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Ruled fibrations on normal surfaces

Dedicated to Professor M. Nagata on his 60th birthday

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Let Y be a normal projective surface over $\boldsymbol{C}$. A *ruled fibration* on Y over a smooth curve B is a surjective morphism $p: Y\rightarrow B$ such that the general fibre is isomorphic to $\boldsymbol{P}^1$. We have the notion of exceptional curves of the first kind in the category of normal surfaces. Namely, an irreducible curve C on Y is called an *exceptional curve of the first kind* if $K_Y C<0$ and $C^2<0$, where the K_Y denotes a canonical divisor on Y. Cf. [**S3**]. A *minimal ruled fibration* will mean a ruled fibration whose fibres contain no exceptional curves of the first kind. Given a ruled fibration on Y, contract successively all exceptional curves of the first kind in fibres, then we obtain a minimal ruled fibration. In this paper we study the structure of a normal surface Y having a minimal ruled fibration over a curve B of genus g.

In § 1 we consider the structure of singular fibres. It turns out that every singular fibre is necessarily a multiple fibre and contains one or two singular points of Y. To describe a singular fibre, we observe the weighted dual graph of the inverse image of the singular fibre on the minimal resolution of Y. In § 2 we introduce a nonnegative rational number τ, which measures the amount of Sing(Y). We have the formula: $K_Y^2=8(1-g)-4\tau$. Suppose that Y has singular fibres f_i with multiplicities m_i, $i=1, \cdots, k$. Then we show that $\tau\geqq\sum(1-1/m_i)$. In § 3 we define the invariants $s_n\in\boldsymbol{Q}$ for positive integers n. The first invariant $s=s_1$ is defined to be the minimum of the self-intersection numbers of all sections in the ruled fibration. Provided that Y is singular, we prove the inequality: $s\leqq g+\tau-1$. Recall that for the smooth case a theorem of Nagata [**N**] says that $s\leqq g$. Similarly, we define the invariants s_n to be $1/n^2$ of the minimum of the self intersection numbers of all effective divisors of degree n over B. We show that $s_n\leqq 2g/(n+1)+\tau$. The invariant $s_*=\inf\{s_n\}$ plays an important role in the numerical criterion for an ample divisor. In § 4 we consider the anti-Kodaira dimension $\kappa^{-1}(Y)$. We give a classification of Y in terms of $\kappa^{-1}(Y)$ together with the numerical type of the anticanonical divisor $-K_Y$. For the smooth case, this was done in [**S1**], [**S3**]. We also deal with the question when Y admits another ruled fibration or an elliptic fibration. We finally prove that Y becomes a normal del Pezzo surface (i. e., a normal surface with ample anticanonical divisor) if and only if either Y admits another minimal ruled fibration, or Y contains an exceptional curve of the first kind in the above sense.

NOTATION AND CONVENTIONS. We use the notation and the results in the previous papers [**S2**], [**S3**]. Let Y be a normal surface. A *divisor* will mean a Weil divisor. Let $\mathrm{Div}(Y)$ denote the group of divisors on Y. We employ the $\boldsymbol{Q}$-valued intersection theory on $\mathrm{Div}(Y)$, which was introduced by Mumford. We denote by $\sim$ (resp. $\equiv$) the linear equivalence (resp. numerical equivalence) on $\mathrm{Div}(Y)$. For a divisor D, we denote by $\mathcal{O}(D)$ the corresponding divisorial sheaf. We mean by $\kappa(D, Y)$ the D-dimension of Y. A divisor D is *nef* if $DC \geqq 0$ for all irreducible curves C on Y, and is *pseudoeffective* if $DP \geqq 0$ for all nef divisors P on Y. We say that D is *ample* if some positive multiple of D becomes an ample Cartier divisor in the usual sense.

In the previous papers [**S3**], [**S4**], a minimal ruled fibration is also called a $\boldsymbol{P}^1$-fibration. But some authors use it to mean a ruled fibration. To avoid confusion we employ "minimal ruled fibration" in this paper. A smooth projective surface with a minimal ruled fibration is known to be a $\boldsymbol{P}^1$-bundle over the base curve. As usual, such a surface is called a *geometrically ruled surface*. See [**H2**], [**M**] for the general theory of geometrically ruled surfaces.

§ 1. Singular fibres.

Let $\boldsymbol{D}$ be the unit disc. Let us consider a normal surface Y having a minimal ruled fibration $p: Y \to \boldsymbol{D}$. In this section, we describe the structure of singular fibres. Let f denote the fibre over 0. More precisely, we define f to be the Cartier divisor $p^*(0)$ where (0) is regarded as a divisor on $\boldsymbol{D}$. We say that f is a *regular fibre* if f does not meet $\mathrm{Sing}(Y)$ and $f \cong \boldsymbol{P}^1$. Otherwise, we say that f is a *singular fibre*. We have seen in [**S3**] that f contains no exceptional curves of the first kind if and only if $\mathrm{Supp}(f)$ is irreducible. The argument is as follows. Suppose that $\mathrm{Supp}(f)$ is reducible, so that $f = \sum m_i F_i$ where the F_i are irreducible. The connectedness of $\mathrm{Supp}(f)$ implies that $F_i^2 < 0$ for all i. Since $K_Y(\sum m_i F_i) = K_Y f = -2$, there must exist at least one component F_i with $K_Y F_i < 0$. This F_i would be an exceptional curve of the first kind. Thus the fibre f has the form:

$$f = mF \qquad (F \text{ is irreducible}) \tag{1.1}$$

where the positive integer m is called the *multiplicity* of f. The fibre f is a *multiple fibre* if $m \geqq 2$. If $m=1$, then we get $(K_Y + f)f = -2$ and so we infer from Lemma 1 in [**S4**] that f is a regular fibre. We conclude therefore that there are only multiple singular fibres.

To describe singular fibres, we fix the notation:

$$\underset{-a}{\circ} : (-a)\text{-curve}, \qquad \bullet : (-2)\text{-curve}, \qquad * : (-1)\text{-curve}.$$

Here a $(-a)$-curve is a smooth rational curve with self-intersection number $-a$. Given positive integers $a_1, \cdots, a_n$, we define the continued fraction:

$$[a_1, \cdots, a_n] = a_1 - \cfrac{1}{a_2 - \cfrac{\ddots}{-\cfrac{1}{a_n}}}.$$

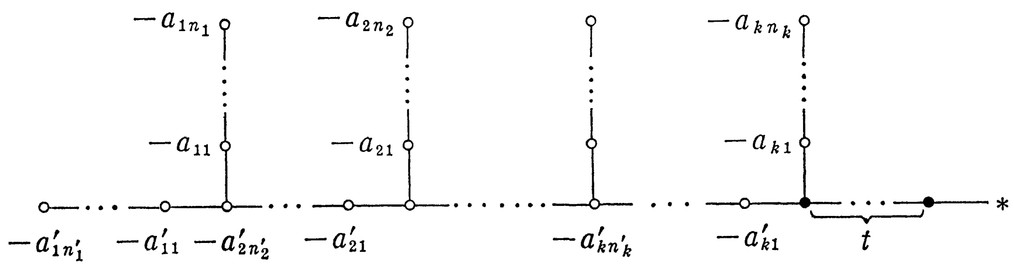
We write $[a_1, \cdots, a_n] = d/e$ where the d and e are mutually prime positive integers. If $a_i \geqq 2$ for all i, then the sequence $\{a_1, \cdots, a_n\}$ is uniquely determined by the pair (d, e) with $0<e<d$. Consider the linear equations of indeterminates $X_0, \cdots, X_{n+1}$:

$$X_{j+1} = a_j X_j - X_{j-1}, \qquad j=1, \cdots, n.$$

Let $\{w_j\}$ be the solution satisfying the conditions: $w_n = c$, $w_{n+1}=0$. Then we find that $w_0 = cd$.

