J. Math. Soc. Japan
Vol. 40, No. 3, 1988

Completeness of Boolean powers of Boolean algebras

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(Received March 10, 1986)
(Revised Jan. 14, 1987)

Introduction.

In [4], Dwinger considered the completeness of Boolean powers of complete Boolean algebras. Dwinger obtained a necessary and sufficient condition in algebraic form:

THEOREM (Dwinger [4]). *Let A and B be complete Boolean algebras. The Boolean power $A[B]$ is complete if and only if*

$$\bigvee_{x\in A}\Big(\bigwedge_{y\nleq x}\sim f(y)\wedge\bigwedge_{x\nleq z}\bigvee_{u\nleq z}f(u)\Big)=\mathbf{1}\qquad\text{for each}\quad f:A\longrightarrow B.$$

Our main purpose is to consider some relationship which exists among the completeness of $A[B]$, the distributive-like properties of B and the saturation number of A.

In the notation of Boolean valued models of set theory, the Boolean power $A[B]$ is isomorphic to

$$\hat{\check{A}}=\{f\in V^{(B)}\mid[\![f\in\check{A}]\!]^{(B)}=\mathbf{1}\}$$

where $\check{A}$ is an element of $V^{(B)}$ such that $\check{A}=\{\check{a}\mid a\in A\}\times\{\mathbf{1}\}$. We can have a better perspective, if we deal with $\check{A}$ in $V^{(B)}$ instead of $A[B]$. By virtue of 5.5 of Solovay and Tennenbaum [13], $A[B]$ is complete if and only if $[\![\check{A}$ is complete$]\!]^{(B)}=\mathbf{1}$. Since

$$\begin{aligned}&[\![\check{A}\text{ is complete}]\!]^{(B)}\\&=[\![\forall X\subset\check{A}\,\exists x\in\check{A}[\forall y\in X[y\leqq x]\wedge\forall z\in\check{A}\,[\forall u\in X[u\leqq z]\Rightarrow x\leqq z]]]\!]^{(B)}\\&=\bigwedge_{f:A\to B}\Big(\bigvee_{x\in A}\Big(\bigwedge_{y\nleq x}\sim f(y)\wedge\bigwedge_{x\nleq z}\bigvee_{u\nleq z}f(u)\Big)\Big),\end{aligned}$$

we can obtain a proof of Dwinger's theorem which uses Boolean valued models of set theory. This suggests why we are going to work in $V^{(B)}$. We assume that the reader is familiar with the technique of Boolean valued models of set theory, as presented, e.g. [6, 7, 13]. We assume that $V^{(B)}$ is separated, i.e.,

$[\![x=y]\!]^{(B)}=\mathbf{1}$ is equivalent to $x=y$ for every $x, y\in V^{(B)}$. When no confusion appears possible, we shall write $[\![\Phi]\!]=\mathbf{1}$ in place of $[\![\Phi]\!]^{(B)}=\mathbf{1}$.

The following is fundamental in the present paper.

PROPOSITION 1. *Let A and B be complete Boolean algebras.*

(1) *If B satisfies the* $(<\mathrm{sat}(A), |A|)$-DL *and* $[\![\widecheck{\mathrm{sat}(A)}=\mathrm{sat}(\check{A})]\!]^{(B)}=\mathbf{1}$, *then* $[\![\check{A}$ *is complete*$]\!]^{(B)}=\mathbf{1}$.

(2) *If A is well decomposable and* $[\![\check{A}$ *is complete*$]\!]^{(B)}=\mathbf{1}$, *then B satisfies the* $(<\mathrm{sat}(A), |A|)$-DL.

(For precise definitions see below.)

Let F be a free Boolean algebra with κ generators. A completion $\bar{F}$ of F satisfies the c.c.c., so that $\bar{F}$ is well decomposable. Since "F is a free Boolean algebra" is absolute, $[\![\check{F}$ is free$]\!]^{(B)}=\mathbf{1}$ for every complete Boolean algebra B. $[\![\check{F}\subset\check{\bar{F}}\subset\bar{\bar{\check{F}}}]\!]=\mathbf{1}$ where $\bar{\bar{\check{F}}}$ is a completion of $\check{F}$ in $V^{(B)}$. Hence

$$[\![\bar{\bar{\check{F}}} \text{ satisfies the c.c.c.}]\!] \geqq [\![\check{\bar{F}} \text{ satisfies the c.c.c.}]\!] = \mathbf{1}.$$

There is an (ω, ∞)-distributive complete Boolean algebra which is not $(\omega_1, 2)$-distributive (see [10]). Hence, Proposition 1 shows the negative answer to the question in [4] whether $|A|=\kappa$ and completeness of $A[B]$ imply that B satisfies the (κ, κ)-DL. We can not remove the assumption $[\![\widecheck{\mathrm{sat}(A)}=\mathrm{sat}(\check{A})]\!]^{(B)}=\mathbf{1}$ of Proposition 1(1). There exist complete Boolean algebras A and B such that $[\![\widecheck{\mathrm{sat}(A)}=\mathrm{sat}(\check{A})]\!]^{(B)}=[\![\check{A}$ is complete$]\!]^{(B)}=\mathbf{0}$ and B satisfies the $(<\mathrm{sat}(A), \infty)$-DL (see Example 1).

Suppose that B is complete. In view of Proposition 1, the following question appears:

Is $[\![\widecheck{\mathrm{sat}(A)}=\mathrm{sat}(\check{A})]\!]^{(B)}=\mathbf{1}$ *a necessary condition for* $[\![A$ is complete$]\!]^{(B)}=\mathbf{1}$?

We have a partial answer to this question.

