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The modified analytic trivialization of real analytic families via blowing-ups

Dedicated to Professor Yukihiro Kodama on his 60th birthday

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Introduction.

One of the most important and interesting problems in the theory of real analytic function-germs (or singularities) is to search for "nice and natural" equivalence relations in the set of germs of analytic functions.

I am sure that the notion of blow-analytic equivalence relation defined by Professor T.-C. Kuo ([**3, 4**]), is one of them.

Let $F(x;p):(\boldsymbol{R}^n\times P, 0\times P)\longrightarrow(\boldsymbol{R}, 0)$ be an analytic function, where P is a subanalytic subset of some Euclidean space. Then, T.-C. Kuo ([**4**]) proves the classification theorem: if for fixed p, $f_p(x):=F(x;p)$ has an isolated singularity at the origin, then there exists a finite filtration $\{P^i\}$ by subanalytic subsets P^i of the parameter space P of an analytic family $F(x;p)$ such that the functions $f_p(x)$ parameterized by elements p of a connected component of P^i form a blow-analytic equivalence class.

The next problem to be considered would be the following: can we construct concretely the filtration $\{P^i\}$ of P for a given analytic family $F(x;p)$ in the classification theorem or what kind of singularities form a blow-analytic equivalence class?

Several authors studied this problem, see e.g. [**1, 3, 5**].

In [**5**], it is proved that if a real analytic family $F(x;t)$ of real analytic function-germs $f_t(x):=F(x;t):(\boldsymbol{R}^n, 0)\longrightarrow(\boldsymbol{R}, 0)$ admits a simultaneous resolution ϕ, then it admits a $\pi\circ\phi$-MAT (see the definition (1.1)), where π is a finite succession of blowing-ups with non-singular centers of $\boldsymbol{R}^n$. So, the family $f_t(x)$ forms a blow-analytic equivalence class.

In [**1**] (resp. [**3**]), it is proved that if an analytic family $F(x;t)$ is non-degenerate in some sense, it admits a π-MAT along the parameter space via the blowing-up π of $\boldsymbol{R}^n$ at the origin (resp. a so-called toroidal embedding π). Here, it should be emphasized that the mapping π is concretely constructible from the Newton boundary of $F(x;t)$.

In this paper, we also study this problem. The subblowing-ups and the blowing-ups of $\boldsymbol{R}^n$ with the ideal centers defined by families are made use of to prove our results. Main results are formulated in (1.2), (1.3) and (1.4). I think that these results are ones of direct generalizations of the theorem formulated by making use of blowing-ups in [**1**]. The families in (1.4) are not treated in [**1, 3**].

I would like to thank T.-C. Kuo and S. Koike for valuable communications. The idea of singular Riemannian metric on Euclidean space used here, is originally suggested by T.-C. Kuo. I would like to thank the referee also. I revised the earlier draft of the introduction part of this paper according to the suggestion made by the referee.

1. Main Theorems.

Let $F(x\,;t):(\boldsymbol{R}^n\times\boldsymbol{R}^m,\ 0\times I)\longrightarrow(\boldsymbol{R},\ 0)$ be a real analytic function of $n+m$ variables $(x\,;t)=(x_1, x_2, \cdots, x_n\,; t_1, t_2, \cdots, t_m)$ in a neighbourhood of $\{0\}\times I$ where I is a compact cube $\times_{i=1}^{m}[a_i, b_i]$ in $\boldsymbol{R}^m$. Assume that $F(0\,;t)=0$ for any $t\in I$. We call the function $F(x\,;t)$ a *real analytic family* of functions

$$f_t(x):=F(x\,;t):(\boldsymbol{R}^n,\ 0)\longrightarrow(\boldsymbol{R},\ 0)\,.$$

Let $\pi: X\longrightarrow\boldsymbol{R}^n$ be a proper analytic modification of $\boldsymbol{R}^n$.

(1.1) DEFINITION. A real analytic family $F(x\,;t)$ admits an *almost π-modified analytic trivialization* (abbreviated to an *almost* π-MAT) along I if there exist a neighbourhood U of the origin of $\boldsymbol{R}^n$ and t-level preserving analytic isomorphism

$$\tilde{H}:\tilde{U}_1\times I\longrightarrow\tilde{U}_2\times I$$

where $\tilde{U}_1, \tilde{U}_2$ are two small neighbourhoods of $\pi^{-1}(0)$ in $\pi^{-1}(U)$ such that $F\circ(\pi\times\mathrm{id}_I)\circ\tilde{H}$ is independent of t. Here id_I is the identity map of I. Moreover, if the map $\tilde{H}$ induces a t-level preserving homeomorphism H between $\pi(\tilde{U}_1)\times I$ and $\pi(\tilde{U}_2)\times I$, then we say that the family $F(x\,;t)$ admits a *π-modified analytic trivialization* (abbreviated to a π-MAT) along I. Namely, the following diagram (1.1.1) is commutative:

(1.1.1) Commutative diagram.

$$\begin{tikzcd}
(X\times I,\ \pi^{-1}(0)\times I) \arrow[rr, "\pi\times\mathrm{id}_I"] \arrow[dr, "p_1"] \arrow[dd, "\tilde{H}"'] & & (\boldsymbol{R}^n\times I,\ 0\times I) \arrow[rr, "p_3"] \arrow[dl, "p_2"'] \arrow[dd, "H"] & & (\boldsymbol{R}^n,\ 0) \arrow[dd, "f_a"] \\
& I & & & \\
(X\times I,\ \pi^{-1}(0)\times I) \arrow[rr, "\pi\times\mathrm{id}_I"] \arrow[ur, "p_1"] & & (\boldsymbol{R}^n\times I,\ 0\times I) \arrow[rr, "F"] \arrow[ul, "p_2"'] & & (\boldsymbol{R},\ 0).
\end{tikzcd}$$

Where p_i, $1\leqq i\leqq 3$, are canonical projections and $a=(a_1, a_2, \cdots, a_m)$ is an element of I.

An $F(x;t)$ is called to be a family with a *fixed Newton polygon* Γ_+ if the Newton polygons of the all germs $f_t(x):=F(x;t)$ are simultaneously equal to $\Gamma_+=\Gamma_+(f_a)$.

For any real analytic family $F(x;t)$, we can choose finite analytic functions $c_j(x;t)$ with $c_j(0;t)\neq 0$ and monomials $x^{i_j}:=x_1^{i_{j1}}x_2^{i_{j2}}\cdots x_n^{i_{jn}}$, $0\leqq j\leqq k$, so that

$$F(x;t)=\sum_{j=0}^{k} c_j(x;t)x^{i_j}.$$

Let $\boldsymbol{R}\pi: \boldsymbol{R}X\longrightarrow \boldsymbol{R}^n$ (resp. $\boldsymbol{C}\pi|\boldsymbol{R}: \boldsymbol{C}X|\boldsymbol{R}\longrightarrow \boldsymbol{R}^n$) be the subblowing-up (resp. the blowing-up) of $\boldsymbol{R}^n$ with center $\boldsymbol{R}W$, where

$$\boldsymbol{R}W:=\{x\in \boldsymbol{R}^n \mid x^{i_0}=x^{i_1}=\cdots=x^{i_k}=0\}$$

(see §2). We can find also the definitions of notion of non-degeneracy in (3.1), (3.5), (3.6), the notion of coordinate face in (2.10) and the notion of transversal direction J_γ of a face γ of a Newton polygon in (2.7).

Now, we state the main results.

(1.2) THEOREM. *Let $F(x;t)$ be an $\boldsymbol{R}$ (resp. $\boldsymbol{C}$)-non-degenerate real analytic family. Then the family $F(x;t)$ admits an almost $\boldsymbol{R}\pi$*-MAT (*resp. an almost $\boldsymbol{C}\pi|\boldsymbol{R}$*-MAT) *along I.*

(1.3) THEOREM. *Let $F(x;t)$ be an $\boldsymbol{R}$ (resp. $\boldsymbol{C}$)-non-degenerate real analytic family. Suppose that $F_\gamma(x;t)$ is independent of t for any non-compact, non-coordinate face γ of Γ_+. Then the family $F(x;t)$ admits an $\boldsymbol{R}\pi$*-MAT (*resp. a $\boldsymbol{C}\pi|\boldsymbol{R}$*-MAT) *along I.*

(1.4) THEOREM. *Let $F(x;t)$ be a strongly $\boldsymbol{R}$ (resp. $\boldsymbol{C}$)-non-degenerate real analytic family. Then the family $F(x;t)$ admits an $\boldsymbol{R}\pi$*-MAT (*resp. a $\boldsymbol{C}\pi|\boldsymbol{R}$*-MAT) *along I.*

2. Subblowing-ups and blowing-ups.

We let

$K:=\boldsymbol{R}$ or $\boldsymbol{C}$,

$x^{i_j}:=x_1^{i_{j1}}x_2^{i_{j2}}\cdots x_n^{i_{jn}}$, $0\leqq j\leqq k$, monomials,

