ R_{c(0)^{-1}c(t)} \}_* v(0).$$

LEMMA 2.1. *Let v be a vector field along c. Then*
(1) *v is left invariant along c iff $D_{\dot{c}} v = \dot{c} \times v$,*
(2) *v is right invariant along c iff $D_{\dot{c}} v = v \times \dot{c}$.*

PROOF. Set $f_i(t) = \langle v(t), E_i(c(t)) \rangle$. Then by (2.1) and (2.2) we obtain

$$(2.5) \qquad D_{\dot{c}} v = \sum_{i=1}^{3} f_i' E_i(c) + \dot{c} \times v,$$

where $f_i' = df_i/dt$. Since E_i is left invariant, it follows that v is left invariant along c iff f_1, f_2 and f_3 are constant. Hence (2.5) implies the assertion of (1).

To prove (2) we consider a map $\tau : S^3 \to S^3$ given by

(2.6) $$\tau(a) = a^{-1}.$$

Since τ is an orientation reversing isometry of S^3, we obtain

$$D_{\tau_* \dot{c}} \tau_* v = \tau_*(D_{\dot{c}} v), \qquad \tau_* \dot{c} \times \tau_* v = \tau_*(v \times \dot{c}).$$

Thus it follows from (1) that $\tau_* v$ is left invariant along $\tau(c)$ iff $D_{\dot{c}} v = v \times \dot{c}$. It is easy to see that $\tau_* v$ is left invariant along $\tau(c)$ iff v is right invariant along c. Hence we have (2). Q.E.D.

Let $f : M \to S^3$ be an isometric immersion of a complete connected flat surface M into S^3. In Moore [1] it is shown that there exists a covering $T : R^2 \to M$ such that

$$g\left(\frac{\partial T}{\partial t_i}, \frac{\partial T}{\partial t_i}\right) = 1, \qquad \sigma\left(\frac{\partial T}{\partial t_i}, \frac{\partial T}{\partial t_i}\right) = 0$$

for $i = 1, 2$, where g denotes the Riemannian metric on M and σ denotes the second fundamental form on M induced by the immersion f. The covering T is called an *asymptotic Tchebychef net* of M. Moreover $f \circ T : R^2 \to S^3$ becomes a flat asymptotic Tchebychef immersion. Here we give the following

DEFINITION. An immersion $F : R^2 \to S^3$ is said to be a *flat asymptotic Tchebychef immersion* (abrreviated as FAT) if F induces a flat metric on R^2 and satisfies the following

(2.7) $$\langle F_i, F_i \rangle = 1, \qquad \langle D_{F_i} F_i, \xi \rangle = 0$$

for $i = 1, 2$, where $F_i = \partial F / \partial t_i$, and $\xi = F_1 \times F_2 / \|F_1 \times F_2\|$.

Following [4] and [5], we summarize basic properties of FATs. For a FAT $F : R^2 \to S^3$, we set

(2.8) $$g_{ij} = \langle F_i, F_j \rangle, \qquad h_{ij} = \langle D_{F_i} F_j, \xi \rangle.$$

By (2.7) we have $1 - g_{12}^2 > 0$, and so there exists a real valued function ω on R^2 such that

(2.9) $$g_{12} = \cos \omega, \qquad 0 < \omega < \pi.$$

Then the Gaussian curvature K of the metric g_{ij} satisfies

(2.10) $$K = -\frac{1}{\sin \omega}\left(\frac{\partial^2 \omega}{\partial t_1 \partial t_2}\right).$$

Since $K = 0$, we obtain

(2.11) $$\frac{\partial^2 \omega}{\partial t_1 \partial t_2} = 0.$$

The Gauss equation implies that $h_{12}^2 = \sin^2 \omega > 0$. Now we assume that $h_{12} > 0$.

Then

(2.12) $$h_{12} = \sin \omega.$$

We set $n_i = \xi \times F_i$. Then it follows from (2.7) that there exist real valued functions κ_i and τ_i on R^2 which satisfy the following Frenet formulas.

(2.13) $$\begin{cases} D_{F_i}F_i = \kappa_i n_i, \\ D_{F_i}n_i = -\kappa_i F_i + \tau_i \xi, \\ D_{F_i}\xi = -\tau_i n_i. \end{cases}$$

By [4] we obtain

(2.14) $$\kappa_1 = -\partial \omega/\partial t_1, \qquad \kappa_2 = \partial \omega/\partial t_2,$$

(2.15) $$\tau_1 = 1, \qquad \tau_2 = -1.$$

By (2.13) and (2.15) we obtain

(2.16) $$D_{F_1}\xi = F_1 \times \xi, \qquad D_{F_2}\xi = \xi \times F_2.$$

It follows from (2.11) and (2.14) that $\kappa_1(t_1, t_2) = \kappa_1(t_1)$ and $\kappa_2(t_1, t_2) = \kappa_2(t_2)$. Hence the Frenet formulas imply that all curves $t_1 \mapsto F(t_1, t_2)$ are congruent each other, and all curves $t_2 \mapsto F(t_1, t_2)$ are congruent each other. Furthermore we have the following lemma which is proved in [5, pp. 152-154].

LEMMA 2.2. $F(t_1, t_2) = F(t_1, 0) \cdot F(0, 0)^{-1} \cdot F(0, t_2)$ *for all* $(t_1, t_2) \in R^2$.

Using this lemma, we prove the following

THEOREM 2.3. *Let* F *be a* FAT *and let* $\rho : R^2 \to R^2$ *be a diffeomorphism such that* $F \circ \rho = F$. *If* $\rho(0, 0) = (r_1, r_2)$, *then* $\rho(t_1, t_2) = (t_1 + r_1, t_2 + r_2)$ *for all* $(t_1, t_2) \in R^2$.

PROOF. Let τ be an orientation reversing isometry of S^3. Replacing F by $\tau \cdot F$, if necessary, we may assume that h_{12} is positive. To establish the theorem it is sufficient to show that

(2.17) $$\frac{\partial \rho}{\partial t_j} = (\delta_{1j}, \delta_{2j}).$$

Let $(x_1, x_2) \in R^2$ and let $\rho(x_1, x_2) = (y_1, y_2)$. Then it is easy to see that the curve $t \mapsto \rho(x_1 + t, x_2)$ is a unit speed asymptotic curve on R^2 starting from (y_1, y_2). Hence the following four cases (2.18)-(2.21) may occur.

(2.18) $$\rho(x_1 + t, x_2) = (y_1 + t, y_2),$$

(2.19) $$\rho(x_1 + t, x_2) = (y_1 - t, y_2),$$

(2.20) $$\rho(x_1 + t, x_2) = (y_1, y_2 + t),$$

(2.21) $$\rho(x_1 + t, x_2) = (y_1, y_2 - t).$$

Now we show that (2.19)-(2.21) are impossible. Assume (2.19). Then it follows that

(2.22) $F(x_1+t, x_2) = F(y_1-t, y_2)$

In particular we have $F((x_1+y_1)/2, x_2)=F((x_1+y_1)/2, y_2)$. Hence Lemma 2.2 implies that $F(t, x_2)=F(t, y_2)$, and so it follows from (2.22) that

$$F(x_1+t, x_2) = F(y_1-t, x_2).$$

Differentiating the above relation at $t=(y_1-x_1)/2$, we have $F_1((x_1+y_1)/2, x_2)=0$. This is a contradiction.