PROPOSITION 2. *Let T be a splitting tree and A be a completion of the partially ordered set P that is obtained from $(T, <_T)$ by reversing the order. Then*

$$[\![\check{A} \textit{ is complete}]\!]^{(B)}=\mathbf{1} \quad \textit{implies that} \quad [\![\widecheck{\mathrm{sat}(A)}=\mathrm{sat}(\check{A})]\!]^{(B)}=\mathbf{1}.$$

Our main results are as follows:

COROLLARY 1. *Let κ be a regular cardinal. The following are equivalent.*

(1) $[\![\check{A}$ *is complete*$]\!]^{(B)}=\mathbf{1}$ *for every κ-saturated complete Boolean algebra A.*

(2) *B satisfies the* $(<\kappa, \infty)$-DL *and* $[\![\check{A}$ *is $\check{\kappa}$-saturated*$]\!]^{(B)}=\mathbf{1}$ *for every κ-saturated Boolean algebra A.*

COROLLARY 3. *Suppose that κ is a regular cardinal and $2^{<\kappa}=\kappa$. The following are equivalent.*

(1) *B is κ-representable.*

(2) *B satisfies the $(<\kappa, \kappa)$-DL and $[\![\check{A}$ is $\check{\kappa}$-saturated$]\!]^{(B)}=\mathbf{1}$ for every κ-saturated Boolean algebra A with $|A|\leqq\kappa$.*

(3) *$[\![\check{A}$ is complete$]\!]^{(B)}=\mathbf{1}$ for every κ-saturated complete Boolean algebra A with $|A|\leqq\kappa$.*

Let A and C be complete Boolean algebras. Then we let $C\leqq A$ if there is a function i from C to A such that

$$\bigvee i(W)=\bigvee i(W') \text{ implies } \bigvee W=\bigvee W' \text{ for every } W, W'\subset C.$$

THEOREM 2. *If $C\leqq A$ and $[\![\check{A}$ is complete$]\!]^{(B)}=\mathbf{1}$, then $[\![\check{C}$ is complete$]\!]^{(B)}=\mathbf{1}$.*

In Section 1, we give basic definitions and notation. In Section 2, we prove Theorem 1 and its corollaries. Theorem 2 is proved in Section 3.

The author would like to express his thanks to the referee for his careful reading of the paper and his valuable advice.

1. Preliminaries.

We denote the first infinite cardinal by ω and the first uncountable cardinal by ω_1. We use letters α, β for ordinals and δ, κ, λ, μ for cardinals. We denote the least cardinal greater than κ by κ^+. We use letters A, B, C for infinite Boolean algebras. We denote the finite Boolean operations by $\wedge_B$, $\vee_B$, $\sim_B$, the infinitary joins and meets by $\bigvee_B$ and $\bigwedge_B$, the least element by $\mathbf{0}_B$ and the greatest element by $\mathbf{1}_B$. $\leqq_B$ is the canonical ordering of B. We shall omit subscripts if there is no confusion. Let $B^+=\{b\in B\,|\,b>\mathbf{0}\}$. For each $b\in B^+$, $\{a\in B\,|\,a\leqq b\}$ is a Boolean algebra with the restricted operations and we denote it by $B\restriction b$. Let $\mathcal{P}(X)$ be the Boolean algebra of all subsets of a set X. The cardinality of a set S is denoted by $|S|$. B is *κ-complete* if $\bigvee S$ exists for every subset S of B with $|S|<\kappa$. B is *countably complete* if it is ω_1-complete. B is *complete* if it is λ-complete for every λ. A subset X of B is *dense* if for each $b\in B^+$, there is $x\in X$ such that $\mathbf{0}<x\leqq b$. A separative partially ordered set $(P, \leqq)$ determines uniquely (up to isomorphism) a complete Boolean algebra B such that $(P, \leqq)$ is isomorphic to a dense subset of B (see [**6**, **7**]). It is called a *completion* of P. In particular, every Boolean algebra B has a unique (up to isomorphism) completion. We denote it by $\bar{B}$. Two elements a, b of B are *disjoint* if $a\wedge b=\mathbf{0}$. A *partition* of B is a maximal pairwise disjoint family of B. The set of all partitions of B is denoted by PART(B). B is *κ-saturated* (or satisfies the *κ-chain condition*) if there is no partition P of B with $|P|=\kappa$.

The least cardinal κ such that B is κ-saturated is called the saturation number of B and we denote it by sat(B). The ω_1-chain condition is called the *countable chain condition* (c. c. c.). We note that A is complete if and only if A is sat(A)-complete (see [20.5, **11**]). Let σ be a function from κ to λ. A complete Boolean algebra B satisfies the (κ, σ)-DL (or is (κ, σ)-*distributive*) if

$$\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\sigma(\alpha)} b_{\alpha,\beta} = \bigvee_{f\in \prod_{\alpha<\kappa}\sigma(\alpha)} \bigwedge_{\alpha<\kappa} b_{\alpha,f(\alpha)}$$

for every $\{\{b_{\alpha,\beta} | \beta<\sigma(\alpha)\} | \alpha<\kappa\} \subset \mathrm{PART}(B)$. If σ is a constant function such that $\sigma(\alpha)=\delta$ for every $\alpha<\kappa$, then the (κ, σ)-DL is equivalent to the usual (κ, δ)-distributive law ((κ, δ)-DL). B satisfies the ($<\delta$, λ)-DL if it satisfies the (κ, λ)-DL for every $\kappa<\delta$. And B satisfies the (κ, ∞)-DL if it satisfies the (κ, λ)-DL for every λ. We note that B satisfies the (κ, σ)-DL if and only if

$$\left[\!\!\left[\Big(\prod_{\alpha<\kappa} \sigma(\alpha)\Big)^{\check{}} = \prod_{\alpha<\check{\kappa}} \check{\sigma}(\alpha) \right]\!\!\right]^{(B)} = \mathbf{1}.$$

A complete Boolean algebra B satisfies the condition P(κ, λ, μ) if for each $a\in B^+$ and any partitions $p_\alpha \in \mathrm{PART}(B\restriction a)$ ($\alpha<\kappa$) with $|p_\alpha| \leqq \lambda$, there is a function $f\in \prod_{\alpha<\kappa} p_\alpha$ such that $\bigwedge_{\alpha\in X} f(\alpha)>\mathbf{0}$ for every $X\subset\kappa$ with $|X|<\mu$. Thus, P(κ, κ, ω) is the κ-representability; P(κ, κ, κ) is the property P_κ defined by Smith [**12**]; P(κ, λ, κ^+) is the (κ, λ)-DL. Let $\kappa<$sat(A) and $\lambda \leqq |A|$. A is (κ, λ)-*decomposable* if there is $P\in \mathrm{PART}(A)$ such that $|P|\geqq\kappa$ and $|A\restriction a|\geqq\lambda$ for every nonzero $a\in P$. A is ($<\delta$, λ)-decomposable if it is (κ, λ)-decomposable for every $\kappa<\delta$. A is *well decomposable* if it is ($<$sat(A), $|A|$)-decomposable. Each complete Boolean algebra A is (ω, $|A|$)-decomposable and ($<$sat(A), 2)-decomposable. In particular, every complete Boolean algebra with the c. c. c. is well decomposable. If $|A| = |A\restriction a|$ for every $a\in A^+$, then A is well decomposable. So each complete Boolean algebra is isomorphic to a product of well decomposable complete Boolean algebras (see [**9**]). For more details on Boolean algebras, see [**11**]. With respect to basic facts on partially ordered sets we refer the reader to [**6**, **7**].

