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Fractional powers of operators

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1. Introduction.

The study of fractional powers of operators has an extensive history. The semigroup of fractional powers of a bounded operator was studied by Hille [**5**] (1939). Fractional powers for the negatives of the infinitesimal generators of bounded strongly continuous semigroups have been discussed by Bochner [**3**], Phillips [**19**], Yosida [**22**] and Balakrishnan [**1**], who afterwards (1960) [**2**] gives a new definition and extends this theory to closed linear operators A in a Banach space X such that $]-\infty, 0[$ is contained in the resolvent set $\rho(A)$, and the resolvent satisfies

$$\|\mu(\mu+A)^{-1}\| \leqq M < \infty, \qquad 0<\mu<+\infty \tag{1.1}$$

(operators which we shall call non-negatives, following the terminology used by H. Komatsu [**13**]).

Balakrishnan [**2**] defines the power with base A and exponent α ($\mathrm{Re}\,\alpha>0$) as the closure of a closable operator, J^{α}, whose expression is:

$$\left\{\begin{array}{l} \text{For } 0<\mathrm{Re}\,\alpha<1,\ D(J^{\alpha})=D(A), \\ \qquad J^{\alpha}\varphi = \dfrac{\sin\alpha\pi}{\pi}\displaystyle\int_0^{+\infty}\mu^{\alpha-1}(\mu+A)^{-1}A\varphi d\mu . \\ \text{For } 0<\mathrm{Re}\,\alpha<2,\ D(J^{\alpha})=D(A^2), \\ \qquad J^{\alpha}\varphi = \dfrac{\sin\alpha\pi}{\pi}\displaystyle\int_0^{+\infty}\mu^{\alpha-1}\Big[(\mu+A)^{-1}-\frac{\mu}{1+\mu^2}\Big]A\varphi d\mu+\sin\Big(\alpha\frac{\pi}{2}\Big)A\varphi . \\ \text{For } n<\mathrm{Re}\,\alpha<n+1,\ \ D(J^{\alpha})=D(A^{n+1}), \\ \qquad J^{\alpha}\varphi = J^{\alpha-n}A^{n}\varphi . \\ \text{For } n<\mathrm{Re}\,\alpha\leqq n+1,\ \ D(J^{\alpha})=D(A^{n+2}), \\ \qquad J^{\alpha}\varphi = J^{\alpha-n}A^{n}\varphi . \end{array}\right. \tag{1.2}$$

This author proves that

$$\overline{\overline{J^{\alpha}}\,\overline{J^{\beta}}} = \overline{J^{\alpha+\beta}} .$$

Almost at the same time two different definitions appear, introduced by Krasnosel'skii-Sobolevskii [**16**] and T. Kato [**9**], limited to particular cases, and equivalent to the one given by Balakrishnan when A is densely defined as Nollau proves in [**18**] (1967).

In his first work out of a series of six on fractional powers, Komatsu proves the additivity for the first time, namely:

$$\overline{J^\alpha}\,\overline{J^\beta} = \overline{J^{\alpha+\beta}} \tag{1.3}$$

with no restriction for A or for the exponent. Other ways to prove (1.3) can be found in [**11**], [**18**], [8], although all of them are long and difficult.

When $D(A)$ is dense in X, the resolvent of $\overline{J^\alpha}$ has been studied in several papers ([**2**], [**9**], [**21**], [**11**]) and consequently the spectral mapping theorem has been proved. Another property, deduced from the resolvent of $\overline{J^\alpha}$ was the multiplicativity (equality between the power of power and the power with base A and product exponent) also limited to the case in which A is densely defined.

An extensive and annotated bibliography can be found in Fattorini [**4**], Hirsch [**7**] and Krein [**17**].

We have to point out that the $\overline{J^\alpha}$ power has its domain and range contained in $\overline{D(A)}$, which makes the resolvent set of $\overline{J^\alpha}$ to be empty when A is not densely defined. Consequently, with this notion of power, there is no sense in establishing the two properties mentioned above (the spectral mapping theorem and multiplicativity). Another obvious fault of the $\overline{J^\alpha}$ power is that $\overline{J^1} \neq A$ when $\overline{D(A)} \neq X$.

These disadvantages can be avoided when A^{-1} is bounded considering the definition

$$A^\alpha = [(A^{-1})^\alpha]^{-1}$$

well known in the literature (for example see Krein [**17**] and Krasnosel'skii-Sovolevskii [**16**]).

Nevertheless, there is not yet a definition which could be satisfactory in that sense when

$$\overline{D(A)} \neq X \quad \text{and} \quad 0 \in \sigma(A) \tag{1.4}$$

(where $\sigma(A)$ is the spectrum of A).

This is one of the purposes of this paper. The definition we shall take is original only when we are precisely under assumption (1.4) and it will allow us to develop a basic theory about fractional powers with natural and direct proofs, in which the fact of $D(A)$ being dense or not is irrelevant. Henceforth we shall keep the notation A^α for this definition.

On the other hand, we shall introduce the operator

$$T(\varepsilon, n) = A^\alpha - \sum_{p=0}^{n}(-1)^p \varepsilon^p \binom{\alpha}{p}[(A+\varepsilon)^{-1}]^p (A+\varepsilon)^\alpha$$

$\left(\text{where } \binom{\alpha}{p} = \frac{1}{p!}\alpha(\alpha-1)\cdots(\alpha-p+1)\right)$ about which there is no bibliographical background. At most, some authors (see [**9**] and [**18**]) have considered estimates and integral representations of the difference

$$J_A^\alpha - J_{A+\varepsilon}^\alpha$$

(where the subindex indicates the base operator).

This paper is divided as follows:

In Section 2, we give the definition of the A^α fractional power, we prove that the domain $D[(A+\varepsilon)^\alpha]$ does not depend on $\varepsilon \geqq 0$ and the fundamental property

$$[A(A+\varepsilon)^{-1}]^\alpha = A^\alpha[(A+\varepsilon)^{-1}]^\alpha.$$

As a consequence A^α turns out to be closed.

The additivity ($A^{\alpha+\beta} = A^\alpha A^\beta$) and the connection between A^α and $\overline{J^\alpha}$ are discussed in Section 3. In particular we offer a very easy way to demonstrate property (1.3).

Section 4 is devoted to the resolvent of A^α, the spectral mapping theorem and the multiplicativity.

