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On the structure of locally convex filtrations on complete manifolds

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Introduction.

Let M be a connected complete Riemannian manifold without boundary. It is known that the existence of a convex function imposes a strong restriction on the topology of M. In fact, according to Greene and Shiohama [7], [8] and to Bangert [1] for a special case, if M admits a locally nonconstant convex function, then it is diffeomorphic to the normal bundle of a submanifold. In particular, M is noncompact. The author has shown in the previous work [22] that the same conclusion is still valid for a locally quasiconvex function, which is a generalization of convex functions. In this paper, we consider more general function concerning convexity. We say that a continuous function $f: M \to \boldsymbol{R}$ is a *locally convex filtration* if it is locally nonconstant and if all sublevel sets $M^a = \{x ; f(x) \leqq a\}$ are locally convex in M. This is a natural generalization of convex functions and locally quasiconvex functions. It should be noted that some compact manifolds admit such filtrations. For example, the function f on the unit sphere S^n in $\boldsymbol{R}^{n+1}$ defined by $f(x^1, \cdots, x^{n+1}) = -(x^{n+1})^2$ provides such an example. The purpose of the present paper is to characterize the geometric structure of the filtrations, and to classify the topological structure of manifolds admitting the filtrations of a certain type.

Let H_f^* be the union of level components of f intersecting the closure of the local maximum set H_f of f. Under a certain regularity condition on f, we shall prove that H_f *is* (*if it is not empty*) *a locally finite union of totally geodesic hypersurfaces and the complement* $H_f^* - H_f$ *is a Lipschitz submanifold* (Theorem 2.3), and that *each connected component of* $M - H_f^*$ *is homeomorphic* (*diffeomorphic if the boundary of the component is smooth*) *to the normal bundle of a submanifold* (Theorem 3.1). This is an extension of the works [1], [7], [8] and [22] stated in the beginning.

On the other hand, to treat the classification problem, it will be needed to restrict our filtrations to have a nice property because any complete surface has

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a regular filtration (Example 1.3). We say that f is *nice* if H_f^* has no branch points, or equivalently, if $H_f^*=H_f$ and it is a manifold. Then we shall obtain that the *diffeomorphism classes of compact Riemannian manifolds admitting nice filtrations can be classified into certain four types* (Theorem 4.3). In the noncompact case, however, it seems to us that the nicety condition does not interfere with the topological structure so as in the compact case. In fact, it will be seen that every noncompact surface has a nice filtration for a Riemannian metric. In this situation, we shall prove that *the absolute value of the Euler characteristic of a noncompact surface not homeomorphic to* $\boldsymbol{R}^2$ *is equal to the minimum number of connected components of* H_f *when* f *runs over all nice filtrations for all metrics admitting such filtrations* (Theorem 4.4). For *strict* nice filtrations, as corollaries to Theorem 4.3 and 4.4, we shall obtain that *a compact Riemannian manifold admitting a strict nice filtration is homeomorphic to a sphere or its quotient by a* $\boldsymbol{Z}_2$ *action* (Corollary 5.2), and that *a noncompact orientable surface admits a strict nice filtration for a metric if and only if the genus of the surface is zero* (Corollary 5.3).

In the proof, we shall develop convex analysis and Morse theory for the filtrations.

§ 1. Definition.

Let A be a subset of M. A is called *convex* if every two points in A are joined by a unique minimal geodesic and if it lies in A. A is called *locally convex* if every point in the closure $\bar{A}$ has a neighborhood U such that $A\cap U$ is convex. For subsets $A\subset B\subset M$, A is called *totally convex* in B if every two points in A can be joined by at least one geodesic in B and if each of these geodesics lies in A. We summarize some local properties of a locally convex set. See [**6**], [**19**], [**20**] and also [**4**] for the details.

Let A be a closed connected locally convex set in M. Then in the induced topology, A carries a manifold structure with (possibly empty) Lipschitz boundary ∂A. The interior Int$A$ is a smooth totally geodesic submanifold. The tangent cone $C_p(A)$ of A at a point p in A is by definition the set

$$\{0\}\cup\{v\in T_pM\,;\, \exp_p tv\in \mathrm{Int}A \text{ for all sufficiently small } t>0\},$$

where $\exp_p\colon T_pM\to M$ is the exponential mapping on the tangent space T_pM. The cone $C_p(A)$ is a convex cone. If the point p lies in ∂A, then the cone $C_p(A)$ is included in an open half space of the subspace spanned by $C_p(A)$. If $C_p(A)$ has the form of an open half space, then p is called a *smooth* point of ∂A. Since ∂A is Lipschitz, almost all points in ∂A are smooth with respect to the $(m-1)$-dimensional Hausdorff measure, $m=\dim A$. A tangent vector v in

T_pM, $p \in A$, is called *normal* to A if $\langle v, w\rangle \leqq 0$ for all w in $C_p(A)$. The set $\nu(A)$ of all normal vectors to A, which is equipped with the induced topology from the tangent bundle of M, is called the *normal bundle* of A in M. A closed locally convex set A has an open neighborhood U with the following properties:

(i) The boundary ∂U is of class C^1.