Let B be complete. The Boolean power of A by B is the Boolean algebra $A[B]$ such that

$$A[B] = \{f\in B^A | f(A)\in \mathrm{PART}(B)\},$$

$$f\vee g(a) = \bigvee\{f(b)\wedge g(c) | b\vee c=a\},$$

$$f\wedge g(a) = \bigvee\{f(b)\wedge g(c) | b\wedge c=a\},$$

$$\sim f(a) = f(\sim a)$$

for every $f, g \in A[B]$ and $a \in A$,

$$\mathbf{0}_{A[B]}(\mathbf{0}_A) = \mathbf{1}_B \quad \text{and}$$

$$\mathbf{1}_{A[B]}(\mathbf{1}_A) = \mathbf{1}_B .$$

With respect to basic results on Boolean powers we refer the reader to [**3**]. In the following section 2 and section 3, we assume that B is complete.

2. Relationships of $\mathrm{P}(\kappa, \lambda, \mu)$ to completeness of A[B].

LEMMA 1 ([18.11, **6**]). *Let $\Phi(x_1, \cdots, x_n)$ be a bounded set-theoretical formula (i.e., it uses only bounded quantifiers).*

$$\Phi(a_1, \cdots, a_n) \textit{ if and only if } [\![\Phi(\check{a}_1, \cdots, \check{a}_n)]\!] = \mathbf{1}.$$

We shall use Lemma 1 without any mention.

LEMMA 2. *Let $\{a_\alpha \in A^+ \mid \alpha < \kappa\} \in \mathrm{PART}(A)$. The following are equivalent.*

(1) $[\![\check{A}$ *is complete*$]\!]^{(B)} = \mathbf{1}$.

(2) $[\![(A \restriction a_\alpha)^\vee$ *is complete*$]\!]^{(B)} = \mathbf{1}$ *for every $\alpha < \kappa$ and B satisfies the (κ, σ)-DL where $\sigma(\alpha) = |A \restriction a_\alpha|$ for every $\alpha < \kappa$.*

PROOF. "If C is a Boolean algebra and $\{b_\alpha \in C^+ \mid \alpha < \lambda\} \in \mathrm{PART}(C)$, then C is complete if and only if $C \restriction b_\alpha$ is complete for every $\alpha < \lambda$ and the canonical embedding $e: C \to \prod_{\alpha<\lambda} C \restriction b_\alpha$ defined by $e(c)(\alpha) = c \wedge b_\alpha$ is an isomorphism." This statement is a theorem of Zermelo-Fraenkel set theory (ZFC) and every theorem of ZFC has Boolean value **1**. Hence,

$$[\![\check{A} \text{ is complete}]\!] = \mathbf{1} \quad \text{if and only if} \quad [\![\forall \alpha < \check{\kappa} [\check{A} \restriction \check{a}_\alpha \text{ is complete}]]\!] = \mathbf{1} \text{ and}$$

$$[\![\text{the canonical embedding } e: \check{A} \longrightarrow \prod_{\alpha \in \check{\kappa}} \check{A} \restriction \check{a}_\alpha \text{ is an isomorphism}]\!] = \mathbf{1}.$$

Since, $[\![e(\check{A}) = (\prod_{\alpha<\kappa} A \restriction a_\alpha)^\vee]\!] = \mathbf{1}$,

$$[\![e \text{ is an isomorphism}]\!] = \left[\!\!\left[\prod_{\alpha<\check{\kappa}} \check{A} \restriction \check{a}_\alpha = \Big(\prod_{\alpha<\kappa} A \restriction a_\alpha \Big)^\vee \right]\!\!\right].$$

This completes the proof.

Each complete Boolean algebra is a product of well decomposable complete Boolean algebras. So, in view of our purpose, it is enough to deal with well decomposable complete Boolean algebras by virtue of Lemma 2.

LEMMA 3. *If A is (κ, λ)-decomposable and $[\![\check{A}$ is complete$]\!]^{(B)} = \mathbf{1}$, then B satisfies the (κ, λ)-DL. In particular, $[\![\check{A}$ is complete$]\!]^{(B)} = \mathbf{1}$ implies that B satisfies the $(<\mathrm{sat}(A), 2)$-DL and the $(\omega, |A|)$-DL.*

PROOF. There is a family $\{a_\alpha \in A^+ \mid \alpha<\kappa\} \in \mathrm{PART}(A)$ such that $|A \restriction a_\alpha| \geqq \lambda$ for every $\alpha<\kappa$. Hence, by Lemma 2, B satisfies the (κ, λ)-DL.

PROPOSITION 1. *Suppose that A is complete.*

(1) *If B satisfies the $(<\mathrm{sat}(A), |A|)$-DL and $[\![\check{\mathrm{sat}(A)}=\mathrm{sat}(\check{A})]\!]=\mathbf{1}$, then $[\![\check{A}$ is complete$]\!]^{(B)}=\mathbf{1}$.*

(2) *If A is well decomposable and $[\![\check{A}$ is complete$]\!]^{(B)}=\mathbf{1}$, then B satisfies the $(<\mathrm{sat}(A), |A|)$-DL.*

