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Isotropy representations of semisimple symmetric spaces and homogeneous hypersurfaces

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The isotropy representation of a symmetric space is the linear action of the isotropy group of a point on the tangent space. This paper deals with the case that the orbits have codimension 2, i.e. they are homogeneous hypersurfaces in the pseudo-riemannian sphere. In the Riemannian case all homogeneous hypersurfaces of the sphere are orbits under isotropy representations and Takagi/Takahashi [**13**] have studied their geometry in detail. These investigations will be extended to the isotropy representations of semisimple symmetric spaces.

The first section gives the algebraic prerequisites about semisimple symmetric Lie algebras and their isotropy representations.

Section 2 contains the geometric results. Hypersurfaces in the pseudo-riemannian sphere occur as orbits under the isotropy representation if the symmetric Lie algebra is of rank 2. The hypersurfaces have 2, 3, 4 or 6 distinct principal curvatures. Depending on the rank of the maximal compact symmetric subalgebra, either the orbits form one family of homogeneous hypersurfaces with complex principal curvatures and at most one focal variety, or they form several families, one with real principal curvatures and at least 2 focal varieties and the other families with only one focal variety that coincides with a focal variety of the first family.

The last section gives a list of the examples and their geometric data: principal curvatures and the numbers of focal varieties and of families of homogeneous hypersurfaces. Finally an example of a homogeneous hypersurface is displayed that is *not* orbit under an isotropy representation.

The material is taken from the author's Bonn University doctoral dissertation [**5**], which may be consulted for detailed proofs.

1. Semisimple symmetric Lie algebras.

A *semisimple symmetric Lie algebra* $(\underline{g}, \sigma)$ consists of a real semisimple Lie algebra $\underline{g}$ and an involution σ. Let $\underline{g}=\underline{h}+\underline{q}$ be the decomposition into eigenspaces of σ, i.e. $\underline{h}=\{X\in\underline{g} \mid \sigma X=X\}$ and $\underline{q}=\{X\in\underline{g} \mid \sigma X=-X\}$. Then $[\underline{h}, \underline{h}]\subset\underline{h}$, i.e. $\underline{h}$ is a subalgebra of $\underline{g}$, $[\underline{h}, \underline{q}]\subset\underline{q}$ and $[\underline{q}, \underline{q}]\subset\underline{h}$. The notation $(\underline{g}, \sigma)=\underline{g}/\underline{h}$

will be used frequently. $\underline{h}$ and $\underline{q}$ are orthogonal with respect to the Killing form $B(\cdot, \cdot)$ of $\underline{g}$ and $\langle X, Y\rangle := -B(X, Y)$ is a nondegenerate metric on $\underline{q}$.

H is the analytic subgroup of the adjoint group of $\underline{g}$ with Lie algebra $\mathrm{ad}(\underline{h})$ (ad is the adjoint representation of $\underline{g}$). H acts $\langle\cdot, \cdot\rangle$-orthogonally on $\underline{q}$ and the representation $H \rightarrow SO(\underline{q})$ is called the *isotropy representation* of $(\underline{g}, \sigma)$.

$\underline{g}^C = \underline{g} + i\underline{g}$ is the complexification of $\underline{g}$ and τ denotes the conjugation of $\underline{g}^C$ with respect to $\underline{g}$. The semilinear extension of σ on $\underline{g}^C$ is also denoted by σ. Real forms of $\underline{g}^C$ will be indicated by their conjugation, e.g. $\underline{g}_\tau = \underline{g}$, $\underline{g}_\sigma = \underline{h} + i\underline{q}$.

A *Cartan subspace* is a subspace $\underline{a} \subset \underline{q}$ that is maximal abelian in $\underline{q}$ and consists only of semisimple elements.

A maximal abelian subspace of $\underline{q}$ is a Cartan subspace if and only if its induced metric $\langle\cdot, \cdot\rangle$ is nondegenerate. This is a consequence of the Jordan decomposition.

1.1. MAIN LEMMA. *If $\underline{a}$ is a Cartan subspace of $(\underline{g}, \sigma)$, then there is a Cartan subalgebra $\underline{c}$ that contains $\underline{a}$, and a conjugation κ of $\underline{g}^C$ such that:*

(1) *The real form $\underline{g}_\kappa$, determined by the conjugation κ, is compact.*

(2) *The conjugations κ, σ, τ commute.*

(3) *$\underline{a}$ and $\underline{c}$ are invariant under the conjugations κ, σ, τ.*

PROOF. Since $\underline{a}$ is abelian and consists only of semisimple elements, it can be extended to a Cartan subalgebra $\underline{c}$. By the maximality of $\underline{a}$, $\underline{c}$ is σ-stable, cf. [**7**, p. 259].

Now a Cartan involution of $\underline{g}$, that commutes with σ and leaves $\underline{c}$ invariant, can be constructed as in [**7**, p. 182f], cf. [**9**, p. 337]. Take κ to be the semilinear extension of this Cartan involution and (1)-(3) follow. ////

From now on let $\underline{a}$, $\underline{c}$, κ be as in the main lemma.

$$\underline{g} = \underline{k} + \underline{p}, \qquad \text{where } \underline{k} := \{X \in \underline{g} \mid \kappa X = X\} \text{ and } \underline{p} := \{X \in \underline{g} \mid \kappa X = -X\},$$

is a Cartan decomposition of $\underline{g}$ and $\underline{g}$ has the simultaneous decomposition $\underline{g} = \underline{hk} + \underline{hp} + \underline{qk} + \underline{qp}$, where $\underline{hk} := \underline{kh} := \underline{h} \cap \underline{k}$ etc. Similarly $\underline{a} = \underline{ak} + \underline{ap}$.

Now set $\theta = \kappa\sigma\tau$ and define new symmetric Lie algebras:

$$(\underline{g}_\kappa, \theta) = (\underline{k} + i\underline{p})/(\underline{hk} + i\underline{hp}), \qquad \text{the } \textit{compact version} \text{ of } (\underline{g}, \sigma),$$

$$(\underline{g}_\sigma, \tau) = (\underline{h} + i\underline{q})/\underline{h}, \qquad \text{the } \textit{dual symmetric Lie algebra}$$

and

$$(\underline{k}, \sigma) = \underline{k}/\underline{hk}, \qquad \text{the } \textit{maximal compact symmetric subalgebra}.$$

In general $\underline{k}$ is not semisimple but merely reductive, i.e. $\underline{k} = [\underline{k}, \underline{k}] + \underline{z}$, $[\underline{k}, \underline{k}] \cap \underline{z} = 0$, where $\underline{z}$ is the center of $\underline{k}$ and $[\underline{k}, \underline{k}]$ is semisimple. The center decomposes with respect to σ as $\underline{z} = \underline{zh} + \underline{zq}$ and $([\underline{k}, \underline{k}], \sigma)$ is a semisimple symmetric Lie algebra. If $\underline{b}$ is a Cartan subspace of $([\underline{k}, \underline{k}], \sigma)$ then $\underline{b} + \underline{zq}$ is a Cartan subspace of $(\underline{k}, \sigma)$.

A Cartan subspace $\underline{a}$ of $(\underline{g}, \sigma)$ is said to have *maximal compact part*, if $\underline{ak}$ is a Cartan subspace of $(\underline{k}, \sigma)$.

1.2. PROPOSITION. (1) *Every Cartan subspace of $(\underline{g}, \sigma)$ is H-conjugate to a κ-invariant Cartan subspace.*

(2) *All Cartan subspaces of $(\underline{g}, \sigma)$ have the same dimension, called the rank of $(\underline{g}, \sigma)$. It coincides with the rank of the compact version.*

(3) *All Cartan subspaces with maximal compact part are H-conjugate.*

PROOF. (1) See [**10**, p. 406].