We establish two integral representations of the $T(\varepsilon, n)$ operator and we examine its range in Section 5. As an application we offer a counterexample for the separation of the domains $D(A^\alpha)$ and $D(A^\beta)$ even if $\mathrm{Re}\,\alpha = \mathrm{Re}\,\beta$.

Throughout this paper we assume that A is a non-negative operator in a Banach space X and α is a complex number with $\mathrm{Re}\,\alpha > 0$.

2. Construction of fractional powers.

DEFINITION 2.1. If A is bounded we define $A^\alpha = J^\alpha$.

If A is unbounded and $0 \in \rho(A)$, we define $A^\alpha = [(A^{-1})^\alpha]^{-1}$.

If A is unbounded and $0 \in \sigma(A)$, we define A^α by the formula

$$A^\alpha \varphi = \lim_{\varepsilon \to 0+} (A+\varepsilon)^\alpha \varphi \tag{2.1}$$

where the domain of A^α is the set of all $\varphi \in X$ which the limit (2.1) exists.

REMARK 2.1. It's obvious that $A^1 = A$.

2.2. If A is bounded and $\rho(A)$ contains $]-\infty, 0]$, A^α is a bounded operator which can be obtained from functional calculus for bounded operators (see W. Rudin [**20**]).

2.3. If A is bounded and non-negative, it is easy to prove that A^α is the limit, in the uniform operator topology of $(A+\varepsilon)^{\alpha}$ as $\varepsilon\to 0+$, which we shall denote henceforth by

$$A^{\alpha}=\|\ \|-\lim_{\varepsilon\to 0+}(A+\varepsilon)^{\alpha}. \tag{2.2}$$

2.4. If A is unbounded and $0\in\rho(A)$, the adopted definition can be found in [**16**], and it is known that A^{α} is an extension of J^{α}.

2.5. If A is unbounded and $0\in\sigma(A)$, then for each $\varepsilon>0$, $A+\varepsilon$ is non-negative and $0\in\rho(A+\varepsilon)$, therefore applying the latter remark and making use the dominated convergence theorem, it follows that A^{α} is an extension of J^{α}.

THEOREM 2.1. *Let* $\mathrm{Re}\,\alpha>0$ *and* $\varepsilon>0$. *Then we have that* $A(A+\varepsilon)^{-1}$ *is non-negative and*

$$D[(A+\varepsilon)^{\alpha}]=D(A^{\alpha}), \tag{2.3}$$

$$[A(A+\varepsilon)^{-1}]^{\alpha}=A^{\alpha}[(A+\varepsilon)^{-1}]^{\alpha}. \tag{2.4}$$

PROOF. The first statement is a consequence of the following identity

$$A(A+\varepsilon)^{-1}+\eta=(\eta+1)\Big(A+\frac{\varepsilon\eta}{\eta+1}\Big)(A+\varepsilon)^{-1} \tag{2.5}$$

(valid for each $\eta>0$).

In order to prove (2.3) and (2.4) we split the argument into four steps.

I. If A is bounded and $0\in\rho(A)$, (2.4) is a consequence of functional calculus.

II. Let A be bounded and $0\in\sigma(A)$. Since $(A+\varepsilon\eta/(\eta+1))$ satisfies the assumption of I, for each $\eta>0$, it is enough to take powers of exponent α on both sides in (2.5) and then to apply (2.4) replacing A by $A+\varepsilon\eta/(\eta+1)$. Taking $\|\ \|-\lim_{\eta\to 0+}$ we obtain (2.4).

III. Let A be unbounded and $0\in\rho(A)$. Through the identity

$$(A+\varepsilon)^{-1}=\varepsilon^{-1}A^{-1}(A^{-1}+\varepsilon^{-1})^{-1}$$

and using that A^{-1} satisfies the assumption of II we have

$$[(A+\varepsilon)^{-1}]^{\alpha}=\varepsilon^{-\alpha}(A^{-1})^{\alpha}[(A^{-1}+\varepsilon^{-1})^{-1}]^{\alpha}. \tag{2.6}$$

Hence it follows that the ranges of $(A^{-1})^{\alpha}$ and $[(A+\varepsilon)^{-1}]^{\alpha}$ are equal, therefore (2.3) is valid.

On the other hand, operating A^{α} on both sides in (2.6) we obtain (2.4).

IV. The general case when A is unbounded and $0\in\sigma(A)$ can be proved in the same way as II.

COROLLARY 2.1. *A^{α} is a closed linear operator.*

PROOF. In view of (2.4) and the commutativity of A^{α} with $[(A+\varepsilon)^{-1}]^{\alpha}$, for any $\varepsilon>0$, we obtain:

$$A^{\alpha}=(A+\varepsilon)^{\alpha}[A(A+\varepsilon)^{-1}]^{\alpha} \tag{2.7}$$

and taking into account that $[A(A+\varepsilon)^{-1}]^{\alpha}$ is bounded and $(A+\varepsilon)^{\alpha}$ is closed (since it is the inverse of a bounded operator) we conclude the closedness of A^{α}.

REMARK 2.6. If A is unbounded and $0\in\rho(A)$, identity (2.7) implies that

$$(A+\varepsilon)^{\alpha}=(\varepsilon A^{-1}+1)^{\alpha}A^{\alpha}.$$

Moreover, from the integral expressions (1.2) through a simple calculation we obtain

$$1=\| \quad \|-\lim_{\varepsilon\to 0+}(\varepsilon A^{-1}+1)^{\alpha}$$

and therefore (for each $\varphi\in D(A^{\alpha})=D[(A+\varepsilon)^{\alpha}]$)

$$A^{\alpha}\varphi=\lim_{\varepsilon\to 0+}(A+\varepsilon)^{\alpha}\varphi$$

thus the definition given in (2.1) for unbounded operators with $0\in\sigma(A)$ is valid for any unbounded operator.

3. Additivity.

When A is bounded and $0\in\rho(A)$ the additivity is a consequence of functional calculus and if $0\in\sigma(A)$ it follows immediately using (2.2). Thus, according to Definition 2.1, it is verified that

$$A^{\alpha}A^{\beta}=A^{\beta}A^{\alpha}=A^{\alpha+\beta}$$

for all α, β such that $\operatorname{Re}\alpha$, $\operatorname{Re}\beta>0$ if A is bounded or if A is unbounded with $0\in\rho(A)$.