Now we assume (2.20). We set

$$c(t) = F(x_1+t, x_2), \qquad v(t) = \xi(x_1+t, x_2),$$

where $\xi=F_1\times F_2/\|F_1\times F_2\|$. Then it follows from (2.16) that $D_{\dot c}v=\dot c\times v$. Since $\xi\circ\rho=\pm\xi$, the assumption (2.20) implies that

$$c(t) = F(y_1, y_2+t), \qquad v(t) = \pm\xi(y_1, y_2+t).$$

So it follows from (2.16) that $D_{\dot c}v=v\times\dot c$. Hence $\dot c\times v=v\times\dot c=0$, which is a contradiction. Similarly we see that (2.21) is impossible, and so we have (2.18) which shows that (2.17) holds for $j=1$.

By the same way we see that (2.17) holds for $j=2$. This completes the proof of Theorem 2.3.   Q.E.D.

Now we return to the isometric immersion $f:M\to S^3$, where M is a complete and connected flat surface. Let T be an asymptotic Tchebychef net of M and let $\rho:R^2\to R^2$ be a covering transformation of T. Then it follows that $f\circ T$ is a FAT and $(f\circ T)\circ\rho=f\circ T$. By Theorem 2.3 we see that $\rho^*(dt_1\wedge dt_2)=dt_1\wedge dt_2$. So we have the following

THEOREM 2.4. *Any non-orientable complete flat surface cannot be isometrically immersed in S^3.*

3. Asymptotic curves on Hopf cylinders.

In this section we study the behavior of the asymptotic curves on Hopf cylinders. The results of this section will be used in the subsequent sections.

We begin with a description of the Hopf fibration. Let $S^2=\{x\in\mathfrak{su}(2):\|x\|=1\}$. Then the Hopf fibration $p:S^3\to S^2$ is defined by

$$p(a) = \mathrm{Ad}(a)e_3.$$

For each $x\in S^2$, the tangent space T_xS^2 is canonically identified with a linear

subspace $x^\perp$ of $\mathfrak{su}(2)$ which is given by

$$x^\perp = \{y \in \mathfrak{su}(2) : \langle x, y \rangle = 0\}.$$

Then the standard Riemannian metric on S^2 is induced by the inner product $\langle , \rangle$ on $\mathfrak{su}(2)$. We denote by ∇ the Riemannian connection on S^2 with respect to the standard metric. Let S^1 be a closed subgroup of S^3 given by

$$S^1 = \{a \in S^3 : \mathrm{Ad}(a)e_3 = e_3\}.$$

Note that $S^1 = \{\exp(te_3) : 0 \le t < 2\pi\}$. The group S^1 acts on S^3 by the right translation and the Hopf fibration p has a structure of principal S^1-bundle. For each $a \in S^3$, let H_a be a linear subspace of $T_a S^3$ given by

$$H_a = \{v \in T_a S^3 : \langle v, E_3 \rangle = 0\}.$$

Then the correspondence $a \mapsto H_a$ defines a connection in the principal S^1-bundle. The following lemma is easily verified.

LEMMA 3.1. *If $v \in H_a$, then $\|p_*(v)\| = 2\|v\|$.*

Let $c(t)$ be a curve in S^3 and let $\dot{c}(t)$ be the tangent vector of $c(t)$. We denote by $c'(t)$ the 2×2 matrix $(c'_{ij}(t))$, where $c_{ij}(t)$ is the (i, j)-component of the matrix $c(t)$ and $c'_{ij} = dc_{ij}/dt$. Then $c(t)^{-1} \cdot c'(t) \in \mathfrak{su}(2)$ and we have the follwing

LEMMA 3.2. $\dot{c}(t) \in H_{c(t)}$ *iff* $\langle c(t)^{-1} \cdot c'(t), e_3 \rangle = 0$.

PROOF. Set $a(s) = c(t) \cdot \exp\{s(c(t)^{-1} \cdot c'(t))\}$. Since $a(0) = c(t)$ and $a'(0) = c'(t)$, we have $\dot{a}(0) = \dot{c}(t)$. So we see that $\langle \dot{c}(t), E_3 \rangle = \langle \dot{a}(0), E_3 \rangle = \langle c(t)^{-1} \cdot c'(t), e_3 \rangle$.
                                                                                    Q.E.D.

LEMMA 3.3. *Let X and Y be vector fields on S^2 and let $\tilde{X}$ and $\tilde{Y}$ be the horizontal lifts of X and Y, respectively. Then $D_{\tilde{X}}\tilde{Y}$ and $\nabla_X Y$ are p-related.*

PROOF. Consider a new metric on S^2 which is homothetic to the standard metric and has constant Gaussian curvature 4. Then it follows from Lemma 3.1 that $p : S^3 \to S^2$ is a Riemannian submersion with respect to the new metric. Since the new metric and the standard one induce the same Riemannian connection ∇, the assertion of Lemma 3.3 follows from [2, Lemma 1].   Q.E.D.

Let J be a $(1, 1)$-tensor field on S^2 defined by

$$J(v) = \frac{1}{2}[x, v]$$

for $v \in T_x S^2$, where $T_x S^2$ is identified with $x^\perp$. Note that $\|J(v)\| = \|v\|$ and $\langle J(v), v \rangle = 0$. Let $\tilde{J}$ be a $(1, 1)$-tensor field on S^3 defined by

$$\tilde{J}(E_i) = E_3 \times E_i.$$

Then it is easy to see the following

LEMMA 3.4. $p_* \circ \tilde{J} = J \circ p_*$.

Now we consider a regular curve $\gamma : R \to S^2$. It is known that the inverse image $p^{-1}(\gamma)$ of the curve γ is an immersed flat surface in S^3. The surface $p^{-1}(\gamma)$ is called a *Hopf cylinder* corresponding to γ (see Pinkall [3]). We introduce the following

DEFINITION. A curve $c : R \to S^3$ is said to be an *asymptotic lift* of γ if $p \circ c = \gamma$ and c is an asymptotic curve on $p^{-1}(\gamma)$.

Let $h : R \to S^3$ be a horizontal lift of γ and let θ be a real valued function on R. We consider a curve c in S^3 defined by

(3.1) $$c(t) = h(t) \cdot \exp\{\theta(t)e_3\} .$$

We recall the geodesic curvature k of γ which is given by

$$k = \langle \nabla_{\dot{\gamma}} \dot{\gamma}, J(\dot{\gamma}) \rangle / \|\dot{\gamma}\|^3 .$$

Then we have the following

LEMMA 3.5. *The curve c is an asymptotic lift of γ iff* $\theta' = k\|\dot{\gamma}\|/2$, *where* $\theta' = d\theta/dt$.