(2) If $\underline{a}$ is a κ-invariant Cartan subspace, then $\underline{ak}+i\,\underline{ap}$ is a Cartan subspace of the compact version. Its dimension is $\mathrm{rank}(\underline{g}_\kappa, \theta)$.

(3) Let $\underline{a}, \underline{b}$ be Cartan subspaces with maximal compact part. By (1) it can be assumed that they are κ-invariant. Then $i\,\underline{ak}+\underline{ap}$, $i\,\underline{bk}+\underline{bp}$ are Cartan subspaces of $(\underline{g}_\theta, \kappa)$, the dual of the compact version, whose $i\,\underline{kq}$-parts are maximal abelian in $i\,\underline{kq}$. By [**9**, p. 341] there is an element h of the analytic subgroup of the adjoint group of $\underline{g}_\theta$ with Lie algebra $\mathrm{ad}(\underline{hk})$ such that $h\cdot(i\,\underline{ak}+\underline{ap})=i\,\underline{bk}+\underline{bp}$. Since h respects the $\kappa\sigma$-eigenspace decomposition of $\underline{q}$ and $h\in H$, the subspaces $\underline{a}, \underline{b}$ are H-conjugate. ////

For a $\boldsymbol{C}$-linear form λ on $\underline{a}^C$ set

$$\underline{h}^C_\lambda := \{X\in\underline{h}^C \mid (\mathrm{ad}A)^2X=\lambda(A)^2X \quad \text{for all } A\in\underline{a}\}$$
$$\underline{q}^C_\lambda := \{X\in\underline{q}^C \mid (\mathrm{ad}A)^2X=\lambda(A)^2X \quad \text{for all } A\in\underline{a}\}$$

and $\underline{h}_\lambda$, $\underline{q}_\lambda$, $\underline{kq}^C_\lambda$, $\underline{pq}^C_\lambda$ are defined similarly.

$m(\lambda):=\dim_C \underline{q}^C_\lambda$ is called the *multiplicity* of λ.

$\Delta:=\{\lambda \mid 0\neq\lambda,\ \underline{q}^C_\lambda\neq\{0\}\}$ is the set of restrictions to $\underline{a}^C$ of the roots of $\underline{g}^C$ with respect to $\underline{c}^C$ [**7**, p. 288]. Δ is a — possibly non-reduced — root system [**7**, p. 456], called the *root system of $(\underline{g}, \sigma)$ with respect to $\underline{a}$.*

Δ^+ denotes the set of positive roots according to an ordering of Δ.

The following lemma corresponds to [**13**, p. 474].

1.3. LEMMA. (1) *The following decompositions are orthogonal with respect to the Killing form of $\underline{g}^C$:*

$$\underline{h}^C = \underline{h}^C_0 + \sum_{\lambda\in\Delta^+} \underline{h}^C_\lambda, \qquad \underline{q}^C = \underline{a}^C + \sum_{\lambda\in\Delta^+} \underline{q}^C_\lambda .$$

(2) *If $A\in\underline{a}$, $\lambda(A)\neq 0$, then $\mathrm{ad}A\colon \underline{q}^C_\lambda\to\underline{h}^C_\lambda$ and $\mathrm{ad}A\colon \underline{h}^C_\lambda\to\underline{q}^C_\lambda$ are isomorphisms. If $\lambda(A)=0$, then $\mathrm{ad}A(\underline{q}^C_\lambda)=\{0\}$ and $\mathrm{ad}A(\underline{h}^C_\lambda)=\{0\}$.*

(3) *If $A, B\in\underline{a}$ and $X\in\underline{q}^C_\lambda$(or $\underline{h}^C_\lambda$), then*

$$[A, [B, X]] = \lambda(A)\lambda(B)X .$$

(4) *If $\underline{a}$ is positive definite, then the decompositions are real and orthogonal:*

$$\underline{h}=\underline{h}_0+\sum_{\lambda\in\Delta^+}\underline{h}_\lambda,\qquad \underline{q}=\underline{a}+\sum_{\lambda\in\Delta^+}\underline{q}_\lambda .$$

PROOF. (1) is a direct consequence of [7, p. 335].

(2), (3) follow from the definitions of $\underline{q}_\lambda^C$, $\underline{h}_\lambda^C$.

(4) If $A\in\underline{a}\subset\underline{g}_\kappa$, then ad$A$ has purely imaginary eigenvalues and $(\mathrm{ad}A)^2$ has a real eigenspace decomposition. ////

Now let $\underline{a}$ be a Cartan subspace with maximal compact part. Similar as above put $\underline{qk}_\mu^C:=\{X\in\underline{qk}^C\mid(\mathrm{ad}A)^2X=\mu(A)^2X$, for all $A\in\underline{ak}\}$ etc. for a C-linear form μ on $\underline{ak}^C$. The multiplicity of μ is $n(\mu):=\dim_C\underline{qk}_\mu^C$. Γ denotes the root system of $(\underline{k},\sigma)$ with respect to $\underline{ak}$ (if $\underline{k}$ is merely reductive, Γ is set to consist of the roots of $([\underline{k},\underline{k}],\sigma)$ with respect to $\underline{ak}\cap[\underline{k},\underline{k}]$, extended by 0 on the center).

If $\underline{a}$ is a positive definite Cartan subspace, then Γ is a subsystem of Δ and for $\lambda\in\Delta$ is $n(\lambda)\leqq m(\lambda)$.

1.4. LEMMA. (1) *If two points $X, Y\in\underline{qk}$ are H-conjugate, then they are even conjugate with respect to K, the analytic subgroup of H with Lie algebra* ad$(\underline{hk})$.

(2) *Two points $X, Y\in\underline{ak}$ are H-conjugate if and only if they are in the same orbit under the action of the Weyl group $W(\Gamma)$. ($W(\Gamma)$ is generated by the reflections at the hyperplanes $\mu(X)=0$, $\mu\in\Gamma$.)*

PROOF. Cf. [11, p. 160], [12, p. 404].

(1) Let $Y=h\cdot X$, $h\in H$. If $h=k\cdot e^{\mathrm{ad}P}$, $k\in K$, $P\in\underline{hp}$, is the Cartan decomposition of h, then $Z:=k^{-1}\cdot Y=e^{\mathrm{ad}P}\cdot X\in\underline{qk}$. Claim: $Z=X$. It is $Z=\kappa Z=e^{\mathrm{ad}\kappa P}\cdot\kappa X=e^{-\mathrm{ad}P}\cdot X$, therefore $(e^{\mathrm{ad}P})^2\cdot X=X$. Since $e^{\mathrm{ad}P}$ is semisimple with only positive eigenvalues, even $e^{\mathrm{ad}P}\cdot X=X$, i.e. $Z=X$.

(2) Because of (1) the situation is reduced to the well known compact case, see [7, p. 285 ff]. ////

2. Geometry of the orbits under isotropy representations.

The orbits of the isotropy representation of $(\underline{g},\sigma)$ are homogeneous submanifolds. Here attention is restricted to orbits that are hypersurfaces in the pseudo-riemannian sphere $S(\underline{q}):=\{X\in\underline{q}\mid\langle X, X\rangle=+1\}$. It is assumed that $S(\underline{q})\neq\emptyset$ (this excludes the duals of the compact symmetric Lie algebras). For a Cartan subspace $\underline{a}$ put $S(\underline{a}):=S(\underline{q})\cap\underline{a}$.

If the orbit is a hypersurface (with nondegenerate metric) in $S(\underline{q})$ its *shape operator* S is an important geometrical object. It can be computed by $S\,\mathrm{ad}h(X)=-\mathrm{ad}h(N)$, where $h\in\underline{h}$, $X\in S(\underline{q})$ and N is the unit normal vector at X, cf.

[**13**, p. 473]. Its eigenvalues are the *principal curvatures.*

The orbits along a normal geodesic form a family of *homogeneous hypersurfaces,* see [**6**, Section 1]. The family is called *elliptic,* if the normal geodesic is a circle ($\langle N, N\rangle=+1$), and *hyperbolic,* if the normal geodesic is a hyperbola ($\langle N, N\rangle=-1$) (this corresponds to type $+1$, -1 resp. in the notation of [**6**]).

