J. Math. Soc. Japan
Vol. 40, No. 3, 1988

Compactness of the moduli space of Yang-Mills connections in higher dimensions

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(Received Nov. 17, 1986)

§1. Introduction and statement of results.

In analytical aspect of the Yang-Mills theory one of the most fundamental results is the K. Uhlenbeck's compactness theorem on the moduli space of Yang-Mills connections.

The purpose of the present paper is to generalize the theorem of Uhlenbeck to higher dimensions. More precisely, let G be a compact Lie group, and $\{D(i)\}$ a sequence of Yang-Mills connections on a G-principal P over an n-dimensional Riemannian manifold M such that for some constant R

$$\int_M |R(i)|^2 dV \leqq R < \infty .$$

Then we can state the theorem of K. Uhlenbeck:

(1.1) FACT ([8], [2]). *Let* $2 \leqq n \leqq 4$. *Then there exist a subsequence* $\{j\} \subset \{i\}$, *a subset* M' $(\subset M)$, *and a Yang-Mills connection* $D(\infty)$ *on* P *over* M' *such that* $M - M'$ *consists of at most finitely many points* $\{p_1, \cdots, p_l\}$, *and that for each compact subset* $K \subset M'$ *there exist gauge transformations* $g_K(j)$ *of* P *over* K *so that*

$$g_K(j)^*(D(j)) \longrightarrow D(\infty) \qquad \textit{in } C^\infty\textit{-topology on } K.$$

Furthermore,

a) *when* $n=2, 3$, *we have* $M' = M$,

b) *when* $n=4$, *in a neighborhood of each* p_k, *the following happens:*
If $x = (x_1, x_2, x_3, x_4)$, $|x| < \delta$, *denote normal coordinates of* M *at* p_k, *then there are rescalings* $\rho(j)(x) = (1/r_j)x$ *of this coordinates with* $r_j \to 0$ *such that for each compact subset* $H \subset \boldsymbol{R}^n$ *there exist gauge transformations* $\gamma_H(j)$ *of* $\rho(j)^*P$ *over* H *so that*

$$\gamma_H(j)^* \rho(j)^* D(j) \longrightarrow D \qquad \textit{in } C^\infty\textit{-topology on } H$$

where D *is a non-flat Yang-Mills connection on* $\boldsymbol{R}^4$ *with respect to the standard metric of* $\boldsymbol{R}^4$ *with finite action*

$$\int_{R^4}|R|dy<\infty.$$

Our result now reads:

(1.2) THEOREM. *Let $n\geqq 4$. Then there exist a subsequence $\{j\}\subset\{i\}$, a compact subset $\mathcal{S}$ with finite $(n-4)$-dimensional Hausdorff measure $H_{n-4}(\mathcal{S})<\infty$, a G-principal bundle Q over $M-\mathcal{S}$, and a Yang-Mills connection $D(\infty)$ on Q such that for each compact subset $K\subset M-\mathcal{S}$ there are bundle equivalences*

$$g_K(j):P|K\longrightarrow Q|K$$

so that

$$g_K(j)^*(D(j))\longrightarrow D(\infty)\qquad \text{in } C^\infty\text{-topology on } K.$$

Furthermore, in a neighborhood of $p\in\mathcal{S}$ the following happens. Let $x=(x_1,\cdots,x_n)$, $|x|<\delta$, denote normal coordinates of M at p. Then there are rescalings $\rho(j)(x)=(1/r_j)x$ of this coordinates, where $r_j\to 0$ such that there exist gauge transformations $\gamma(j)$ of $\rho(j)^*P$ over $B_1(0)$ (n-dimensional unit ball with center at 0), so that

$$\gamma(j)^*\rho(j)^*(D(j))\longrightarrow D\qquad \text{in } C^\infty\text{-topology on } B_1(0)$$

where D is a non-flat Yang-Mills connection on $B_1(0)$ with respect to the standard metric of $\boldsymbol{R}^n$.