THEOREM 1.2. *Let $p: Y \to \boldsymbol{D}$ be a minimal ruled fibration of a normal surface Y over the unit disc $\boldsymbol{D}$. Suppose that it has a singular fibre f over $0 \in \boldsymbol{D}$. If $\pi: X \to Y$ is the minimal resolution of Y, then*

(i) *the curves in $\pi^{-1}(f)$ consist of a tree of $\boldsymbol{P}^1$'s with the following weighted dual graph:*



where $a_{ij} \geqq 2$, $a'_{ij} \geqq 2$ for all i, j and $t \geqq 0$,

(ii) *if $[a_{11}, \cdots, a_{1n_1}] = d_1/e_1$, then*

$$[a'_{11}, \cdots, a'_{1n'_1}] = d_1/(d_1 - e_1),$$

and for $i \geqq 2$, if $[a_{i1}, \cdots, a_{in_i}] = d_i/e_i$, then

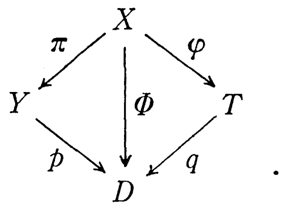
$$[a'_{i1}, \cdots, a'_{in'_i-1}, a'_{in'_i} - 1] = d_i/(d_i - e_i),$$

(iii) *the multiplicity of f is equal to the product $\prod_{i=1}^k d_i$.*

DEFINITION 1.3. In the above case, the singular fibre f is said to be of type $\{(d_1, e_1), \cdots, (d_k, e_k), t\}$.

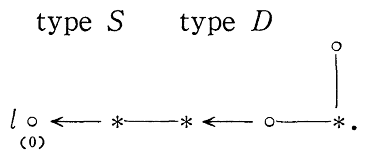
PROOF. We may assume that there are no singular fibres other than f. Since $\Phi = p \circ \pi : X \to \boldsymbol{D}$ is a ruled fibration, by contracting (-1)-curves in its fibres, it factors through a $\boldsymbol{P}^1$-bundle $T \to \boldsymbol{D}$:

(1.4)



Let l be the fibre of $T \to \boldsymbol{D}$ over 0. Then $\pi^* f = \varphi^* l$ and $\pi^{-1}(f) = \varphi^{-1}(l)$. We observe the process of blowing ups in $X \to T$. Following Fujita [**F1**], p. 520, a blowing up over l is called *subdivisional* (*type D*, for short) if it is performed at one of the points where two curves over l meet together, otherwise it is called *sprouting* (*type S*, for short).

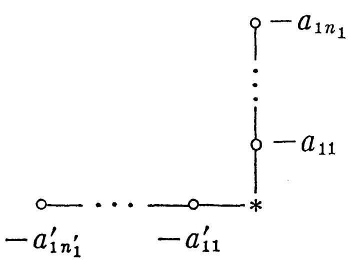
Write $f = mF$ as in (1.1). Let $\bar{F}$ be the strict transform of F by π. We see that $\bar{F}$ is a (-1)-curve. Indeed, since $\pi^{-1}(f)$ is reducible, $\bar{F}^2 < 0$, also $K_X \bar{F} \leqq K_Y F = -2/m < 0$, hence $\bar{F}$ is a (-1)-curve. Therefore, in every intermediate step of $X \to T$, there are no mutually disjoint (-1)-curves over l. By this reason, the first two blowing ups should be the following:



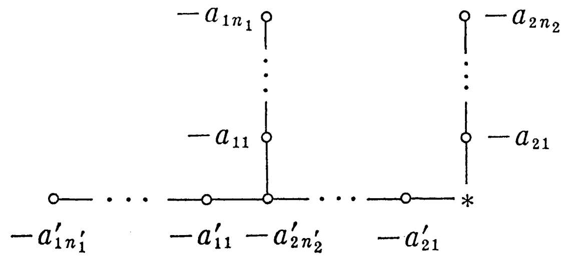
After this step, there is only one (-1)-curve over l, and every blowing up must be performed on that (-1)-curve. We write the order of types of blowing ups over l in φ:

$$\underbrace{SD\cdots D}_{r_1}\underbrace{S\cdots}_{t_2}\underbrace{SD\cdots D}_{r_2}\cdots\cdots\underbrace{S\cdots}_{t_k}\underbrace{SD\cdots D}_{r_k}\underbrace{S\cdots S}_{t}$$

where $r_1 \geqq 1$ and $t \geqq 0$. After the first r_1-times type D blowing ups, one has the dual graph:



where $a_{1j}\geqq 2$, $a'_{1j}\geqq 2$, and $n_1+n'_1=1+r_1$. Next, after t_2-times type S blowing ups followed by r_2-times type D blowing ups, we arrive at the following dual graph:



where $a_{2j}\geqq 2$, $a'_{2j}\geqq 2$ and $n_2+n'_2=t_2+r_2$. Continuing the process of blowing ups in this way, we finally obtain the assertion (i).

By induction, the assertion (ii) follows from the following

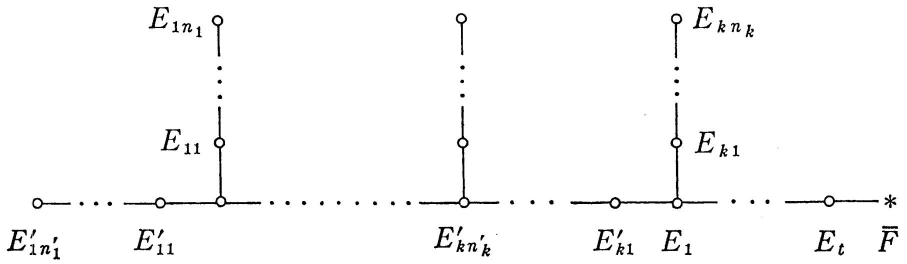
LEMMA 1.5. *If positive integers* $a_1, \cdots, a_n, a'_1, \cdots, a'_{n'}$ *satisfy the condition:*

$$[a_1, \cdots, a_n]^{-1}+[a'_1, \cdots, a'_{n'}]^{-1}=1,$$

then the following equality holds:

$$[a_1+1, a_2, \cdots, a_n]^{-1}+[2, a'_1, \cdots, a'_{n'}]^{-1}=1.$$

To prove (iii), we name the curves as follows



Since $f=\pi_*(\pi^*f)=\pi_*(\varphi^*l)$, the multiplicity m is equal to the coefficient of $\bar{F}$ in the divisor φ^*l. Write

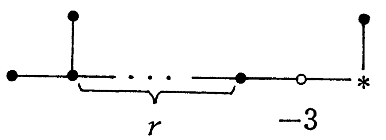
$$\varphi^*l=\sum m_{ij}E_{ij}+\sum m'_{ij}E'_{ij}+\sum m_iE_i+m\bar{F}.$$

By checking step by step, we see the following relations:

$$\begin{cases} m_1=\cdots=m_t=m \\ m_{1n_1}=m'_{1n'_1}=1 \\ m_{in_i}=m'_{in'_i} \qquad \text{for } i=2, \cdots, k. \end{cases}$$

Since $(\varphi^* l)E_{1j}=0$ for all j, the sequence of integers $\{m_{1j}\}$ with $m_{10}=m'_{2n'_2}$ is a solution of the equations: $X_{j+1}=a_{1j}X_j-X_{j-1}$ with $m_{1n_1+1}=0$, $m_{1n_1}=1$. As we have seen before, we get $m_{10}=d_1$. Thus $m_{2n_2}=m'_{2n'_2}=d_1$. Similarly, the equations: $(\varphi^* l)E_{2j}=0$ imply that $\{m_{2j}\}$ with $m_{20}=m'_{3n'_3}$ is a solution of the equations: $X_{j+1}=a_{2j}X_j-X_{j-1}$ with $m_{2n_2+1}=0$, $m_{2n_2}=d_1$. Hence $m_{20}=d_1 d_2$, and it follows that $m_{3n_3}=d_1 d_2$. Repeating the calculation in this way, we can show that $m=\prod_{i=1}^{k} d_i$. □

REMARK 1.6. In case $k=1$, the weighted dual graph is uniquely determined by the type. But in case $k\geqq 2$, this is not the case. For instance, the following is of type $\{(2, 1), (2, 1), 0\}$ for every $r\geqq 1$.