$KW:=\{x\in K^n \mid x^{i_j}=0,\ 0\leqq j\leqq k\}$,

$KS:=\{x\in K^n \mid x_1x_2\cdots x_n=0\}$,

$KX_W^*:=\{(x,\zeta)\in(K^n-KW)\times K\boldsymbol{P}^k \mid \zeta_0:\zeta_1:\cdots:\zeta_k=x^{i_0}:x^{i_1}:\cdots:x^{i_k}\}$,

$KX_S^* := \{(x, \zeta)\in(K^n - KS)\times \boldsymbol{KP}^k \mid \zeta_0:\zeta_1:\cdots:\zeta_k = x^{i_0}:x^{i_1}:\cdots:x^{i_k}\}$,

$KX := KX_W :=$ the topological closure of KX_W^* in the Hausdorff space $K^n\times \boldsymbol{KP}^k$,

$KX_S :=$ the topological closure of KX_S^* in the Hausdorff space $K^n\times \boldsymbol{KP}^k$

and

$\boldsymbol{C}X|\boldsymbol{R} := \boldsymbol{C}X\cap(\boldsymbol{R}^n\times \boldsymbol{RP}^k)$.

(2.1) DEFINITIONS.

(2.1.1) We call the canonical projection

$$\boldsymbol{R}\pi : \boldsymbol{R}X \longrightarrow \boldsymbol{R}^n$$

(or simply, $\boldsymbol{R}X$) *the subblowing-up of* $\boldsymbol{R}^n$ *with center* $\boldsymbol{R}W$.

(2.1.2) We call the canonical projection

$$\boldsymbol{C}\pi : \boldsymbol{C}X \longrightarrow \boldsymbol{C}^n$$

(or simply, $\boldsymbol{C}X$) *the blowing-up of* $\boldsymbol{C}^n$ *with center* $\boldsymbol{C}W$.

(2.1.3) We call the restriction map $\boldsymbol{C}\pi|\boldsymbol{R}$ of $\boldsymbol{C}\pi$ to the real part $\boldsymbol{C}X|\boldsymbol{R} := \boldsymbol{C}X\cap(\boldsymbol{R}^n\times \boldsymbol{RP}^k)$

$$\boldsymbol{C}\pi|\boldsymbol{R} : \boldsymbol{C}X|\boldsymbol{R} \longrightarrow \boldsymbol{R}^n$$

(or simply, $\boldsymbol{C}X|\boldsymbol{R}$) *the blowing-up of* $\boldsymbol{R}^n$ *with center* $\boldsymbol{R}W$.

It is well-known that the subblowing-up $\boldsymbol{R}X$ is a semi-algebraic set and the blowing-up $\boldsymbol{C}X$ (resp. $\boldsymbol{C}X|\boldsymbol{R}$) is the Zariski closure of $\boldsymbol{C}X_W^*$ (resp. $\boldsymbol{R}X_W^*$) in $\boldsymbol{C}^n\times \boldsymbol{CP}^k$ (resp. $\boldsymbol{R}^n\times \boldsymbol{RP}^k$).

The subblowing-up $\boldsymbol{R}\pi : \boldsymbol{R}X \longrightarrow \boldsymbol{R}^n$ is a proper surjective modification of $\boldsymbol{R}^n$ in the sense that the map $\boldsymbol{R}\pi$ is a proper surjective analytic map and the restriction

$$\boldsymbol{R}\pi|\boldsymbol{R}X_W^* : \boldsymbol{R}X_W^* \longrightarrow \boldsymbol{R}-\boldsymbol{R}W$$

is an analytic isomorphism of real analytic manifolds.

The blowing-up $\boldsymbol{C}\pi$ (resp. $\boldsymbol{C}\pi|\boldsymbol{R}$) is, of course, a proper surjective modification of $\boldsymbol{C}^n$ (resp. $\boldsymbol{R}^n$).

(2.2) LEMMA. $KX_W = KX_S$.

PROOF. Let (x^0, ζ^0) be a point of KX_W^*. There exists a monomial x^{i_j} such that $(x^0)^{i_j}\neq 0$ for $x^0=(x_1^0, x_2^0, \cdots, x_n^0)\notin KW$.

Define

$$x_p^m := \begin{cases} x_p^0 & \text{if } x_p^0 \neq 0\,, \\ 1/m & \text{otherwise} \end{cases}$$

and

$$\zeta^m := (x^m)^{i_0} : (x^m)^{i_1} : \cdots : (x^m)^{i_k}\,.$$

Then, $(x^m, \zeta^m) \in KX_S^*$ and

$$(x^m, \zeta^m) \longrightarrow (x^0, \zeta^0) \qquad \text{if } m \to \infty\,.$$

So, $(x^0, \zeta^0) \in KX_S$. This implies $KX_W^* \subset KX_S$ and $KX_W \subset KX_S$.