We remark here that when dimension $n=4$, the singular set $\mathcal{S}$ is a finite set, $M-\mathcal{S}$ is diffeomorphic to $S^3\times(0,1)$ in each neighborhood of $p\in\mathcal{S}$. Thus we can extend the bundle map $g_K(j)$ over K to a bundle map over $M-\mathcal{S}$ if we take a sufficiently large K. Hence the bundle Q is isomorphic to P over $M-\mathcal{S}$. But we do not know whether it holds in higher dimensions: Indeed, the topology of $M-\mathcal{S}$ is not so clear that our method can be successfully applied. Moreover in dimension $n=4$, by removable singularities theorem ([7]) we can extend the bundle $P|M-\mathcal{S}$ and the connection D over $M-\mathcal{S}$ to a bundle Q (not necessarily equivalent to P) and a Yang-Mills connection $\tilde{D}$ over the whole M, and the connection D can be extended on S^4 through the stereographic projection $S^4-\{\text{north pole}\}\to\boldsymbol{R}^4$.

If we replace the C^∞ convergence by the weak L_1^2 convergence, Fact (1.1) holds for general (not necessarily Yang-Mills) connections $D(i)$ (see [6]). Indeed, these results follows from the existence of the Coulomb gauges in case that $L^{n/2}$-norm of curvature is small (see K. Uhlenbeck ([8])). We shall derive an estimate for the C^0-norm of curvature for Yang-Mills connections in case that $r^{4-n}\int_{B_r(x)}|R|^2dV$ is sufficiently small. Combining this with K. Uhlenbeck's result on the existence of the Coulomb gauge we can prove Theorem (1.2).

Our method to get the estimate is analogous to that for harmonic maps which is obtained by R. Schoen ([5, Th. 2.2]). He has derived this estimate using a Bochner-Weitzenböck formula and a monotonicity formula. Since we have these formulas also in the Yang-Mills case, we can follow his proof.

The author would like to thank Prof. T. Ochiai for his advice, and Prof. Y. Matsumoto for his lectures at the University of Tokyo which stimulated the author's interest in Yang-Mills theory.

§ 2. Notation.

In this section we summarize the notation. Let G be a compact Lie group, and $\mathfrak{g}$ its Lie algebra.

We define two bundles $\mathrm{Aut}(P)$ and $\mathrm{Ad}(P)$ associated with a G-principal bundle P over an n-dimensional Riemannian manifold M. $\mathrm{Aut}(P)$ is the automorphism bundle of P, i.e.

$$\mathrm{Aut}(P) = P \times_{\mathrm{Ad}} G,$$

$\mathrm{Ad}(P)$ the adjoint bundle with fiber $\mathfrak{g}$, i.e.

$$\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} \mathfrak{g},$$

and we put a fiber metric on $\mathrm{Ad}(P)$ by some Ad-invariant metric on $\mathfrak{g}$.

Let $\Omega^k(\mathrm{Ad}(P))$ denote the space of k-forms on M with values in $\mathrm{Ad}(P)$.

$\mathcal{C}(P)$ is the space of G-connections on P. We fix a connection D_0, and identify $\mathcal{C}(P)$ with $\Omega^1(\mathrm{Ad}(P))$ since the difference $A = D - D_0$ is a 1-form with values in $\mathrm{Ad}(P)$.

Let $R = R^D$ be the curvature form of D, which is a 2-form with values in $\mathrm{Ad}(P)$. We define the Yang-Mills functional over $\mathcal{C}(P)$ by

$$\mathcal{YM}(D) = \frac{1}{2}\int_M |R^D|^2 dV$$

where dV is the volume element of M. A critical point of the Yang-Mills functional is called a Yang-Mills connection.

We call the group of all inner automorphisms of P the gauge group of P, and denote it by $\mathcal{G}(P)$. This is nothing but the space of global cross sections of $\mathrm{Aut}(P)$. The gauge group $\mathcal{G}(P)$ acts on the $\mathcal{C}(P)$ as follows;

given g in $\mathcal{G}(P)$ and D in $\mathcal{C}(P)$ we define

$$g^*(D) = g \circ D \circ g^{-1}.$$

Then we have $R^{g^*(D)} = g \circ R^D \circ g^{-1}$. Thus the Yang-Mills functional $\mathcal{YM}$ is invariant under the action of $\mathcal{G}(P)$. Especially the space of Yang-Mills connection is invariant under the action of $\mathcal{G}(P)$. We say two connections D_1 and D_2 to be gauge equivalent when there exists a gauge transformation g in $\mathcal{G}(P)$ such that

$D_2=g^*(D_1)$.