REMARK 1.7. If $t=0$, f contains two singularities of Y, and if $t\geqq 1$, then f contains one singularity of Y. Note that f contains only rational double points if and only if f is of type $\{(2, 1), t\}$ with $t\geqq 0$.

§ 2. The invariant τ.

Let Y be a normal projective surface having a minimal ruled fibration $p: Y \to B$ over a smooth curve B of genus g. We know that Y carries only rational singularities ([S3], Lemma 4.6). Let $\pi: X \to Y$ be the minimal resolution of Y. Let $\mathrm{Sing}(Y)=\{y_1, \cdots, y_t\}$ and $A=\sum A_i$ where each A_i denotes the exceptional set $\pi^{-1}(y_i)$. Let r_i be the determinant of the intersection matrix of all irreducible components of A_i, and let $r=\mathrm{l.c.m.}(r_i)$.

LEMMA 2.1. *Let r be as above. Then*
(i) *$DD'\in(1/r)\boldsymbol{Z}$ for $D, D'\in\mathrm{Div}(Y)$,*
(ii) *rD is a Cartier divisor for every $D\in\mathrm{Div}(Y)$.*

PROOF. (i) follows directly from the definition of intersection numbers ([S2]). (ii) follows from Theorem (4.2) in [S2]. □

There exists an effective $\boldsymbol{Q}$-divisor Δ supported on A satisfying the relation: $\pi^* K_Y=K_X+\Delta$. Cf. [S2]. Decompose $\Delta=\sum\Delta_i$ as $\mathrm{Supp}(\Delta_i)\subset A_i$. For each singular point y_i, we define

$$\tau(y_i)=\frac{1}{4}(\rho(A_i)+\Delta_i^2)$$

where $\rho(A_i)$ denotes the number of irreducible components of A_i. Note that $\tau(y_i)\in \boldsymbol{Q}$, which is possibly negative and that $\tau(y_i)$ depends only on the weighted dual graph of A_i. Define

$$\tau=\tau(Y)=\sum\tau(y_i)$$

where the summation is taken over all singularities. Since each y_i is a rational singularity, $4\tau(y_i)$ is equal to the (generalized) Milnor number $\mu(y_i)$ defined in [**S2**]. The Noether formula (4.7) in [**S2**] gives

$$K_Y^2=8(1-g)-4\tau. \tag{2.2}$$

LEMMA 2.3. $\tau\geqq 0$.

PROOF. See [**S4**], Proposition 5, where it is shown that $K_Y^2\leqq 8(1-g)$. In Remark 2.10 below we give another simple proof. □

Each singular fibre contains one or two singular points of Y. Cf. §1. For a singular fibre f, define

$$\tau(f)=\sum_{y_j\in f}\tau(y_i).$$

EXAMPLE 2.4. (i) If f is of type $\{(d, e), 0\}$, then $\tau(f)=1-1/d$. To see this, consider the following action of $G=\boldsymbol{Z}/d\boldsymbol{Z}$ on $\boldsymbol{P}^1\times\boldsymbol{P}^1$.

$$\begin{array}{ccc}\boldsymbol{P}^1\times\boldsymbol{P}^1 & \longrightarrow & \boldsymbol{P}^1\times\boldsymbol{P}^1\\ \Cup & & \Cup\\ (z, w) & \longrightarrow & (\zeta z, \zeta^e w)\end{array}$$

where ζ is a primitive d-th root of unity. The action has four fixed points. The induced ruled fibration on the quotient $Y=\boldsymbol{P}^1\times\boldsymbol{P}^1/G$ is minimal and has two singular fibres f_1, f_2 of type $\{(d, e), 0\}$. It follows from (2.2) that $K_Y^2=8-4(\tau(f_1)+\tau(f_2))$. But

$$K_Y^2=(1/d)K_{\boldsymbol{P}^1\times\boldsymbol{P}^1}^2=8/d.$$

So this implies that $\tau(f_1)=\tau(f_2)=1-1/d$.

(ii) If f is of type $\{(d, 1), t\}$, then $\tau(f)=(d+t)(d-1)/d^2$.

THEOREM 2.5. *Let Y be a normal projective surface with a minimal ruled fibration. Let f be a singular fibre of the ruled fibration, and let m denote its multiplicity. Then*

$$\tau(f)\geqq 1-\frac{1}{m}.$$

The equality holds if and only if f is of type $\{(m, e), 0\}$ for some e.

PROOF. Since the question is local, it suffices to consider the case in which $p: Y\to\boldsymbol{P}^1$ has one singular fibre f of the given type and one singular fibre f'

of type $\{(m, 1), 0\}$. Choose inhomogeneous coordinate z on $\boldsymbol{P}^1$ so that f is over 0 and f' is over ∞. Take an m-fold covering $\boldsymbol{P}^1 \ni w \to z = w^m \in \boldsymbol{P}^1$. Let $\tilde{Y}$ be the normalization of the fibre product $Y \times_{\boldsymbol{P}^1} \boldsymbol{P}^1$. Then $\tilde{Y}$ has an induced ruled fibration (not necessarily minimal) without multiple fibres. We see that $K_{\tilde{Y}}^2 \leqq 8$. Indeed, let $\tilde{Y} \to \tilde{Y}_0$ be successive contractions of exceptional curves of the first kind in fibres, so that $\tilde{Y}_0$ has a minimal ruled fibration. Then $K_{\tilde{Y}}^2 < K_{\tilde{Y}_0}^2$ unless $\tilde{Y} = \tilde{Y}_0$. But by Lemma 2.3, $K_{\tilde{Y}_0}^2 \leqq 8$. Note that the cyclic group $G = \boldsymbol{Z}/m\boldsymbol{Z}$ acts on $\tilde{Y}$ and $Y = \tilde{Y}/G$. By construction G has only a finite number of points with nontrivial stabilizers, and so $K_Y^2 = (1/m)K_{\tilde{Y}}^2$. Since $\tau(f') = 1 - 1/m$, it follows that

$$\frac{8}{m} \geqq K_Y^2 = 8 - 4\left(1 - \frac{1}{m}\right) - 4\tau(f),$$

and hence $\tau(f) \geqq 1 - 1/m$ as desired. In case $\tau(f) = 1 - 1/m$, we have $K_{\tilde{Y}}^2 = 8$ in the above argument. We infer from this that $\tilde{Y}$ is a geometrically ruled surface and that f has two cyclic quotient singularities. It follows easily that f is of type $\{(m, e), 0\}$ for some e. Conversely, if f is of type $\{(m, e), 0\}$, then the multiplicity of f is equal to m (Theorem 1.2) and $\tau(f) = 1 - 1/m$ (Example 2.4). □

Let $f_1, \cdots, f_k$ be the set of singular fibres, and let m_i denote the multiplicity of f_i for each i. If f_i is over $x_i \in B$, then $f_i = p^*(x_i) = m_i F_i$. Of course

$$\tau = \sum \tau(f_i).$$

COROLLARY 2.6. $$\tau \geqq \sum\left(1 - \frac{1}{m_i}\right).$$

In particular, $\tau = 0$ *if and only if* Y *is smooth.*

A divisor D on Y is said to be of *degree* n over B if $Df = n$ where f is a fibre. An irreducible curve is called an *n-section* ($n > 0$) if it is of degree n over B. A *section* will mean a 1-section.