Therefore there remains to prove the additivity when A is unbounded and $0\in\sigma(A)$. In order to prove it first we give the following:

LEMMA 3.1. *Let* $\operatorname{Re}\gamma>0$, $\operatorname{Re}\delta>0$ *and* $\varepsilon>0$. *Then we have*

$$\varphi\in D(A^{\gamma}) \text{ if and only if } [A(A+\varepsilon)^{-1}]^{\delta}\varphi\in D(A^{\gamma})$$

and

$$(A+\varepsilon)^{\gamma}[A(A+\varepsilon)^{-1}]^{\delta}\varphi=[A(A+\varepsilon)^{-1}]^{\delta}(A+\varepsilon)^{\gamma}\varphi. \tag{3.1}$$

PROOF. Since $[(A+\varepsilon)^{-1}+\lambda]^{-1}$ and $[A(A+\varepsilon)^{-1}+\mu]^{-1}$ commute for λ, $\mu>0$, it is easily shown that the bounded operators $[A(A+\varepsilon)^{-1}]^{\delta}$ and $[(A+\varepsilon)^{-1}]^{\gamma}$ commute. Hence, it follows that the condition is necessary and (3.1) holds.

Conversely, if $[A(A+\varepsilon)^{-1}]^{\delta}\varphi\in D(A^{\gamma})$, taking an integer n such that $n>\operatorname{Re}\delta$ and using the additivity of the powers of $A(A+\varepsilon)^{-1}$ and the first part of this lemma, we conclude

$$[A(A+\varepsilon)^{-1}]^n\varphi \in D(A^\gamma).$$

Writing

$$\psi=[A(A+\varepsilon)^{-1}]^{n-1}\varphi$$

we have

$$\psi-\varepsilon(A+\varepsilon)^{-1}\psi \in D(A^\gamma).$$

Therefore

$$\prod_{p=0}^{m-1}(1+\varepsilon^{2^p}[(A+\varepsilon)^{-1}]^{2^p})(\psi-\varepsilon(A+\varepsilon)^{-1}\psi) \in D(A^\gamma)$$

for $m=1, 2, \cdots$. That is:

$$\psi-\varepsilon^{2^m}[(A+\varepsilon)^{-1}]^{2^m}\psi \in D(A^\gamma).$$

Hence taking m such that $2^m>\mathrm{Re}\,\gamma$ we have $\psi\in D(A^\gamma)$ and by recurrence we obtain that

$$[A(A+\varepsilon)^{-1}]^{n-2}\varphi, \cdots, A(A+\varepsilon)^{-1}\varphi, \varphi$$

belong to $D(A^\gamma)$.

THEOREM 3.1. *Let A be unbounded and $0\in\sigma(A)$ and let* $\mathrm{Re}\,\alpha>0$, $\mathrm{Re}\,\beta>0$. *Then*

$$A^{\alpha+\beta} = A^\beta A^\alpha = A^\alpha A^\beta.$$

PROOF. Let $\varphi\in D(A^{\alpha+\beta})$. From (2.3) we know that

$$\varphi \in D[(A+\varepsilon)^{\alpha+\beta}] \qquad (\text{for any } \varepsilon>0)$$

and since

$$(A+\varepsilon)^{\alpha+\beta}=(A+\varepsilon)^\beta(A+\varepsilon)^\alpha$$

we have, by (2.3) and (2.4), that $\varphi\in D(A^\alpha)$ and

$$A^\alpha\varphi = [A(A+\varepsilon)^{-1}]^\alpha(A+\varepsilon)^\alpha\varphi.$$

Hence, using Lemma 3.1, $A^\alpha\varphi\in D[(A+\varepsilon)^\beta]$ and

$$(A+\varepsilon)^\beta A^\alpha\varphi = [A(A+\varepsilon)^{-1}]^\alpha(A+\varepsilon)^{\alpha+\beta}\varphi.$$

Applying $[A(A+\varepsilon)^{-1}]^\beta$ to both sides we obtain

$$A^\beta A^\alpha\varphi = A^{\alpha+\beta}\varphi.$$

Conversely, let $\varphi\in D(A^\alpha)$ such that $A^\alpha\varphi\in D(A^\beta)$. In view of (2.4) and the first part of this proof we have

$$\begin{aligned}A^\beta A^\alpha[(A+\varepsilon)^{-1}]^{\alpha+\beta}\varphi &= A^{\alpha+\beta}[(A+\varepsilon)^{-1}]^{\alpha+\beta}\varphi \\ &= [A(A+\varepsilon)^{-1}]^{\alpha+\beta}\varphi.\end{aligned} \tag{3.2}$$

On the left hand side of this equality the operator $[(A+\varepsilon)^{-1}]^{\alpha+\beta}$ can be placed on the left, since it commutes with the powers of A, as it is easily followed from Definition 2.1.

Thus, from (2.4), we find that

$$[A(A+\varepsilon)^{-1}]^{\alpha+\beta}\varphi \in D[(A+\varepsilon)^{\alpha+\beta}]$$

and

$$A^{\beta}A^{\alpha}\varphi = (A+\varepsilon)^{\alpha+\beta}[A(A+\varepsilon)^{-1}]^{\alpha+\beta}\varphi .$$

Hence, using Lemma 3.1 and Theorem 2.1 we conclude that $\varphi \in D(A^{\alpha+\beta})$ and

$$A^{\beta}A^{\alpha}\varphi = A^{\alpha+\beta}\varphi .$$

In the following theorem we give a very simple proof of the well-known property (1.3):

$$\overline{J^{\alpha}}\,\overline{J^{\beta}} = \overline{J^{\alpha+\beta}}$$

(see [11], [8] and [18]).

THEOREM 3.2. *Let* $\mathrm{Re}\,\alpha>0$ *and* $\mathrm{Re}\beta>0$. *Then*

(i) $\varphi \in D(\overline{J^{\alpha}})$ *if and only if* $\varphi \in D(A^{\alpha})$ *and* $A^{\alpha}\varphi \in \overline{D(A)}$.

(ii) $\overline{J^{\alpha}}=A^{\alpha}$ *if and only if* $\overline{D(A)}=X$.

(iii) $\overline{J^{\alpha}}\,\overline{J^{\beta}}=\overline{J^{\alpha+\beta}}$.

PROOF. A^{α} is an extension of $\overline{J^{\alpha}}$ in view of Corollary 2.1 and Remark 2.5. The necessary condition of statement (i) is evident since $\mathrm{Ran}\overline{J^{\alpha}} \subset \overline{D(A)}$.