A connection D on P naturally induces a connection on $\mathrm{Ad}(P)$ (also denoted by D). On the other hand we have Levi-Civita connection ∇ on the tangent bundle TM. Thus we have naturally the connection (also denoted by D) on $\otimes^k T^*M\otimes\mathrm{Ad}(P)$ induced from D and ∇. We define the exterior differential operator $d^D: \Omega^k(\mathrm{Ad}(P))\to\Omega^{k+1}(\mathrm{Ad}(P))$ by

$$(d^D\psi)(X_0, \cdots, X_k) = \sum_{i=0}^{k}(-1)^k(D_{X_i}\psi)(X_0, \cdots, \hat{X}_i, \cdots, X_k).$$

We denote by $\delta^D: \Omega^{k+1}(\mathrm{Ad}(P))\to\Omega^k(\mathrm{Ad}(P))$ the formal adjoint operator of d^D. We can write it as

$$(\delta^D\psi)(X_2, \cdots, X_k) = -\sum_{i=1}^{n}(D_{e_i}\psi)(e_i, X_2, \cdots, X_k)$$

where $(e_1, \cdots, e_n)$ is an orthonormal frame of TM.

Then it is well known that a connection D is a Yang-Mills connection if and only if

$$\delta^D R^D = 0.$$

Now we recall two useful formulas which will be used in the later sections.

(2.1) FACT (Bochner-Weitzenböck formula [1]). *If D is a Yang-Mills connection on P, then we have*

$$\mathrm{trace}\, D^2R^D = R^D\circ(\mathrm{Ric}\,\Lambda I)+\mathfrak{R}^T(R^D)+\mathfrak{R}^P(R^D),$$

where $\mathrm{Ric}\,\Lambda I$ *is the Ricci transformation on 2-forms defined by*

$$(\mathrm{Ric}\,\Lambda I)(X, Y) = \mathrm{Ric}(X)\Lambda Y+X\Lambda\,\mathrm{Ric}(Y)\,; \quad \textit{and}$$

$\mathfrak{R}^T(R^D)$ *and* $\mathfrak{R}^P(R^D)$ *are the curvature operators of TM and P respectively defined by*

$$\mathfrak{R}^T(R^D)(X, Y) = \sum_j R^D(e_j, R^M(X, Y)e_j),$$
$$\mathfrak{R}^P(R^D)(X, Y) = 2\sum_j [R^D(e_j, X), R^D(e_j, Y)].$$

Here $(e_1, \cdots, e_n)$ is an orthonormal frame of TM and R^M the Riemannian curvature tensor, and $[\,,\,]$ the bracket operation induced from that of Lie algebra $\mathfrak{g}$.

(2.2) FACT (Monotonicity formula [4]). *Let $B_r=\{x\in\boldsymbol{R}^n : |x|<r\}$ be the r-ball in $\boldsymbol{R}^n$, and g a metric on B_r which satisfies*

$$\frac{1}{\Lambda}(\delta_{ij}) \leqq (g_{ij}) \leqq \Lambda(\delta_{ij}), \qquad r\left|\frac{\partial g_{ij}}{\partial x_k}\right| \leqq \Lambda, \quad \textit{and} \quad r^2\left|\frac{\partial^2 g_{ij}}{\partial x_k\partial x_l}\right| \leqq \Lambda.$$

Then there exists a constant $C=C(n, \Lambda)$ depending only on n and Λ, such that if

D is a Yang-Mills connection on a G-principal bundle P with respect to the metric g, then we have

$$\sigma^{4-n}\int_{B_\sigma}|R^D|^2 dV \leqq C\rho^{4-n}\int_{B_\rho}|R^D|^2 dV \qquad \textit{for } \sigma\leqq\rho\leqq r.$$

§ 3. Local estimate.