LEMMA 2.7. *Let* D *be a divisor of degree* 0 *over* B. *Then there exists a* $\boldsymbol{Q}$*-divisor* $\mathfrak{d}$ *on* B *such that*

$$D \sim p^*\mathfrak{d}.$$

In this case, $\mathfrak{d}$ *has the form:*

$$\mathfrak{d} = \mathfrak{d}_0 + \sum\left(\frac{n_i}{m_i}\right)x_i$$

where $\mathfrak{d}_0 \in \mathrm{Div}(B)$ *with* $\mathcal{O}(\mathfrak{d}_0) \cong p_*\mathcal{O}(D)$ *and* $0 \leqq n_i < m_i$ *for all* i.

PROOF. Consider a commutative diagram:

$$
\begin{array}{ccccc}
 & & X & & \\
 & \pi\swarrow & \Big\downarrow & \searrow\varphi & \\
Y & & \Phi & & T \\
 & p\searrow & \Big\downarrow & \swarrow q & \\
 & & B & &
\end{array}
\tag{2.8}
$$

where T is a geometrically ruled surface over B. Namely, φ consists of successive contractions of (-1)-curves contained in fibres of Φ. Cf. (1.4). By definition ([**S2**]), $\pi^*D=\bar{D}+Z$ where $\bar{D}$ is the strict transform of D and the Z is a $\boldsymbol{Q}$-divisor supported on A. Write $\bar{D}=\varphi^*D'+G$ where D' is a divisor on T of degree 0 over B and the G is a divisor supported on the exceptional set of φ. It is well known that there is a divisor $\mathfrak{d}'$ on B such that $D'\sim q^*\mathfrak{d}'$. Note that $q_*\mathcal{O}(D')\cong\mathcal{O}(\mathfrak{d}')$. Cf. [**H2**]. Thus $\pi^*D\sim\Phi^*\mathfrak{d}'+G+Z$. It follows that $D\sim p^*\mathfrak{d}'+\pi_*G$. Since $\mathrm{Supp}(G)\subset\pi^{-1}(\cup f_i)$, we have $\pi_*G=\sum n_i'F_i$ for some $n_i'\in\boldsymbol{Z}$. Write $n_i'\equiv n_i \bmod m_i$ with $0\leqq n_i<m_i$ for each i, and set $\mathfrak{d}_0=\mathfrak{d}'+\sum((n_i'-n_i)/m_i)x_i \in\mathrm{Div}(B)$. Setting $\mathfrak{d}=\mathfrak{d}^0+\sum(n_i/m_i)x_i$, we get the required linear equivalence: $D\sim p^*\mathfrak{d}$. Clearly, $p_*\mathcal{O}(D)\cong\mathcal{O}(\mathfrak{d}_0)$. □

PROPOSITION 2.9. *Let $p: Y\to B$ be a minimal ruled fibration on a normal surface Y over a curve B of genus g. Let D be a divisor on Y of degree n (>0) over B. Then there exists a $\boldsymbol{Q}$-divisor $\mathfrak{e}(D)$ on B satisfying:*

$$nK_Y\sim -2D+p^*(n(\mathfrak{k}+\mathfrak{e}(D)))$$

where $\mathfrak{k}$ is a canonical divisor on B. In particular, we have

$$K_YD=n\left(2g-2+\tau-\frac{D^2}{n^2}\right)$$

and

$$\deg\mathfrak{e}(D)=\frac{D^2}{n^2}+\tau\,.$$

PROOF. Since nK_Y+2D is of degree 0 over B, the existence of $\mathfrak{e}(D)$ follows from Lemma 2.7. Since $(nK_Y+2D)^2=0$, it follows that

$$nK_YD=-\frac{1}{4}n^2K_Y^2-D^2=n^2(\deg(\mathfrak{k})+\tau)-D^2 \qquad \text{(by (2.2))}\,.$$

Thanks to the definition of $\mathfrak{e}(D)$ we have

$$nK_YD=-2D^2+n^2(\deg(\mathfrak{k}+\mathfrak{e}(D)))\,.$$

Combining these together we obtain the remaining formulae. □

REMARK 2.10. We give a simple proof of the fact: (i) $\tau \geqq 0$, (ii) $\tau=0$ if and only if Y is smooth. Cf. Lemma 2.3 and Corollary 2.6. Take a section C on Y, then by Proposition 2.9, $(K_Y+C)C=2g-2+\tau$. To see (i) it is sufficient to show that $(K_Y+C)C \geqq 2g-2$. Let $\bar{C}$ be the strict transform of C on the minimal resolution X of Y. We have seen in [**S4**], Lemma 1 that $(K_Y+C)C \geqq (K_X+\bar{C})\bar{C}$. This gives the required inequality, because $\bar{C}$ is smooth and so $(K_X+\bar{C})\bar{C}=2g-2$. (ii) Suppose that $\tau=0$. Then $(K_Y+C)C=(K_X+\bar{C})\bar{C}$, which implies that C does not meet Sing(Y) ([**S4**], Lemma 1). This is however possible only if Y is smooth, for otherwise there would be multiple fibres.

In the subsequent sections we use the following

LEMMA 2.11. *Let Y be a normal surface with a minimal ruled fibration over a curve B. Let D be a divisor on Y of nonnegative degree over B. Suppose that $D^2=0$, $K_YD \leqq 0$. Then*

(i) *there exists an effective $\boldsymbol{Q}$-divisor D' such that $D' \equiv D$,*

(ii) *furthermore, in case $B=\boldsymbol{P}^1$, we have $\kappa(D, Y) \geqq 0$.*

PROOF. Let X, π, Φ be as in (2.8), and let r be as in Lemma 2.1. Applying the proof of Claim 6.5 in [**S3**] to $\mathcal{L}=\mathcal{O}(\pi^*(rD))$, we see that there exists a degree zero divisor $\mathfrak{a}$ on B such that $H^0(X, \mathcal{L}\otimes\mathcal{O}(\Phi^*\mathfrak{a})) \neq 0$. Take $\Gamma \in |\mathcal{L}\otimes\mathcal{O}(\Phi^*\mathfrak{a})|$, and let $D'=(1/r)\pi_*\Gamma$. Since $\deg \mathfrak{a}=0$, we have $D' \equiv D$. If in addition $B=\boldsymbol{P}^1$, then $\mathfrak{a}=0$, and so $|rD| \neq \emptyset$. □

§ 3. The invariants s_n.

Let Y, p, B have the same meaning as in § 2. For a positive integer n, we define a rational number s_n by

$$s_n = s_n(Y) = \min\left\{\frac{D^2}{n^2}\right\}$$

where the minimum is taken over all effective divisors D of degree n over B. For simplicity write $s=s_1$, so s is equal to the minimum of the self-intersection numbers of all sections. A section b attaining the minimum s is called a *base section* (or a minimal section).