If $k=\cot t$ (in the elliptic case) or $\coth t$ (in the hyperbolic case) is a principal curvature, then the orbit through $X\cos t+N\sin t$ or $X\cosh t+N\sinh t$, resp., has higher codimension and is called the *focal variety associated to k.*

2.1. PROPOSITION. (1) *If the orbit of a point $X\in S(\underline{q})$ under the isotropy representation $H\rightarrow SO(\underline{q})$ is a hypersurface with nondegenerate metric, then the normal space in $\underline{q}$ is a Cartan subspace of $(\underline{g}, \sigma)$.*

(2) *If $\underline{a}$ is a Cartan subspace of $(\underline{g}, \sigma)$ and $X\in S(\underline{a})$, then the complexified tangent space of the orbit $H\cdot X$ is*

$$T_X^C(H\cdot X)=\sum_{\lambda\in\Delta^+,\,\lambda(X)\neq 0}\underline{q}_\lambda^C$$

and the complexified normal space in $S(\underline{q})$ is

$$\perp_X^C(H\cdot X)=\{Y\in\underline{a}^C\,|\,\langle X, Y\rangle=0\}+\sum_{\lambda\in\Delta^+,\,\lambda(X)=0}\underline{q}_\lambda^C,$$

where $\langle\cdot, \cdot\rangle$ denotes the complex metric.

If $\operatorname{rank}(\underline{g}, \sigma)=2$ *and if $\lambda(X)\neq 0$ for all $\lambda\in\Delta^+$, then the orbit is a homogeneous hypersurface in the pseudo-riemannian sphere $S(\underline{q})$. The hypersurface is elliptic, if $\underline{a}$ is positive definite, and is hyperbolic, if $\underline{a}$ is indefinite.*

(3) *If $\underline{a}$ is positive definite, then the* (*real*) *tangent space of the orbit is*

$$T_X(H\cdot X)=\sum_{\lambda\in\Delta^+,\,\lambda(X)\neq 0}\underline{q}_\lambda$$

and the normal space in $S(\underline{q})$ is

$$\perp_X(H\cdot X)=\{Y\in\underline{a}\,|\,\langle X, Y\rangle=0\}+\sum_{\lambda\in\Delta^+,\,\lambda(X)=0}\underline{q}_\lambda.$$

PROOF. (1) The tangent space at X is $T_X(H\cdot X)=[\underline{h}, X]$. $Z\in\underline{q}$ is in the normal space if and only if $0=\langle Z, [\underline{h}, X]\rangle=B([Z, X], \underline{h})$. Since $[Z, X]\in\underline{h}$, it follows $[Z, X]=0$ and the normal space is maximal abelian. Since it carries a nondegenerate metric, it is also a Cartan subspace.

(2), (3) are similar to [**13**, p. 475] and follow directly from (1.3). ////

In order to study orbit hypersurfaces only symmetric Lie algebras of rank 2 have to be investigated. From now on it is assumed that $\operatorname{rank}(\underline{g}, \sigma)=2$.

2.2. PROPOSITION. *If the orbit of $X\in S(\underline{a})$ is a hypersurface with normal vector N at X, then the* (*complexified*) *shape operator is given by*

$$SY = \sum_{\lambda \in \Delta^+} -\frac{\lambda(N)}{\lambda(X)} Y_\lambda,$$

where $Y = \sum_{\lambda \in \Delta^+} Y_\lambda$ *is the decomposition according to* (2.1.2). *S is diagonalizable and the hypersurface has the* $\sharp\Delta^{*+}$ *distinct principal curvatures* $k_\lambda = -\lambda(N)/\lambda(X)$, $\lambda \in \Delta^{*+}$, *with multiplicity* $m_\lambda = m(\lambda) + m(2\lambda)$. *Here* $\Delta^* := \{\lambda \in \Delta \mid \lambda/2 \notin \Delta\}$ *is the reduced root system of* $(\underline{g}, \sigma)$, *with the ordering induced by* Δ.

PROOF. Cf. [13, p. 476f]. Put $h := \sum_{\lambda \in \Delta^+} (1/\lambda(X)^2)[Y_\lambda, X]$, then $h \in \underline{h}^C$ and $[h, X] = Y$. Therefore $SY = -[h, N] = -\sum_\lambda (\lambda(N)/\lambda(X)) Y_\lambda$ (1.3.3). The principal curvatures are $k_\lambda = -\lambda(N)/\lambda(X)$ $(\lambda \in \Delta^+)$. These are all distinct, if λ varies only in Δ^{*+}. The eigenspace belonging to the principal curvature k_λ is $\underline{q}_\lambda^C + \underline{q}_{2\lambda}^C$ $(\lambda \in \Delta^{*+})$. ////

The root system Δ is one of the following:

$$A_1 \times A_1,\ A_2,\ B_2,\ BC_2,\ G_2.$$

If $\Delta = BC_2$ then $\Delta^* = B_2$ and in all other cases $\Delta^* = \Delta$. The orbit hypersurface has 2 (if $\Delta = A_1 \times A_1$), 3 $(\Delta = A_2)$, 4 $(\Delta = B_2, BC_2)$ or 6 $(\Delta = G_2)$ distinct principal curvatures.

For $\lambda \in \Delta$ choose $A_\lambda \in \underline{a}_* := \underline{ak} + i\,\underline{ap}$, such that $\langle A_\lambda, X \rangle = i\,\lambda(X)$ $(X \in \underline{a})$ and set $\|\lambda\|^2 := \langle A_\lambda, A_\lambda \rangle$. Then $(A_\lambda)_{\lambda \in \Delta}$ is a root system in $\underline{a}_*$.

If $\underline{a}$ has maximal compact part and $\mu \in \Gamma$, then define $A_\mu \in \underline{ak}$ by $\langle A_\mu, X \rangle = i\,\mu(X)$ $(X \in \underline{ak})$. $(A_\mu)_{\mu \in \Gamma}$ is a root system in $\underline{ak}$. The mapping $\Gamma \to \underline{a}_*$, $\mu \to A_\mu$ is called the *embedding* of Γ. The roots of Δ^* are enumerated in cyclic order, starting with a short root:

$$\Delta^* = \{\lambda_j \mid j = 1, \cdots, 2d\}, \quad d := \sharp\Delta^{*+}, \quad \|\lambda_1\| \leqq \|\lambda_j\|.$$

Set $m_j := m(\lambda_j) + m(2\lambda_j)$, $A_j := A_{\lambda_j}$. Note that $m_j = m_{j+2}$ (indices mod d) [7, p. 523 ff].

Now the cases $\mathrm{rank}(\underline{k}, \sigma) = 2$ or 1 will be studied separately.

In case $\mathrm{rank}(\underline{k}, \sigma) = 2$, the Cartan subspace $\underline{a}$ with maximal compact part is also a Cartan subspace of $(\underline{k}, \sigma)$. $\underline{a}$ is positive definite and $A_\lambda \in \underline{a}$ for $\lambda \in \Delta$. Γ is a subsystem of Δ and the embedding of Γ is the inclusion (via the identification $\lambda \leftrightarrow A_\lambda$). Choose an orthonormal basis (A, B) of $\underline{a}$ by

$$A := A_1/\|A_1\|, \qquad \langle A_d, B \rangle > 0,$$

and define $B_\lambda \in \boldsymbol{S}(\underline{a})$ $(\lambda \in \Delta^*)$ by: $\langle A_\lambda, B_\lambda \rangle = 0$ and (A_λ, B_λ) has the same orientation as (A, B).