(3.1) LEMMA. *Let the notation be as in Fact* (2.2). *There exist constants* $\varepsilon=\varepsilon(n, \Lambda, G)$ *and* $C=C(n, \Lambda, G)$ *such that if* D *is a Yang-Mills connection on* P *satisfying* $r^{4-n}\int_{B_r}|R^D|^2 dV\leqq\varepsilon$, *then*

$$\sup_{B_{r/4}}|R^D|^2 \leqq Cr^{-n}\int_{B_r}|R^D|^2 dV.$$

PROOF. From Fact (2.1) we have

$$\Delta|R|^2 \geqq -C(|R|+r^{-2})|R|^2 \qquad \text{on } B_r. \tag{3.2}$$

We take $x_0\in\text{Closure}(B_{r/2})$ such that

$$\left(\frac{r}{2}-|x_0|\right)^2|R(x_0)| = \sup_{B_{r/2}}\left(\frac{r}{2}-|x|\right)^2|R(x)|.$$

Let $\rho=(1/2)(r/2-|x_0|)$. We may assume $\rho>0$; If $\rho=0$ i.e. $|R|=0$ on $B_{r/2}$, our assertion is clearly true. Then we have

$$\sup_{B_\rho(x_0)}|R| \leqq \sup_{B_{\rho+|x_0|}}|R| \leqq \rho^{-2}\sup_{B_{\rho+|x_0|}}\left(\frac{r}{2}-|x|\right)^2|R(x)| \leqq 4|R(x_0)|. \tag{3.3}$$

We shall study two cases $|R(x_0)|\leqq\rho^{-2}$ and $|R(x_0)|>\rho^{-2}$ separately. Now suppose $|R(x_0)|\leqq\rho^{-2}$. Then we have from (3.2) and (3.3)

$$\Delta|R|^2 \geqq -C\rho^{-2}|R|^2 \qquad \text{on } B_\rho(x_0).$$

By the mean value theorem (cf. [3], Th. 9.20) we have

$$|R(x_0)|^2 \leqq C\rho^{-n}\int_{B_\rho(x_0)}|R^D|^2 dV. \tag{3.4}$$

Here we should remark that the constant C of mean value theorem depends on (coefficient of $|R|^2$)$\times$(radius of ball). Thus in our case the constant depends only on n, Λ and G.

On the other hand from Fact (2.2)

$$\rho^{4-n}\int_{B_\rho(x_0)}|R|^2 dV \leqq C\left(\rho+\frac{r}{2}\right)^{4-n}\int_{B_{\rho+r/2}(x_0)}|R|^2 dV \leqq Cr^{4-n}\int_{B_r}|R|^2 dV.$$

Thus combining with (3.4) we obtain

$$\sup_{B_{r/4}}|R|^2 \leqq \left(\frac{r}{4}\right)^{-4}\sup_{B_{r/4}}\left(\frac{r}{2}-|x|\right)^4|R(x)|^2 \leqq \left(\frac{r}{4}\right)^{-4}(2\rho)^4|R(x_0)|^2$$
$$\leqq Cr^{-n}\int_{B_r}|R|^2dV,$$

which shows our assertion is true for the case $\|R(x_0)\|\leqq\rho^{-2}$.

Next suppose $|R(x_0)|>\rho^{-2}$. Set $r_0=|R(x_0)|^{-1/2}$. Then we have $r_0<\rho$. So we have got from (3.2) and (3.3),

$$\Delta|R|^2 \geqq -Cr_0^{-2}|R|^2 \qquad \text{on } B_{r_0}(x_0).$$

Using mean value theorem again we obtain

$$|R(x_0)|^2 \leqq Cr_0^{-n}\int_{B_{r_0}(x_0)}|R|^2dV. \tag{3.5}$$

Also in this case the constant C depends only on n, Λ and G by the same reason as the previous case. We use Fact (2.2) again to get

$$r_0^{4-n}\int_{B_{r_0}(x_0)}|R|^2dV \leqq C\left(r_0+\frac{r}{2}\right)^{4-n}\int_{B_{r_0+r/2}(x_0)}|R|^2dV \leqq Cr^{4-n}\int_{B_r}|R|^2dV \leqq C\varepsilon. \tag{3.6}$$

From (3.5) and (3.6), we obtain

$$r_0^{-4} = |R(x_0)|^2 \leqq r_0^{-4}C\varepsilon,$$

which is impossible for sufficiently small ε. Q.E.D.

This lemma corresponds to Theorem (2.2) in [5] and the above proof is similar to his proof.