LEMMA 3.1. *The above minimum actually exists.*

PROOF. By Lemma 2.1, $D^2/n^2 \in (1/rn^2)\boldsymbol{Z}$. So it suffices to show that D^2/n^2 is bounded below. This is clear if $D^2 \geqq 0$ for all D. We therefore consider the case in which there exists an irreducible curve C_0 with $C_0^2<0$. Let n_0 be the degree of C_0 over B. Let D be an arbitrary effective divisor of degree n over B. We can write $D=kC_0+D'$ with $k \geqq 0$, where the D' does not contain C_0 as its component. If n' denotes the degree of D' over B, then of course, $n'=n-kn_0$. Since $D'C_0\geqq 0$ and $(n_0D'-n'C_0)^2=0$, we have $n_0^2D'^2\geqq -n'^2C_0^2$. Thus

$$D^2\geqq k^2C_0^2+D'^2\geqq (n_0^2k^2-n'^2)\frac{C_0^2}{n_0^2},$$

and hence

$$\frac{D^2}{n^2}\geqq\left(1-\frac{2n'}{n}\right)\frac{C_0^2}{n_0^2}\geqq\frac{C_0^2}{n_0^2}. \qquad \square$$

LEMMA 3.2. *With the above notation, we have*

(i) *there exists at most one irreducible curve with negative self-intersection number,*

(ii) *if there is an n_0-section C_0 with $C_0^2\leqq 0$, then $s_n\geqq s_{n_0}$ for all n and $s_n=s_{n_0}$ if $n_0|n$,*

(iii) *if $s\leqq 0$, then $s_n=s$ for all $n>0$,*

(iv) *if $s>0$, then $s_n\geqq -s$ for all $n\geqq 2$,*

(v) *if $s>0$, then $s_n\geqq -\tau$ for all $n\geqq 2$.*

PROOF. (i)-(iv) follow immediately from the proof of Lemma 3.1. We prove (v). If $s_n\geqq 0$ for all $n\geqq 2$, then (v) holds trivially. Suppose that $s_{n_0}<0$ for some $n_0\geqq 2$. Choose n_0 minimal with this property. By the proof of Lemma 3.1, there is an n_0-section C_0 with $C_0^2<0$, so that $s_{n_0}=C_0^2/n_0^2$. Apply the Hurwitz formula to the ramified covering map $\tilde{C}_0\to B$ where $\tilde{C}_0$ is the normalization of C_0. Then we infer that $(K_Y+C_0)C_0\geqq n_0(2g-2)$. By Proposition 2.9, we have

$$(K_Y+C_0)C_0=n_0(2g-2+\tau)+\left(1-\frac{1}{n_0}\right)C_0^2.$$

It follows that

$$s_{n_0}=\frac{C_0^2}{n_0^2}\geqq\frac{-\tau}{n_0-1}\geqq -\tau \qquad (\text{because } n_0\geqq 2).$$

With the help of (ii) we conclude that $s_n\geqq s_{n_0}\geqq -\tau$ if $n\geqq n_0$. By the choice of n_0, of course $s_n\geqq 0$ if $n<n_0$. $\square$

EXAMPLE 3.3. We give an example with $s>0$, $s_2<0$. On the rational ruled surface $\boldsymbol{F}_1=\boldsymbol{P}(\mathcal{O}\oplus\mathcal{O}(-1))$ over $\boldsymbol{P}^1$, there is a smooth 2-section $C\in|2b+2f|$ where the b is the base section. Let P be a point on C where $C\to\boldsymbol{P}^1$ ramifies. Blow up 7-times over P at the points where the strict transforms of C meet the (-1)-curves. Contract all curves over the fibre passing through P except the remaining last (-1)-curve. Then we get a minimal ruled fibration $Y\to\boldsymbol{P}^1$. We see that Y has a singular fibre of type $\{(2,1),5\}$, so that $s=3/4$, $\tau=7/4$. If C_0 denotes the strict transform of C on Y, then C_0 is again a 2-section with $C_0^2=-3$, and so $s_2=-3/4$. In this example, $s=\tau-1$. See Theorem 3.5 below.

REMARK 3.4. In case Y is smooth, if $s>0$, then $s_n \geqq 0$ for all $n>0$ (for instance by Lemma 3.2, (v)). However, in the positive characteristic case, this is not the case. See [**H2**], Exercise 2.15, where an example ($\mathrm{ch}(k)=3$) with $s=1$, $s_3=-1$ can be found.

THEOREM 3.5. *Let Y be a normal projective surface with a minimal ruled fibration over a curve B of genus g. Then*

$$\text{(i)} \quad s_n \leqq \tau + \begin{cases} \dfrac{1}{n}\left[\dfrac{2ng}{n+1}\right] & (\textit{if } n \textit{ is odd}), \\[2ex] \dfrac{2}{n}\left[\dfrac{ng}{n+1}\right] & (\textit{if } n \textit{ is even}), \end{cases}$$

(ii) *if Y is singular, then*

$$s \leqq g+\tau-1 .$$

PROOF. We first consider the smooth case. Let T be a geometrically ruled surface $\boldsymbol{P}(\mathcal{E})$ defined by a rank 2 vector bundle $\mathcal{E}$ on B. By virtue of the observation in [**H1**], p. 51, there is a one to one correspondence between effective devisors D, having no fibre components, of degree n over B and invertible sheaves $\mathcal{L}$ on B which is a subline bundle of the n-th symmetric power $S^n\mathcal{E}$. The correspondence is given by

$$D \longrightarrow \mathcal{L} = p_*(\mathcal{O}_T(n) \otimes \mathcal{O}(-D)) \subsetneq S^n\mathcal{E} .$$

Furthermore, by using the computation in [**H1**], p. 52, we obtain

$$\frac{D^2}{n^2} = \deg\mathcal{E} - \frac{2}{n}\deg\mathcal{L} . \tag{3.6}$$

Choose D so that D^2/n^2 attains the minimum $s_n(T)$. In this case, D contains no fibre components, and the corresponding $\mathcal{L}$ is a maximal subline bundle of $S^n\mathcal{E}$. Note that $\operatorname{rank} S^n\mathcal{E}=n+1$, $\deg S^n\mathcal{E}=(1/2)n(n+1)\deg\mathcal{E}$. The Theorem in [**MS**] applied to $S^n\mathcal{E}$ yields the inequality:

$$\frac{n+1}{2}(n\deg\mathcal{E}-2\deg\mathcal{L}) \leqq ng .$$

Thus

$$n\deg\mathcal{E}-2\deg\mathcal{L} \leqq \left[\frac{2ng}{n+1}\right].$$

Also if n is even, we have

$$\frac{1}{2}(n\deg\mathcal{E}-2\deg\mathcal{L}) \leqq \left[\frac{ng}{n+1}\right].$$

Substituting (3.6) to these inequalities, we get

$$(3.7) \qquad s_n(T) = D^2/n^2 \leqq \begin{cases} \dfrac{1}{n}\left[\dfrac{2ng}{n+1}\right] & \text{if } n \text{ is odd,} \\ \dfrac{2}{n}\left[\dfrac{ng}{n+1}\right] & \text{if } n \text{ is even.} \end{cases}$$

Now we pass to the singular case. Let X, π, T, φ have the same meaning as in (2.8).

CLAIM 3.8. $$s_n(Y) \leqq s_n(T)+\tau .$$

PROOF. Let D be an effective divisor on T of degree n over B such that $s_n(T)=D^2/n^2$. Let $\bar{D}$ be the strict transform of D on X, and let D' denote the image of $\bar{D}$ on Y. Then

$$\begin{aligned} s_n(Y) &\leqq \frac{D'^2}{n^2} = 2g-2+\tau-\frac{1}{n}K_Y D' \\ &= 2g-2+\tau-\frac{1}{n}(K_X+\Delta)\bar{D} \\ &\leqq 2g-2+\tau-\frac{1}{n}K_X\bar{D} \\ &\leqq \frac{D^2}{n^2}+\tau = s_n(T)+\tau . \end{aligned}$$

This claim together with (3.7) yields the assertion (i).