Set $B_j := B_{\lambda_j}$ and $n_j := n(\lambda_j) + n(2\lambda_j)$. It can be assumed that the roots of Δ^* are enumerated such that either $n_1 > 0$ (if there is a short root λ_j with $n_j > 0$)

or $n_2>0=n_1$ (if there are only long roots λ_j with $n_j>0$), or $n_1=n_2=0$.

2.3. THEOREM. *Case* rank($\underline{k}$, σ)=2.

(1) *There is exactly one elliptic family of homogeneous hypersurfaces among the orbits. The orbit hypersurface through* $X=-A\sin x+B\cos x$ $(x\neq j\pi/d,\ j\in \boldsymbol{Z})$ *has the (real) principal curvatures* $k_j=\cot((j-1)\pi/d-x)$, $j=1, \cdots, d$. *The eigenspace belonging to the principal curvature* k_j *has signature* (m_j, m_j-n_j). *The orbits through* B_j, $-B_j$ *are focal varieties associated to the pricipal curvature* k_j $(j=1, \cdots, d)$. *The number of focal varieties equals the number of points* B_λ, $\lambda\in\Delta^*$ *in a closed Weyl chamber of* Γ, *cf. Table* 2.1.

(2) *Every hyperbolic family has exactly one focal variety, which is also a focal variety of the elliptic family. The focal variety through* B_j *is focal variety of a hyperbolic family if and only if* $n_j<m_j$. *If* $\boldsymbol{R}A_k+\boldsymbol{R}A_p(A_k\in\underline{qk},\ A_p\in\underline{qp}$ *orthonormal) is a normal space (in* $\underline{q}$*) of a hyperbolic family, then the orbit hypersurface through* $X=A_k\cosh x+A_p\sinh x$ $(x\neq0)$ *has the principal curvatures* $\coth((j-1)(\pi/d)i-x)$, $j=1, \cdots d$, *and the orbit through* A_k *is the focal variety.*

Table 2.1. Number of focal varieties of the elliptic family.

Γ \ Δ	$A_1\times A_1$	A_2	B_2	BC_2	G_2
$A_1\times A_1$	2	/	3	3	4
A_2	/	2	/	/	3*
B_2	/	/	2	2	/
BC_2	/	/	/	2	/
G_2	/	/	/	/	2
$A_1^{(2)}$	3	4*	5	5	7*
$BC_1^{(2)}$	3	4*	5	5	7*
$A_0^{(2)}$	4	6*	8*	8*	12*

$A_1^{(2)}$, $BC_1^{(2)}$ denotes the root system $A_1\subset\boldsymbol{R}^2$, $BC_1\subset\boldsymbol{R}^2$, resp.

$A_0^{(2)}$ is the empty root system in $\boldsymbol{R}^2$; this occurs if $\underline{a}$ is in the center of $\underline{k}$.

/: Γ cannot be a subsystem of Δ.

*: these cases do not occur in the tables 3.I-3.III.

PROOF. (1) The normal spaces of elliptic families are positive definite Cartan subspaces by (2.1.1) and are mutually H-conjugate by (1.2.3), i.e. there is exactly one elliptic family. The orbit through X is a hypersurface if $\lambda_j(X)\neq0$, i.e. $x\neq(j-1)\pi/d$, with normal vector $N=-A\cos x-B\sin x$. The principal curvatures are (2.2) $k_j=-\langle A_j, N\rangle/\langle A_j, X\rangle=\cot((j-1)\pi/d-x)\in\boldsymbol{R}$, since $A_j=\|A_j\|(A\cos(j-1)\pi/d+B\sin(j-1)\pi/d)$. The eigenspace belonging to k_j is $\underline{q}_{\lambda_j}+\underline{q}_{2\lambda_j}$. Its decomposition into eigenspaces with respect to $\kappa\tau$ shows that its signature is (m_j, m_j-n_j). $\pm B_j$ are focal points associated to k_j since the normal circle through X meets B_j at distance $t_j=(j-1)\pi/d-x$ and $-B_j$ at $t_j+\pi$. By (1.4), the number of focal varieties equals the number of points B_λ in a fundamental domain of the operation of $W(\Gamma)$, i.e. in a closed Weyl chamber.

(2) If $\boldsymbol{R}A_k+\boldsymbol{R}A_p$ is a normal space of a hyperbolic family, then A_k is contained in a Cartan subspace of $(\underline{k}, \sigma)$ that is also a positive definite Cartan subspace of $(\underline{g}, \sigma)$. Therefore A_k is contained in two different normal spaces and must be a focal point.

The normal space at B_j is indefinite, if $n_j<m_j$ (2.1.3). In this case B_j is a focal point of a hyperbolic family. The principal curvatures can be computed similarly as in (1). ////

In case rank$(\underline{k}, \sigma)=1$, the Cartan subspace $\underline{a}$ with maximal compact part is indefinite and $\underline{ak}$ is a Cartan subspace of $(\underline{k}, \sigma)$. Choose an orthonormal basis (A, B) of $\underline{a}$ such that $A\in\underline{ak}$, $B\in\underline{ap}$.

2.4. THEOREM. *Case* rank$(\underline{k}, \sigma)=1$.

(1) *The orbits under the isotropy representation form exactly one hyperbolic family of homogeneous hypersurfaces.*

(2) a) *If* $\langle A_\lambda, A\rangle=0$ *for some* $\lambda\in\Delta$, *then the orbit through* A *is the only focal variety. If the roots are enumerated such that* $\langle A_1, A\rangle=0$, *then the orbit hypersurface through* $X=A\cosh x+B\sinh x$ $(x\neq 0)$ *has the principal curvatures* $k_j=\coth((j-1)(\pi/d)i-x)$ *with multiplicities* m_j, $j=1, \cdots, d$.

b) *If* $\langle A_\lambda, A\rangle\neq 0$ *for all* $\lambda\in\Delta$, *then there is no focal variety. This is only possible for* $\Delta=A_1\times A_1$, A_2. *The orbit hypersurface through* $X=A\cosh x+B\sinh x$ *has the principal curvatures* $k_j=\coth(((2j-1)/2d)\pi i-x)$, $j=1, \cdots, d$ *with the same multiplicity* m_1.

PROOF. (1) By (1.2.3) all indefinite normal spaces are H-conjugate. If $\Gamma=A_1$ or BC_1, then the 2 components of $S(\underline{a})$ are conjugate under $W(\Gamma)$ and there is only one family of hypersurfaces. If $\Gamma=A_0^{(1)}$, then the orbits along the components of $S(\underline{a})$ form 2 families. But this is only possible if $\underline{ak}=\underline{qk}$. Then also $S(\underline{q})$ is not connected and there is only one family in each component of $S(\underline{q})$.

(2) The set $(A_\lambda)_{\lambda\in\Delta}$ is $\kappa\tau$-invariant and $\kappa\tau|_{\underline{a}_*}$ is the reflection at $\underline{ak}$. Axes of symmetry, that are not orthogonal to a root, exist only for the root systems $A_1\times A_1$, A_2. Then the multiplicities of the roots coincide. It can be assumed, that the roots are enumerated such that $A_j=\|A_j\|(A\sin((j-1)\pi/d+u)+iB\cos((j-1)\pi/d+u))$ and either $u=0$ (a) or $u=\pi/2d$ (b).

The orbit through $X(x\neq 0$ in (a)) is a hypersurface with normal vector $N=A\sinh x+B\cosh x$ and the principal curvatures are $k_j=-\langle A_j, N\rangle/\langle A_j, X\rangle=\coth(((j-1)\pi/d+u)i-x)$.

If $u=0$, then only the principal curvature k_j has an associated focal variety, that is just the orbit through A. If $u=\pi/2d$, then there is no focal variety. ////

3. Classification and examples.

In this section a list of all homogeneous hypersurfaces that are orbits under isotropy representations of semisimple symmetric Lie algebras will be given.