PROOF. (1) Since A is complete, $[\![\forall f \in \check{A}^{\kappa}[\bigvee f(\check{\kappa})$ exists$]]\!]=\mathbf{1}$ for every $\kappa<\mathrm{sat}(A)$. B satisfies the $(<\mathrm{sat}(A), |A|)$-DL, so that $[\![\forall f \in \check{A}^{\check{\kappa}}[\bigvee f(\check{\kappa})$ exists$]]\!]=\mathbf{1}$ for every $\kappa<\mathrm{sat}(A)$. Hence,

$$\begin{aligned} [\![\check{A} \text{ is complete}]\!] &\geqq [\![\check{A} \text{ is } \mathrm{sat}(\check{A})\text{-complete}]\!] \\ &= [\![\check{A} \text{ is } \check{\mathrm{sat}(A)}\text{-complete}]\!] \\ &= [\![\forall \kappa<\check{\mathrm{sat}(A)}\forall f \in \check{A}^{\kappa}[\bigvee f(\kappa) \text{ exists}]]\!] \\ &= \bigwedge_{\kappa<\mathrm{sat}(A)} [\![\forall f \in \check{A}^{\check{\kappa}}[\bigvee f(\check{\kappa}) \text{ exists}]]\!] \\ &= \mathbf{1}. \end{aligned}$$

(2) It is easily obtained from Lemma 3.

Let us recall some terminology concerning trees. A *tree* is a partially ordered set $(T, <_T)$, such that for each $x \in T$, $\{y \in T \mid y<x\}$ is well ordered by $<$. The order type of $\{y \in T \mid y<x\}$ is denoted by $\mathrm{ht}(x, T)$. The *height* of T is the least α such that $T_\alpha=\{x \in T \mid \mathrm{ht}(x, T)=\alpha\}=\emptyset$. It is denoted by $\mathrm{ht}(T)$. T is *splitting* if for any $t \in T$, there are at least two immediate successors to t. A *chain* of T is a set $X \subset T$ which is linearly ordered by $<$. Two elements x, y of T are *incomparable* $(x \perp y)$ if neither $x \leqq_T y$ nor $y \leqq_T x$. An *antichain* of T is a set $X \subset T$ such that any two distinct elements of X are incomparable. Let $\mathrm{sat}(T)$ be the least cardinal κ such that there is no antichain X of T with $|X|=\kappa$.

PROPOSITION 2. *Let T be a splitting tree and A be a completion of the partially ordered set P that is obtained from $(T, <_T)$ by reversing the order. Then*

$$[\![\check{A} \text{ is complete}]\!]^{(B)}=\mathbf{1} \textit{ implies that } [\![\check{\mathrm{sat}(A)}=\mathrm{sat}(\check{A})]\!]^{(B)}=\mathbf{1}.$$

PROOF. Since $[\![\check{T}$ is dense in $\check{A}]\!]^{(B)}=\mathbf{1}$, to see that $[\![\check{\mathrm{sat}(A)}=\mathrm{sat}(\check{A})]\!]^{(B)}=\mathbf{1}$, it is enough to show that $[\![\check{\mathrm{sat}(T)}=\mathrm{sat}(\check{T})]\!]^{(B)}=\mathbf{1}$. Let $\kappa=\mathrm{sat}(T)$. Since $[\![\check{A}$ is complete$]\!]^{(B)}=\mathbf{1}$, B satisfies the $(<\kappa, 2)$-DL by Lemma 3. Hence $[\![\check{\kappa}$ is a cardinal$]\!]^{(B)}=\mathbf{1}$. If $|T|<\kappa$, then there exists no new antichain of $\check{T}$ in $V^{(B)}$ by the $(<\kappa, 2)$-DL and so $[\![\mathrm{sat}(\check{T})=\check{\kappa}]\!]^{(B)}=\mathbf{1}$. Otherwise, $|T|\geqq\kappa$. On the other hand, $\mathrm{ht}(T)\leqq\kappa$ because $\kappa=\mathrm{sat}(T)$. Suppose that $\mathrm{ht}(T)<\kappa$, then there exists $\alpha<\mathrm{ht}(T)$ such that $|T_\alpha|\geqq\kappa$, which contradicts to $\kappa=\mathrm{sat}(T)$. Therefore $\mathrm{ht}(T)=\kappa$. Suppose that $[\![\widecheck{\mathrm{sat}(T)}<\mathrm{sat}(\check{T})]\!]>\mathbf{0}$. Then there exists $X\in V^{(B)}$ such that $[\![|X|=\check{\kappa}$ and X is an antichain of $\check{T}]\!]>\mathbf{0}$. Let $X=\{p_\alpha\}_{\alpha<\check{\kappa}}$. Without loss of generality, we can assume that $\mathrm{ht}(p_\alpha, \check{T})>\alpha$. Let $t_{2\alpha}$ and $t_{2\alpha+1}$ be two distinct immediate successors to p_α. Since $[\![\check{A}$ is complete$]\!]=\mathbf{1}$, $[\![\bigvee_{\alpha<\check{\kappa}}t_{2\alpha}=\check{c}]\!]>\mathbf{0}$ for some $c\in A$. $[\![\bigvee\{t\in\check{T}\,|\,t\leqq\check{c}\}=\check{c}]\!]>\mathbf{0}$. Hence we have $\bigvee\{t\in T\,|\,t\leqq c\}=c$. Since $\kappa=\mathrm{sat}(A)$, $\bigvee\{t\in\bigcup_{\alpha<\lambda}T_\alpha\,|\,t\leqq c\}=c$ for some $\lambda<\kappa$. Therefore

$$
\begin{aligned}
\mathbf{0} &< [\![\textstyle\bigvee_{\alpha<\check{\kappa}}t_{2\alpha}=\check{c}]\!] \\
&\leqq [\![\exists\alpha<\check{\lambda}\,\exists t\in\check{T}_\alpha[t\leqq\check{c} \text{ and } t_{2\check{\lambda}}\wedge t\neq\mathbf{0}]]\!] \\
&\leqq [\![\exists\alpha<\check{\lambda}\,\exists t\in\check{T}_\alpha[t\leqq\check{c} \text{ and } p_{\check{\lambda}}\wedge t\neq\mathbf{0}]]\!] \\
&\leqq [\![\exists\alpha<\check{\lambda}\,\exists t\in\check{T}_\alpha[t\leqq\check{c} \text{ and } p_{\check{\lambda}}\leqq t]]\!] \\
&\leqq [\![\mathbf{0}<t_{2\check{\lambda}+1}\leqq p_{\check{\lambda}}\leqq\check{c}]\!]\,.
\end{aligned}
$$

On the other hand, $\mathbf{0}<[\![\bigvee_{\alpha<\check{\kappa}}t_{2\alpha}=\check{c}]\!]\leqq[\![t_{2\check{\lambda}+1}\wedge\check{c}=\mathbf{0}]\!]$. This is a contradiction.