(ii) $\bar{U}$ is simply covered by minimal geodesics from points in ∂U to A, and these geodesics are transversal to ∂U.

An open set with these properties is called a *tubular neighborhood* of A. If A is compact, the proof of the existence of such U is standard. For the non-compact case, see [**21**], §6, proof of Theorem A. By the property (ii), U is homeomorphic to $\nu(A)$. We impose a differentiable structure on $\nu(A)$ by the requirement that the homeomorphism is a diffeomorphism. This is independent of the choice of U.

The following elementary lemma will be needed in the next section.

LEMMA 1.1. *Let A be locally convex and B convex. Then each connected component of $A \cap B$ is convex.*

Now we define our filtration once again for completeness.

DEFINITION 1.2. We say that a function $f: M \to R$ is a *locally convex filtration* if the following conditions are satisfied:

(i) f is continuous and locally nonconstant.

(ii) Each sublevel set M^a is locally convex,

where a function is called locally nonconstant if it is nonconstant on every non-empty open subset.

EXAMPLE 1.3. Let M have dimension two or constant sectional curvature. Then M can be triangulated with small simplices and with totally geodesic $(n-1)$-simplices, where $n = \dim M$. We can easily construct a locally convex filtration on M such that the maximum set coincides with the $(n-1)$-skelton.

We do not know whether every manifold has a metric on which a locally convex filtration exists. For a locally convex filtration f on M, we consider the function $m_f: M \to N \cup \{\infty\}$ defined by

$$m_f(p) = \limsup_{a \uparrow f(p),\, \varepsilon \downarrow 0} \#\{\text{components of } B(p, \varepsilon) \cap M^a\}.$$

We denote by $\#S$ the order of a set S, and by $B(p, \varepsilon)$ the open ball of radius ε around p. Here we also set $m_f(p) = 1$ for local minimum point p.

EXAMPLE 1.4. Let M be one dimensional. We can consider a Cantor set C in M. From the standard construction of Cantor set, we can define a locally convex filtration f on M such that the maximum set is equal to C and that $m_f(p)=\infty$ for p in C and $m_f(p)=1$ for p in $M-C$.

We say that f is *regular* if $m_f(p)<\infty$ for all p in M. Since irregular locally convex filtrations cause technical difficulty, we shall consider only regular locally convex filtrations, which we abbreviate as regular filtrations. In the next section, we shall examine the structure of local maximum set.

§ 2. Structure of local maximum set.

Let f be a regular filtration on M. For a point $p\in M$ that is not a local minimum point for f, we set $\lambda=f(p)$, $m=m_f(p)$ for simplicity. Choose $\varepsilon>0$ and $c<\lambda$ such that $M^\lambda\cap B(p, \varepsilon)$ as well as $B(p, \varepsilon)$ is convex and that the number of components of $B(p, \varepsilon)\cap M^a$ is equal to m for all a, $c\leqq a<\lambda$. Let $\{M_i^a\}_{i=1,\dots,m}$ be the collection of components of $B(p, \varepsilon)\cap M^a$. Rearranging the indices, we may assume $M_i^a\subset M_i^b$ for all a, b, $c\leqq a<b<\lambda$. We set $A_i(p, \varepsilon)=\bigcup_{c\leq a<\lambda}M_i^a$. Note that the closure $\bar{A}_i(p, \varepsilon)$ are convex sets containing p, and that $M^\lambda\cap B(p, \varepsilon)=\bigcup_{i=1}^m\bar{A}_i(p, \varepsilon)$. We shall use the notation $A_i(p, \varepsilon)$ implicitly.

LEMMA 2.1. *If $m_f(p)=1$, then there is $\delta>0$ such that $M^a\cap B(p, \delta)$ is convex for every $a\in\boldsymbol{R}$ if it is non-empty.*

PROOF. We choose ε and $c<\lambda:=f(p)$ as above. Take $\delta_1<\varepsilon$ so small that $M^a\cap B(p, \delta_1)$ is empty for every $a<c$. Hence by Lemma 1.1, $M^a\cap B(p, \delta_1)$ is convex or empty for every $a\leqq\lambda$. Since $M^\lambda\cap B(p, \delta_1)$ is convex, we have easily that there is $\alpha>\lambda$ such that $M^a\cap B(p, \delta_1)$ is convex for every a, $\lambda\leqq a\leqq\alpha$. For the proof, it suffices to take δ so small that $B(p, \delta)\subset M^\alpha$.