(3.7) LEMMA. *Let the notation be as in Fact* (2.2). *There exist constants* $\varepsilon=\varepsilon(n, \Lambda, G)$ *and* $C=C(n, \Lambda, G)$ *such that if D is a Yang-Mills connection on P satisfying* $r^{4-n}\int_{B_r}|R|^2dV\leqq\varepsilon$ *then there exists a Yang-Mills connection* $\tilde{D}=d+A$ *on* $P|B_{r/4}$ *which is gauge equivalent to D on* $B_{r/4}$, *and satisfies*

1) $r\sup_{B_{r/4}}|A| \leqq Cr^2\sup_{B_{r/4}}|R|$
2) $\delta A=0$

where d is the exterior differential operator with respect to the flat connection on $P=B_r\times G$, *and δ its adjoint operator.*

PROOF. Combining Lemma (3.1) with K. Uhlenbeck's result ([7]), we obtain the conclusion immediately. Q.E.D.

§4. Proof of main theorem.

We take $\varepsilon=\varepsilon(M, A, G)$ as in Lemma (3.7). We define the singular set $\mathcal{S}$ by $\mathcal{S}=\bigcap_{0<r<\delta}\left\{x\in M:\liminf_{i\to\infty} r^{4-n}\int_{B_r(x)}|R(i)|^2dV\geqq\varepsilon\right\}$, where δ is the injectivity radius of M.

First we show $\mathcal{S}$ is closed. Suppose $\{x_j\}$ is a sequence in $\mathcal{S}$ converging to x in M, and r is arbitrary positive number. Then for all $r_1<r$, $B_{r_1}(x_j)\subset B_r(x)$ holds for sufficiently large j. Hence

$$r_1^{4-n}\liminf_{i\to\infty}\int_{B_r(x)}|R(i)|^2dV\geqq r_1^{4-n}\liminf_{i\to\infty}\int_{B_{r_1}(x_j)}|R(i)|^2dV\geqq\varepsilon.$$

Taking limit $r_1\to r$, we get $x\in\mathcal{S}$.

Next we claim $H_{n-4}(\mathcal{S})<\infty$. For all r, we take finitely many balls $\{B_r(x_k): x_k\in\mathcal{S}\}$ such that

1) $\mathcal{S}\subset\bigcup_{k=1}^{N}B_{2r}(x_k)$

2) $B_r(x_k)\cap B_r(x_l)=\emptyset$ for $k\neq l$.

Since $x_k\in\mathcal{S}$, for sufficiently large i we get

$$r^{4-n}\int_{B_r(x_k)}|R(i)|^2dV\geqq\frac{\varepsilon}{2}\qquad\text{for all } k=1,\cdots,N.$$

And so

$$\sum_{k=1}^{N}r^{n-4}\leqq 2\varepsilon^{-1}\sum_k\int_{B_r(x_k)}|R(i)|^2dV\leqq C\int_M|R(i)|^2dV\leqq CR,$$

from which it follows that $H_{n-4}(\mathcal{S})<\infty$.

From the definition of $\mathcal{S}$ if we take a subsequence $D(j)$, we can take a countable covering $\{B_\alpha=B_{r_\alpha/4}(x_\alpha)\}$ of $M-\mathcal{S}$ so that

$$r_\alpha^{4-n}\int_{B_{r_\alpha}(x_\alpha)}|R(j)|^2dV<\varepsilon\qquad\text{for all } \alpha \text{ and } j.$$

From Lemma (3.2) we can take $\tilde{D}(j)=d+A_\alpha(j)$ such that

1) $D(j)$ and $\tilde{D}(j)$ are gauge equivalent on B_α,

2) $r_\alpha\sup_{B_\alpha}|A_\alpha(j)|\leqq C$,

3) $r_\alpha^2\sup_{B_\alpha}|R(j)|\leqq C$,

4) $\delta A_\alpha(j)=0$.