PROPOSITION 3. $[\![\check{B}$ is complete$]\!]^{(B)}=\mathbf{1}$ *if and only if B is atomic.*

PROOF. ($\Leftarrow$) Obvious.
($\Rightarrow$) Suppose that B is not atomic. Let $u=\bigvee\{b\in B\,|\,b$ is an atom$\}$ and $A=B\restriction\sim u$. Then A is atomless. Since $\mathbf{1}_B=[\![\check{B}$ is complete$]\!]^{(B)}\leqq[\![\check{A}$ is complete$]\!]^{(B)}$, we obtain $[\![\check{A}$ is complete$]\!]^{(A)}=\mathbf{1}_A$. Let $\kappa=\mathrm{sat}(A)$. Then A satisfies the $(<\kappa, 2)$-DL by Lemma 3. For each $a\in A^+$, pick $\mathbf{0}<a^*<a$. We define T_α $(\alpha<\kappa)$ by induction. Let $T_0=\{\mathbf{1}\}$ and $T_{\alpha+1}=\{a^*, a\wedge\sim a^*\,|\,a\in T_\alpha\}$. If α is limit, then let $T_\alpha=\{\bigwedge_{\beta<\alpha}f(\beta)\,|\,f\in\prod_{\beta<\alpha}T_\beta,\ \bigwedge_{\beta<\alpha}f(\beta)>\mathbf{0}\}$. Since A is κ-saturated and satisfies the $(<\kappa, 2)$-DL, $\bigvee T_\alpha=\mathbf{1}$ and $|T_\alpha|<\kappa$ for every $\alpha<\kappa$. Put $T=\bigcup_{\alpha<\kappa}T_\alpha$. $(T, <_T)$ is a splitting tree with $\mathrm{ht}(T)=\kappa$ where $<_T$ is the inverse of the partial ordering of T inherited from A. Let G be the canonical generic filter of $\check{A}$. Since G is $\check{V}$-complete, $|G\cap\check{T}_\alpha|=1$ for every $\alpha<\check{\kappa}$. Hence $[\![\check{T}$ has a chain of size $\check{\kappa}]\!]^{(A)}=\mathbf{1}$. So we have $[\![\check{T}$ has an antichain of size $\check{\kappa}]\!]^{(A)}=\mathbf{1}$. But this contradicts to $[\![\check{A}$ is complete$]\!]^{(A)}=\mathbf{1}$ as in the proof of Proposition 2.

Let P_T be the set of all finite antichains of T ordered by reversed inclusion and B_T be the complete Boolean algebra associated to P_T. The following lemma is a slight modification of Theorem 3 in [**1**] or Lemma 24.3 of [**6**]. We omit the proof, since it seems to be a folklore and can be proved by a straightforward generalization of the methods in the indicated parts in [**1**, **6**].

LEMMA 4. *Let κ be a regular cardinal and T be a tree with* $\mathrm{ht}(T)=\kappa$ *which has no chain of size κ. Then, P_T has no antichain of size κ; i.e. B_T satisfies the κ-chain condition.*

LEMMA 5. *If B satisfies* $\mathrm{P}(\kappa, \lambda, 3)$, *then* $[\![\check{A}$ *is $\check{\kappa}$-saturated*$]\!]=\mathbf{1}$ *for every κ-saturated Boolean algebra A with $|A|\leqq\lambda$.*

PROOF. Suppose that there exists a κ-saturated Boolean algebra A with $|A|\leqq\lambda$ such that $[\![\check{A}$ is $\check{\kappa}$-saturated$]\!]<\mathbf{1}$. There exists $g\in V^{(B)}$ such that

$$\begin{aligned} b &= [\![\check{A} \text{ is not } \check{\kappa}\text{-saturated}]\!] \\ &= [\![g:\check{\kappa}\longrightarrow \check{A}^{+} \text{ and } \forall\alpha,\ \beta<\kappa[\alpha\neq\beta\Longrightarrow g(\alpha)\wedge g(\beta)=\mathbf{0}]]\!] \\ &> \mathbf{0}. \end{aligned}$$

Put $p_{\alpha,a}=[\![\check{a}=g(\check{\alpha})]\!]\wedge b$ for every $a\in A$ and $\alpha<\kappa$. Let $P_\alpha=\{p_{\alpha,a}\mid a\in A\}\in\mathrm{PART}(A\restriction b)$. Since $|P_\alpha|\leqq|A|\leqq\lambda$, there is $f\in\prod_{\alpha<\kappa}P_\alpha$ such that $f(\alpha)\wedge f(\beta)>\mathbf{0}$ for every α, $\beta<\kappa$. Put $f(\alpha)=p_{\alpha,a_\alpha}$. Since $p_{\alpha,0}=[\![\check{\mathbf{0}}=g(\check{\alpha})]\!]\wedge b\leqq[\![\check{\mathbf{0}}\in\check{A}^{+}]\!]\wedge b=\mathbf{0}$, $a_\alpha>\mathbf{0}$ for every $\alpha<\kappa$. If $\alpha\neq\beta<\kappa$, then

$$\begin{aligned} [\![\check{a}_\alpha\wedge\check{a}_\beta=\mathbf{0}]\!] &\geqq [\![\check{a}_\alpha=g(\check{\alpha})]\!]\wedge[\![\check{a}_\beta=g(\check{\beta})]\!]\wedge b \\ &= f(\alpha)\wedge f(\beta)>\mathbf{0}. \end{aligned}$$

Therefore $\{a_\alpha\mid\alpha<\kappa\}$ is a pairwise disjoint family of size κ. This contradicts that A is κ-saturated.

THEOREM 1. *Let κ be a regular cardinal. The following are equivalent.*

(1) *B satisfies* $\mathrm{P}(\kappa, \lambda^{<\kappa}, \kappa)$.

(2) *B satisfies* $\mathrm{P}(\kappa, \lambda^{<\kappa}, 3)$ *and the* $(<\kappa, \lambda^{<\kappa})$-DL.