We denote by H_f the local maximum set of f, and by H_f^* the union of level sets components of f which meet $\bar{H}_f$. Clearly H_f^* includes $\bar{H}_f$. However, the case $H_f^*\supsetneqq\bar{H}_f$ may be occur, as is illustrated in Figure 1.

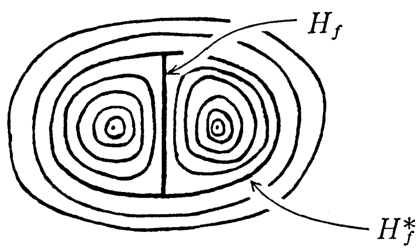

Figure 1.

LEMMA 2.2. $$\bar{H}_f=\{p\,;\,m_f(p)\geqq 2\}.$$

PROOF. Suppose $m_f(p)\geqq 2$ and take the mutually disjoint convex sets $\{A_i(p, \varepsilon)\}_{i=1,\dots,m}$, $m=m_f(p)$. Since $M^\lambda\cap B(p, \varepsilon)=\bigcup_{i=1}^m \bar{A}_i(p, \varepsilon)$, $\lambda=f(p)$, the boundary $\partial\bar{A}_i(p, \varepsilon)$ must intersect $\mathrm{Int}M^\lambda$. It is clear that some point in the intersection $\partial\bar{A}_i(p, \varepsilon)\cap\mathrm{Int}M^\lambda$ is a local maximum point. Since ε is taken arbitrary small, we have certainly that p is contained in $\bar{H}_f$. Suppose $m_f(p)=1$. Let δ be as in Lemma 2.1. By the lemma, $\partial M^a\cap B(p, \delta)=f^{-1}(a)\cap B(p, \delta)$ for all a. This implies that $q\in\partial M^{f(q)}$ for all q in $B(p, \delta)$, that is, q is not local maximum point. Hence p is not contained in $\bar{H}_f$.

THEOREM 2.3. *Suppose f is regular and H_f is not empty. Then the following statements are true:*

(1) *$H_f^*-H_f$ is (if it is not empty) a Lipschitz submanifold of codimension one.*

(2) *$\bar{H}_f$ is a locally finite union of totally geodesic hypersurfaces with (possibly empty and non smooth) boundary.*

PROOF. (1) Let K be a component of $H_f^*-H_f$. The function f takes a constant, say λ, on K. Since no point in K is local maximal for f and since f is locally nonconstant, we see that K is included in ∂M^λ. Hence by the definition of H_f^*, K coincides with a component of ∂M^λ, which is a Lipschitz submanifold of M of codimension one as is remarked in Section 1.

(2) We have to show that any point p in $\bar{H}_f$, there is ε such that $\bar{H}_f\cap B(p, \varepsilon)$ consists of a finite union of totally geodesic hypersurfaces. Take the mutually disjoint convex sets $\{A_i(p, \varepsilon)\}_{i=1,\dots,m}$, $m=m_f(p)\geqq 2$, and set $A_i=A_i(p, \varepsilon)$, $A_i'=\bar{A}_i\cap B(p, \varepsilon)$. Let Σ denote the set of all pairs (i, j), $1\leqq i, j\leqq m$ such that the dimension of convex set $A_i'\cap A_j'$ is equal to $n-1$, $n=\dim M$. For $(i, j)\in\Sigma$, we set $N_{ij}=A_i'\cap A_j'$ $(=\partial A_i'\cap\partial A_j')$, which is a totally geodesic hypersurface with (possibly empty) boundary. We show $\bar{H}_f\cap B(p, \varepsilon)=\bigcup_{(i,j)\in\Sigma}N_{ij}$. Clearly each N_{ij} is included in $\bar{H}_f\cap B(p, \varepsilon)$. We show the converse inclusion. To do this, we consider two cases.