In order to compute their geometric data: principal curvatures, multiplicities, signature of the eigenspaces, number of elliptic and hyperbolic families and the number of focal varieties, the root systems Δ, Γ, the multiplicities and the embedding of Γ must be determined.

Refering to the tables of Araki concerning the root systems of the compact symmetric Lie algebras [7, p. 532 ff], only the embedding of Γ remains to be computed. Therefore criteria for this embedding are given at first.

3.1. LEMMA. *If* $\operatorname{rank}(\underline{k}, \sigma)=2$, *then* Γ *is a subsystem of* Δ *and*

(1) $n(\lambda)\leqq m(\lambda)$, *for* $\lambda\in\Delta$,

(2) *if* $\kappa\tau$ *is an inner automorphism of* $\underline{g}_\kappa$, $\underline{g}_\kappa$ *simple and* $n(\lambda)$ *odd* ($\lambda\in\Gamma$), *then also* $m(\lambda)$ *is odd. If* $\operatorname{rank}(\underline{k}, \sigma)=1$, *then* $(A_\lambda)_{\lambda\in\Delta}$ *is symmetric with respect to the reflection at* $\underline{ak}$ *and*

(3) *if* $A\to\bar{A}$ *denotes the orthogonal projection onto* $\underline{ak}$ *and* $\mu\in\Gamma$, *then there is* $\lambda\in\Delta$ *such that* $A_\mu=\bar{A}_\lambda$ *and* $n(\mu)\leqq\sum_{\bar{A}_\lambda=A_\mu}m(\lambda)$,

(4) *if the maximal compact subalgebra of the dual symmetric Lie algebra has rank* 2, *then there is* $\lambda\in\Delta$ *with* $A_\lambda\in\underline{ak}$.

PROOF. (1) is clear.

(2) $\underline{a}\subset\underline{qk}$ can be extended to a κ-invariant Cartan subalgebra $\underline{c}$ of $\underline{k}$. Since $\kappa\tau$ is inner, $\underline{c}$ is also a Cartan subalgebra of $\underline{g}_\kappa$ [4, p. 416], invariant under the conjugations κ, σ, τ, θ and the roots of $\underline{k}^C$ with respect to $\underline{c}^C$ are contained in the root system of $\underline{g}^C$. If $n(\lambda)$ is odd, then there is a root α of $\underline{k}^C$, whose restriction to $\underline{a}^C$ is λ and $(\kappa\theta)^*\alpha=-\alpha$ [7, p. 429]. Since α is also a root of $\underline{g}^C$, $m(\lambda)$ is odd by the same argument.

(3) $\underline{qk}_\mu^C\subset\underline{q}_\mu^C=\sum\underline{q}_\lambda^C$, where the sum is taken over those $\lambda\in\Delta$, whose restriction to $\underline{ak}^C$ is μ. The latter is equivalent to $\bar{A}_\lambda=A_\mu$.

(4) By assumption, $i\,\underline{ap}$ is contained in a 2-dimensional Cartan subspace in $i\,\underline{qp}$. Therefore there is $\lambda\in\Delta$ with $\lambda(i\,\underline{ap})=\{0\}=\langle A_\lambda, \underline{ap}\rangle$, i.e. $A_\lambda\in\underline{ak}$. ////

Towards a classification of the orbit hypersurfaces it suffices to assume that $[\underline{q}, \underline{q}]=\underline{h}$, since the orthogonal complement of $[\underline{q}, \underline{q}]$ in $\underline{h}$ acts trivially on $\underline{q}$. According to Berger [1, pp. 97, 110] the following cases have to be considered:

I. $\underline{g}$ is absolutely simple.

II. $\underline{g}$ is isomorphic to $\underline{h}\times\underline{h}$, $\underline{h}$ absolutely simple, and the involution is the reflection at the diagonal.

III. $\underline{g}$ is a complex simple Lie algebra, $\underline{h}$ a real form of $\underline{g}$ and the involution is the conjugation with respect to $\underline{h}$.

IV. $\underline{g}$ is a complex simple Lie algebra and $\underline{h}$ is complex.

V. $(\underline{g}, \sigma)$ is the direct sum of two semisimple symmetric Lie algebras.

Case I ($\underline{g}$ absolutely simple). The absolutely simple symmetric Lie algebras of rank 2 are taken from Berger's list [**1**, p. 157 ff]. Table 3.I gives the list of examples and their geometric data. The root systems Δ, Γ and the multiplicities are obtained from Araki's table [**7**, p. 532 ff]. Lemma 3.1 allows to determine the embedding of Γ as far as is needed to compute the geometric data using theorems 2.3, 2.4—with only 2 exceptions: No. I.10, I.20. No. I.10 is treated separately in Example 3.3. and No. I.20 is similar.

Case II (product with diagonal reflection). Here the isotropy representation is equivalent to the adjoint representation [**1**, p. 98]. The list of examples is given in Table 3.II.

Case III (complex simple Lie algebra with conjugation). The symmetric Lie algebras of this case are dual to those of case II. See Table 3.III.

The tables indicate:

$(\underline{g}, \sigma)$	
$(\underline{g}_\kappa, \theta)$	the type of the compact version of $(\underline{g}, \sigma)$.
$(\underline{k}, \sigma)$	the type of the maximal compact symmetric subalgebra.
sign $\underline{q}$	the signature of the space on which the isotropy representation acts; the first number is the dimension of $\underline{q}$ and the second is the number of "—" signs in the metric.
Δ, Γ	the root systems of $(\underline{g}, \sigma)$, $(\underline{k}, \sigma)$, resp.
d	the number of distinct principal curvatures.
m_1, m_2	the multiplicities of the principal curvatures k_1, k_2 ($m_{j+2}=m_j$, indices mod d).
$s_1, \cdots\cdots, s_d$	the signatures (m_j, s_j) of the eigenspaces belonging to the real principal curvatures k_j, $j=1, \cdots\cdots, d$ (only if an elliptic family occurs); "=" means that $s_j=s_{j-2}$.
f	the number of focal varieties.
e, h	the numbers of elliptic, hyperbolic families.

The Lie algebras and the types of the compact simple symmetric Lie algebras are denoted as in Helgason [**7**, p. 446, 518]. A(1), A(2), B(2), G(2), resp. stand for the type of the symmetric Lie algebras belonging to the compact simple Lie groups (of rank 1 or 2). The 1-dimensional Lie algebra is written as T, $so(2)$, $u(1)$ if it is contained in $\underline{k}$, and is written as $\boldsymbol{R}$, $so(1, 1)$ if it is contained in $\underline{p}$.

Table 3.I.