The converse inclusion $KX_W \supset KX_S$ follows the inclusion $KX_W^* \supset KX_S^*$.

This completes the proof of (2.2).

(2.3) LEMMA. *The blowing-up $\boldsymbol{C}\pi : \boldsymbol{C}X \longrightarrow \boldsymbol{C}^n$ (resp. the blowing-up $\boldsymbol{C}\pi | \boldsymbol{R} : \boldsymbol{C}X | \boldsymbol{R} \longrightarrow \boldsymbol{R}^n$, the subblowing-up $\boldsymbol{R}\pi : \boldsymbol{R}X \longrightarrow \boldsymbol{R}^n$) is independent of the choice of the generators of the ideal*

$$(x^{i_0}, x^{i_1}, \cdots, x^{i_k})K[x_1, x_2, \cdots, x_n]$$

for $K=\boldsymbol{C}$ (resp. $K=\boldsymbol{C}$, $K=\boldsymbol{R}$) up to an analytic isomorphism where $K[x_1, x_2, \cdots, x_n]$ is the ring of polynomials with coefficients K.

PROOF. It is well-known that the blowing-up $\boldsymbol{C}\pi$ and $\boldsymbol{C}\pi | \boldsymbol{R}$ are independent of the choice of the generators up to analytic isomorphism ([2]). The uniqueness of the subblowing-up $\boldsymbol{R}\pi$ follows the uniqueness of the blowing-up $\boldsymbol{C}\pi | \boldsymbol{R}$.

Let Γ_+ be the Newton polygon of the polynomial

$$x^{i_0} + x^{i_1} + \cdots + x^{i_k}\,,$$

namely Γ_+ be the convex hull of the set

$$\{i_j + \boldsymbol{R}_+{}^n \mid 0 \leqq j \leqq k\}$$

in $\boldsymbol{R}^n$, where $\boldsymbol{R}_+ := \{x \in \boldsymbol{R} \mid x \geqq 0\}$.

Adding some monomials x^{i_j}, $k+1 \leqq j \leqq l$, of the ideal $(x^{i_0}, x^{i_1}, \cdots, x^{i_k})K[x_1, x_2, \cdots, x_n]$ to the set of generators x^{i_j}, $0 \leqq j \leqq k$, if necessary, we may assume that the following properties (2.4) are satisfied.

(2.4) PROPERTIES.

(2.4.1) The Newton polygon of the polynomial $x^{i_0} + x^{i_1} + \cdots + x^{i_l}$ is Γ_+.

(2.4.2) The rank of $\{i_s - i_t \mid i_s, i_t \in \gamma,\ 0 \leqq s, t \leqq l\}$ is equal to $\dim \gamma$ for any face γ of Γ_+.

In the sequal of this section, we assume that the blowing-up and the subblowing-up of K^n with center KW satisfy the properties (2.4). Namely, let

KX_l be the topological closure of

$$KX^*_{W,l} := \{(x, \zeta) \in (K^n - KW) \times KP^l \mid \zeta_0 : \zeta_1 : \cdots : \zeta_l = x^{i_0} : x^{i_1} : \cdots : x^{i_l}\}$$

in the Hausdorff space $K^n \times KP^l$ and $K\pi_l : KX_l \longrightarrow K^n$ be the canonical projection. By (2.3), $K\pi$ is analytically isomorphic to $K\pi_l$, namely there exists an analytic isomorphism $\psi : KX \longrightarrow KX_l$ such that $K\pi = K\pi \circ \psi$. Define

$$KX^*_{S,l} := \{(x, \zeta) \in (K^n - KS) \times KP^l \mid \zeta_0 : \zeta_1 : \cdots : \zeta_l = x^{i_0} : x^{i_1} : \cdots : x^{i_l}\}.$$

(2.5) LEMMA. *For any $(x^0, \zeta^0) \in KX_l - KX^*_{S,l}$, there exists a unique face $\gamma = \gamma(x^0, \zeta^0)$ of the Newton polygon Γ_+ such that $\zeta^0_j \neq 0$ if and only if $i_j \in \gamma(x^0, \zeta^0)$.*