PROOF. Let ξ be a vector field along the curve c defined by

(3.2) $$\xi = -\tilde{J}(\dot{c})/\|\tilde{J}(\dot{c})\| .$$

Note that c is an asymptotic lift of γ iff $\langle D_{\dot{c}}\dot{c}, \xi \rangle = 0$. So it is sufficient to show that

(3.3) $$\langle D_{\dot{c}}\dot{c}, \xi \rangle = \theta'\|\dot{\gamma}\| - \frac{1}{2}k\|\dot{\gamma}\|^2 .$$

By (3.1) we obtain

(3.4) $$\dot{c} = \{R_{\exp(\theta e_3)}\}_* \dot{h} + \theta' E_3(c) .$$

We set $a_i = \langle \dot{c}, E_i(c) \rangle$ and $b_i = \langle \dot{h}, E_i(h) \rangle$. Since $\tilde{J}(\dot{c}) = -a_2 E_1(c) + a_1 E_2(c)$ and $\|\tilde{J}(\dot{c})\| = \|\dot{h}\|$, we obtain

(3.5) $$\langle D_{\dot{c}}\dot{c}, \xi \rangle = (a_1' a_2 - a_1 a_2') / \|\dot{h}\| .$$

A calculation shows that

(3.6) $$\begin{cases} \mathrm{Ad}(\exp(\theta e_3))e_1 = (\cos 2\theta)e_1 + (\sin 2\theta)e_2, \\ \mathrm{Ad}(\exp(\theta e_3))e_2 = -(\sin 2\theta)e_1 + (\cos 2\theta)e_2. \end{cases}$$

This implies that

$$\{R_{\exp(\theta e_3)}\}_* E_1(h) = (\cos 2\theta) E_1(c) - (\sin 2\theta) E_2(c),$$

$$\{R_{\exp(\theta e_3)}\}_* E_2(h) = (\sin 2\theta) E_1(c) + (\cos 2\theta) E_2(c).$$

Thus it follows from (3.4) that

$$(3.7) \qquad \begin{cases} a_1 = b_1 \cos 2\theta + b_2 \sin 2\theta, \\ a_2 = -b_1 \sin 2\theta + b_2 \cos 2\theta, \\ a_3 = \theta'. \end{cases}$$

Since $\dot{h}$ is horizontal, we have $\|\dot{h}\|^2 = b_1^2 + b_2^2$. So it follows from (3.5) and (3.7) that

$$\langle D_{\dot{c}}\dot{c}, \xi \rangle = 2\theta' \|\dot{h}\| + (b_1' b_2 - b_1 b_2')/\|\dot{h}\|.$$

By Lemmas 3.1, 3.3 and 3.4, we see that

$$k\|\dot{\gamma}\|^3 = \langle \nabla_{\dot{\gamma}}\dot{\gamma}, J(\dot{\gamma}) \rangle = \langle p_*(D_{\dot{h}}\dot{h}), p_* \circ \tilde{J}(\dot{h}) \rangle$$

$$= 4\langle D_{\dot{h}}\dot{h}, \tilde{J}(\dot{h}) \rangle = -4(b_1' b_2 - b_1 b_2').$$

Since $\|\dot{\gamma}\| = 2\|\dot{h}\|$, we have (3.3). Q.E.D.

REMARK 3.6. It follows from Lemma 3.5 that there exists an asymptotic lift of γ. If c_1 and c_2 are asymptotic lifts of γ, then $c_2 = R_a(c_1)$ for some $a \in S^1$.

LEMMA 3.7. *Suppose that the curve c given by (3.1) is an asymptotic lift of γ. Let ξ be the vector field along c given by (3.2), and let $\alpha(t)$ denote the angle between $\dot{c}(t)$ and E_3 such that $0 < \alpha(t) < \pi$. Then*
  (1) $\|\dot{c}\| \cos \alpha = k\|\dot{\gamma}\|/2$, $\|\dot{c}\| \sin \alpha = \|\dot{\gamma}\|/2$,
  (2) ξ *is left invariant along c,*
  (3) *if $\|\dot{\gamma}\|^2(1+k^2) = 4$, then $\|\dot{c}\| = 1$ and $D_{\dot{c}}\dot{c} = \alpha'(\dot{c} \times \xi)$.*

PROOF. It follows from (3.4) that $\|\dot{c}\| \cos \alpha = \theta'$ and $\|\dot{c}\| \sin \alpha = \|\dot{h}\|$. Thus Lemmas 3.1 and 3.5 imply (1).

Since $\langle D_{\dot{c}}\xi, \dot{c} \rangle = 0$ and $\langle D_{\dot{c}}\xi, \xi \rangle = 0$, there exists a real valued function λ on R such that $D_{\dot{c}}\xi = \lambda(\dot{c} \times \xi)$. Then we see that

$$\langle \xi, \dot{c} \times E_3 \rangle = \langle \xi, D_{\dot{c}}E_3 \rangle = -\langle D_{\dot{c}}\xi, E_3 \rangle$$

$$= \lambda \langle \xi \times \dot{c}, E_3 \rangle = \lambda \langle \xi, \dot{c} \times E_3 \rangle.$$

Since $\langle \xi, \dot{c} \times E_3 \rangle = \|\dot{c} \times E_3\| > 0$, we have $\lambda = 1$. Hence the assertion of (2) follows from Lemma 2.1.

Suppose that $\|\dot{\gamma}\|^2(1+k^2) = 4$. Then $\|\dot{c}\| = 1$ by (1). Since $\langle D_{\dot{c}}\dot{c}, \dot{c} \rangle = 0$ and $\langle D_{\dot{c}}\dot{c}, \xi \rangle = 0$, there exists a real valued function μ on R such that $D_{\dot{c}}\dot{c} = \mu(\dot{c} \times \xi)$. Differentiating $\cos \alpha = \langle \dot{c}, E_3 \rangle$, we obtain

$$-\alpha' \sin \alpha = \langle D_{\dot{c}}\dot{c}, E_3 \rangle + \langle \dot{c}, D_{\dot{c}}E_3 \rangle$$

$$= \mu \langle \dot{c} \times \xi, E_3 \rangle + \langle \dot{c}, \dot{c} \times E_3 \rangle$$

$$= -\mu \langle \xi, \dot{c} \times E_3 \rangle = -\mu \| \dot{c} \times E_3 \|.$$

Since $\| \dot{c} \times E_3 \| = \sin \alpha$, we have $\mu = \alpha'$.   Q.E.D.

Now we discuss the periodicity of asymptotic lifts of γ. Let $U(S^2)$ be the unit tangent bundle of S^2. We identify $U(S^2)$ with a subset of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ in the usual way. Then

$$U(S^2) = \{(x, y) : \|x\| = \|y\| = 1, \langle x, y \rangle = 0\},$$

and the canonical projection $p_1 : U(S^2) \to S^2$ is given by $p_1(x, y) = x$. Define $p_2 : S^3 \to U(S^2)$ by

$$p_2(a) = (\mathrm{Ad}(a)e_3, \mathrm{Ad}(a)e_1).$$

Note that p_2 is a double covering and $p = p_1 \circ p_2$.

LEMMA 3.8. *Let $c : R \to S^3$ be a curve in S^3 such that $p_2(c) = \dot{\gamma}/\|\dot{\gamma}\|$. Then c is an asymptotic lift of γ.*

PROOF. Without loss of generality we may assume that $\|\dot{\gamma}\| = 1$. Since $\dot{\gamma} = \langle \gamma, \gamma' \rangle$, we obtain

(3.8) $\mathrm{Ad}(c)e_3 = \gamma,$

(3.9) $\mathrm{Ad}(c)e_1 = \gamma'.$

By (3.8) there exists a real valued function θ on R such that $c(t) = h(t) \cdot \exp\{\theta(t)e_3\}$, where h is a horizontal lift of γ. Then due to Lemma 3.5 we only have to show that the geodesic curvature k of γ must be $2\theta'$. By (3.6) we obtain

$$\mathrm{Ad}(c)e_1 = \mathrm{Ad}(h)\{(\cos 2\theta)e_1 + (\sin 2\theta)e_2\},$$

$$\mathrm{Ad}(c)e_2 = \mathrm{Ad}(h)\{-(\sin 2\theta)e_1 + (\cos 2\theta)e_2\}.$$