Conversely, let $\varphi \in D(A^{\alpha})$ such that $A^{\alpha}\varphi \in \overline{D(A)}$ and let us take an integer n such that $n>\mathrm{Re}\,\alpha$. We know

$$\lim_{\lambda \to +\infty}[\lambda(\lambda+A)^{-1}]^{n}\varphi = \varphi$$

and

$$\lim_{\lambda \to +\infty} J^{\alpha}[\lambda(\lambda+A)^{-1}]^{n}\varphi = \lim_{\lambda \to +\infty}[\lambda(\lambda+A)^{-1}]^{n}A^{\alpha}\varphi = A^{\alpha}\varphi$$

since A is non-negative and $A^{\alpha}\varphi$ belongs to $\overline{D(A)}$. Therefore $\varphi \in D(\overline{J^{\alpha}})$ and the statement (i) is proved.

If $\overline{D(A)}=X$, it is obvious by (i) that $A^{\alpha}=\overline{J^{\alpha}}$. Conversely, if $D(A)\neq X$ and n is an integer such that $n>\mathrm{Re}\,\alpha$, there is some $\varphi \in D(A^{n})$ such that $A^{n}\varphi \notin \overline{D(A)}$. Consequently from $A^{\alpha}A^{n-\alpha}\varphi = A^{n}\varphi$ we obtain that A^{α} is a strict extension of $\overline{J^{\alpha}}$.

Statement (iii) follows at once from (i) and Theorem 3.1.

4. Spectral theory. Multiplicativity.

Balakrishnan on [2] proves the spectral mapping theorem for $\overline{J^{\alpha}}$, based on the integral representation of the resolvent of $\overline{J^{\alpha}}$ with a suitable choice of the exponent.

In this section we shall give a similar theorem for the resolvent of A^{α}.

Theorem 4.1. *Let $0<|\beta|^2<\mathrm{Re}\beta$ and the non-empty set*

$$U_\beta=\{\mu\in \boldsymbol{C}\sim\{0\}:\ \mu+\lambda^\beta e^{i\theta\beta}\neq 0,\ \lambda>0,\ -\pi\leqq\theta\leqq\pi\}.$$

We define the function f_β: $U_\beta\times[0, +\infty[\longmapsto \boldsymbol{C}$ as

$$f_\beta(\mu, \lambda)=\frac{1}{2\pi i}\left(\frac{1}{\mu+\lambda^\beta e^{-i\pi\beta}}-\frac{1}{\mu+\lambda^\beta e^{i\pi\beta}}\right).$$

Then, for each $\mu\in U_\beta$, we have that $-\mu\in\rho(A^\beta)$ and

$$(\mu+A^\beta)^{-1}=\int_0^{+\infty}f_\beta(\mu, \lambda)(\lambda+A)^{-1}d\lambda. \tag{4.1}$$

Proof. Let us denote the right hand side of (4.1) by $R(\mu)$. It is clear that the integral converges in the uniform operator topology and therefore $R(\mu)$ is a bounded operator.

The proof will be given in four steps.

I. If A is bounded and $0\in\rho(A)$, (4.1) follows easily from functional calculus for bounded operators (see [**20**]).

II. Let A be bounded and $0\in\sigma(A)$. Then $A+\varepsilon$ satisfies the assumption of I for any $\varepsilon>0$. Therefore $-\mu\in\rho[(A+\varepsilon)^\beta]$ and (4.1) holds for the operator $A+\varepsilon$. An easy calculation gives

$$\|R(\mu)-[\mu+(A+\varepsilon)^\beta]^{-1}\|\leqq C\left(\frac{2M}{\mathrm{Re}\beta}+\frac{M^2}{1-\mathrm{Re}\beta}\right)\varepsilon^{\mathrm{Re}\beta}$$

(C not depending on ε). Hence we conclude that

$$R(\mu)(\mu+A^\beta)=(\mu+A^\beta)R(\mu)=1.$$

III. Let A be unbounded and $0\in\rho(A)$. Given $\mu\in U_\beta$ then $\mu^{-1}\in U_\beta$ and taking into account that A^{-1} is an operator of the type considered in II we get

$$(\mu^{-1}+(A^{-1})^\beta)^{-1}=\int_0^{+\infty}f_\beta(\mu^{-1}, \lambda)(\lambda+A^{-1})^{-1}d\lambda$$

replacing this expression in the identity

$$(\mu+A^\beta)^{-1}=\mu^{-1}[1-\mu^{-1}[\mu^{-1}+(A^{-1})^\beta]^{-1}]$$

and through a simple transformation in this integral we obtain (4.1).

IV. Let A be unbounded and $0\in\sigma(A)$. Taking into account that for all $\varepsilon>0$

$$[\mu+(A+\varepsilon)^\beta]^{-1}=\int_0^{+\infty}f_\beta(\mu, \lambda)(\lambda+A+\varepsilon)^{-1}d\lambda$$

just like in II, that

$$\lim_{\varepsilon\to 0+}\|[\mu+(A+\varepsilon)^\beta]^{-1}-R(\mu)\|=0$$

which yields that for all $\varphi\in D(A^\beta)$

$$R(\mu)(\mu+A^{\beta})\varphi=\varphi.$$

On the other hand, given $\varphi\in X$ and $\varepsilon>0$, the element

$$R(\mu)\varphi-[\mu+(A+\varepsilon)^{\beta}]^{-1}\varphi=\varepsilon\int_{0}^{+\infty}f_{\beta}(\mu,\lambda)(\lambda+A+\varepsilon)^{-1}(\lambda+A)^{-1}\varphi d\lambda$$

belongs to $D(A)$ and furthermore

$$\lim_{\varepsilon\to 0+}A[R(\mu)\varphi-(\mu+(A+\varepsilon)^{\beta})^{-1}\varphi]=0$$

which yields that

$$\lim_{\varepsilon\to 0+}[\mu+(A+\varepsilon)^{\beta}][R(\mu)\varphi-[\mu+(A+\varepsilon)^{\beta}]^{-1}\varphi]=0$$

and consequently $R(\mu)\varphi\in D(A^{\beta})$ and

$$(\mu+A^{\beta})R(\mu)\varphi=\varphi.$$

COROLLARY 4.1. *Let* $\operatorname{Re}\alpha>0$. *Then*

$$\sigma(A^{\alpha})=\{z^{\alpha}:\ z\in\sigma(A)\}. \tag{4.2}$$

PROOF. This result can be proved from Theorem 4.1 in a similar way to the corresponding well known statement for $\overline{J^{\alpha}}$ when $\overline{D(A)}=X$, given by Balakrishnan in [2].