PROOF. By (2.2) and the curve selection lemma ([6]), there exists an analytic map $\phi : [0, \varepsilon) \longrightarrow KX_l$ such that $\phi(0) = (x^0, \zeta^0)$, $\phi(s) \in KX^*_{S,l}$ for $s \in (0, \varepsilon)$. Let $K\pi_l \circ \phi = (\phi_1, \phi_2, \cdots, \phi_n)$ and

$$\phi_p(s) := d_p s^{r_p} + (\text{higher terms}), \quad d_p \neq 0$$

be the Taylor expansion of ϕ_p for $1 \leqq p \leqq n$. Here, $\sum_{p=1}^n r_p > 0$ for $x^0 \in KS$. Define

$$\Phi(s) := (d_1 s^{r_1}, d_2 s^{r_2}, \cdots, d_n s^{r_n}).$$

Then,

$$(K\pi_l \circ \phi(s))^{i_0} : (K\pi_l \circ \phi(s))^{i_1} : \cdots : (K\pi_l \circ \phi(s))^{i_l} \longrightarrow \zeta_0$$

if $s \longrightarrow 0$. This implies that

$$(\Phi(s))^{i_0} : (\Phi(s))^{i_1} : \cdots : (\Phi(s))^{i_l} \longrightarrow \zeta_0$$

if $s \longrightarrow 0$.

Let $\gamma(x^0, \zeta^0)$ be the face of the Newton polygon Γ_+ such that the restriction to Γ_+ of the linear function $L(\nu_1, \nu_2, \cdots, \nu_n) := r_1\nu_1 + r_2\nu_2 + \cdots + r_n\nu_n$ takes the minimum value if and only if $(\nu_1, \nu_2, \cdots, \nu_n) \in \gamma(x^0, \zeta^0)$. Noting that $(r_1, r_2, \cdots, r_n) \neq (0, 0, \cdots, 0)$ for $x^0 \in KS$. Then, $\zeta^0_j \neq 0$ if and only if $i_j \in \gamma(x^0, \zeta^0)$. The property (2.4.2) guarantees the uniqueness of the face with this property.

(2.6) DEFINITION. We say that the point $(x^0, \zeta^0) \in KX_l - KX^*_{S,l}$ is *supported by the face* $\gamma(x^0, \zeta^0)$ of (2.5).

(2.7) DEFINITION. For a face γ of the Newton polygon Γ_+, a subset J_γ of $\{1, 2, \cdots, n\}$ is called the *transversal direction* of the face γ if J_γ is the subset of all indexes p of x_p-axis, each of which is transversal to the face γ, namely there is no parallel translation τ of $\boldsymbol{R}^n$ such that the affine space determined by γ contains $\tau(x_p\text{-axis})$.

(2.8) LEMMA. *Suppose that a point $(x^0, \zeta^0) \in KX_l - KX^*_S$ is supported by a face $\gamma = \gamma(x^0, \zeta^0)$ of Γ_+. Then, $x^0_p = 0$ if and only if $p \in J_\gamma$.*

PROOF. Let $\Phi(s)=(d_1s^{r_1}, d_2s^{r_2}, \cdots, d_ns^{r_n})$ be the analytic map in the proof of (2.5). Then, the restriction to Γ_+ of the linear function $L(\nu_1, \nu_2, \cdots, \nu_n)=r_1\nu_1+r_2\nu_2+\cdots+r_n\nu_n$ takes its minimum value just on the face $\gamma(x^0, \zeta^0)$. So, $r_p\neq 0$ if and only if $p\in J_\gamma$. Note that $\Phi(0)=x^0$. This completes the proof of (2.8).

(2.9) COROLLARY. *A face $\gamma(x^0, \zeta^0)$ is compact if and only if $x^0=0$.*

PROOF. Note that a face γ is compact if and only if the transversal direction J_γ of γ is equal to $\{1, 2, \cdots, n\}$. So, (2.8) implies (2.9).

(2.10) DEFINITION. We call a non-compact face γ of Γ_+ a *coordinate face* if γ is contained in some coordinate space and contains a non-empty open subset of the coordinate space.

(2.11) LEMMA.

(2.11.1) *Suppose that a point (x^0, ζ^0) is supported by a coordinate face $\gamma=\gamma(x^0, \zeta^0)$. Then*

$$K\pi_l^{-1}(x^0)=\{(x^0, \zeta^0)\}.$$

(2.11.2) *If $\# K\pi_l^{-1}(x^0)>1$ and $x^0\neq 0$, then a point $(x^0, \zeta^0)\in KX_l$ is supported by a non-compact, non-coordinate face.*

PROOF. (2.11.2) is an immediate corollary of (2.9) and (2.11.1).