First consider the case $p\in H_f$. Then we have easily

$$\bar{H}_f\cap B(p, \varepsilon) = H_f\cap B(p, \varepsilon) = \partial A_1'\cup\cdots\cup\partial A_m' .$$

For every point x in $H_f\cap B(p, \varepsilon)$, take i with $x\in\partial A_i'$. For each positive integer k, we put $D_k(x)=B(x, 1/k)\cap S(\partial A_i')$, where we denote by $S(\partial A)$ the set of all smooth points of the boundary of a locally convex set A. Note that each point in $\partial A_i'$ must be contained in another $\partial A_j'$, and that $S(\partial A_j')$ has full measure in $\partial A_j'$ with respect to the $(n-1)$-dimensional Hausdorff measure. This implies there is $j(k)\neq i$ such that $D_k(x)\cap S(\partial A_{j(k)}')$ has a positive measure, in particular, it is non-empty. Let x_k be a point in $D_k(x)\cap S(\partial A_{j(k)}')$. Then the tangent cones $C_{x_k}(A_i')$, $C_{x_k}(A_{j(k)}')$ have a unique supporting hyperplane in common, which is the boundary of the cones. This yields $(i, j(k))\in\Sigma$ and $x_k\in N_{ij(k)}$. Passing to a subsequence, we may assume $j(k)=j$ for all k. Thus we conclude $x=\lim x_k\in N_{ij}$.

For the case $p\in\bar{H}_f$, take any point x in $\bar{H}_f\cap B(p, \varepsilon)$ and a sequence x_k in H_f converging to x. From the above argument, x_k is contained in $N_{ij(k)}$ for some $(i, j(k))\in\Sigma$. Passing to a subsequence, we obtain $x\in N_{ij}$ for some j.

§ 3. Structure of $M-H_f^*$.

Let f be a regular filtration on M. In this section, we prove the following theorem.

THEOREM 3.1. *Each connected component of $M-H_f^*$ is homeomorphic (diffeomorphic if the boundary of the component is smooth) to the normal bundle of a submanifold.*

Cut M open along H_f^*. Let $\{N_i\}_{i=1,2,\dots}$ be the resulting connected Riemannian manifolds with (possibly non smooth) boundary. Note that N_i does not need to coincide with the closure of the corresponding component of $M-H_f^*$ (see Example 4.2 (2)). There is a natural extension $\tilde{N}_i$ of N_i, a Riemannian manifold without boundary, with the metric inherited from M. Note that N_i is locally convex in $\tilde{N}_i$. We set $N=N_i$ for simplicity. We have to show that $\mathrm{Int}N$ admits a normal bundle structure.

Let d_N is the distance function on N. By definition, $d_N(x, y)$ is the infimum of lengths of curves in N joining x and y. Since N is a closed connected locally convex set of $\tilde{N}$, we have immediately

LEMMA 3.2. *For every two points x and y in N, there is a geodesic in N which realize the distance $d_N(x, y)$.*

We put $N^a=\{x\in N; f(x)\leqq a\}$.

LEMMA 3.3. *N^a is totally convex in N.*

PROOF. Suppose there is a geodesic $\gamma: [0, 1]\to N$ such that γ is not included in N^a and $\gamma(0)\in N^a$, $\gamma(1)\in N^a$. The maximum b, $b>a$, of $f\circ\gamma$ is realized at a point $\gamma(t)$. If $\gamma(t)$ is a local maximum point of $f|N$, that is, if $\gamma(t)\in\partial N$, then γ would be broken at t. Since this is a contradiction, $\gamma(t)$ is not a local maximum point of $f|N$. Hence $\gamma(t)$ is contained in ∂N^b. But this is also a contradiction because $\gamma(0), \gamma(1)\in\mathrm{Int}N^b$.

For two points p, q in N, let $\mathfrak{V}^N(q, p)$ denote the set of all initial vectors to unit speed minimal geodesics in N from q to p. Then q is called a *non-critical point* of the distance function dist_p from p, in the sense of Gromov [11] (see also Grove-Shiohama [**12**]), if the set $\mathfrak{B}^N(q, p)$ is included in an open half space of T_qM. For a set S of N, we set $\mathfrak{B}^N(q, S)=\bigcup_{p\in S}\mathfrak{B}^N(q, p)$. Then we also say formally that q is a non-critical point of dist_S if $\mathfrak{B}^N(q, S)$ is included in an open half space of T_qM.

LEMMA 3.4. *If $q\in\partial N^a$, then there is a vector v in $C_q(N^a)$ such that $\mathfrak{B}^N(q, \mathrm{Int}N^a)$ is included in the open half space supported by v. In particular, q is a non-critical point of* $\mathrm{dist}_{(\mathrm{Int}N^a)}$.

PROOF. Since Lemma 3.3 shows $\mathfrak{B}^N(q, \mathrm{Int}N^a)\subset C_q(N^a)$, it suffices to take a vector v in the convex cone $C_q(N^a)$ such that $\langle v, w\rangle>0$ for all $w\in C_q(N^a)$.