If we put $\delta A_\alpha(j)=0$ and Yang-Mills equations $\delta R(j)=0$ together, we get an uniform elliptic system on $A_\alpha(j)$. Therefore we can use the standard technique to obtain C^k-estimates of $A_\alpha(j)$ for all k. Thus if we take a subsequence once again (we use same letter $A_\alpha(j)$), we conclude

$$A_\alpha(j)\to A_\alpha(\infty)\quad\text{in } C^\infty \text{ topology, with } A_\alpha(\infty) \text{ in } \Omega^1(\mathrm{Ad}(P)|B_\alpha).$$

Next we observe that $\{A_\alpha(\infty)\}$ fits together to define a connection on some G-principal bundle Q over $M-\mathcal{S}$. From the above construction we have the transition functions $\gamma_{\alpha\beta}(j): B_\alpha\cap B_\beta\to G$ for each j such that

$$A_\alpha(j) = -d\gamma_{\alpha\beta}(j)\gamma_{\alpha\beta}(j)^{-1}+\gamma_{\alpha\beta}(j)A_\beta(j)\gamma_{\alpha\beta}(j)^{-1}$$
$$\text{for all } \alpha \text{ and } \beta \text{ with } B_\alpha\cap B_\beta\neq\emptyset.$$

Since G is compact, it follows that $\{\gamma_{\alpha\beta}(j)\}$ is uniformly bounded. Combining this with the estimate of $A_\alpha(j)$ mentioned above, we can get C^k estimates of $\gamma_{\alpha\beta}(j)$ for all k by bootstrapping method; we may assume

$$\gamma_{\alpha\beta}(j)\to\gamma_{\alpha\beta}(\infty) \quad \text{in } C^\infty \text{ topology with } \gamma_{\alpha\beta}(\infty): B_\alpha\cap B_\beta\to G.$$

Since for each j, $\{\gamma_{\alpha\beta}(j)\}$ satisfies the cocycle condition, $\{\gamma_{\alpha\beta}(\infty)\}$ also satisfies the cocycle condition, and $\{\gamma_{\alpha\beta}(\infty)\}$ defines a G-principal bundle Q over $M-\mathcal{S}$, and $\{A_\alpha(\infty)\}$ defines a Yang-Mills connection $D(\infty)$ on Q.

Next we shall construct a bundle map $g_K(j): P|K\to Q|K$ for each compact subset $K\subset M-\mathcal{S}$ so that $g_K(j)^*D(j)$ converges to $D(\infty)$ in C^∞ topology on K. This can be done by the induction on the number of balls B_α covering K just as in [8, §3]. We shall show that we can take a smaller cover $U_\alpha\subset B_\alpha$ and smooth functions $\rho_\alpha(j): U_\alpha\to G$ for sufficiently large j satisfying the consistency condition

$$\gamma_{\alpha\beta}(\infty)=\rho_\alpha(j)\gamma_{\alpha\beta}(j)\rho_\beta(j)^{-1} \qquad \text{on } U_\alpha\cap U_\beta.$$

We introduce an ordering in $\{B_\alpha\}$, and represent this by $1, \cdots, \alpha, \cdots, N$.

At first we set $\rho_1(j)=e\in G$ on B_1. If we have constructed U_β and $\rho_\beta(j)$ for $\beta\leqq\alpha-1$, we define $\rho_\alpha(j): B_\alpha\to G$ as follows; We define $X_\alpha(j)$ by

$$\exp X_\alpha(j)=\gamma_{\alpha\beta}(\infty)\rho_\beta(j)\gamma_{\alpha\beta}(j)^{-1} \qquad \text{on } B_\alpha\cap\Big(\bigcup_{\beta<\alpha}U_\beta\Big).$$

Taking j sufficiently large so that $\gamma_{\alpha\beta}(j)\gamma_{\alpha\beta}(\infty)^{-1}$ is sufficiently near to $e\in G$, we can construct $\rho_\beta(j)$ sufficiently near to $e\in G$. Hence $X_\alpha(j)$ can be defined by the above equation which is independent of β by the consistency condition. Take a smooth function f on K which is 0 on $B_\alpha-(\bigcup_{\beta<\alpha}U_\beta)$ and 1 on an appropriate subset of $\bigcup_{\beta<\alpha}U_\beta$ (we fix later). We set

$$\rho_\alpha(j)=\exp(fX_\alpha(j)) \qquad \text{on } B_\alpha.$$

If we replace U_β by $U_\beta\cap\operatorname{int}\{f=1\}$ for $\beta<\alpha$, and define U_α by B_α, the consistency condition will be satisfied between $\rho_\alpha(j)$ and $\rho_\beta(j)$ for $\beta<\alpha$. And if we take an appropriate f, then U_β $(\beta\leqq\alpha)$ and B_γ $(\gamma>\alpha)$ still cover K. Thus by induction we can construct a smaller cover $\{U_\alpha\}$ and $\rho_\alpha(j)$. And $\rho_\alpha(j)$ converges to $e\in G$ in C^∞ topology on U_α.