No.	$(\underline{g}, \sigma)$	$(\underline{g}_\kappa, \theta)$	$(\underline{k}, \sigma)$	sign $\underline{q}$	Δ	Γ	d	m_1,	m_2	$s_1, \cdots, s_d$	f	e	h
1	$su(3)/so(3)$	compact	AI(3)	5, 0	A_2	A_2	3	1	1	0 = =	2	1	0
2	$sl(3,\boldsymbol{R})/sl(2,\boldsymbol{R})$	AI(3)	BI(1,2)	5, 3	A_2	A_1	3	1	1		0	0	1
3	$su(2,1)/so(2,1)$	AI(3)	BI(1,2) × T	5, 2	A_2	$A_1^{(2)}$	3	1	1	0 1 1	4	1	2
4	$su(6)/sp(3)$	compact	AII(3)	14, 0	A_2	A_2	3	4	4	0 = =	2	1	0
5	$sl(6,\boldsymbol{R})/sp(3,\boldsymbol{R})$	AII(3)	AIII(1,3)	14, 8	A_2	BC_1	3	4	4		0	0	1
6	$su^*(6)/sp(2,1)$	AII(3)	CII(1,2)	14, 6	A_2	BC_1	3	4	4		0	0	1
7	$su(4,2)/sp(2,1)$	AII(3)	AII(2) × T	14, 8	A_2	$A_1^{(2)}$	3	4	4	0 4 4	4	1	2
8	$su(3,3)/sp(3,\boldsymbol{R})$	AII(3)	A(2)	14, 6	A_2	A_2	3	4	4	2 = =	2	1	2
9	$su(2+n)/su(2) \times su(n) \times T$, $n \geqq 3$	compact	AIII(2, n)	$4n$, 0	BC_2	BC_2	4	$2n-3$	2	0 0 = =	2	1	0
10	$sl(2+n,\boldsymbol{R})/sl(2,\boldsymbol{R}) \times sl(n,\boldsymbol{R}) \times \boldsymbol{R}$, $n \geqq 3$	AIII(2, n)	BDI(2, n)	$4n$, $2n$	BC_2	B_2	4	$2n-3$	2	$n-1$ 1 = =	2	1	2
11	$su^*(2p+2)/su^*(2) \times su^*(2p) \times \boldsymbol{R}$, $p \geqq 2$	AIII(2, $2p$)	CII(1, p)	$8p$, $4p$	BC_2	BC_1	4	$4p-3$	2		1	0	1
12	$su(p+2,q)/su(2) \times su(p,q) \times T$, $p \geqq 3, q \geqq 1$	AIII(2, $p+q$)	AIII(2, p)	$4(p+q)$, $4q$	BC_2	BC_2	4	$2(p+q)-3$	2	$2q$ 0 = =	2	1	1
13	$su(4,q)/su(2) \times su(2,q) \times T$, $q \geqq 1$	AIII(2, $q+2$)	AIII(2,2)	$4q+8$, $4q$	BC_2	B_2	4	$2q+1$	2	$2q$ 0 = =	2	1	1
14	$su(3,q)/su(2) \times su(1,q) \times T$, $q \geqq 2$	AIII(2, $q+1$)	AIII(1,2)	$4q+4$, $4q$	BC_2	BC_1	4	$2q-1$	2		1	0	1
15	$su(p+1,q-1)/su(1,1) \times su(p,q) \times T$, $p, q \geqq 2$	AIII(2, $p+q$)	AIII(1, p) × AIII(1, q)	$4(p+q)$, $2(p+q)$	BC_2	$BC_1 \times BC_1$	4	$2(p+q)-3$	2	$2p-2$ 2 $2q-2$ =	3	1	3

Table 3.I. (continued)

No.	$(\underline{g}, \sigma)$	$(\underline{g}_\kappa, \theta)$	$(\underline{k}, \sigma)$	sign $\underline{q}$	Δ	Γ	d	$m_1,$	m_2	$s_1, \cdots, s_d$				f	e	h
16	$su(2,q)/su(1,1) \times su(1,q-1) \times T,\ q \geqq 3$	AIII$(2,q)$	AI$(1) \times$ AIII$(1,q-1)$	$4q,\ 2q$	BC_2	$A_1 \times BC_1$	4	$2q-3$	2	0	2	$2q-4$	=	3	1	2
17	$su(1,q)/su(1,1) \times su(q-1) \times T,\ q \geqq 4$	AIII$(2,q-1)$	AIII$(1,q-1)$	$4q-4,\ 2q-2$	BC_2	BC_1	4	$2q-7$	2					1	0	1
18	$so(2+n)/so(2) \times so(n),\ n \geqq 3$	compact	BDI$(2,n)$	$2n,\ 0$	B_2	B_2	4	$n-2$	1	0	0	=	=	2	1	0
19	$so(p+2,q)/so(2) \times so(p,q),\ p \geqq 3, q \geqq 1$	BDI$(2,p+q)$	BDI$(2,p)$	$2(p+q),\ 2q$	B_2	B_2	4	$p+q-2$	1	q	0	=	=	2	1	1
20	$so(4,q)/so(2) \times so(2,q),\ q \geqq 1$	BDI$(2,2+q)$	AI$(2) \times$ AI(2)	$2q+4,\ 2q$	B_2	$A_1 \times A_1$	4	q	1	q	0	=	=	3	1	1
21	$so(3,q)/so(2) \times so(1,q),\ q \geqq 2$	BDI$(2,1+q)$	AI(2)	$2q+2,\ 2q$	B_2	A_1	4	$q-1$	1					1	0	1
22	$so(p+1,q+1)/so(1,1) \times so(p,q),\ p \geqq 3, q \geqq 2$	BDI$(2,p+q)$	BDI$(1,p) \times$ BDI$(1,q)$	$2(p+q),\ p+q$	B_2	$A_1 \times A_1$	4	$p+q-2$	1	$q-1$	1	$p-1$	=	3	1	3
23	$so(3,3)/so(1,1) \times so(2,2)$	DI$(2,4)$	AI$(2) \times$ AI(2)	$8,\ 4$	B_2	$A_1 \times A_1$	4	2	1	2	0	=	=	3	1	1
24	$so(2,n)/so(1,1) \times so(1,n-1),\ n \geqq 4$	BDI$(2,n)$	BDI$(1,n-1) \times T$	$2n,\ n$	B_2	$A_1^{(2)}$	4	$n-2$	1	0	1	$n-2$	=	5	1	3
25	$so(2,3)/so(1,1) \times so(1,2)$	BI$(2,3)$	AI$(2) \times T$	$6,\ 3$	B_2	$A_1^{(2)}$	4	1	1	0	1	1	=	5	1	3
26	$so^*(2p+2)/so(2) \times so^*(2p),\ p \geqq 2$	DI$(2,2p)$	AIII$(1,p)$	$4p,\ 2p$	B_2	BC_1	4	$2p-2$	1					1	0	1
27	$so(10)/u(5)$	compact	DIII(5)	$20,\ 0$	BC_2	BC_2	4	5	4	0	0	=	=	2	1	0
28	$so(5,5)/sl(5,\boldsymbol{R}) \times \boldsymbol{R}$	DIII(5)	B(2)	$20,\ 10$	BC_2	B_2	4	5	4	3	2	=	=	2	1	2
29	$so(2,8)/su(1,4) \times T$	DIII(5)	DI$(2,6)$	$20,\ 8$	BC_2	B_2	4	5	4	4	0	=	=	2	1	1

Table 3.I. (continued)