Finally we prove (ii). We can choose T as $s(T)\leqq g-1$ under the assumption that Y is singular. By (3.7), we have always $s(T)\leqq g$. Suppose that $s(T)=g$. Since Y is singular, there must be a point P on T over which φ is not isomorphic. In case $s(T)=g$, Lemma 4.4 in [**LN**] (see also [**M**]) guarantees that there exists a base section passing through P. Let $T \dashrightarrow T'$ be the elementary transformation of T at P. It is easy to check that $X \to T'$ is still a morphism, and that $s(T')=g-1$. Therefore, by replacing T with T', we can make $s(T)\leqq g-1$. Consequently, the assertion (ii) follows from Claim 3.8. □

COROLLARY 3.9. *When $g\leqq 1$, we have*

$$s_n \leqq \tau - \begin{cases} 0 & \textit{in case } g=1 \\ 1 & \textit{in case } g=0 \end{cases}$$

for every n under the condition that Y is singular.

PROOF. In the proof of (ii), if $g\leqq 1$, we can make as $s(T)\leqq 0$. It follows from Lemma 3.2, (iii) that $s_n(T)=s(T)$ for all $n>0$. So by the inequality (ii), $s_n(Y)\leqq\tau$ (if $g=1$), $\leqq\tau-1$ (if $g=0$). □

Now we define the following invariant:

$$s_* = \inf\{s_n\}$$

where the infimum is taken over all positive integers n. The following properties of s_* are immediate from Lemma 3.2.

LEMMA 3.10. (i) *If there is an* n_0*-section* C_0 *with* $C_0^2 \leqq 0$, *then* $s_* = s_{n_0}$. *In particular, if* $s \leqq 0$, *then* $s_* = s$,

(ii) *if* $s > 0$, *then* $s_* \geqq -s$ *and* $s_* \geqq -\tau$,

(iii) *if* $s_* < 0$, *then there exists a unique irreducible curve* C_0 *with* $C_0^2 < 0$, *and in this case,* $s_* = s_{n_0}$ *where* $n_0 =$ *the degree of* C_0 *over* B.

LEMMA 3.11. *Let* D *be a divisor of degree* n $(n > 0)$ *over* B. *Then* D *is nef if and only if* $D^2/n^2 \geqq -s_*$.

PROOF. Clearly, $DF > 0$ for a fibre component F. So D is nef if $DC \geqq 0$ for all irreducible curves C of positive degree over B. Let C be an effective divisor of degree k $(k > 0)$ over B. Then

$$DC = \frac{nk}{2}\left(\frac{D^2}{n^2} + \frac{C^2}{k^2}\right).$$

If $D^2/n^2 \geqq -s_*$, then it follows that $DC \geqq 0$. Conversely, assume that D is nef. By the definition of s_*, for any $\varepsilon > 0$, there exists an effective divisor C such that $s_* \leqq C^2/k^2 < s_* + \varepsilon$ where $k =$ the degree of C over B. Since D is nef, $DC \geqq 0$, and so $D^2/n^2 \geqq -C^2/k^2 > -s_* - \varepsilon$. Letting $\varepsilon \to 0$, we find that $D^2/n^2 \geqq -s_*$. □

PROPOSITION 3.12. *The invariant* s_* *is a nonpositive rational number.*

PROOF. First we show that $s_* \leqq 0$. Assume to the contrary that $s_* > 0$. We can find a divisor D of positive degree over B such that $0 > D^2/n^2 > -s_*$, where $n =$ the degree of D over B. To see this, take an ample divisor H on Y. Choose a rational number α as $H^2 < 2\alpha h < H^2 + s_* h^2$ where h is the degree of H over B. Let N be a positive integer such that $N\alpha$ is integral. Then the divisor $D = N(H - \alpha f)$ satisfies the above condition. By Lemma 3.11, this D is nef, and hence we must have $D^2 \geqq 0$. This is a contradiction. The rationality of s_* is now clear from (iii) in Lemma 3.10. □

We say that Y is of *finite type* if $s_* = s_{n_0}$ for some n_0, and is of *infinite type* otherwise. Note that if $s_* < 0$, then Y is of finite type. In case $s_* = 0$, there occur both types.

EXAMPLE 3.13. Let B be a curve of genus $\geqq 2$. It is known that there exists a rank 2 vector bundle $\mathcal{E}$ on B such that all its symmetric powers $S^n\mathcal{E}$

are stable. Cf. [**H1**], Theorem 10.5. Let $T=\boldsymbol{P}(\mathcal{E})$. In this case, $s_n(T)>0$ for all n, and so T is of infinite type.

LEMMA 3.14. *If $-K_Y$ is pseudoeffective, then Y is of finite type.*

PROOF. We have only to consider the case: $s_*=0$. Take a divisor D of positive degree over B such that $D^2=0$. By Lemma 3.11, D is nef, and so $K_Y D\leqq 0$, because $-K_Y$ is pseudoeffective. By Lemma 2.11, there exists an effective $\boldsymbol{Q}$-divisor D' such that $D'\equiv D$, and hence $D'^2=0$ in this situation, which implies that Y is of finite type. □

A divisor D is *numerically positive* if $DC>0$ for all irreducible curves C on Y. Also D is *numerically ample* if D is numerically positive and $D^2>0$. In our case, since Y has only rational singularities, D is ample if and only if it is numerically ample (Nakai criterion).

PROPOSITION 3.15. *Let Y be a normal surface with a minimal ruled fibration. Let b be a base section, and f a fibre. Let $D\equiv nb+\alpha f$ be a divisor on Y. Then*

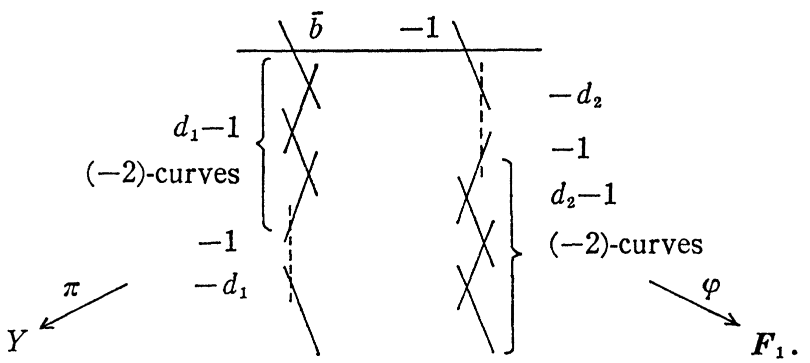
(i) *D is numerically positive if and only if $n>0$, $\alpha>-(n/2)(s+s_*)$ (in case Y is of finite type), or $n>0$, $\alpha\geqq -ns/2$ (in case Y is of infinite type, and so $s_*=0$).*

(ii) *D is ample* (resp. *nef*) *if and only if $n>0$, $\alpha>-(n/2)(s+s_*)$* (resp. *$n\geqq 0$, $\alpha\geqq -(n/2)(s+s_*)$*).