Thus it follows from (3.9) that

$$\gamma'' = 2\theta' \mathrm{Ad}(c)e_2 + (\cos 2\theta)(\mathrm{Ad}(h)e_1)' + (\sin 2\theta)(\mathrm{Ad}(h)e_2)'.$$

Then the geodesic curvature k of γ is given by

$$k = \frac{1}{2}\langle \gamma'', [\gamma, \gamma'] \rangle = \langle \gamma'', \mathrm{Ad}(c)e_2 \rangle = 2\theta' + P,$$

where $P = \langle (\cos 2\theta)(\mathrm{Ad}(h)e_1)' + (\sin 2\theta)(\mathrm{Ad}(h)e_2)', \mathrm{Ad}(c)e_2 \rangle$. We see that

$$P = \langle (\mathrm{Ad}(h)e_1)', \mathrm{Ad}(h)e_2 \rangle = \langle \mathrm{Ad}(h)[h^{-1}h', e_1], \mathrm{Ad}(h)e_2 \rangle$$

$$= \langle [h^{-1}h', e_1], e_2 \rangle = \langle h^{-1}h', [e_1, e_2] \rangle = 2\langle h^{-1}h', e_3 \rangle.$$

Thus Lemma 3.2 implies that $P=0$, and so $\theta'=k/2$. Q.E.D.

THEOREM 3.9. *Let γ be a regular curve on S^2 and let c be an asymptotic lift of γ. If γ is l-periodic, then c is $2l$-periodic.*

PROOF. By Remark 3.6 and Lemma 3.8, we may assume that $p_2(c)=\dot{\gamma}/\|\dot{\gamma}\|$. This implies that $p_2(c)$ is l-periodic. Since p_2 is a double covering, c is $2l$-periodic. Q.E.D.

4. Construction of flat surfaces in S^3.

For $i=1, 2$, let $a_i:\boldsymbol{R}\to S^3$ be a curve in S^3 such that

$$a_i(0) = e, \quad \|\dot{a}_i\| = 1, \quad \dot{a}_1(0)\times\dot{a}_2(0) \neq 0,$$

where e denotes the unit element of the group S^3. Define a map $F:\boldsymbol{R}^2\to S^3$ by

$$F(t_1, t_2) = a_1(t_1)\cdot a_2(t_2).$$

We set

$$(4.1) \quad \begin{cases} \xi_0 = \dot{a}_1(0)\times\dot{a}_2(0)/\|\dot{a}_1(0)\times\dot{a}_2(0)\|, \\ \xi_1(t) = \{L_{a_1(t)}\}_*\xi_0, \quad \xi_2(t) = \{R_{a_2(t)}\}_*\xi_0, \\ n_i = \xi_i\times\dot{a}_i, \quad \kappa_i = \langle D_{\dot{a}_i}\dot{a}_i, n_i\rangle \quad (i=1, 2), \\ \omega(t_1, t_2) = \omega_0 - \int_0^{t_1}\kappa_1(t)dt + \int_0^{t_2}\kappa_2(t)dt, \end{cases}$$

where ω_0 denotes the angle between $\dot{a}_1(0)$ and $\dot{a}_2(0)$ such that $0<\omega_0<\pi$. The following lemma is essentially due to [4] and [5].

LEMMA 4.1. *If $0<\omega<\pi$ and $\langle\dot{a}_i, \xi_i\rangle=0$ for $i=1, 2$, then the map F is a FAT such that $g_{12}=\cos\omega$ and $h_{12}=\sin\omega$.*

PROOF. Let X_i and Y_i be vector fields along a_i given by

$$(4.2) \quad \begin{cases} X_1(t) = \{L_{a_1(t)}\}_*\dot{a}_1(0), & X_2(t) = \{R_{a_2(t)}\}_*\dot{a}_2(0), \\ Y_1(t) = \{L_{a_1(t)}\}_*n_1(0), & Y_2(t) = \{R_{a_2(t)}\}_*n_2(0). \end{cases}$$

Since $\xi_i=X_i\times Y_i$ and $\langle\dot{a}_i, \xi_i\rangle=0$, there exists a real valued function θ_i on $\boldsymbol{R}$ such that

$$(4.3) \quad \dot{a}_i = (\cos\theta_i)X_i+(\sin\theta_i)Y_i, \quad \theta_i(0) = 0.$$

So $n_i=-(\sin\theta_i)X_i+(\cos\theta_i)Y_i$. Then we see that

$$D_{\dot{a}_i}\dot{a}_i = \theta_i'n_i+(\cos\theta_i)D_{\dot{a}_i}X_i+(\sin\theta_i)D_{\dot{a}_i}Y_i$$

$$= \theta_i'n_i\pm\dot{a}_i\times\{(\cos\theta_i)X_i+(\sin\theta_i)Y_i\} = \theta_i'n_i.$$

It follows from (4.1) that $\kappa_i=\theta_i'$, and so we obtain

(4.4) $\omega(t_1, t_2) = \omega_0 - \theta_1(t_1) + \theta_2(t_2)$.

By (4.2) and (4.3) we see that

$$F_i(t_1, t_2) = \{\cos \theta_i(t_i)\} \Phi_* \dot{a}_i(0) + \{\sin \theta_i(t_i)\} \Phi_* n_i(0),$$

where $F_i = \partial F / \partial t_i$ and $\Phi = L_{a_1(t_1)} \circ R_{a_2(t_2)}$. So it follows from (4.4) that the angle between $F_1(t_1, t_2)$ and $F_2(t_1, t_2)$ is equal to $\omega(t_1, t_2)$. By the assumption that $0 < \omega < \pi$, the map F is an immersion such that $g_{11} = g_{22} = 1$ and $g_{12} = \cos \omega$. Since $\partial^2 \omega / \partial t_1 \partial t_2 = 0$, it follows from (2.10) that the map F induces a flat metric on R^2. We define a vector field ξ along F by

$$\xi(t_1, t_2) = \{L_{a_1(t_1)} \circ R_{a_2(t_2)}\}_* \xi_0.$$

Then Lemma 2.1 implies that $D_{F_1}\xi = F_1 \times \xi$ and $D_{F_2}\xi = \xi \times F_2$. Since $\langle F_i(t_1, t_2), \xi(t_1, t_2) \rangle = \langle \dot{a}_i(t_i), \xi_i(t_i) \rangle = 0$, we have $\xi = F_1 \times F_2 / \|F_1 \times F_2\|$. Therefore we see that

$$h_{ii} = -\langle D_{F_i}\xi, F_i \rangle = 0 \qquad (i=1, 2),$$

$$h_{12} = -\langle D_{F_1}\xi, F_2 \rangle = \langle \xi, F_1 \times F_2 \rangle = \sin \omega. \qquad \text{Q.E.D.}$$

Combining the notion of asymptotic lifts defined in the previous section with the method of construction of FATs described in Lemma 4.1, we establish more geometric method of constructing FATs. For the purpose we introduce the following

DEFINITION. For $i=1, 2$, let $\gamma_i : R \to S^2$ be a regular curve on S^2. The pair $\Gamma = (\gamma_1, \gamma_2)$ is said to be *admissible* if the following conditions (4.5)-(4.7) are satisfied.

(4.5) $\gamma_i(0) = e_3, \qquad \gamma_i'(0) / \|\gamma_i'(0)\| = e_1,$

(4.6) $\|\gamma_i'\|^2 (1 + k_i^2) = 4,$

(4.7) $k_1(t_1) > k_2(t_2)$ for all $(t_1, t_2) \in R^2$,

where k_i denotes the geodesic curvature of γ_i.