(3) *B satisfies the* $(<\kappa, \lambda^{<\kappa})$-DL *and* $[\![\check{A}$ *is $\check{\kappa}$-saturated*$]\!]=\mathbf{1}$ *for every κ-saturated Boolean algebra A with $|A|\leqq\lambda^{<\kappa}$.*

(4) $[\![\check{A}$ *is complete*$]\!]=\mathbf{1}$ *for every κ-saturated complete Boolean algebra A with $|A|\leqq\lambda^{<\kappa}$.*

PROOF. (1)$\Rightarrow$(2) follows from definitions.

(2)$\Rightarrow$(3) follows from Lemma 5.

(3)$\Rightarrow$(4) follows from the same argument as in the proof of Proposition 1 (1).

(4)$\Rightarrow$(1): We first show that there exists a well decomposable complete Boolean algebra A with $\mathrm{sat}(A)=\kappa$ and $|A|=\lambda^{<\kappa}$. Then, it follows from $[\![\check{A}$ is complete$]\!]=\mathbf{1}$ that B satisfies the $(<\kappa, \lambda^{<\kappa})$-DL. Let C be a completion of a free product $\bigoplus_{\alpha<\kappa}\mathcal{P}(\alpha)$, F be a free Boolean algebra with λ generators and A be a completion of $F[C]$. Since $[\![\check{F}$ satisfies the c.c.c.$]\!]^{(C)}=\mathbf{1}$, we have $\mathrm{sat}(A)=\mathrm{sat}(C)=\kappa$. It is easy to show that A is well decomposable. Hence $|A|=|A|^{<\kappa}$, $\lambda=|F|\leqq|A|\leqq(\lambda\times2^{<\kappa})^{<\kappa}$. Since κ is regular, $(\lambda^{<\kappa})^{<\kappa}=\lambda^{<\kappa}$. Therefore we have

$|A|=\lambda^{<\kappa}$. Suppose that B does not satisfy $\mathrm{P}(\kappa, \lambda^{<\kappa}, \kappa)$. There exist $a>\mathbf{0}$ and $P_\alpha \in \mathrm{PART}(B\restriction a)$ $(\alpha<\kappa)$ with $|P_\alpha| \leqq \lambda^{<\kappa}$ such that, for every $f\in\prod_{\alpha<\kappa}P_\alpha$, $\bigwedge_{\alpha\in X} f(\alpha)=\mathbf{0}$ for some $X\subset\kappa$ with $|X|<\kappa$. Without loss of generality, we may assume that $a=\mathbf{1}$. Since B satisfies the $(<\kappa, \lambda^{<\kappa})$-DL, we may also assume that P_α is a common refinement of $\{P_\beta | \beta<\alpha\}$ for each $\alpha<\kappa$ and $P_\alpha \cap P_\beta=\emptyset$ for every $\alpha\neq\beta<\kappa$. Put $T=\bigcup\{P_\alpha | \alpha<\kappa\}$. $(T, <_T)$ is a tree with $\mathrm{ht}(T)=\kappa$ and has no chain of size κ, where $<_T$ is the inverse of the partial ordering of T inherited from B. Then, by Lemma 4, B_T satisfies the κ-chain condition. Let G be the canonical generic filter of $\check{B}$. Since G is $\check{V}$-complete, $|G\cap\check{P}_\alpha|=1$. Put $q_\alpha=\{p_\alpha\}=G\cap\check{P}_\alpha$ for each $\alpha<\check{\kappa}$. We note that $[\![\check{\kappa}$ is a cardinal$]\!]=\mathbf{1}$ and $[\![G\cap\check{T}$ is a chain of $\check{T}$ of size $\check{\kappa}]\!]=\mathbf{1}$, so that $[\![\check{B}_T$ is $\check{\kappa}$-saturated$]\!]=\mathbf{0}$. Since $|T|\leqq\lambda^{<\kappa}$, $|B_T| \leqq(\lambda^{<\kappa})^{<\kappa}=\lambda^{<\kappa}$. Hence, by the assumtion, $[\![\check{B}_T$ is complete$]\!]=\mathbf{1}$, so that $[\![\bigvee_{\alpha<\check{\kappa}} q_{2\alpha}=\check{c}]\!]>\mathbf{0}$ for some $c\in B_T$. Since B_T is κ-saturated, there exist $\{b_\alpha\}_{\alpha<\mu}\subset P_T$ $(\mu<\kappa)$ such that $c=\bigvee_{\alpha<\mu} b_\alpha$. Put $\delta=\sup\{\mathrm{ht}(t, T) | \exists\alpha<\mu[t\in b_\alpha]\}<\kappa$. Then,

$$\mathbf{0} < [\![\bigvee_{\alpha<\check{\kappa}} q_{2\alpha}=\check{c}]\!] \leqq [\![\exists\alpha<\check{\mu}[q_{2\check{\delta}}\wedge\check{b}_\alpha\neq\mathbf{0}]]\!] .$$

Hence we obtain $[\![\exists\alpha<\check{\mu}\, \forall t\in\check{b}_\alpha[t\perp p_{2\check{\delta}}]]\!]>\mathbf{0}$. Since $[\![p_{2\check{\delta}}<_T p_{2\check{\delta}+1}]\!]=\mathbf{1}$, we have $[\![\exists\alpha<\check{\mu}\, \forall t\in\check{b}_\alpha[t\perp p_{2\check{\delta}+1}]]\!]>\mathbf{0}$. Therefore $[\![\exists\alpha<\check{\mu}[q_{2\check{\delta}+1}\wedge\check{b}_\alpha\neq\mathbf{0}]]\!]>\mathbf{0}$. On the other hand, by the definition of b_α, we obtain $\mathbf{1}=[\![q_{2\check{\delta}+1}\wedge\check{c}=\mathbf{0}]\!]\leqq[\![\forall\alpha>\check{\kappa}[q_{2\check{\delta}+1}\wedge\check{b}_\alpha=\mathbf{0}]]\!]$. This is a contradiction.