THEOREM 4.2 (Multiplicativity). *Let* $0<\alpha<1$. *Then* A^{α} *is a non-negative operator and for all* β *such that* $\operatorname{Re}\beta>0$ *we have*

$$(A^{\alpha})^{\beta}=A^{\alpha\beta}. \tag{4.3}$$

PROOF. As a consequence from Theorem 4.1 we can assert that $\rho(A^{\alpha})\supset]-\infty, 0[$ and for all $\mu>0$

$$(\mu+A^{\alpha})^{-1}=\int_{0}^{+\infty}f_{\alpha}(\mu,\lambda)(\lambda+A)^{-1}d\lambda.$$

Hence it is easily obtained that

$$\|\mu(\mu+A^{\alpha})^{-1}\|\leqq M,\qquad 0\leqq\mu<+\infty$$

(where $M=\sup_{\mu>0}\|\mu(\mu+A)^{-1}\|$).

On the other hand, since $(A^{\alpha})^{n}=A^{\alpha n}$ for positive integers, by additivity, it is enough to show (4.3) for $0<\operatorname{Re}\beta=s<1$. In order to prove it we shall consider four cases.

I. When A is bounded and $0\in\rho(A)$, (4.3) is an immediate consequence from functional calculus for bounded operators.

II. If A is bounded and $0\in\sigma(A)$, we obtain, for all $\varepsilon>0$, the well known estimate

$$\|[(A+\varepsilon)^{\alpha}]^{\beta}-(A^{\alpha})^{\beta}\| \leqq \frac{|\sin\beta\pi|}{\pi}\left[\frac{2M(M+1)}{s}+\frac{M^2}{1-s}\right]\|A^{\alpha}-(A+\varepsilon)^{\alpha}\|$$

which implies

$$(A^{\alpha})^{\beta} = \lim_{\varepsilon\to 0+}[(A+\varepsilon)^{\alpha}]^{\beta}$$

and consequently

$$(A^{\alpha})^{\beta} = A^{\alpha\beta}.$$

III. When A is unbounded and $0\in\rho(A)$, (4.3) is evident, since A^{-1} is an operator of the type considered in II and therefore A^{-1} fulfils it.

IV. If A is unbounded and $0\in\sigma(A)$, in the same way as II we obtain

$$(4.4) \qquad (A^{\alpha})^{\beta}\varphi = A^{\alpha\beta}\varphi \quad \text{for all} \quad \varphi\in D(A^{\alpha}).$$

Let $\phi\in D(A^{\alpha\beta})$ and $\mu>0$ and let us denote $R(\mu)=(\mu+A^{\alpha})^{-1}$. We have

$$(4.5) \qquad (A^{\alpha})^{\beta}R(\mu)\phi = A^{\alpha\beta}R(\mu)\phi = R(\mu)A^{\alpha\beta}\phi.$$

Since $A^{\alpha}=(A^{\alpha})^{\beta}(A^{\alpha})^{1-\beta}$ we obtain that

$$A^{\alpha}R(\mu)\phi \in D[(A^{\alpha})^{\beta}].$$

Hence we conclude that $\phi\in D[(A^{\alpha})^{\beta}]$ and commuting $R(\mu)$ with $(A^{\alpha})^{\beta}$ in (4.5) we get

$$(A^{\alpha})^{\beta}\phi = A^{\alpha\beta}\phi.$$

Conversely, let $\phi\in D[(A^{\alpha})^{\beta}]$. From (4.4) we have

$$A^{\alpha\beta}R(\mu)\phi = (A^{\alpha})^{\beta}R(\mu)\phi = R(\mu)(A^{\alpha})^{\beta}\phi.$$

Since $A^{\alpha}=A^{\alpha\beta}A^{\alpha-\alpha\beta}$ we obtain that $A^{\alpha}R(\mu)\phi\in D(A^{\alpha\beta})$ which implies that $\phi\in D(A^{\alpha\beta})$.

5. A Taylor rest.

In this section we shall give two integral representations of the vector

$$(5.1) \qquad T(\varepsilon, n)\varphi = A^{\alpha}\varphi - \sum_{p=0}^{n}\binom{\alpha}{p}(-1)^{p}\varepsilon^{p}(A+\varepsilon)^{\alpha-p}\varphi$$

where $\mathrm{Re}\,\alpha>0$, $\varepsilon>0$, n is a positive integer, $\varphi\in D(A^{\alpha})$ and by $(A+\varepsilon)^{\alpha-p}$ we denote the operator $[(A+\varepsilon)^{-1}]^{p}(A+\varepsilon)^{\alpha}$.

The first one of them (Theorem 5.1) is a consequence of Taylor's formula and it is less interesting than the second one (Theorem 5.2) due to the fact that the integrand contains a fractional power, dependent on the variable of integration. As a consequence of these results we can study the range of the operator $T(\varepsilon, n)$.

In order to prove the first integral representation we need the following lemma.

LEMMA 5.1. *Let* $\mathrm{Re}\,\alpha>0$ *and* $\varphi\in D(A^{\alpha})$. *Then the function*

$$\varepsilon\longmapsto(A+\varepsilon)^{\alpha}\varphi$$

defined from $]0, +\infty[$ *into* X *is continuous.*

PROOF. When $0<\mathrm{Re}\,\alpha<1$, the continuity is a consequence of the estimate (valid for all $\varphi\in D(A^{\alpha})$)

$$\|(A+\varepsilon)^{\alpha}\varphi-(A+\varepsilon')^{\alpha}\varphi\|\leqq K(\alpha, M)\frac{|\varepsilon-\varepsilon'|^{\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha(1-\mathrm{Re}\,\alpha)}\|\varphi\|$$

easily obtained from the integral formula (1.2) (where $K(\alpha, M)$ is constant in relation to ε and ε').

When $0<\mathrm{Re}\,\alpha<2$ the continuity is a consequence of the identity:

$$\begin{aligned}(A+\varepsilon)^{\alpha}\varphi-(A+\varepsilon')^{\alpha}\varphi &= (A+\varepsilon)^{\alpha/2}(A+\varepsilon)^{\alpha/2}\varphi-(A+\varepsilon')^{\alpha/2}(A+\varepsilon)^{\alpha/2}\varphi\\ &\quad+(A+\varepsilon)^{\alpha/2}(A+\varepsilon')^{\alpha/2}\varphi-(A+\varepsilon')^{\alpha/2}(A+\varepsilon')^{\alpha/2}\varphi\end{aligned}$$

and the latest estimate. Finally, case $\mathrm{Re}\,\alpha\geqq 2$ comes down with no difficulty to the former two.