Let $\Gamma = (\gamma_1, \gamma_2)$ be an admissible pair and let $f_\Gamma : R^2 \to S^3$ be a map defined by

(4.8) $f_\Gamma(t_1, t_2) = c_1(t_1) \cdot c_2(t_2)^{-1},$

where c_i denotes the asymptotic lift of γ_i such that $c_i(0) = e$. Then we have the following

THEOREM 4.2. *The map f_Γ is a FAT.*

PROOF. Let $F = f_\Gamma$ and let $\alpha_i(t)$ be the angle between $\dot{c}_i(t)$ and E_3 such that $0 < \alpha_i(t) < \pi$. Then it follows from Lemma 3.7 that $\|\dot{c}_i\| = 1$ and $\cot \alpha_i = k_i$. By (4.7) we obtain

(4.9) $$0 < \alpha_2(t_2) - \alpha_1(t_1) < \pi.$$

Set $a_1(t) = c_1(t)$ and $a_2(t) = c_2(t)^{-1}$. Then it follows that $F(t_1, t_2) = a_1(t_1) \cdot a_2(t_2)$, $a_i(0) = e$ and $\|\dot{a}_i\| = 1$. By (4.5) we obtain

$$\dot{c}_i(0) = \{\sin \alpha_i(0)\} E_2(e) + \{\cos \alpha_i(0)\} E_3(e).$$

This implies that the angle ω_0 between $\dot{a}_1(0)$ and $\dot{a}_2(0)$ is equal to $\pi - \alpha_2(0) + \alpha_1(0)$, and so $\dot{a}_1(0) \times \dot{a}_2(0) \neq 0$ by (4.9). Define ξ_0, $\xi_i(t)$, $n_i(t)$, $\kappa_i(t)$ and $\omega(t_1, t_2)$ by (4.1). Since $\xi_0 = -\tilde{J}(\dot{c}_i(0))/\|\tilde{J}(\dot{c}_i(0))\|$, Lemma 3.7 (2) implies that

(4.10) $$\xi_1 = -\tilde{J}(\dot{c}_1)/\|\tilde{J}(\dot{c}_1)\|, \qquad \xi_2 = \tau_*(\tilde{J}(\dot{c}_2)/\|\tilde{J}(\dot{c}_2)\|),$$

where τ is given by (2.6). Hence it follows from Lemma 3.7 (3) that $D_{\dot{a}_i}\dot{a}_i = -\alpha_i'(\xi_i \times \dot{a}_i) = -\alpha_i' n_i$, and so $\kappa_i = -\alpha_i'$. Thus we see that $\omega(t_1, t_2) = \pi + \alpha_1(t_1) - \alpha_2(t_2)$, and so $0 < \omega < \pi$ by (4.9). Since $\langle \dot{a}_i, \xi_i \rangle = 0$ by (4.10), Lemma 4.1 implies that F is a FAT. **Q.E.D.**

THEOREM 4.3. *Let* $F: \boldsymbol{R}^2 \to S^3$ *be a FAT. If the mean curvature of* F *is bounded, then there exists an admissible pair* Γ *such that* $f_\Gamma = \Phi \circ F$ *for some isometry* Φ *of* S^3.

PROOF. We set $F_i = \partial F/\partial t_i$, $\xi = F_1 \times F_2/\|F_1 \times F_2\|$, $g_{ij} = \langle F_i, F_j \rangle$ and $h_{ij} = \langle D_{F_i} F_j, \xi \rangle$. Recall the map τ given by (2.6). Replacing F by $\tau \circ F$, if necessary, we may assume that $h_{12} > 0$. Let ω be the function on $\boldsymbol{R}^2$ defined by (2.9). Then we obtain

$$g_{11} = g_{22} = 1, \qquad g_{12} = \cos \omega,$$

$$h_{11} = h_{22} = 0, \qquad h_{12} = \sin \omega.$$

So the mean curvature H of F satisfies $H = -\cot \omega$. Since H is bounded, there exists a positive number δ such that $\delta \leq \omega \leq \pi - \delta$. Since ω satisfies the equation (2.11), there exist real valued functions ω_1 and ω_2 such that

$$\omega(t_1, t_2) = \omega_1(t_1) + \omega_2(t_2), \qquad \frac{\delta}{2} \leq \omega_i \leq \pi - \frac{\delta}{2}.$$

Replacing F by $\Phi \circ F$ for some orientation preserving isometry Φ of S^3, we may assume that

(4.11) $$F(0, 0) = e, \qquad E_1(e) = \frac{F_1(0, 0) \times E_3}{\sin \omega_1(0)} = \frac{E_3 \times F_2(0, 0)}{\sin \omega_2(0)}.$$

Then by Lemma 2.2 we obtain

(4.12) $$F(t_1, t_2) = F(t_1, 0) \cdot F(0, t_2).$$

We set

$$c_1(t) = F(t, 0), \quad \xi_1(t) = \xi(t, 0), \quad \alpha_1(t) = \omega_1(t),$$

$$c_2(t) = \tau \circ F(0, t), \quad \xi_2(t) = -\tau_* \xi(0, t), \quad \alpha_2(t) = \pi - \omega_2(t).$$

Then it is easy to see that

$$(4.13) \quad \begin{cases} c_i(0) = e, \quad \|\dot{c}_i\| = 1, \quad \|\dot{c}_i(0) \times E_3\| = \sin \alpha_i(0), \\ \xi_i(0) = \dot{c}_i(0) \times E_3 / \|\dot{c}_i(0) \times E_3\|, \\ 0 < \alpha_i < \pi, \quad \langle \dot{c}_i, \xi_i \rangle = 0. \end{cases}$$

By (2.16) we obtain

$$(4.14) \qquad\qquad D_{\dot{c}_i}\xi_i = \dot{c}_i \times \xi_i.$$

It follows from (2.13) and (2.14) that $D_{F_1}F_1 = \omega_1'(F_1 \times \xi)$ and $D_{F_2}F_2 = \omega_2'(\xi \times F_2)$. So we obtain

$$(4.15) \qquad\qquad D_{\dot{c}_i}\dot{c}_i = \alpha_i'(\dot{c}_i \times \xi_i).$$

Now we recall the Hopf fibration $p: S^3 \to S^2$ and define a curve γ_i on S^2 by $\gamma_i = p \circ c_i$. Then by the lemma below, it follows from (4.13)-(4.15) that the curve γ_i is regular and the curve c_i is an asymptotic lift of γ_i and the following relations hold.

$$\|\dot{\gamma}_i\| = 2 \sin \alpha_i, \qquad k_i = \cot \alpha_i,$$

where k_i denotes the geodesic curvature of γ_i. This implies that $\|\dot{\gamma}_i\|^2(1+k_i^2) = 4$ and $k_1 > k_2$, since $\alpha_2 - \alpha_1 = \pi - \omega > 0$. By (4.11) we see that $\gamma_i(0) = e_3$ and $\dot{\gamma}_i(0)/\|\dot{\gamma}_i(0)\| = e_1$. Hence $\Gamma = (\gamma_1, \gamma_2)$ is an admissible pair. Then it follows from (4.12) that $f_\Gamma(t_1, t_2) = c_1(t_1) \cdot c_2(t_2)^{-1} = F(t_1, t_2)$.   Q.E.D.