We are interested in the cases $\lambda=2$ or $\lambda=\infty$. For $\lambda=\infty$, we obtain the following:

COROLLARY 1. *Let κ be a regular cardinal. The following are equivalent.*

(1) $[\![\check{A}$ *is complete*$]\!]^{(B)}=\mathbf{1}$ *for every κ-saturated complete Boolean algebra A.*

(2) *B satisfies the* $(<\kappa, \infty)$-DL *and* $[\![\check{A}$ *is $\check{\kappa}$-saturated*$]\!]^{(B)}=\mathbf{1}$ *for every κ-saturated Boolean algebra A.*

In the case of $\lambda=2$, it is related to the Smith's condition P_κ, and consequently to the κ-representability as follows.

LEMMA 6. *If B satisfies* $\mathrm{P}(\kappa, \kappa, \kappa)$, *then B satisfies* $\mathrm{P}(\kappa, 2^\mu, \kappa)$ *for every $\mu<\kappa$.*

PROOF. Let $a\in B^+$ and $P_\alpha\in\mathrm{PART}(B\restriction a)$ $(\alpha<\kappa)$ with $|P_\alpha|\leqq 2^\mu$. Without loss of generality, we can assume that $|P_\alpha|=2^\mu$ for every $\alpha<\kappa$. Let τ_α be a bijection from P_α to $\mathscr{P}(\mu)$. We define $q_{\alpha,\beta}\in B\restriction a$ for $\alpha<\kappa$ and $\beta<\mu$ by $q_{\alpha,\beta}=\bigvee_{\beta\in\tau_\alpha(p)} p$.

Put $q^0_{\alpha,\beta}=q_{\alpha,\beta}$ and $q^1_{\alpha,\beta}=\sim q_{\alpha,\beta}\wedge a$.

Let $Q_{\alpha,\beta}=\{q^0_{\alpha,\beta}, q^1_{\alpha,\beta}\}\in\mathrm{PART}(B\restriction a)$. Note that

$$p = \bigwedge_{\beta\in\tau_\alpha(p)} q^0_{\alpha,\beta}\wedge \bigwedge_{\beta\notin\tau_\alpha(p)} q^1_{\alpha,\beta} \qquad \text{for every } p\in P_\alpha .$$

Since B satisfies $\mathrm{P}(\kappa, \kappa, \kappa)$,

$$\exists f\in\prod_{\substack{\alpha<\kappa\\ \beta<\mu}} Q_{\alpha,\beta} \text{ such that } \forall X\subset\kappa\times\mu[|X|<\kappa \text{ implies } \bigwedge_{(\alpha,\beta)\in X} f(\alpha, \beta)>\mathbf{0}].$$

We define $g\in\prod_{\alpha<\kappa}P_\alpha$ by $g(\alpha)=\bigwedge_{\beta<\mu}f(\alpha, \beta)$. Let $Y=\{\beta<\mu \mid f(\alpha, \beta)=q^0_{\alpha,\beta}\}$. There is $p\in P_\alpha$ such that $\tau_\alpha(p)=Y$. Then

$$g(\alpha)=\bigwedge_{\beta<\mu} f(\alpha, \beta)=\bigwedge_{\beta\in\tau_\alpha(p)} q^0_{\alpha,\beta}\wedge\bigwedge_{\beta\notin\tau_\alpha(p)} q^1_{\alpha,\beta}=p\in P_\alpha.$$

Hence $g\in\prod_{\alpha<\kappa}P_\alpha$. For every $Z\subset\kappa$ with $|Z|<\kappa$, $\bigwedge_{\alpha\in Z}g(\alpha)=\bigwedge_{\substack{\alpha\in Z\\ \beta<\mu}}f(\alpha, \beta)>\mathbf{0}$. Therefore B satisfies $\mathrm{P}(\kappa, 2^\mu, \kappa)$.

If κ is a regular cardinal such that $2^{<\kappa}=2^\mu$ for some $\mu<\kappa$ or $2^{<\kappa}=\kappa$, then B satisfies the Smith's condition P_κ i.e. $\mathrm{P}(\kappa, \kappa, \kappa)$ if and only if B satisfies $\mathrm{P}(\kappa, 2^{<\kappa}, \kappa)$ as above. Hence we obtain the following:

COROLLARY 2. *Let κ be a regular cardinal such that $2^{<\kappa}=2^\mu$ for some $\mu<\kappa$ or $2^{<\kappa}=\kappa$. The following are equivalent.*

(1) *B satisfies the Smith's condition P_κ.*

(2) *B satisfies the $(<\kappa, 2^{<\kappa})$-DL and $[\![\check{A}$ is $\check{\kappa}$-saturated$]\!]^{(B)}=\mathbf{1}$ for every κ-saturated Boolean algebra A with $|A|\leqq 2^{<\kappa}$.*

(3) *$[\![\check{A}$ is complete$]\!]^{(B)}=\mathbf{1}$ for every κ-saturated complete Boolean algebra A with $|A|\leqq 2^{<\kappa}$.*

In [12], Smith showed that if $\kappa^{<\kappa}=\kappa$, then P_κ is equivalent to the κ-representability. Hence we obtain the following:

COROLLARY 3. *Suppose that κ is regular and $2^{<\kappa}=\kappa$. The following are equivalent.*

(1) *B is κ-representable.*

(2) *B satisfies the $(<\kappa, \kappa)$-DL and $[\![\check{A}$ is $\check{\kappa}$-saturated$]\!]^{(B)}=\mathbf{1}$ for every κ-saturated Boolean algebra A with $|A|\leqq\kappa$.*

(3) *$[\![\check{A}$ is complete$]\!]^{(B)}=\mathbf{1}$ for every κ-saturated complete Boolean algebra A with $|A|\leqq\kappa$.*

In [5], assuming $2^\omega<2^{\omega_1}$, Gregory showed that there exists an (ω, ω)-distributive complete Boolean algebra that does not satisfy P_{ω_1}. The following example, shooting a closed unbounded set [2], enables us to remove the assumption $2^\omega<2^{\omega_1}$.