So, let us prove (2.11.1). Since $\gamma(x^0, \zeta^0)$ is coordinate face, we may assume that $\dim\gamma=p$ and $\phi(s)=(d_1, \cdots, d_p, d_{p+1}s^{r_{p+1}}, \cdots, d_ns^{r_n})\to x^0=(d_1, \cdots, d_p, 0, \cdots, 0)$ if $s\to 0$.

If $i_j\in\gamma(x^0, \zeta^0)$, then $i_{jp+1}=\cdots=i_{jn}=0$ because $\gamma(x^0, \zeta^0)$ is a coordinate face. And $x^{i_j}\circ\phi(s)=d_1^{i_{j1}}\cdots d_p^{i_{jp}}$ if $i_j\in\gamma(x^0, \zeta^0)$ and $\operatorname{ord}(x^{i_j}\circ\phi(s))\geqq 1$ otherwise. Therefore ζ^0 is uniquely determined by $x^0=(d_1, \cdots, d_p, 0, \cdots, 0)$.

3. Non-degeneracy.

Let $f(x)=\sum_i c_ix^i$ be a germ of real analytic function at the origin of $\boldsymbol{R}^n$. Suppose $f(0)=0$. Define

$$f_\gamma(x)=\sum_{i\in\gamma} c_ix^i$$

for any subset γ of the Newton polygon $\Gamma_+(f)$ of $f(x)$. Let J be a subset of $\{1, 2, \cdots, n\}$ and $K=\boldsymbol{R}$ or $\boldsymbol{C}$.

(3.1) DEFINITION. A germ $f(x)$ is *K-J-non-degenerate* if the following equation

(3.1.1) $$x_p(\partial f_\gamma/\partial x_p)=0\,,\qquad p\in J$$

has no solutions in K^n-KS for any compact face γ of $\Gamma_+(f)$.

In particular, we call a germ $f(x)$ being *K-non-degenerate* if it is K-J-non-degenerate for $J=\{1, 2, \cdots, n\}$.

The following Lemma (3.2) is clear.

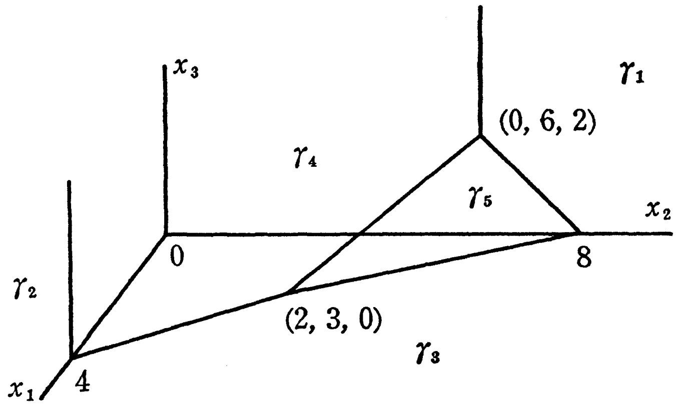
(3.2) LEMMA.

(3.2.1) *The* $\boldsymbol{C}$-*J-non-degeneracy of* $f(x)$ *implies the* $\boldsymbol{R}$-*J-non-degeneracy of* $f(x)$.

(3.2.2) *Suppose* $J_1\subset J_2\subset\{1, 2, \cdots, n\}$. *If a germ* $f(x)$ *is* K-J_1-*non-degenerate, then* $f(x)$ is K-J_2-*non-degenerate.*

(3.3) EXAMPLE. Let $f(x_1, x_2, x_3):=x_1^4+x_1^2x_2^3+x_2^8+x_2^6x_3^2$. The following figure (3.3.1) represents the Newton polygon $\Gamma_+(f)$ of $f(x)$.

(3.3.1) Figure.



The Newton polygon $\Gamma_+(f)$ has five faces of dimension two. They are $\gamma_1, \cdots, \gamma_5$ in the figure (3.3.1). Define

$$\gamma_{ij}:=\gamma_i\cap\gamma_j \quad\text{and}\quad \gamma_{ijk}:=\gamma_i\cap\gamma_j\cap\gamma_k\,.$$

The all compact faces of $\Gamma_+(f)$ are

$$\gamma_5,\ \gamma_{15},\ \gamma_{35},\ \gamma_{45},\ \gamma_{34},\ \gamma_{135},\ \gamma_{145},\ \gamma_{345} \text{ and } \gamma_{234}\,.$$

The all coordinate faces are

$$\gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_{13} \text{ and } \gamma_{23}\,.$$

The faces γ_4, γ_{14}, γ_{24} are non-compact and non-coordinate faces.