No.	$(\underline{g}, \sigma)$	$(\underline{g}_\varepsilon, \theta)$	$(\underline{k}, \sigma)$	sign $\underline{q}$	Δ	Γ	d	m_1,	m_2	$s_1, \cdots, s_d$				f	e	h
30	$so(4,6)/su(2,3) \times T$	DIII(5)	BI(1,2) × AI(1,3)	20, 12	BC_2	$A_1 \times BC_1$	4	5	4	0	4	4	=	3	1	2
31	$so^*(10)/su(1,4) \times T$	DIII(5)	AIII(1,4)	20, 12	BC_2	BC_1	4	5	4					1	0	1
32	$so^*(10)/su(2,3) \times T$	DIII(5)	AIII(2,3)	20, 8	BC_2	BC_2	4	5	4	2	2	=	=	2	1	2
33	$sp(2+n)/sp(2) \times sp(n)$, $n \geqq 3$	compact	CII(2, n)	$8n$, 0	BC_2	BC_2	4	$4n-5$	4	0	0	=	=	2	1	0
34	$sp(2+n, \boldsymbol{R})/sp(2, \boldsymbol{R}) \times sp(n, \boldsymbol{R})$, $n \geqq 3$	CII(2, n)	AIII(2, n)	$8n$, $4n$	BC_2	BC_2	4	$4n-5$	4	$2n-2$	2	=	=	2	1	2
35	$sp(2+p, q)/sp(2) \times sp(p, q)$, $p \geqq 3$, $q \geqq 1$	CII(2, $p+q$)	CII(2, p)	$8(p+q)$, $8q$	BC_2	BC_2	4	$4(p+q)-5$	4	$4q$	0	=	=	2	1	1
36	$sp(4, q)/sp(2) \times sp(2, q)$, $q \geqq 1$	CII(2, $q+2$)	CII(2, 2)	$8q+16$, $8q$	BC_2	B_2	4	$4q+3$	4	$4q$	0	=	=	2	1	1
37	$sp(3, q)/sp(2) \times sp(1, q)$, $q \geqq 2$	CII(2, $q+1$)	CII(1, 2)	$8q+8$, $8q$	BC_2	BC_1	4	$4q-1$	4					1	0	1
38	$sp(p+1, q+1)/sp(1,1) \times sp(p, q)$, $p, q \geqq 2$	CII(2, $p+q$)	CII(1, p) × CII(1, q)	$8(p+q)$, $4(p+q)$	BC_2	$BC_1 \times BC_1$	4	$4(p+q)-5$	4	$4q-4$	4	$4p-4$	=	3	1	3
39	$sp(2, q+1)/sp(1,1) \times sp(1, q)$, $q \geqq 2$	CII(2, $q+1$)	BI(1, 4) × CII(1, q)	$8q+8$, $4q+4$	BC_2	$A_1 \times BC_1$	4	$4q-1$	4	$4q-4$	4	0	=	3	1	2
40	$sp(1, n+1)/sp(1,1) \times sp(n)$, $n \geqq 3$	CII(2, n)	CII(1, n)	$8n$, $4n$	BC_2	BC_1	4	$4n-5$	4					1	0	1
41	$sp(4)/sp(2) \times sp(2)$	compact	CII(2, 2)	16, 0	B_2	B_2	4	4	3	0	0	=	=	2	1	0
42	$sp(4, \boldsymbol{R})/sp(2, \boldsymbol{R}) \times sp(2, \boldsymbol{R})$	CII(2, 2)	AIII(2, 2)	16, 8	B_2	B_2	4	4	3	2	2	=	=	2	1	2
43	$sp(4, \boldsymbol{R})/sp(2, \boldsymbol{C})$	CII(2, 2)	AII(2) × T	16, 10	B_2	$A_1^{(2)}$	4	4	3	0	3	4	=	5	1	4

Table 3.I. (continued)

No.	$(\underline{g}, \sigma)$	$(\underline{g}_\kappa, \theta)$	$(\underline{k}, \sigma)$	sign $\underline{q}$	Δ	Γ	d	m_1,	m_2	$s_1, \cdots, s_d$	f	e	h
44	$sp(2,2)/sp(2,\boldsymbol{C})$	CII(2,2)	B(2)	16, 6	B_2	B_2	4	4	3	2 1 = =	2	1	2
45	$sp(2,2)/sp(1,1)\times sp(1,1)$	CII(2,2)	CII(1,1) × CII(1,1)	16, 8	B_2	$A_1\times A_1$	4	4	3	4 0 = =	3	1	1
46	$sp(3,1)/sp(2)\times sp(1,1)$	CII(2,2)	CII(1,2)	16, 8	B_2	BC_1	4	4	3		1	0	1
47	$e_{6(-78)}/f_{4(-52)}$	compact	EIV	26, 0	A_2	A_2	3	8	8	0 = =	2	1	0
48	$e_{6(6)}/f_{4(4)}$	EIV	CII(1,3)	26, 14	A_2	BC_1	3	8	8		1	0	1
49	$e_{6(2)}/f_{4(4)}$	EIV	AII(3)	26, 12	A_2	A_2	3	8	8	4 = =	2	1	2
50	$e_{6(-14)}/f_{4(-20)}$	EIV	DI(1,9) × T	26, 16	A_2	$A_1^{(2)}$	3	8	8	0 8 8	4	1	2
51	$e_{6(-26)}/f_{4(-20)}$	EIV	FII	26, 10	A_2	BC_1	3	8	8		0	0	1
52	$e_{6(-78)}/so(10)\times T$	compact	EIII	32, 0	BC_2	BC_2	4	9	6	0 0 = =	2	1	0
53	$e_{6(6)}/so(5,5)\times \boldsymbol{R}$	EIII	CII(2,2)	32, 16	BC_2	B_2	4	9	6	5 3 = =	2	1	2
54	$e_{6(2)}/so(4,6)\times T$	EIII	AIII(2,4)	32, 16	BC_2	BC_2	4	9	6	4 4 = =	2	1	2
55	$e_{6(2)}/so^*(10)\times T$	EIII	AIII(1,5) × AI(2)	32, 20	BC_2	$BC_1\times A_1$	4	9	6	0 6 8 =	3	1	2
56	$e_{6(-14)}/so(2,8)\times T$	EIII	DI(2,8)	32, 16	BC_2	B_2	4	9	6	8 0 = =	2	1	1
57	$e_{6(-14)}/so^*(10)\times T$	EIII	DIII(5)	32, 12	BC_2	BC_2	4	9	6	4 2 = =	2	1	2
58	$e_{6(-26)}/so(1,9)\times \boldsymbol{R}$	EIII	FII	32, 16	BC_2	BC_1	4	9	6		1	0	1
59	$g_{2(-14)}/su(2)\times su(2)$	compact	G	8, 0	G_2	G_2	6	1	1	0 0 = = = =	2	1	0
60	$g_{2(2)}/sl(2,\boldsymbol{R})\times sl(2,\boldsymbol{R})$	G	AI(2) × AI(2)	8, 4	G_2	$A_1\times A_1$	6	1	1	0 1 1 0 1 1	4	1	2

Table 3. II.

No.	$(\underline{g}, \sigma)$	$(\underline{g}_\kappa, \theta)$	$(\underline{k}, \sigma)$	sign $\underline{q}$	Δ	Γ	d	m_1, m_2	$s_1, \cdots, s_d$	f e h
1	$su(3) \times su(3)/su(3)$	compact	A(2)	8, 0	A_2	A_2	3	2 2	0 = =	2 1 0
2	$sl(3, \boldsymbol{R}) \times sl(3, \boldsymbol{R})/sl(3, \boldsymbol{R})$	A(2)	A(1)	8, 5	A_2	A_1	3	2 2		0 0 1
3	$su(2,1) \times su(2,1)/su(2,1)$	A(2)	A(1) × T	8, 4	A_2	$A_1^{(2)}$	3	2 2	0 2 2	4 1 2
4	$so(5) \times so(5)/so(5)$	compact	B(2)	10, 0	B_2	B_2	4	2 2	0 0 = =	2 1 0
5	$so(4,1) \times so(4,1)/so(4,1)$	B(2)	A(1) × A(1)	10, 4	B_2	$A_1 \times A_1$	4	2 2	0 2 = =	3 1 1
6	$so(3,2) \times so(3,2)/so(3,2)$	B(2)	A(1) × T	10, 6	B_2	$A_1^{(2)}$	4	2 2	2 2 0 2	5 1 3
7	$\underline{g}_{2(-14)} \times \underline{g}_{2(-14)}/\underline{g}_{2(-14)}$	compact	G(2)	14, 0	G_2	G_2	6	2 2	0 0 = = = =	2 1 0
8	$\underline{g}_{2(2)} \times \underline{g}_{2(2)}/\underline{g}_{2(2)}$	G(2)	A(1) × A(1)	14, 8	G_2	$A_1 \times A_1$	6	2 2	0 2 2 0 2 2	4 1 2

Table 3. III.