LEMMA 4.4. *Let α be a real valued function on $\mathbf{R}$ such that $0 < \alpha(t) < \pi$ and let $c: \mathbf{R} \to S^3$ be a curve such that $c(0) = e$, $\|\dot{c}(0) \times E_3\| = \sin \alpha(0)$ and $\|\dot{c}\| = 1$. Suppose that there exists a vector field ξ along c such that $\xi(0) = \dot{c}(0) \times E_3 / \|\dot{c}(0) \times E_3\|$, $\langle \dot{c}, \xi \rangle = 0$, $D_{\dot{c}}\dot{c} = \alpha'(\dot{c} \times \xi)$ and $D_{\dot{c}}\xi = \dot{c} \times \xi$. Then the curve $\gamma = p \circ c$ on S^2 is regular and c is an asymptotic lift of γ. Furthermore $\|\dot{\gamma}\| = 2 \sin \alpha$ and the geodesic curvature k of γ satisfies $k = \cot \alpha$.*

PROOF. We set $a_1(t) = c(t)$ and $a_2(t) = \exp(te_3)$. Then it follows that $a_i(0) = e$, $\|\dot{a}_i\| = 1$ and $\dot{a}_1(0) \times \dot{a}_2(0) \neq 0$. Define ξ_0, $\xi_i(t)$, $n_i(t)$, $\kappa_i(t)$ and $\omega(t_1, t_2)$ by (4.1). Since $\xi(0) = \xi_0$ and ξ is left invariant along c, we have $\xi_1(t) = \xi(t)$, and so $\langle \dot{a}_1, \xi_1 \rangle = \langle \dot{c}, \xi \rangle = 0$. Since E_3 is invariant under the action of S^1, we see that $\langle \dot{a}_2(t), \xi_2(t) \rangle = \langle E_3, \xi_0 \rangle = 0$. Thus we obtain

$$(4.16) \qquad\qquad \langle \dot{a}_i, \xi_i \rangle = 0.$$

It follows that $\kappa_1 = \langle D_{\dot{c}}\dot{c}, \xi \times \dot{c} \rangle = -\alpha'$ and $\kappa_2 = 0$. By the assumption that $\sin \alpha(0) = \|\dot{c}(0) \times E_3\|$, the angle ω_0 between $\dot{a}_1(0)$ and $\dot{a}_2(0)$ is equal to $\alpha(0)$. So we see

that

$$\omega(t_1, t_2) = \alpha(0) + \int_0^{t_1} \alpha'(t)dt = \alpha(t_1).$$

Since $0 < \alpha(t) < \pi$, we obtain

(4.17) $$0 < \omega(t_1, t_2) < \pi.$$

Define a map $F: R^2 \to S^3$ by $F(t_1, t_2) = a_1(t_1) \cdot a_2(t_2)$. By (4.16) and (4.17) it follows from Lemma 4.1 that F is a FAT such that $g_{12} = \cos \omega$ and $h_{12} = \sin \omega$. Since $F(t_1, t_2) = c(t_1) \cdot \exp(t_2 e_3)$ and $\omega(t_1, t_2) = \alpha(t_1)$, the angle between $\dot{c}(t)$ and E_3 is equal to $\alpha(t)$. Hence by Lemma 3.1 we have $\|\dot{\gamma}\| = \|p_* \dot{c}\| = 2\|\dot{c}\| \sin \alpha = 2 \sin \alpha > 0$, and so the curve γ is regular. Since $h_{11} = 0$, it follows that c is an asymptotic lift of γ. Furthermore, Lemma 3.7 (1) implies that $k = \cot \alpha$.   Q.E.D.

REMARK 4.5. The admissible pair Γ constructed in the proof of Theorem 4.3 satisfies

(4.18) $$\inf\{k_1(t_1) - k_2(t_2)\} > 0, \quad \sup\{k_1(t_1) - k_2(t_2)\} < \infty.$$

Conversely let Γ be an admissible pair which satisfies (4.18). Then it is easy to see that the mean curvature of f_Γ is bounded and the metric on R^2 induced by f_Γ is complete.

5. Periodicity of admissible pairs.

Let $\Gamma = (\gamma_1, \gamma_2)$ be an admissible pair and let f_Γ be a FAT defined by (4.8). We consider a group $G(\Gamma)$ given by

$$G(\Gamma) = \{\rho \in \text{Diff}(R^2) : f_\Gamma \circ \rho = f_\Gamma\},$$

where $\text{Diff}(R^2)$ denotes the group of all diffeomorphisms of R^2. By Theorem 2.3 the group $G(\Gamma)$ is naturally identified with a subgroup of the additive group R^2. It is easy to see that $G(\Gamma)$ is a discrete subgroup of R^2, and so the quotient space $R^2/G(\Gamma)$ is a 2-dimensional manifold. Let $\pi: R^2 \to R^2/G(\Gamma)$ be the canonical projection. Then the immersion f_Γ induces a flat immersion $\bar{f}_\Gamma: R^2/G(\Gamma) \to S^3$ such that $\bar{f}_\Gamma \circ \pi = f_\Gamma$.

In this section we give a criterion for the quotient space $R^2/G(\Gamma)$ to be compact. To state the result we introduce the following

DEFINITION. An admissible pair $\Gamma = (\gamma_1, \gamma_2)$ is said to be *periodic* if there exist positive numbers l_1 and l_2 such that γ_i is l_i-periodic for $i = 1, 2$.

THEOREM 5.1. *The quotient space $R^2/G(\Gamma)$ is compact iff Γ is periodic.*

To establish the theorem we need some lemmas. Let c_i be the asymptotic

lift of γ_i such that $c_i(0)=e$. Set $F=f_\Gamma$ and $\xi=F_1\times F_2/\|F_1\times F_2\|$. Then we obtain

$$(5.1) \qquad \mathrm{Ad}(c_i)e_3 = \gamma_i, \qquad \mathrm{Ad}(c_i)e_1 = \gamma_i'/\|\gamma_i'\|,$$

$$(5.2) \qquad F(t_1, t_2) = c_1(t_1)\cdot c_2(t_2)^{-1}.$$

LEMMA 5.2. *Let* $(l_1, l_2)\in\mathbf{R}^2$. *If* $F(l_1, l_2)=F(0, 0)$ *and* $\xi(l_1, l_2)=\xi(0, 0)$, *then* $\gamma_1(l_1)=\gamma_2(l_2)$ *and* $e_1=\gamma_i'(l_i)/\|\gamma_i'(l_i)\|$ *for* $i=1, 2$.

PROOF. Since $F(l_1, l_2)=F(0, 0)=e$, (5.2) implies that $c_1(l_1)=c_2(l_2)$, and so $\gamma_1(l_1)=\gamma_2(l_2)$ by (5.1). It follows from the proof of Theorem 4.2 that F satisfies $h_{12}>0$. By (2.16) we see that ξ is left invariant along $t_1\mapsto F(t_1, t_2)$ and right invariant along $t_2\mapsto F(t_1, t_2)$. Thus we obtain

$$\xi(l_1, l_2) = \{R_{c_2(l_2)^{-1}}\}_*\{L_{c_1(l_1)}\}_*\xi(0, 0).$$

Since $\xi(0, 0)=E_1(e)$ and $c_1(l_1)=c_2(l_2)$, it follows that

$$E_1(e) = \xi(l_1, l_2) = \{R_{c_2(l_2)^{-1}}\}_*E_1(c_1(l_1))$$

$$= \frac{d}{dt}\{c_1(l_1)\cdot\exp(te_1)\cdot c_2(l_2)^{-1}\}|_{t=0}$$

$$= \frac{d}{dt}\{c_i(l_i)\cdot\exp(te_1)\cdot c_i(l_i)^{-1}\}|_{t=0}$$

$$= \frac{d}{dt}\exp\{t\,\mathrm{Ad}(c_i(l_i))e_1\}|_{t=0}.$$

This shows that $e_1=\mathrm{Ad}(c_i(l_i))e_1$, and so $e_1=\gamma_i'(l_i)/\|\gamma_i'(l_i)\|$ by (5.1). Q.E.D.