THEOREM 5.1. *Let* n *be positive integer and* $\mathrm{Re}\,\alpha>0$, $\varepsilon>0$ *and* $\varphi\in D(A^{\alpha})$. *Then, the function* $t\mapsto t^{n}(A+\varepsilon t)^{\alpha-n-1}\varphi$ *is integrable over* $[0, 1]$ *and*

$$T(\varepsilon, n)\varphi=(-1)^{n+1}\varepsilon^{n+1}(\alpha-n)\binom{\alpha}{n}\int_{0}^{1}t^{n}(A+\varepsilon t)^{\alpha-n-1}\varphi dt. \tag{5.2}$$

PROOF. Let $q>\mathrm{Re}\,\alpha$ with q an integer and $\psi\in D(A^{q})$. Differentiating under the sign of integration in formulae (1.2) (refering to the operator $A+\varepsilon$), it is easy to obtain that the function $\varepsilon\mapsto(A+\varepsilon)^{\alpha}\psi$ is differentiable for $\varepsilon>0$ and

$$\frac{d}{d\varepsilon}(A+\varepsilon)^{\alpha}\psi=\alpha(A+\varepsilon)^{\alpha-1}\psi.$$

By induction on n, it follows that this function is infinitely differentiable and

$$\frac{d^{n}}{d\varepsilon^{n}}(A+\varepsilon)^{\alpha}\psi=\alpha(\alpha-1)\cdot\cdots\cdot(\alpha-n+1)(A+\varepsilon)^{\alpha-n}\psi.$$

Therefore using Taylor's formula we obtain

$$\begin{aligned}(A+\eta)^{\alpha}\psi &= \sum_{p=0}^{n}\binom{\alpha}{p}(\eta-\varepsilon)^{p}(A+\varepsilon)^{\alpha-p}\psi\\ &\quad+(\alpha-n)\binom{\alpha}{n}(\eta-\varepsilon)^{n+1}\int_{0}^{1}(1-t)^{n}[A+\varepsilon+t(\eta-\varepsilon)]^{\alpha-n-1}\psi dt\end{aligned} \tag{5.3}$$

(for $\varepsilon>0$ and $\eta>0$).

Applying (5.3) to vector $\psi=(A+\mu)^{-q}\varphi$ (where $\mu\in\rho(-A)$) and commuting

$(A+\mu)^{-1}$ with operators $(A+\varepsilon)^{-1}$, $(A+\varepsilon)^{\alpha}$ and $(A+\eta)^{\alpha}$ we obtain

$$\begin{aligned}(5.4)\qquad &(A+\mu)^{-q}\Big[(A+\eta)^{\alpha}\varphi-\sum_{p=0}^{n}\binom{\alpha}{p}(\eta-\varepsilon)^{p}(A+\varepsilon)^{\alpha-p}\varphi\Big]\\ &=(\alpha-n)\binom{\alpha}{n}(\eta-\varepsilon)^{n+1}\int_0^1(1-t)^n[A+\varepsilon+t(\eta-\varepsilon)]^{\alpha-n-1}(A+\mu)^{-q}\varphi dt.\end{aligned}$$

Lemma 5.1 allows us to commute operator $(A+\mu)^{-q}$ with the integral sign on the right hand side and hence we obtain (5.3) with φ replacing ψ and passing to the limit as $\eta\to 0+$ we get

$$(5.5)\quad T(\varepsilon,\, n)\varphi=(-1)^{n+1}\varepsilon^{n+1}(\alpha-n)\binom{\alpha}{n}\lim_{\eta\to 0+}\int_0^1(1-t)^n[A+\varepsilon+t(\eta-\varepsilon)]^{\alpha-n-1}\varphi dt.$$

We now show that $(1-t)^n\|[A+\varepsilon+t(\eta-\varepsilon)]^{\alpha-n-1}\varphi\|$ is majorized by a function of t (independent of η) integrable. In order to prove it we shall consider three cases:

If $\mathrm{Re}\,\alpha-n-1<0$, then the norm of the bounded operator

$$[A+\varepsilon+t(\eta-\varepsilon)]^{\alpha-n-1}=\{[A+\varepsilon+t(\eta-\varepsilon)]^{-1}\}^{n+1-\alpha}$$

is majorized by

$$K\varepsilon^{\mathrm{Re}\alpha-n-1}(1-t)^{\mathrm{Re}\alpha-n-1}$$

(where K is a constant), as it can be deduced from the integral expression of the power

$$\{(A+\varepsilon+t(\eta-\varepsilon))^{-1}\}^{1-\alpha/(n+1)}.$$

If $\mathrm{Re}\,\alpha-n-1>0$, the integrand is continuous as function of $(t,\, \eta,\, \varepsilon)$.

If $\mathrm{Re}\,\alpha-n-1=0$, we shall decompose

$$[A+\varepsilon+t(\eta-\varepsilon)]^{\alpha-n-1}\varphi=[A+\varepsilon+t(\eta-\varepsilon)]^{\alpha-(n+1+1/2)}[A+\varepsilon+t(\eta-\varepsilon)]^{1/2}\varphi$$

where $\|[A+\varepsilon+t(\eta-\varepsilon)]^{\alpha-(n+1+1/2)}\|$ can be majorized as it has been done when $\mathrm{Re}\,\alpha-n-1<0$ and the norm of $[A+\varepsilon+t(\eta-\varepsilon)]^{1/2}\varphi$ is, simply, the norm of a continuous function of $(t,\, \eta,\, \varepsilon)$. Therefore we can apply the dominated convergence theorem in (5.5) and we obtain (5.2).

Theorem 5.2. *Let n be a positive integer such that $n+1>\mathrm{Re}\alpha$. Then for each $\varphi\in D(A^{\alpha})$ and $\varepsilon>0$ we have:*

$$(5.6)\qquad T(\varepsilon,\, n)\varphi=-\varepsilon^{n+1}\frac{\sin\alpha\pi}{\pi}\Big[\int_0^{+\infty}\frac{\lambda^{\alpha}}{(1+\lambda)^{n+2}}\Big(A+\frac{\varepsilon\lambda}{1+\lambda}\Big)^{-1}d\lambda\Big](A+\varepsilon)^{\alpha-n}\varphi.$$

Proof. First we shall prove the identity:

$$\begin{aligned}(5.7)\qquad [A(A+\varepsilon)^{-1}]^{\alpha}&=\sum_{p=0}^{n}(-1)^p\varepsilon^p\binom{\alpha}{p}(A+\varepsilon)^{-p}\\ &\quad-\varepsilon^{n+1}\frac{\sin\alpha\pi}{\pi}\Big(\int_0^{+\infty}\frac{\lambda^{\alpha}}{(1+\lambda)^{n+2}}\Big(A+\frac{\varepsilon\lambda}{\lambda+1}\Big)^{-1}d\lambda\Big)(A+\varepsilon)^{-n}.\end{aligned}$$

Applying, then, both sides to $(A+\varepsilon)^\alpha\varphi$ and using (2.4) we shall get (5.6).