No.	$(\underline{g}, \sigma)$	$(\underline{g}_\kappa, \theta)$	$(\underline{k}, \sigma)$	sign $\underline{q}$	Δ	Γ	d	m_1, m_2	$s_1, \cdots, s_d$	f e h
1	$sl(3, \boldsymbol{C})/sl(3, \boldsymbol{R})$	A(2)	AI(3)	8, 3	A_2	A_2	3	2 2	1 = =	2 1 2
2	$sl(3, \boldsymbol{C})/su(2,1)$	A(2)	AIII(1, 2)	8, 4	A_2	BC_1	3	2 2		0 0 1
3	$so(5, \boldsymbol{C})/so(4,1)$	B(2)	BI(1, 4)	10, 6	B_2	A_1	4	2 2		1 0 1
4	$so(5, \boldsymbol{C})/so(3,2)$	B(2)	BI(2, 3)	10, 4	B_2	B_2	4	2 2	1 1 = =	2 1 2
5	$\underline{g}_2^{\boldsymbol{C}}/\underline{g}_{2(2)}$	G(2)	G	14, 6	G_2	G_2	6	2 2	1 1 = = = =	2 1 2

Case IV ($\underline{g}$ and $\underline{h}$ complex, $\underline{g}$ simple). Here the isotropy representation is the same as the complexification of the isotropy representation of the maximal compact symmetric subalgebra. The orbits under the isotropy representation of $(\underline{k}, \sigma)$ are spheres and the orbits under the isotropy representation of $(\underline{g}, \sigma)$ are complex spheres. The homogeneous family is hyperbolic, without focal varieties, and the principal curvatures are $\pm i$.

Case V (direct sum). The isotropy representation is reducible and the invariant subspaces carry nondegenerate metrics. The hypersurface orbits are (components of) products of 2 pseudo-riemannian spheres and have 2 distinct principal curvatures.

3.2. EXAMPLE I. 60. $\underline{g}/\underline{h}=g_{2(2)}/sl(2, \boldsymbol{R})\times sl(2, \boldsymbol{R})$. The compact version is $\underline{g}_{2(-14)}/so(4)$ and the maximal symmetric subalgebra is $so(4)/so(2)\times so(2)$, which is of rank 2. The signature of $\underline{q}$ is (8,4). The root systems are $\Delta=G_2$, $\Gamma=A_1\times A_1$; all roots have multiplicity 1 [7, p. 532 f]. All embeddings of Γ are equivalent; it can be assumed that $\Gamma=\{\pm\lambda_1, \pm\lambda_4\}$. The orbit hypersurfaces in $\boldsymbol{S}(\underline{q})$ have 6 distinct principal curvatures. The signature of the eigenspace belonging to k_j is $(1, s_j)$, where $s_j=0$ if $j=1$, 4 and $s_j=1$ if $j=2$, 3, 5, 6. The closed Weyl chamber $\langle A_4, X\rangle\geqq 0$, $\langle A_7, X\rangle\geqq 0$ contains the focal points B_1, B_2, B_3, B_4, i.e. there are 4 focal varieties. The focal varieties through B_2, B_3 are also focal varieties of hyperbolic families. Figure 3.1 shows a normal circle of the elliptic family.

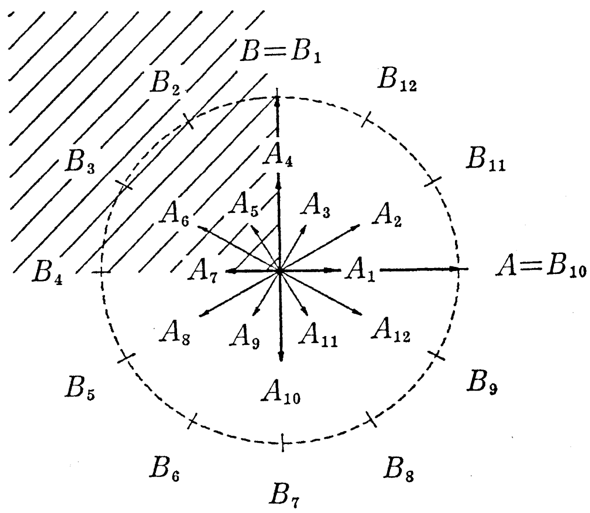

Figure 3.1.

3.3. EXAMPLE I. 10. $\underline{g}/\underline{h}=sl(2+n, \boldsymbol{R})/sl(2, \boldsymbol{R})\times sl(n, \boldsymbol{R})\times\boldsymbol{R}$, $n\geqq 3$. The compact version is $su(2+n)/su(2)\times su(n)\times T$ and the maximal compact symmetric subalgebra is $so(2+n)/so(2)\times so(n)$. The root systems are $\Delta=BC_2$ with multiplicities $m(\lambda_1)=2n-4$, $m(2\lambda_1)=1$, $m(\lambda_2)=2$, and $\Gamma=B_2$ with multiplicities $n(\mu_1)=n-2$, $n(\mu_2)=1$ [7, p. 532 f]. There are 2 possible embeddings of Γ: either $\Gamma=\Delta^*=\{\lambda\in\Delta\,|\,\lambda/2\notin\Delta\}$ or $\Gamma=\{\lambda\in\Delta\,|\,2\lambda\notin\Delta\}$. For $n>4$ the embedding is determined by (3.1.1) and for $n=3$ both possible embeddings yield the same geometric data. Therefore only the case $n=4$ remains open. The involutions of $g_k=su(2+n)$ (cf. [8, p. 110]), $\kappa\theta=\mathrm{Ad}\,\mathrm{diag}(-1, -1, +1, \cdots, +1)$ and $\kappa\tau=$complex conjugation in $su(2+n)$ commute and fix the symmetric Lie algebra (g, σ). Let E_{ij} be the matrix with 1 in position (i, j) and 0 otherwise, $H_1:=E_{1,n+1}-E_{n+1,1}$ and $H_2:=E_{2,n+2}-E_{n+2,2}$. Then $\underline{a}:=\boldsymbol{R}H_1+\boldsymbol{R}H_2\subset \underline{q}k$ is a positive definite Cartan subspace. Let α_j, $j=1, 2$, be the linear forms on $\underline{a}^C$ defined by $\alpha_j(a_1H_1+a_2H_2)=ia_j$. Then $\Delta=\{\pm\alpha_1, \pm\alpha_2, \pm2\alpha_1, \pm2\alpha_2, \pm\alpha_1\pm\alpha_2\}$ and $q^C_{2\alpha_j}=\boldsymbol{C}(E_{j,n+j}+E_{n+j,j})$ [8, p. 111]. Since $E_{j,n+j}+E_{n+j,j}\in \underline{p}^C$, it is $2\alpha_j\notin\Gamma$ and $\Gamma=\Delta^*$.

3.4. NOTE. In the riemannian sphere nearly all homogeneous hypersurfaces with 4 distinct principal curvatures can also be described as Clifford examples [3]. Comparing the geometric data with [6, Theorem 3.1] shows that the examples No. I. 30, I. 32, I. 43, I. 44, I. 55, I. 57, II. 6 and III. 4 can *not* be described as Clifford examples. It should be remarked, that the compact versions of I. 43, I. 44, I. 55 and I. 57 are Clifford.

In contrast to the positive definite case, *not* all homogeneous hypersurfaces in the pseudo-riemannian sphere are orbits under isotropy representations:

3.5. EXAMPLE. Homogeneous hypersurface, that is not orbit under an isotropy representation. Let $\boldsymbol{R}^4_1$ denote the 4-dimensional Lorentz space (with metric $\langle x, x\rangle=-x_1^2+x_2^2+x_3^2+x_4^2$) and $\boldsymbol{S}^3_1:=\{x\in\boldsymbol{R}^4_1\,|\,\langle x, x\rangle=+1\}$. The surface defined by $\langle x, x\rangle=+1$, $x_4=0$ and $x_2-x_1>0$ is a homogeneous hypersurface in $\boldsymbol{S}^3_1$ and not orbit under any isotropy representation. Indeed the surface is an orbit under the analytic subgroups of $SO(1, 3)$ with Lie algebra

$$\left\{\begin{pmatrix} 0 & a & b & 0 \\ a & 0 & b & 0 \\ b & -b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\quad a, b\in\boldsymbol{R}\right\}.$$

It is also not an orbit under an isotropy representation of a general pseudo-riemannian symmetric space. This follows from the classification of Lorentzian symmetric spaces, see [2].

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