PROOF OF THEOREM 3.1. Using Lemma 3.4, for every $a>\inf_N f$ and a compact set K in $\mathrm{Int}N^a$, we can construct a gradient-like vector field of f and dist_K on $N-\mathrm{Int}N^a$ so that any integral curve intersects ∂N^a (cf. [**21**], Proposition 5.4 and 5.5). The integral curves give rise to a homeomorphism between $\mathrm{Int}N-\mathrm{Int}N^a$ and $\partial N^a\times[0, 1)$. In the case when N is compact, take a so close to $\inf_N f$ that N^a is included in a tubular neighborhood of the minimum set of $f|N$. As a result, $\mathrm{Int}N$ is homeomorphic to the normal bundle of the minimum set. In the noncompact case, if $f|N$ has a minimum, then using the argument in [**21**], §6, Proof of Theorem A, we see that $\mathrm{Int}N$ is also homeomorphic to the normal bundle of the minimum set. If $f|N$ has no minimum, then we see that $\mathrm{Int}N$ is homeomorphic to a certain product $L\times(0, 1)$, using the argument in [**22**], §3, Proof of Theorem 2. In any case, the homeomorphism can be replaced by a diffeomorphism if ∂N is smooth.

REMARK 3.5. From Theorem B and C in [**21**], we obtain that every level set of $f|N$ has at most two compact components, and that all such compact components except the minimum set are mutually homeomorphic. On the other hand, in the case when every level set of $f|N$ has only noncompact components, there can be a large number of such components (cf. [**21**], §1, Example 9).

§4. Nice filtrations.

We now proceed to the classification problem of manifolds admitting locally convex filtrations. Example 1.3 implies that to treat the problem, we must impose another restriction to our filtration.

DEFINITION 4.1. We say that a locally convex filtration f on M is *nice* if H_f is closed and $m_f(p)\leqq 2$ for all p in M.

We abbreviate it as a nice filtration. This is equivalent to the requirement that H_f is a manifold and equal to H_f^*.

EXAMPLE 4.2. (1) Let S^n be a round sphere in $\boldsymbol{R}^{n+1}$. The function $f(x^1, \cdots, x^{n+1})=-(x^{n+1})^2$ is a nice filtration on S^n.

(2) Consider the function $f(e^{\sqrt{-1}\theta_1}, e^{\sqrt{-1}\theta_2})=\theta_1{}^2$, $-\pi\leqq\theta_1, \theta_2\leqq\pi$, on the flat torus $S^1\times S^1\subset \boldsymbol{C}^2$. It is also a nice filtration.

We consider a compact connected locally convex set A in a Riemannian manifold, and suppose either $\mathrm{codim}(A)\geqq 2$, or $\partial A\neq\emptyset$ and $\mathrm{codim}(A)=1$. Note that every tubular neighborhood of such A has connected boundary. We call a compact manifold with boundary a *cap* if it is diffeomorphic to the closure of a tubular neighborhood of such a locally convex set A. We exhibit four types of compact manifolds M: We say that M is of

type (I) if M is the quotient manifold of two caps by a diffeomorphism between the two boundaries,

type (II) if M is the quotient manifold of a cap by a free $\boldsymbol{Z}_2$ action on the boundary,

type (III) if M is the total space of a fibre bundle over S^1,

type (IV) if M is the quotient manifold of a product $L\times[0, 1]$, L is a compact manifold, by free $\boldsymbol{Z}_2$ actions on $L\times\{0\}$ and $L\times\{1\}$.

THEOREM 4.3. *If a compact Riemannian manifold has a nice filtration, then it is diffeomorphic to one of the above four types.*

PROOF. For a nice filtration f on a compact manifold M, let $\{N_i\}$ be as in Section 3. Note that N_i are smooth compact manifolds with totally geodesic boundary, and that M is obtained from the disjoint union of $\{N_i\}$ by identifying the boundaries along H_f. Let L_i be the minimum set of the restriction $f|N_i$. If $\mathrm{codim}(L_i)=1$, $\partial L_i=\emptyset$ and if L_i is two-sided, then we call N_i a *cylinder*. If $\mathrm{codim}(L_i)=1$, $\partial L_i=\emptyset$ and if L_i is one-sided, then we call N_i a *Möbius band*. Thus by Theorem 3.1, the possible topological types of N_i are devided into caps, cylinders and Möbius bands. If there are two Möbius bands in $\{N_i\}$, then M is of type (IV). If there is exactly one Möbius band in them, then M is of type (II). Suppose there are no Möbius bands in $\{N_i\}$. Then if there is a cap in them, then M is of type (I), and if there are no caps, then M is of type (III).