EXAMPLE 1. Let $S\subset\omega_1$ be a stationary and co-stationary set. Let $(T, <_T)$ be the tree of closed subset of S under end extension. Let P be the partially ordered set that is obtained from $(T, <_T)$ by reversing the order and B be its completion. It was proved in [2] that B satisfies the (ω, ∞)-DL. T has no uncountable chain and $|\{t\in T \mid \mathrm{ht}(t, T)=\alpha\}|\leqq 2^\omega$ for every $\alpha<\omega_1$, so that B does not satisfy $\mathrm{P}(\omega_1, 2^\omega, \omega_1)$. Hence B satisfies neither P_{ω_1} nor $\mathrm{P}(2^\omega, 2^\omega, \omega)$, i.e. B

is not 2^{ω}-representable. By virtue of the proof of Theorem 1, we have

$$[\![\check{B}_T \text{ satisfies the c.c.c.}]\!]^{(B)} = [\![\check{B}_T \text{ is complete}]\!]^{(B)} = \mathbf{0}.$$

3. Partial order $\leqq$ on complete Boolean algebras.

Let A be a complete Boolean algebra. Put $X(b, f)=\{a\in A \mid b\wedge f(a)>\mathbf{0}\}$ for each $b\in B^+$ and $f: A\to B$. We omit f if there is no confusion. $b\in B^+$ is stable w.r.t. f if $\bigvee X(b)=\bigvee X(c)$ for every $c\leqq b$.

LEMMA 7. $[\![\check{A}$ is complete$]\!]^{(B)}=\mathbf{1}$ *if and only if* $\bigvee\{b\in B \mid b$ *is stable w.r.t.* $f\}=\mathbf{1}$ *for every* $f: A\to B$.

PROOF. ($\Rightarrow$): Let f be a function from A to B. Let $W\in V^{(B)}$ be such that $\mathrm{dom}(W)=\mathrm{dom}(\check{A})$ and $W(\check{a})=f(a)$ for all $a\in A$. Then $[\![W\subset\check{A}]\!]=\mathbf{1}$. Since $[\![\check{A}$ is complete$]\!]^{(B)}=\mathbf{1}$,

$$\bigvee\{b\in B \mid b\leqq[\![\bigvee W=\check{a}]\!] \text{ for some } a\in A\} = \mathbf{1}.$$

Put $b=[\![\bigvee W=\check{a}]\!]>\mathbf{0}$. We show that b is stable w.r.t. f. Let $\mathbf{0}<c\leqq b$. $X(c)=\{x\in A \mid c\wedge f(x)>\mathbf{0}\}=\{x\in A \mid c\wedge[\![\check{x}\in W]\!]>\mathbf{0}\}$. Then $c\leqq[\![W\subset X(c)\check{}\,]\!]$. Hence

$$c\leqq[\![\bigvee W\leqq X(c)\check{}\,]\!]=[\![\bigvee W\leqq(\bigvee X(c))\check{}\,]\!],$$

so that $a\leqq\bigvee X(c)$. On the other hand, for each $x\in X(c)$,

$$\mathbf{0}<c\wedge[\![\check{x}\in W]\!]\leqq[\![\bigvee W=\check{a}]\!]\wedge[\![\check{x}\in W]\!]\leqq[\![\check{x}\leqq\check{a}]\!].$$

Hence we have $x\leqq a$ for every $x\in X(c)$. So we get $\bigvee X(c)\leqq a$. Therefore $\bigvee X(c)=a$ for every $c\leqq b$, so that b is stable w.r.t. f.

($\Leftarrow$): Let $[\![W\subset\check{A}]\!]=\mathbf{1}$ and $f(a)=[\![\check{a}\in W]\!]$ for all $a\in A$. If b is stable w.r.t. f, then

$$b\leqq[\![W\subset X(b)\check{}\,]\!]\leqq[\![\forall w\in W[w\leqq\bigvee X(b)\check{}\,]]\!].$$

Suppose that $b\not\leqq[\![\bigvee W=\bigvee X(b)\check{}\,]\!]$. Then $b\wedge[\![\forall w\in W[w\wedge\check{c}=\mathbf{0}]]\!]>\mathbf{0}$ for some $c<\bigvee X(b)$. Put $d=b\wedge[\![\forall w\in W[w\wedge\check{c}=\mathbf{0}]]\!]$. Since b is stable w.r.t. f, $\bigvee X(d)=\bigvee X(b)>c$. Hence $d\wedge[\![\check{x}\in W]\!]>\mathbf{0}$ for some $x\leqq c$. This is a contradiction.

THEOREM 2. *Let C be a complete Boolean algebra. If $C\leqq A$ and* $[\![\check{A}$ *is complete*$]\!]=\mathbf{1}$, *then* $[\![\check{C}$ *is complete*$]\!]=\mathbf{1}$.

PROOF. Suppose that i makes $C\leqq A$. Let f be a function from C to B. We define $f^*: A\to B$ as follows:

$$f^*(a)=f(c) \quad \text{if } a=i(c)\in i(C),$$

$$f^*(a)=\mathbf{0} \quad \text{if } a\notin i(C).$$

If b is stable w.r.t. f^*, then b is stable w.r.t. f. Hence

$$\mathbf{1} = \bigvee\{b \in B \mid b \text{ is stable w.r.t. } f^*\} \leqq \bigvee\{b \in B \mid b \text{ is stable w.r.t. } f\}.$$

Therefore $[\![\check{C}$ is complete$]\!] = \mathbf{1}$ by Lemma 7.

EXAMPLE 2. Let C be a subalgebra of A. It is well known that, in general, $\bigvee W$ does not coincide with $\bigvee i(W)$ where $W \subset C$ and i is the canonical embedding. It is easy to show that i makes $C \leqq A$. So $[\![\check{A}$ is complete$]\!] = \mathbf{1}$ implies that $[\![\check{C}$ is complete$]\!] = \mathbf{1}$.

EXAMPLE 3. Let X be a topological space and $\mathrm{Reg}(X)$ be the Boolean algebra of all regular open sets of X. Let i be the canonical injection from $\mathrm{Reg}(X)$ to $\mathcal{P}(X)$. Since $\bigvee W = \mathrm{int}(\mathrm{cl}(\cup W) = \mathrm{int}(\mathrm{cl}(\bigvee i(W))$, i makes $\mathrm{Reg}(X) \leqq \mathcal{P}(X)$. Hence, if B satisfies the $(|X|, 2)$-DL, then $[\![\widecheck{\mathrm{Reg}(X)}$ is complete$]\!] = \mathbf{1}$.

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