(3.3.2) *The germ* $f(x)$ *is* $\boldsymbol{R}$-$\{1, 2\}$-*non-degenerate and* $\boldsymbol{C}$-*non-degenerate.*

(3.3.3) *The germ* $f(x)$ *is* K-$\{1, 3\}$-*degenerate and* K-$\{2, 3\}$-*degenerate.*

(3.3.4) *The germ* $f(x)$ *is* $\boldsymbol{C}$-$\{1, 2\}$-*degenerate.*

PROOF. We have the following calculating table.

(3.3.5) Table.

compact face γ	$x_1\dfrac{\partial f_\gamma}{\partial x_1}$	$x_2\dfrac{\partial f_\gamma}{\partial x_2}$	$x_3\dfrac{\partial f_\gamma}{\partial x_3}$
γ_5	$2x_1^2x_2^3$	$3x_1^2x_2^3+8x_2^8+6x_2^6x_3^2$	$2x_2^6x_3^2$
γ_{15}	0	$8x_2^8+6x_2^6x_3^2$	$2x_2^6x_3^2$
γ_{35}	$2x_1^2x_2^3$	$3x_1^2x_2^3+8x_2^8$	0
γ_{45}	$2x_1^2x_2^3$	$3x_1^2x_2^3+6x_2^6x_3^2$	$2x_2^6x_3^2$
γ_{34}	$4x_1^4+2x_1^2x_2^3$	$3x_1^2x_2^3$	0
γ_{135}	0	$8x_2^8$	0
γ_{145}	0	$6x_2^6x_3^2$	$2x_2^6x_3^2$
γ_{345}	$2x_1^2x_2^3$	$3x_1^2x_2^3$	0
γ_{234}	$4x_1^4$	0	0

The table (3.3.5) shows that the equation (3.1.1) for $J=\{1, 2, 3\}$ has no solutions in K^3-KS for any compact face γ. So, the germ $f(x)$ is K-non-degenerate. The equation

$$x_1\frac{\partial f_\gamma}{\partial x_1}=x_2\frac{\partial f_\gamma}{\partial x_2}=0$$

has no solutions in $\boldsymbol{R}^3-\boldsymbol{R}S$. So, the germ $f(x)$ is $\boldsymbol{R}$-$\{1, 2\}$-non-degenerate. This implies (3.3.2).

For the compact face $\gamma=\gamma_{135}$, both the polynomials $x_1\partial f_\gamma/\partial x_1$ and $x_3\partial f_\gamma/\partial x_3$ are identically zeros. So, the germ $f(x)$ is K-$\{1, 3\}$-degenerate.

For the compact face $\gamma=\gamma_{234}$, both of the polynomials $x_2\partial f_\gamma/\partial x_2$ and $x_3\partial f_\gamma/\partial x_3$

are identically zeros. So, the germ $f(x)$ is K-$\{2, 3\}$-degenerate. This implies (3.3.3).

For $\gamma=\gamma_{15}$, $x_1\partial f_\gamma/\partial x_1=0$ and $x_2\partial f_\gamma/\partial x_2=2x_2^6(4x_2^2+3x_3^2)$. Then the equation $x_1\partial f_\gamma/\partial x_1=x_2\partial f_\gamma/\partial x_2=0$ has solutions in $\boldsymbol{C}^3-\boldsymbol{C}S$. This implies (3.3.4).

Let $f(x)$ be a germ of analytic function at the origin of $\boldsymbol{R}^n$. Then, there exist finite analytic functions $c_j(x)$ with $c_j(0)\neq 0$ such that

$$f(x)=\sum_{j=0}^{k} c_j(x)x^{i_j}.$$

Let $KW:=\{x\in K^n \mid x^{i_0}=x^{i_1}=\cdots=x^{i_k}=0\}$ and $K\pi: KX\longrightarrow K^n$ be the sub-blowing-up (resp. the blowing-up) of K^n with center KW for $K=\boldsymbol{R}$ (resp. $K=\boldsymbol{C}$).

(3.4) THEOREM. *A germ $f(x)$ is K-J-non-degenerate if and only if the linear equation*

$$(3.4.1)\qquad i_{0p}c_0(0)\zeta_0+i_{1p}c_1(0)\zeta_1+\cdots+i_{kp}c_k(0)\zeta_k=0, \qquad p\in J$$

has no solutions in $K\pi^{-1}(0)$.