Let $\sigma_\alpha(j): P|U_\alpha\to U_\alpha\times G$ and $\sigma_\alpha(\infty): Q|U_\alpha\to U_\alpha\times G$ be the local trivialization which induce transition functions $\gamma_{\alpha\beta}(j)$ and $\gamma_{\alpha\beta}(\infty)$ respectively. We define bundle maps $g_K(j)$ by

$$g_K(j)=\sigma_\alpha(\infty)^{-1}\rho_\alpha(j)\sigma_\alpha(j).$$

This definition is well defined on K by the consistency condition and

$$\gamma_{\alpha\beta}(j)=\sigma_\alpha(j)\sigma_\beta(j)^{-1}, \qquad \gamma_{\alpha\beta}(\infty)=\sigma_\alpha(\infty)\sigma_\beta(\infty)^{-1}.$$

Then $g_K(j)^*D(j)$ is locally represented as

$$d-d\rho_\alpha(j)\rho_\alpha(j)^{-1}+\rho_\alpha(j)A_\alpha(j)\rho_\alpha(j)^{-1},$$

using the local trivialization $\sigma_\alpha(\infty)$ on U_α. Thus $g_K(j)^*D(j)$ converges to

$$d+A_\alpha(\infty),$$

i.e. $g_K(j)^*D(j)$ converges to $D(\infty)$.

We now prove the second statement of main theorem. We introduce normal coordinates $x=(x_1, \cdots, x_n)$ at $p\in\mathcal{S}$. Then we have

$$\lim_{j\to\infty} r^{4-n}\int_{B_r(p)}|R(j)|^2dV\geqq\varepsilon \qquad \text{for all } r>0.$$

But for fixed j, we know that

$$\lim_{r\to 0} r^{4-n}\int_{B_r(x)}|R(j)|^2dV=0 \qquad \text{uniformly on } x\in M.$$

Hence for all j, there exists a radius $r_j>0$ such that

$$r_j^{4-n}\int_{B_{r_j}(p)}|R(j)|^2dV=\frac{\varepsilon}{4C} \qquad \text{where } C \text{ is as in (2.2).} \tag{4.1}$$

Then we may assume $r_j\to 0$ as $j\to\infty$. In fact, if there exists a positive number r_0 with $r_0\leqq r_j$ for all j,

$$\frac{\varepsilon}{2}\leqq r_0^{4-n}\int_{B_{r_0}(p)}|R(j)|^2dV\leqq Cr_j^{4-n}\int_{B_{r_j}(p)}|R(j)|^2dV\leqq\frac{\varepsilon}{4C}\times C=\frac{\varepsilon}{4},$$

which is a contradiction.

We introduce new coordinates $y=\rho(j)(x)=(1/r_j)x$, and then have, by (4.1),

$$\int_{B_1(0)}|\tilde{R}(j)|^2dV_j(y)=\frac{\varepsilon}{4C} \tag{4.2}$$

where $\tilde{R}(j)=\rho(j)^*R(j)$, and $dV_j(y)$ the volume element of the rescaled metric $(1/r_j^2)\rho(j)^*g$. In the coordinates, the rescaled metric converges to the standard metric in $\boldsymbol{R}^n$ in C^∞ topology. Moreover from (4.2) curvature $\tilde{R}(j)$ does not concentrate in this case. Although the metrics $(1/r_j^2)\rho(j)^*g$ differ each other, they converge to the standard metric. Thus we can argue as in the first part of the proof, and conclude that there exist gauge transformations $\gamma(j)$ over $B_1(0)$ such that $\gamma(j)^*\rho(j)^*D(j)$ converges to a Yang-Mills connection D in C^∞ topology.

Moreover by (4.2)

$$\int_{B_1(0)}|R|^2 dy = \frac{\varepsilon}{4C},$$

showing D is not trivial.

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