We now consider noncompact surfaces. Let $M(h, e)$ denote an orientable surface with h handles and e ends, and let $M'(m, e)$ denote a non-orientable surface with m Möbius caps and e ends, $0\leqq h, e, m\leqq\infty$, $e\neq 0$. As is exhibited in Figure 2 and 3, both $M(h, e)$ and $M'(m, e)$ have nice filtrations for some metrics (compare also [**21**], Example 8). Thus every noncompact surface M admits a metric for which M has a nice filtration f.

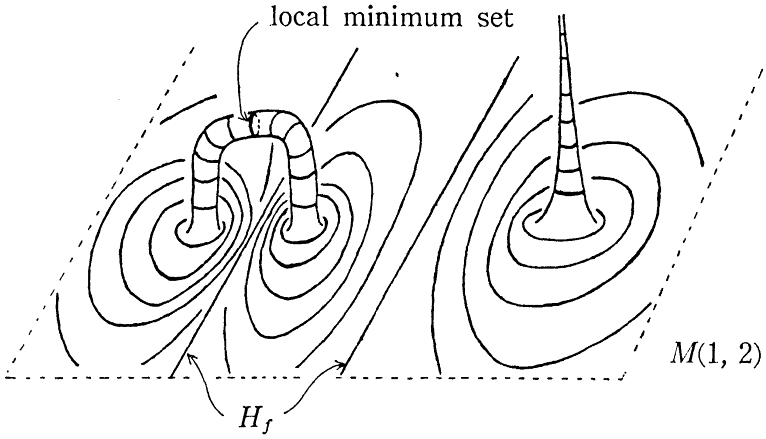

Figure 2.

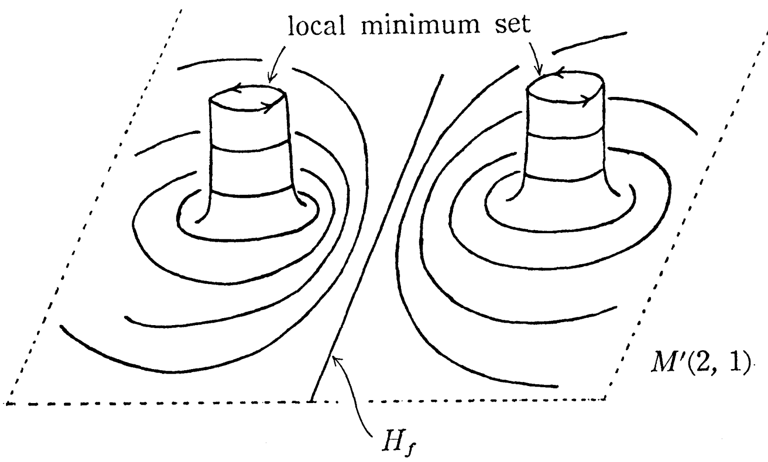

Figure 3.

It is natural, however, to guess that the number of components of H_f is related to the topological complexity of M. Let $\mathfrak{M}_M$ denote the set of all metrics on M for which M has a nice filtration. For a metric g in $\mathfrak{M}_M$, we denote by $k_M(g)$ the infimum of the numbers of components of H_f, where f runs over all nice filtrations on (M, g). Finally we set

$$k_M = \inf\{k_M(g) ; g \in \mathfrak{M}_M\}.$$

Since a plane admits a convex function, we have immediately $k_M=0$ for $M=\boldsymbol{R}^2$.

THEOREM 4.4. *If M is a noncompact surface not homeomorphic to $\boldsymbol{R}^2$, then the number k_M is equal to the absolute value of the Euler characteristic $\chi(M)$ of M.*

PROOF. For the filtration f on $M(h, e)$, $(h, e) \neq (0, 1)$ (resp. on $M'(m, e)$) as in Figure 2 (resp. Figure 3), the number of the components of H_f is equal to $2h+e-2=-\chi(M)$ (resp. $m+e-2=-\chi(M)$). This shows $k_M \leqq |\chi(M)|$. To prove the reverse inequality, let f be any nice filtration on M, and let $\{N_i\}$ be as in Section 3. Note that each N_i has the same topological type as a disk D^2, a cylinder Cy, or a Möbius band Mö. The manifold M is obtained from the disjoint union of $\{N_i\}$ by identifying the boundaries of N_i along each component of H_f. Let S be the connected surface obtained after some times of the identification. Let S' be the connected surface obtained from S and an N_i by identifying a component J_1 of ∂S and a component J_2 of ∂N_i along a component J of H_f. In the next lemma, we shall show that the identification corresponding to a component of H_f does not diminish the Euler characteristic more than two, that is, $\chi(S')\geqq\chi(S)-1$. Since $\chi(N_j)$ is one or zero for any j, and since M is not homeomorphic to $\boldsymbol{R}^2$, the iteration of the identification would yield

$$\chi(M) \geqq -(\text{the number of components of } H_f)$$

and hence $|\chi(M)|\leqq k_M$.