Let's start considering $0<\mathrm{Re}\,\alpha<2$. We obtain due to (1.2)

$$[A(A+\varepsilon)^{-1}]^\alpha = \frac{\sin\alpha\pi}{\pi}\int_0^{+\infty}\lambda^{\alpha-1}\Big[(\lambda+A(A+\varepsilon)^{-1})^{-1}-\frac{\lambda}{1+\lambda^2}\Big]A(A+\varepsilon)^{-1}d\lambda$$

$$+\sin\Big(\alpha\frac{\pi}{2}\Big)A(A+\varepsilon)^{-1}$$

$$=\frac{\sin\alpha\pi}{\pi}\int_0^{+\infty}\frac{\lambda^{\alpha-1}}{(1+\lambda)(1+\lambda^2)}[(1-\lambda)A+\varepsilon]A\Big(A+\frac{\varepsilon\lambda}{1+\lambda}\Big)^{-1}(A+\varepsilon)^{-1}d\lambda$$

$$+\sin\Big(\alpha\frac{\pi}{2}\Big)A(A+\varepsilon)^{-1}$$

after making some operations in the integrand.

Using the identities

$$[(1-\lambda)A+\varepsilon]A\Big(A+\frac{\varepsilon\lambda}{1+\lambda}\Big)^{-1} = (1-\lambda)A+\frac{\lambda^2+1}{\lambda+1}\varepsilon-\frac{(\lambda^2+1)\lambda}{(\lambda+1)^2}\varepsilon^2\Big(A+\frac{\varepsilon\lambda}{1+\lambda}\Big)^{-1}$$

$$\frac{\sin\alpha\pi}{\pi}\int_0^{+\infty}\frac{\lambda^{\alpha-1}(1-\lambda)}{(1+\lambda)(1+\lambda^2)}d\lambda = 1-\sin\Big(\alpha\frac{\pi}{2}\Big)$$

(the integrand is easily calculated by means of the residue theorem). We conclude:

$$[A(A+\varepsilon)^{-1}]^\alpha = 1-\varepsilon\alpha(A+\varepsilon)^{-1}-\varepsilon^2\frac{\sin\alpha\pi}{\pi}\Big[\int_0^{+\infty}\frac{\lambda^\alpha}{(1+\lambda)^3}\Big(A+\frac{\varepsilon\lambda}{1+\lambda}\Big)^{-1}d\lambda\Big](A+\varepsilon)^{-1}.$$

At the same time, if we develop the last integral in the following way:

$$\int_0^{+\infty}\frac{\lambda^\alpha}{(1+\lambda)^3}\Big(A+\frac{\varepsilon\lambda}{\lambda+1}\Big)^{-1}d\lambda = \Big[\int_0^{+\infty}\frac{\lambda^\alpha}{(1+\lambda)^3}(A+\varepsilon)\Big(A+\frac{\varepsilon\lambda}{1+\lambda}\Big)^{-1}d\lambda\Big](A+\varepsilon)^{-1}$$

we obtain, through easy calculations that

$$[A(A+\varepsilon)^{-1}]^\alpha = I-\varepsilon\alpha(A+\varepsilon)^{-1}+\frac{\varepsilon^2}{2!}\alpha(\alpha-1)(A+\varepsilon)^{-2}$$

$$-\varepsilon^3\frac{\sin\alpha\pi}{\pi}\Big[\int_0^{+\infty}\frac{\lambda^\alpha}{(1+\lambda)^4}\Big(A+\frac{\lambda\varepsilon}{1+\lambda}\Big)^{-1}d\lambda\Big](A+\varepsilon)^{-2}$$

after taking into account (by the residue theorem) that

$$\frac{\sin\alpha\pi}{\pi}\int_0^{+\infty}\frac{\lambda^\alpha}{(1+\lambda)^3}d\lambda = \frac{(-1)^3}{2!}\alpha(\alpha-1).$$

Iterating this process we obtain (5.7). Now let $0<\mathrm{Re}\,\alpha<n+1$ with $n>1$, the identity (5.7) can be easily proved by induction on p such that $p\leqq\mathrm{Re}\,\alpha<p+1$.

COROLLARY 5.1. *Let $n+1>\mathrm{Re}\,\alpha$ and $\alpha\neq 1, 2, \cdots, n$. Let $\varphi\in D(A^\alpha)$ and $\varepsilon>0$. Then, if $0\leqq h<1$, the vector $T(\varepsilon, n)\varphi$ belongs to $D(A^{n+1+h})$ if and only if, $\varphi\in D(A^{\alpha+h})$.*

Proof. Since A is a non-negative operator, the integral

$$\int_0^{+\infty} \frac{\lambda^\alpha}{(1+\lambda)^{n+2}} A\left(A+\frac{\varepsilon\lambda}{1+\lambda}\right)^{-1} d\lambda$$

is convergent and (5.6) implies that $T(\varepsilon, n)\varphi \in D(A^{n+1})$.

In the same way, from (5.6) it follows that

$$A(A+\varepsilon)^n T(\varepsilon, n)\varphi = K(A+\varepsilon)^\alpha \varphi + \psi$$

where K is a non-zero complex number, depending on (ε, α, n), and $\psi \in D(A)$.

As $D(A) \subset D(A^h)$ we conclude that $A(A+\varepsilon)^n T(\varepsilon, n)\varphi$ belongs to $D(A^h)$ if and only if $(A+\varepsilon)^\alpha \varphi \in D(A^h)$, namely if and only if $\varphi \in D(A^{\alpha+h})$.

Application to the separation of domains of powers.

From Corollary 5.1 it easily follows that if $0<\operatorname{Re}\alpha<2$ and $\varphi \in D(A^2)$, then

$$(A+\varepsilon)^\alpha \varphi - A^\alpha \varphi - \varepsilon(A+\varepsilon)^{\alpha-1}\varphi \in D(A^{3-\alpha})$$

(vector that will be denoted as $F_\varepsilon \varphi$).