LEMMA 5.3. *Let* $(l_1, l_2)\in\mathbf{R}^2$. *Suppose that* F *satisfies*

$$(5.3) \qquad F(t_1+l_1, t_2+l_2) = F(t_1, t_2)$$

for all $(t_1, t_2)\in\mathbf{R}^2$. *Then there exists an orientation preserving linear isometry* ϕ_i *of* $\mathfrak{su}(2)$ *such that* $\gamma_i(t+l_i)=\phi_i(\gamma_i(t))$ *for all* $t\in\mathbf{R}$.

PROOF. Let $\alpha_i(t)$ denote the angle between $\dot{c}_i(t)$ and E_3 such that $0<\alpha_i(t)<\pi$ and let $\omega(t_1, t_2)=\pi+\alpha_1(t_1)-\alpha_2(t_2)$. As in the proof of Theorem 4.2 we obtain

$$\langle F_1, F_2\rangle = \cos\omega, \qquad 0<\omega<\pi.$$

Then it follows from (5.3) that $\omega(t_1+l_1, t_2+l_2)=\omega(t_1, t_2)$ for all $(t_1, t_2)\in\mathbf{R}^2$. Thus we see that

$$\alpha_1(t_1+l_1)-\alpha_1(t_1) = \alpha_2(t_2+l_2)-\alpha_2(t_2) = \mathrm{constant}.$$

Since α_i is bounded, α_i is l_i-periodic. So it follows from Lemma 3.7 that $\|\dot{\gamma}_i\|$ and k_i are l_i-periodic, where k_i is the geodesic curvature of γ_i. This implies Lemma 5.3. Q.E.D.

LEMMA 5.4. *Let* $(l_1, l_2) \in \boldsymbol{R}^2$. *If* F *satisfies* (5.3), *then* $\gamma_i'(t+l_i) = \gamma_i'(t)$ *for all* $t \in \boldsymbol{R}$.

PROOF. Let $(s_1, s_2) \in \boldsymbol{R}^2$ and let $\phi_i = \mathrm{Ad}(c_i(s_i)^{-1})$. Then by (5.1) we obtain

(5.4) $$\phi_i(\gamma_i(s_i)) = e_3, \qquad \phi_i(\gamma_i'(s_i)/\|\gamma_i'(s_i)\|) = e_1.$$

Let $\tilde{\gamma}_i$ be a regular curve on S^2 defined by

$$\tilde{\gamma}_i(t) = \phi_i(\gamma_i(t+s_i)).$$

Then it follows from (5.4) that $\tilde{\Gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2)$ is an admissible pair. We consider a curve $\tilde{c}_i$ in S^3 defined by

$$\tilde{c}_i(t) = c_i(s_i)^{-1} \cdot c_i(t+s_i).$$

Then we obtain

$$\mathrm{Ad}(\tilde{c}_i)e_3 = \tilde{\gamma}_i, \qquad \mathrm{Ad}(\tilde{c}_i)e_1 = \tilde{\gamma}_i'/\|\tilde{\gamma}_i'\|.$$

Hence Lemma 3.8 implies that $\tilde{c}_i$ is an asymptotic lift of $\tilde{\gamma}_i$. Since $\tilde{c}_i(0) = e$, it follows that $f_{\tilde{\Gamma}}(t_1, t_2) = \tilde{c}_1(t_1) \cdot \tilde{c}_2(t_2)^{-1}$. Set $\tilde{F} = f_{\tilde{\Gamma}}$ and $\tilde{\xi} = \tilde{F}_1 \times \tilde{F}_2 / \|\tilde{F}_1 \times \tilde{F}_2\|$. By (5.2) we obtain

$$\tilde{F}(t_1, t_2) = c_1(s_1)^{-1} \cdot F(t_1+s_1, t_2+s_2) \cdot c_2(s_2).$$

So it follows from (5.3) that $\tilde{F}(t_1+l_1, t_2+l_2) = \tilde{F}(t_1, t_2)$. In particular $\tilde{F}(l_1, l_2) = \tilde{F}(0, 0)$ and $\tilde{\xi}(l_1, l_2) = \tilde{\xi}(0, 0)$. Hence Lemma 5.2 implies that

(5.5) $$\tilde{\gamma}_i'(l_i)/\|\tilde{\gamma}_i'(l_i)\| = e_1.$$

Since $\tilde{\gamma}_i'(l_i) = \phi_i(\gamma_i'(l_i+s_i))$, it follows from (5.1) and (5.5) that

$$\gamma_i'(l_i+s_i) = \mathrm{Ad}(c_i(s_i))\tilde{\gamma}_i'(l_i) = \|\tilde{\gamma}_i'(l_i)\| \, \mathrm{Ad}(c_i(s_i))e_1$$

$$= \|\gamma_i'(l_i+s_i)\| \gamma_i'(s_i)/\|\gamma_i'(s_i)\|.$$

By Lemma 5.3 we have $\|\gamma_i'(l_i+s_i)\| = \|\gamma_i'(s_i)\|$. Hence $\gamma_i'(l_i+s_i) = \gamma_i'(s_i)$. Q.E.D.

LEMMA 5.5. *Let* $(l_1, l_2) \in \boldsymbol{R}^2$. *If* F *satisfies* (5.3), *then* γ_i *is* l_i*-periodic.*

PROOF. By Lemma 5.3 there exists an orientation preserving linear isometry ϕ_i of $\mathfrak{su}(2)$ such that

(5.6) $$\gamma_i(t+l_i) = \phi_i(\gamma_i(t)).$$

Differentiating (5.6), we have $\gamma_i'(t+l_i) = \phi_i(\gamma_i'(t))$. Hence Lemma 5.4 implies that

(5.7) $$\gamma_i'(t) = \phi_i(\gamma_i'(t)).$$

Since γ_i is a regular curve on S^2, there exists $s_i \in \boldsymbol{R}$ such that $\gamma_i'(0)$ and $\gamma_i'(s_i)$ are linearly independent in $\mathfrak{su}(2)$. So it follows from (5.7) that ϕ_i must be identity. Then (5.6) implies that γ_i is l_i-periodic. Q.E.D.

PROOF OF THEOREM 5.1. Suppose that $R^2/G(\Gamma)$ is compact. It is easy to see that there exist positive numbers l_1 and l_2 such that $(l_1, l_2) \in G(\Gamma)$. Then F satisfies (5.3), and so by Lemma 5.5 Γ is periodic. Conversely suppose that Γ is periodic. Then there exist positive numbers l_1 and l_2 such that γ_i is l_i-periodic. By Theorem 3.9 c_i is $2l_i$-periodic. Then (5.2) implies that

$$F(t_1+2l_1, t_2) = F(t_1, t_2+2l_2) = F(t_1, t_2).$$

So the group $G(\Gamma)$ contains $(2l_1, 0)$ and $(0, 2l_2)$. Hence $R^2/G(\Gamma)$ is compact. This completes the proof of Theorem 5.1.

REMARK 5.6. By Theorems 2.4, 4.2, 4.3 and 5.1, the problem of the classification of flat tori in S^3 completely reduces to that of periodic admissible pairs.