(iii) *D is pseudoeffective if and only if $n\geqq 0$, $\alpha\geqq -(n/2)(s-s_*)$.*

PROOF. For the smooth case, see [**H2**], p. 382. See also [**L**], [**S1**]. Of course $Df=n$. Also if C is an effective divisor of degree $k>0$ over B, then $DC=k(\alpha+(n/2)(s+C^2/k^2))$. Therefore, (i) follows from the definition of s_*. Also we see the criterion for the nefness. Since $D^2=2n(\alpha+ns/2)$ and $s_*\leqq 0$, in view of (i), we get the criterion for the ampleness. To see (iii), take a divisor $C=k(b-(1/2)(s+s_*)f)$ for a suitable positive integer k. By (ii), C is nef. Since $DC=k(\alpha+(n/2)(s-s_*))$, the condition: $\alpha\geqq -(n/2)(s-s_*)$ is necessary for the pseudo-effectiveness. The other implication is an easy consequence of (ii). □

REMARK 3.16. We claim that Y contains an exceptional curve of the first kind if and only if $g=0$, $s_*<0$, $\tau<2+s_*$. Indeed, we know that there is an irreducible curve C_0 with $C_0^2<0$ if and only if $s_*<0$. By Proposition 2.9, $K_Y C_0 =n_0(2g-2-s_*+\tau)$ where n_0=the degree of C_0 over B. So $K_Y C_0<0$ if and only if $2g-2-s_*+\tau<0$. Since $s_*<0$, this is equivalent to the condition: $g=0$, $\tau<2+s_*$. We give a series of examples. Consider $\boldsymbol{F}_1$ with a base section b, and construct two singular fibres of types $\{(d_1, 1), 0\}$ and $\{(d_2, 1), 0\}$. Let Y be the resulting normal surface. One can make the configuration as follows:



Here $\bar{b}$ is the strict transform of b. Let C_0 be the image of $\bar{b}$ on Y. Then

$$K_Y C_0 = -2/d_2\,, \qquad C_0^2 = -(d_2-d_1)/d_1 d_2\,.$$

So if $d_2>d_1$, then C_0 is an exceptional curve of the first kind.

§ 4. The anti-Kodaira dimension.

Let Y be a normal surface having a minimal ruled fibration $p: Y\to B$ over a curve B of genus g. We study the *anti-Kodaira dimension* $\kappa^{-1}(Y)$, which is defined to be $\kappa(-K_Y, Y)$. Cf. [**S1**], [**S3**]. Recall the *numerical type* of a divisor D on Y. We say that D is of type (a) if D is not pseudoeffective. In case D is pseudoeffective, let $D=P+N$ be the Zariski decomposition ([**S2**]) where P is a nef $\boldsymbol{Q}$-divisor. We have three types: (b) $P\equiv 0$, (c) $P^2=0$, $P\not\equiv 0$, (d) $P^2>0$.

We first consider the numerical type of the anticanonical divisor $-K_Y$. We fix a base section b on Y. In view of Proposition 2.9, it follows that

$$-K_Y \equiv 2b-(2g-2+s+\tau)f\,.$$

By Proposition 3.15, we obtain the following criteria:

$$\text{(4.1)} \qquad \begin{cases} -K_Y \text{ is pseudoeffective} \iff 2g-2+s_*+\tau \leqq 0 \\ -K_Y \text{ is nef} \qquad\qquad \iff 2g-2-s_*+\tau \leqq 0\,. \end{cases}$$

Suppose now that $-K_Y$ is pseudoeffective, but not nef. This is the case in which $s_*<2g-2+\tau\leqq -s_*$. In particular, $s_*<0$. So there exists an irreducible curve C_0 with $C_0^2<0$. If n_0 is the degree of C_0 over B, then $s_*=C_0^2/n_0^2$. See Lemma 3.10. With the notation of Proposition 2.9, set $\mathfrak{e}_0=\mathfrak{e}(C_0)$. Note that $\deg \mathfrak{e}_0= s_*+\tau$. The Zariski decomposition: $-K_Y=P+N$ is given by

$$\begin{cases} N=\left(1-\dfrac{2g-2+\tau}{s_*}\right)\dfrac{C_0}{n_0}\,, \\ P=-K_Y-N\,. \end{cases}$$

Furthermore, we have the linear equivalence:

$$n_0 P \sim -p^*(n_0(\mathfrak{f}+\mathfrak{e}_0)) + \left(1+\frac{2g-2+\tau}{s_*}\right) C_0 .$$

Also,

$$P^2 = \frac{(2g-2+s_*+\tau)^2}{-s_*} .$$

Therefore, if $P^2=0$, then $2g-2+s_*+\tau=0$, and hence

(4.2) $$n_0 P \sim -p^*(n_0(\mathfrak{f}+\mathfrak{e}_0)) .$$

Suppose next that $-K_Y$ is nef. By (4.1), $2g-2-s_*+\tau \leqq 0$, and so either $g=0$, $\tau - s_* \leqq 2$, or $g=1$, $\tau=0$, $s_*=0$. By (2.2), $K_Y^2=0 \Leftrightarrow \tau=2(1-g)$. So $-K_Y$ is of type (c) in the following cases (i) $g=0$, $\tau=2$, $s_*=0$, (ii) $g=1$, $\tau=0$, $s_*=0$.

As a consequence, we obtain the following

LEMMA 4.3. *The numerical type of $-K_Y$ is given by the following table:*

Type	$2g-2+s_*+\tau$	s_*
(a)	>0	
(b)	0	<0
(c)	0	$0 \begin{cases} g=0, & \tau=2 \\ g=1, & \tau=0 \end{cases}$
(d)	<0	

We now consider the anti-Kodaira dimension $\kappa^{-1}(Y)$.

Type (a). In this case, we have automatically $\kappa^{-1}(Y)=-\infty$.

Type (b). Using (4.2), we see that $\kappa^{-1}(Y)=0$ if $\mathfrak{f}+\mathfrak{e}_0$ is a torsion element, i.e., there exists a positive integer m such that $m(\mathfrak{f}+\mathfrak{e}_0)\sim 0$, and that $\kappa^{-1}(Y)=-\infty$ otherwise.

Type (c). For the case in which $g=1$, $\tau=0$, since Y is smooth, the previous results in [**S1**], [**S3**] imply that $\kappa^{-1}(Y)$ can take 0 and 1. For the case in which $g=0$, $\tau=2$, by Lemma 2.11, we see that $\kappa^{-1}(Y) \geqq 0$. Since $-K_Y$ is nef and $K_Y^2=0$, we see that $\kappa^{-1}(Y) \neq 2$. See Example 4.5 below for examples with $\kappa^{-1}(Y)=0$ and 1.

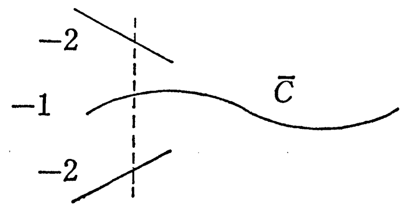
Type (d). It is known that $\kappa^{-1}(Y)=2$.

Summarizing we obtain the following

THEOREM 4.4. *Let Y be a normal projective surface with a minimal ruled fibration over a curve of genus g. Then the classification of Y in terms of $\kappa^{-1}(Y)$ is given as follows:*

$\kappa^{-1}(Y)$	Type	$2g-2+s_*+\tau$	s_*	Structure
$-\infty$	(a)	>0		
	(b)	0	<0	$\mathfrak{f}+\mathfrak{e}_0$ is not a torsion
0	(b)	0	<0	$\mathfrak{f}+\mathfrak{e}_0$ is a torsion
	(c)	0	0	$g=0,\ \tau=2$; $g=1,\ \tau=0$
1	(c)	0	0	$g=0,\ \tau=2$; $g=1,\ \tau=0$
2	(d)	<0		

EXAMPLE 4.5. Take a smooth cubic $C\subset \boldsymbol{P}^2$. Choose a point $P_0\in C$, which is not a flex. There are four distinct points $P_1, \cdots, P_4$ such that the lines $\overline{P_0P_i}$ are tangent to C. Blow up P_0, so that the resulting surface is $\boldsymbol{F}_1$. In this case, every line passing through P_0 corresponds to a fibre. Blow up over each point P_i in the following way. First blow up at P_i and then blow up at the point where the (-1)-curve meets the strict transform of C. Locally we have the following picture:



One of the (-2)-curves is the strict transform of the line $\overline{P_0P_i}$. By contracting the eight (-2)-curves, we get a normal surface Y with a minimal ruled fibration. There are four singular fibres of type $\{(2, 1), 0\}$. The strict transform C_0 of C on Y is a smooth elliptic curve. Note that C_0 is a 2-section with $C_0^2=0$. We have $g=0$, $\tau=2$, $s_*=0$. Let P_∞ be a flex on C. We claim that

$$\kappa^{-1}(Y)=\begin{cases} 1 & \text{if } (P_0-P_\infty) \text{ is a torsion element in } \mathrm{Pic}(C), \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, by construction we find that $K_Y\sim -C_0$. Clearly, C_0 is isomorphic to C, and with this isomorphism the normal sheaf $\mathcal{N}_{C_0}=\mathcal{O}(C_0)\otimes\mathcal{O}_{C_0}$ corresponds to the sheaf $\mathcal{O}(3(P_0-P_\infty))$. The assertion is then a consequence of Proposition 3.3 in [S3].