In a suitable functional space, this fact will allow us to find for each $\alpha \neq 1$ with $\operatorname{Re}\alpha=1$, an element $\varphi_\alpha \in D(A^2)$ so that $F_\varepsilon \varphi_\alpha \in D(A^{3-\alpha})$ and $F_\varepsilon \varphi_\alpha \notin D(A^2)$, being A the negative of the derivative.

Due to simple algebraic considerations, it follows that for such an operator, in this space, we have

$$D(A^{s+it_1}) \backslash D(A^{s+it_2}) \neq \emptyset$$

$\forall s>0$ and $\forall t_1, t_2 \in \boldsymbol{R}$ such that $t_1 \neq t_2$.

We shall use the following notations:

$$E = \{\varphi :\ ]0, +\infty[\longrightarrow \boldsymbol{C}\}$$

$$(P_t \varphi)(x) = \varphi(x+t)$$

for each $x>0$, $t>0$ and $\varphi \in E$.

$\rho :\ ]0, +\infty[\longrightarrow]0, +\infty[$ will be a continuous, increasing, and bounded function and $\lim_{x\to 0+} \rho(x)=0$.

$$Y_\rho = \{\varphi \in E:\ \rho\varphi \text{ is continuous, bounded and such that } \lim_{x\to 0+} \rho(x)\varphi(x) \text{ exists and is finite}\}$$

(understanding by $\rho\varphi$ the usual product).

$$\|\varphi\| = \|\rho\varphi\|_\infty \quad (\text{with } \|\psi\|_\infty = \sup_{x>0} |\psi(x)|) \quad \text{for each } \varphi \in Y_\rho.$$

$$L = \{\varphi \in Y_\rho :\ \varphi \text{ is differentiable and } \varphi' \in Y_\rho\}.$$

$$X_\rho = \bar{L} \text{ (closure with respect to } (Y_\rho, \|\ \|)).$$

COUNTEREXAMPLE. We shall work in the space X_ρ, with the norm induced by $(Y_\rho, \| \ \|)$.

The following statements can be easily verified:

1. $(Y_\rho, \| \ \|)$ is a Banach space (and therefore so is $(X_\rho, \| \ \|)$).

2. $P_t(Y_\rho) \subset Y_\rho$ and $P_t(L) \subset L$. Moreover P_t is contractive on Y_ρ and strongly continuous on L. Consequently strongly continuous and contractive on X_ρ.

3. The semigroup $P_t|_{X_\rho}$ has as generator an operator, A, that in its domain $D(A)$ coincides with the derivative.

4. For all $\varphi \in D(A^2)$ it is fulfilled (with respect to the operator $-A$ as base of powers) (for each $\varepsilon > 0$).

$$(5.8) \qquad (AF_\varepsilon\varphi)(x) = \frac{\sin\alpha\pi}{\pi}\int_0^{+\infty} \lambda^{\alpha-1}\Big(\int_0^{+\infty} e^{-\lambda t}(1-e^{-\varepsilon t})\varphi''(x+t)dt\Big)d\lambda .$$

This is due to an integral expression of $F_\varepsilon\varphi$ obtained from the integral formulae (1.2) and also to the fact that, being A an infinitesimal generator of a bounded strongly continuous semigroup, the resolvent is Laplace's transform of the semigroup.

Let $\alpha \neq 1$ with $\operatorname{Re}\alpha = 1$ and let $f \in E$ such that

$$f(x) = \begin{cases} \dfrac{1}{(\alpha+1)\alpha(\alpha-1)} x^{\alpha+1} & \text{if} \quad x \in]0, b[\quad (b>0) \\ 0 & \text{if} \quad x \in]c, +\infty[\quad (b<c) \end{cases}$$

being $f \in C^3(]0, +\infty[)$.

As a consequence of f being bounded and uniformly continuous in any interval of the form $]\delta, +\infty[$ (with $\delta > 0$) we can prove without difficulties that $f \in D(A^2)$ in every choice of ρ.

We shall prove that $AF_\varepsilon f$ is differentiable but we shall see that its derivative $(AF_\varepsilon f)'$ does not belong to Y_ρ, if ρ is conveniently chosen, therefore $F_\varepsilon f \notin D(A^2)$.

The properties of f allow us to permute the order of integration and differentiating (5.8) under the integral sign, we obtain that $AF_\varepsilon f$ is differentiable and

$$(AF_\varepsilon f)'(x) = \begin{cases} \dfrac{\sin\alpha\pi}{\pi}\Gamma(\alpha)\displaystyle\int_0^{c-x} t^{1-\alpha}\Big(\frac{1-e^{-\varepsilon t}}{t}\Big)f'''(x+t)dt & \text{if} \quad 0<x<c, \\ 0 \qquad \text{if} \quad x \geqq c. \end{cases}$$

Through considerations of elementary calculation we notice that $(AF_\varepsilon f)'$ can be written in the form

$$(AF_\varepsilon f)'(x) = h(x) + K(\alpha, \varepsilon)g(x) \qquad \forall \varepsilon > 0$$

where h is a function with finite limit when x tends to zero, $K(\alpha, \varepsilon)$ is a non-zero constant, and

$$g(x)=\int_0^{b-x} t^{1-\alpha}(x+t)^{\alpha-2}dt \qquad \text{if} \quad x\in]0, b[.$$

Hence, we get

$$xg'(x)=-(b-x)^{1-\alpha}b^{\alpha-1}$$

obtaining afterwards with no difficulty that

$$g(x)=g_0(x)-\ln x \qquad \text{if} \quad x\in]0, b[$$

g_0 being a certain function with finite limit when x tends to zero.

Consequently, once chosen function ρ so that

$$\lim_{x\to 0}\rho(x)\ln x=-\infty$$

we shall obtain that $(AF_\varepsilon f)'\notin Y_\rho$. For example, we can take the function

$$\rho(x)=\begin{cases}\dfrac{1}{1+|\ln x|^{1/2}} & \text{if} \quad 0<x<1 \\ 1 & \text{if} \quad x\geqq 1.\end{cases}$$

Remark. This counterexample, at least with the same choice of function f, cannot be generalized to the space of measurable functions whose product by a suitable function ρ (with the imposed conditions here) is p-summable ($1\leqq p<+\infty$). This is essentially due to the fact that

$$\int_0^\delta |\ln x|^q dx<+\infty \qquad (\delta>0 \quad \text{and} \quad q\in \boldsymbol{R}).$$

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