6. Proof of Theorem A.

Let $f : M \to S^3$ be an isometric immersion of a compact connected flat surface M into S^3 and let $T : R^2 \to M$ be an asymptotic Tchebychef net. To establish Theorem A it is sufficient to show that there exist positive numbers s_1 and s_2 such that

$$T(t_1+s_1, t_2) = T(t_1, t_2+s_2) = T(t_1, t_2)$$

for all $(t_1, t_2) \in R^2$. Since $f \circ T$ is a FAT with bounded mean curvature, it follows from Theorem 4.3 that there exists an admissible pair $\Gamma = (\gamma_1, \gamma_2)$ such that $f \circ T = \Phi \circ f_\Gamma$ for some isometry Φ of S^3. Since the covering transformation group of T is a subgroup of $G(\Gamma)$, the quotient space $R^2/G(\Gamma)$ is compact. So it follows from Theorem 5.1 that there exist positive numbers l_1 and l_2 such that γ_i is l_i-periodic. Then Theorem 3.9 implies that

$$f_\Gamma(t_1+2l_1, t_2) = f_\Gamma(t_1, t_2+2l_2) = f_\Gamma(t_1, t_2).$$

So it follows that

$$f \circ T(0, 0) = f \circ T(2l_1, 0) = f \circ T(4l_1, 0) = \cdots.$$

Since M is compact and f is an immersion, there exist integers m and n such that $m < n$ and $T(2ml_1, 0) = T(2nl_1, 0)$. Let $\rho : R^2 \to R^2$ be a covering transformation of T such that $\rho(2ml_1, 0) = (2nl_1, 0)$. Since $\rho \in G(\Gamma)$, it follows from Theorem 2.3 that $\rho(t_1, t_2) = (t_1+2(n-m)l_1, t_2)$. Now we set $s_1 = 2(n-m)l_1$. Then $s_1 > 0$ and $T(t_1+s_1, t_2) = T(t_1, t_2)$. Similarly we obtain a positive number s_2 such that $T(t_1, t_2+s_2) = T(t_1, t_2)$. This completes the proof of Theorem A.

7. Proof of Theorem B.

Let $\gamma : R \to S^2$ be a regular curve on S^2 defined by

$$\gamma(\theta) = \frac{(\sin\theta)e_1 + (1-\cos\theta)e_2 + e_3}{\sqrt{3-2\cos\theta}}.$$

We introduce a real valued function $\theta(t)$ by the following relation.

$$t = \frac{1}{2}\int_0^{\theta(t)} \|\gamma'\|\sqrt{1+k^2}\,d\theta,$$

where k denotes the geodesic curvature of γ. We set $\gamma_1(t)=\gamma(\theta(t))$. Then it is easy to see the following

LEMMA 7.1. *Let k_1 be the geodesic curvature of γ_1 and let l be a positive number such that $\theta(l)=2\pi$. Then*

(1) $\gamma_1(0)=e_3$, $\gamma_1'(0)/\|\gamma_1'(0)\|=e_1$,

(2) $k_1>0$, $\|\gamma_1'\|^2(1+k_1^2)=4$,

(3) $\gamma_1(s)=\gamma_1(t)$ *iff $(s-t)/l$ is an integer,*

(4) $\langle[\gamma_1(s),\gamma_1'(s)],\gamma_1(t)\rangle\geq 0$, *with equality iff $(s-t)/l$ is an integer,*

(5) k_1 *is not constant.*

Let Φ be an orientation reversing linear isometry of $\mathfrak{su}(2)$ such that

$$\Phi(e_1)=e_1,\quad \Phi(e_2)=-e_2,\quad \Phi(e_3)=e_3.$$

We set $\gamma_2=\Phi\circ\gamma_1$. Since the geodesic curvature k_2 of γ_2 satisfies $k_2(t)=-k_1(t)$, it follows from Lemma 7.1 (1)-(3) that $\Gamma=(\gamma_1,\gamma_2)$ is a periodic admissible pair. By Theorem 5.1 we have an immersed flat torus $\bar{f}_\Gamma:\mathbf{R}^2/G(\Gamma)\to S^3$. The following lemma implies that $\bar{f}_\Gamma$ is an embedding.

LEMMA 7.2. $f_\Gamma(s_1,s_2)=f_\Gamma(t_1,t_2)$ *iff $(s_1-t_1,s_2-t_2)\in G(\Gamma)$.*

PROOF. Let c_i be the asymptotic lift of γ_i such that $c_i(0)=e$. Since $\gamma_i:[0,l]\to S^2$ is a simple closed curve, $\dot{r}_i/\|\dot{r}_i\|:[0,l]\to U(S^2)$ becomes the generator of the fundamental group of $U(S^2)$, where $U(S^2)$ denotes the unit tangent bundle of S^2. Hence Lemma 3.8 shows that

(7.1) $c_i(t+l) = -c_i(t)$ for all $t\in\mathbf{R}$.

Suppose that $f_\Gamma(s_1,s_2)=f_\Gamma(t_1,t_2)$. Then by (4.8) we have $c_1(s_1)=a\cdot c_2(s_2)$, where $a=c_1(t_1)\cdot c_2(t_2)^{-1}$. So it follows from (5.1) that $\gamma_1(s_1)=\mathrm{Ad}(a)\gamma_2(s_2)$, $\gamma_1(t_1)=\mathrm{Ad}(a)\gamma_2(t_2)$ and $\gamma_1'(s_1)=P\,\mathrm{Ad}(a)\gamma_2'(s_2)$, where $P=\|\gamma_1'(s_1)\|/\|\gamma_2'(s_2)\|$. This implies that

$$\langle[\gamma_1(s_1),\gamma_1'(s_1)],\gamma_1(t_1)\rangle = P\langle[\gamma_2(s_2),\gamma_2'(s_2)],\gamma_2(t_2)\rangle$$

$$= -P\langle[\gamma_1(s_2),\gamma_1'(s_2)],\gamma_1(t_2)\rangle.$$

By Lemma 7.1 (4) there exist integers n_1 and n_2 such that $s_i-t_i=n_i l$. Then (7.1) shows that $c_i(s_i)=(-1)^{n_i}c_i(t_i)$. Hence $f_\Gamma(s_1,s_2)=(-1)^{n_1+n_2}f_\Gamma(t_1,t_2)$, and so $(-1)^{n_1+n_2}=1$. By (7.1) we obtain

$$f_\Gamma(x_1+n_1l,\ x_2+n_2l) = f_\Gamma(x_1,\ x_2)$$

for all $(x_1,\ x_2)\in \mathbb{R}^2$. Hence $(s_1-t_1,\ s_2-t_2)\in G(\Gamma)$. Conversely suppose that $(s_1-t_1,\ s_2-t_2)\in G(\Gamma)$. Then by the definition of $G(\Gamma)$ we see that $f_\Gamma(s_1,\ s_2)=f_\Gamma(t_1,\ t_2)$.

Q.E.D.

Let M be the image of the embedding $\bar{f}_\Gamma$. To establish Theorem B it is sufficient to show that M contains no great circle in S^3. Suppose that M contains a great circle c. Then c is an asymptotic curve on M. So it follows from Lemma 2.2 that c is congruent to c_1 or c_2^{-1}. Hence either c_1 or c_2 must be a great circle in S^3. Since $k_2=-k_1$, it follows from Lemma 3.7 that k_1 is constant. This contradicts Lemma 7.1 (5).

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