LEMMA 4.5. $\qquad \chi(S') \geqq \chi(S)-1.$

PROOF. When S is non-orientable, let S_0 be the orientable surface obtained by removing a maximal family of Möbius caps in S. We choose the orientation of ∂S induced from one of S_0. We also note that N_i may be included in S, that is, S' may be obtained by identifying the components J_1 and J_2 of ∂S along J. We denote by $h(S)$ the maximal number of handles in S if S is orientable, by $m(S)$ the maximal number of Möbius caps in S if S is non-orientable, and by $e(S)$ the number of ends of S.

Case (I). J is compact.

Case (I-1). N_i is not included in S.

If $N_i\approx D^2$, then $\chi(S')=\chi(S)+1$. If $N_i\approx$Cy or Mö, then $\chi(S')=\chi(S)$.

Case (I-2). N_i is included in S.

In the case, $e(S')=e(S)-2$. We show $\chi(S')=\chi(S)$. Suppose S is orientable. If J_1 and J_2 are identified in the same orientation (Figure 4), then S' is non-orientable, and we have $m(S')=2h(S)+2$. If J_1 and J_2 are identified in the reverse orientation (Figure 5), then S' is orientable and we have $h(S')=h(S)+1$. Suppose S is non-orientable. Then we have $m(S')=m(S)+2$. Thus in any case, χ is invariant in the case (I-2).

Case (II). J is noncompact.

Case (II-1). N_i is not included in S.

If $N_i\approx D^2$, then $\chi(S')=\chi(S)$. If $N_i\approx$Cy, then the genus is invariant and $e(S')=e(S)+1$. Hence $\chi(S')=\chi(S)-1$. Suppose $N_i\approx$Mö. Then S' is non-orientable and $e(S')=e(S)$. If S is orientable, then $m(S')=2h(S)+1$, and hence $\chi(S')=\chi(S)-1$. If S is non-orientable, then $m(S')=m(S)+1$, and hence $\chi(S')=\chi(S)-1$. Thus in the case (II-1), $\chi(S')$ is equal to $\chi(S)$ or $\chi(S)-1$.

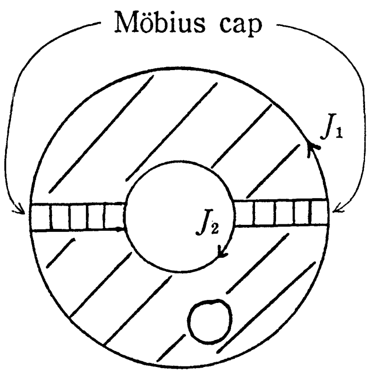

Figure 4.

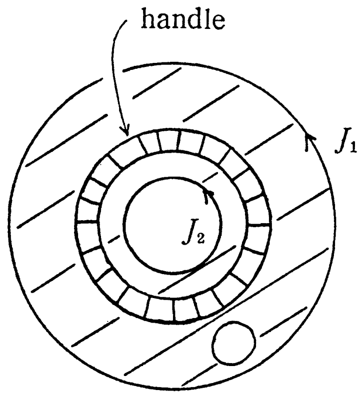

Figure 5.

Case (II-2). N_i is included in S.

In the case, we show $\chi(S')=\chi(S)-1$.

Case (II-2-i). J_1 and J_2 lie in one end of S.

Suppose J_1 and J_2 are identified in the same orientation (Figure 6). Then S' is non-orientable and $e(S')=e(S)$. When S is orientable, we have $m(S')=2h(S)+1$. When S is non-orientable, we have $m(S')=m(S)+1$. Hence $\chi(S')=\chi(S)-1$. Next suppose J_1 and J_2 are identified in the reverse orientations (Figure 7). Then the genus is invariant and $e(S')=e(S)+1$. Hence $\chi(S')=\chi(S)-1$.

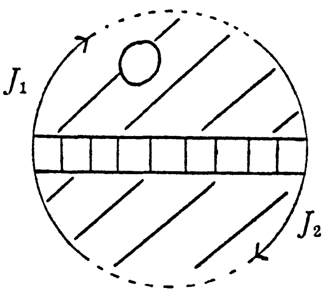

Figure 6.