We give a criterion for the case in which Y admits another ruled fibration or an elliptic fibration. We begin with the following general result.

LEMMA 4.6. *Let D be a nef Cartier divisor on a normal surface Y. Suppose that $\kappa(D, Y)=1$. Then*

(i) *if $K_Y D<0$, then $|mD|$ for some positive integer m defines a ruled fibration on Y,*

(ii) *if $K_Y D=0$, then $|mD|$ for some positive integer m defines an elliptic fibration on Y.*

PROOF. We use a theorem of Zariski in the form in [**F2**], Theorem (4.1), which implies that $\mathcal{O}(mD)$ is generated by global sections for some $m>0$. It follows from this that the map defined by $|mD|$ provides a fibration onto a curve for some large m. Let f denote its general fibre. We find that $K_Y f<0$ or $=0$, according as $K_Y D<0$ or $=0$. Accordingly, f is a smooth rational curve or a smooth elliptic curve. □

PROPOSITION 4.7. *Let $p: Y \to B$ be a minimal ruled fibration on a normal surface Y over a curve B of genus g. Then Y admits another ruled fibration if and only if $g=0$, $\tau<2$ and $s_*=0$. In this case, $\tau=2(1-1/n)$ for some positive integer n.*

PROOF. Suppose that Y has another ruled fibration. Let l be its general fibre. Let f be a fibre of p. Since $l \cong \boldsymbol{P}^1$, we must have $g=0$. If we define $n=fl$, then by Proposition 2.9, $K_Y l=n(\tau-2)$. Since $K_Y l=-2$, it follows that $\tau=2(1-1/n)$. Since $l^2=0$, we infer that $s_n=0$ and $s_*=0$. Cf. Lemma 3.10, (i).

Conversely, assume that $g=0$, $\tau<2$ and $s_*=0$. Thanks to (4.1), we see that $-K_Y$ is nef. It follows from Lemma 3.14 that Y is of finite type. Since $s_*=0$, this means that there exists an n_0-section l_0 with $l_0^2=0$ for some n_0. In particular, $K_Y l_0=n_0(\tau-2)<0$. The Riemann-Roch theorem implies that $\kappa(l_0, Y)=1$. So by Lemma 4.6, there exists a ruled fibration on Y such that l_0 is a fibre. □

EXAMPLE 4.8. (i) In the example in Remark 3.16, if $d_1=d_2=d$, then we have the invariants: $g=0$, $\tau=2(1-1/d)$, $s=s_*=0$.

(ii) Starting from $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, construct a singular fibre of type $\{(d, 1), d\}$. In this case, we have $g=0$, $\tau=2(1-1/d)$ and $s=s_*=0$. Cf. Example 2.4, (ii).

PROPOSITION 4.9. *Let Y be a normal surface with a minimal ruled fibration over a curve B of genus g. Then Y admits an elliptic fibration if and only if $\kappa^{-1}(Y)=1$.*

PROOF. In view of Theorem 4.4, if $\kappa^{-1}(Y)=1$, then $-K_Y$ is nef and $K_Y^2=0$. We infer from Lemma 4.6 that Y has an elliptic fibration.

Conversely, assume that Y has an elliptic fibration. Let C be its general fibre. Set $n=fC>0$. By Proposition 2.9, $K_Y C=n(2g-2+\tau)$, because $C^2=0$. Since $K_Y C=0$, we find that $2g-2+\tau=0$. There occur two cases (i) $g=1$, $\tau=0$, (ii) $g=0$, $\tau=2$. In either case, by Theorem 4.4, $-K_Y$ is of type (c) and $\kappa^{-1}(Y)\geqq 0$. We therefore are able to find an effective divisor $D\in|-mK_Y|$ for some $m>0$. Since $DC=0$, D is contained in fibres of the elliptic fibration. We infer from this that each connected component of D is proportional to a fibre of the elliptic fibration. It follows that $\kappa^{-1}(Y)=1$. □

Let us observe when the anticanonical divisor $-K_Y$ is ample. Recall that in the smooth case, only $\boldsymbol{P}^1\times\boldsymbol{P}^1$ and $\boldsymbol{F}_1$ have this property among geometrically ruled surfaces. We infer from Proposition 3.15 that $-K_Y$ is ample $\Leftrightarrow 2g-2-s_*+\tau<0 \Leftrightarrow g=0$, $\tau<2+s_*$. There are two cases: (i) $s_*=0$, (ii) $s_*<0$. If $s_*=0$, we infer from Proposition 4.7 that Y admits another minimal ruled fibration and that $\tau=2(1-1/n)$ for some positive integer n. If $s_*<0$, by Remark 3.16, Y contains an exceptional curve of the first kind. Summarizing we obtain the following:

THEOREM 4.10. *Let Y be a normal projective surface with a minimal ruled fibration. Then the anticanonical divisor $-K_Y$ is ample if and only if either*
(i) *Y admits two distinct minimal ruled fibrations, or*
(ii) *Y contains an exceptional curve of the first kind.*

CONCLUDING REMARK 4.11. We refer to Fujita [**F1**] and Gurjar-Miyanishi [**GM**] for related topics on open surfaces.

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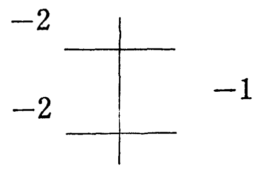
Added in Proof (Correction to the paper [**S4**]). As we have seen in this paper, a minimal ruled fibration on a normal surface may have multiple fibres. For this reason, in the proof of Theorem 1, type (a) in [**S4**], we insert the following: Let $f=mF$ be a fibre with multiplicity m. Since $(K_Y+H)f<0$, $Hf\geqq 1$, we find that $K_YF<-1$, $F^2=0$, which implies that $K_X\bar{F}=-2$, $\bar{F}^2=0$. It follows that $K_YF\geqq -2$. On the other hand, $K_Yf=m(K_YF)=-2$. So we must have $m=1$.

Accordingly, we correct the statement (ii) of Proposition 2 in [**S4**] as follows.

(ii) *the singular fibre is obtained by contracting all* (-2)*-curves in the following configurations:*

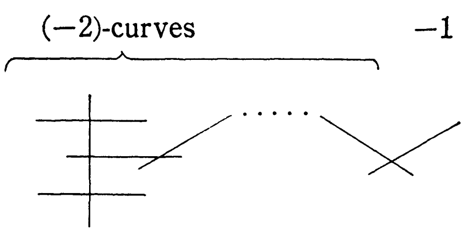
(ii-1) the same as in [**S4**],

(ii-2)



,

(ii-3)



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