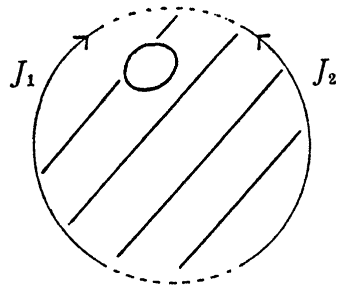

Figure 7.

Case (II-2-ii). J_1 and J_2 lie in distinct ends of N.

In the case, $e(S')=e(S)-1$. Suppose J_1 and J_2 are identified in the same orientation (Figure 8). Then S' is non-orientable and $m(S')=2h(S)+2$ if S is orientable, and $m(S')=m(S)+2$ if S is non-orientable. Thus $\chi(S')=\chi(S)-1$. Suppose J_1 and J_2 are identified in the reverse orientation (Figure 9). When S is orientable we see $h(S')=h(S)+1$. When S is non-orientable, we have $m(S')=m(S)+2$. Thus in any case, $\chi(S')=\chi(S)-1$. This completes the proof of Lemma 4.5.

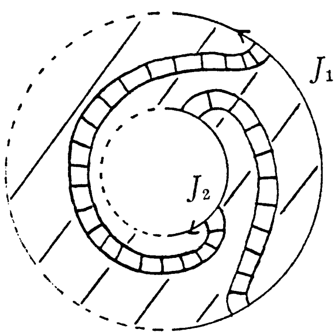

Figure 8.

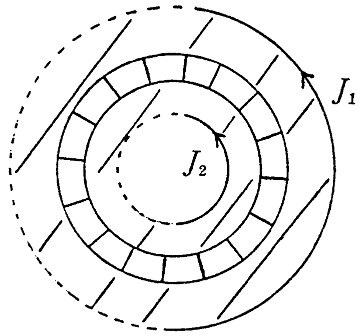

Figure 9.

We do not know whether every higher dimensional noncompact manifold has a nice filtration for a metric or not.

§ 5. Strict nice filtrations.

There is a concept of strictness for convexity. We employ it by the following

DEFINITION 5.1. We say that a locally convex filtration f on M is *strict* if it is nonconstant on every nonconstant geodesic segment in $M-H_f$.

Example 1.3 provides examples of regular strict filtrations on surfaces or manifolds of constant sectional curvature. The filtration on S^n in Example 4.2 is a strict nice filtration. The existence of a strict nice filtration f on M will impose intensive restriction on the topology of M. Note that f has at most one local minimum point in each component of $M-H_f$.

COROLLARY 5.2. *If a compact Riemannian n-manifold has a strict nice filtration, then it is homeomorphic to S^n or its quotient by a $\boldsymbol{Z}_2$ action.*

PROOF. For a strict nice filtration f on a compact manifold M, let $\{N_i\}$ be as in Section 3. Since $f|N_i$ has a unique minimum point, N_i is diffeomorphic to a disk. In particular, the number of the sets $\{N_i\}$ is one or two. Hence M is homeomorphic to S^n or its quotient by a $\boldsymbol{Z}_2$ action.

COROLLARY 5.3. *A noncompact orientable surface M has a strict nice filtration for some metric if and only if the genus of M is zero.*

PROOF. If M has genus zero, it certainly admits a strict nice filtration for a metric (cf. Figure 2). Conversely, for a strict nice filtration f on M, let $\{N_i\}$ be as in Section 3, and let S and S' be as in the proof of Theorem 4.4. If $f|N_i$ has a minimum, then N_i is a cell. Suppose $f|N_i$ has no minimum. If

$f|N_i$ has a compact level set then N_i is homeomorphic to a cylinder, and if $f|N_i$ has no compact level sets, then N_i is a cell (cf. [**21**], Theorem C, and [**22**], Theorem 2). These imply that all the components of ∂N_i are always included in an end of N_i. Hence, if $e(S')<e(S)$, then S' is obtained from D^2 and Cy by identifying the compact components of ∂D^2 and ∂Cy along S^1. This shows that S' coincides with M and it is homeomorphic to $\boldsymbol{R}^2$. Now suppose $e(S')\geqq e(S)$. If $h(S')>h(S)$, then $\chi(S')\leqq\chi(S)-2$, which is a contradiction to Lemma 4.5. Hence $h(S')\leqq h(S)$, and this implies $h(M)=0$, that is, the genus of M is zero.

REMARK 5.4. Probably, every noncompact non-orientable surface has a strict nice filtration for a metric. The metric will be realized by, for instance, one of constant negative curvature.

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