PROOF. Suppose that a germ $f(x)$ is K-J-degenerate. There exist a compact face γ of $\Gamma_+(f)$ and a solution $x^0\in K^n-KS$ of the equation (3.1.1) for γ. Choose n positive integers $r_1, r_2, \cdots, r_n$ so that the restriction to $\Gamma_+(f)$ of the linear function $L(\nu)=r_1\nu_1+r_2\nu_2+\cdots+r_n\nu_n$ takes its minimum value just on the face γ. Let

$$\zeta_j^0:=\begin{cases} d(x^0)^{i_j},\ d\neq 0 & \text{if } i_j\in\gamma, \\ 0 & \text{otherwise.}\end{cases}$$

Then,

$$((s^{r_1}x_1^0, s^{r_2}x_2^0, \cdots, s^{r_n}x_n^0), \zeta)\in KX\longrightarrow(0, \zeta^0)$$

if s tends to zero.

Now,

$$\begin{aligned}\left(x_p\frac{\partial f_\gamma}{\partial x_p}\right)(x^0) &= \sum_{i_j\in\gamma} i_{jp}c_j(0)(x^0)^{i_j} \\ &= d\sum_{i_j\in\gamma} i_{jp}c_j(0)\zeta_j^0 \\ &= d\sum_{j=0}^{k} i_{jp}c_j(0)\zeta_j^0 \\ &= 0 \qquad \text{for } p\in J.\end{aligned}$$

So, $(0, \zeta^0)\in K\pi^{-1}(0)$ is a solution of the equation (3.4.1).

Conversely, suppose that the equation (3.4.1) has a solution $(0, \zeta^0)\in K\pi^{-1}(0)$. Let $\gamma=\gamma(0, \zeta^0)$ be the face supporting the point $(0, \zeta^0)$. Then the face γ is compact by (2.9). Let $\Phi(s)=(d_1s^{r_1}, d_2s^{r_2}, \cdots, d_ns^{r_n})$ be the analytic map in the proof of (2.5). Then, the polynomial $(x_p\partial f_\gamma/\partial x_p)(\Phi(s))$ is the initial term of $(x_p\partial f/\partial x_p)(\Phi(s))$. Let $\alpha:=\deg_s(x_p\partial f_\gamma/\partial x_p)(\Phi(s))$. Then,

$$\left(x_p\frac{\partial f_\gamma}{\partial x_p}\right)(d_1, d_2, \cdots, d_n)=\left(x_p\frac{\partial f_\gamma}{\partial x_p}\right)(\Phi(s))/s^\alpha$$
$$=\sum_{i_j\in\gamma} i_{jp}c_j(0)(\Phi(s)^{i_j}/s^\alpha).$$

The ratio $(\Phi(s)^{i_0}:\Phi(s)^{i_1}:\cdots:\Phi(s)^{i_k})$ tends to ζ^0 if s tends to zero.

On the other hand, $\sum_{i_j\in\gamma}i_{jp}c_j(0)\zeta_j^0=0$ because $\sum_{j=0}^k i_{jp}c_j(0)\zeta_j^0=0$ for $p\in J$ by the assumption and $\zeta_j^0=0$, $i_j\notin\gamma$ by (2.5).

So, $(x_p\partial f_\gamma/\partial x_p)(d_1, d_2, \cdots, d_n)=0$ for $p\in J$ and the equation (3.1.1) has a solution $(d_1, d_2, \cdots, d_n)$ in K^n-KS for the compact face γ.

Let $F(x;t)=\sum c_i(t)x^i$ be a real analytic family with a fixed Newton polygon Γ_+.

(3.5) DEFINITION. For $K=\boldsymbol{R}$ or $\boldsymbol{C}$, we call $F(x;t)$ *a K-J-non-degenerate family* if each function $f_t(x):=F(x;t)$, $t\in I$, is K-J-non-degenerate.

Let γ_p, $1\leqq p\leqq M$, be the all faces of Γ_+ (which contain the face of dimension n) and construct a new family $\tilde{F}(x;T)$, $T=(t_{q,p})\in\tilde{I}$, changing the parameter $t=(t_1, t_2, \cdots, t_m)$ of $F(x;t)$ as follows. Substitute a new parameter $t_{q,p}$ for the parameter t_q (if there exists) in the coefficient $c_i(t)$ of term x^i, $i\in\mathrm{Int}(\gamma_p)$. Then $\tilde{F}(x;T)$ is a real analytic family with the fixed Newton polygon Γ_+ and $\dim\tilde{I}\leqq mM$. We call $\tilde{F}(x;T)$ *the corresponding family* to $F(x;t)$. We denote the boundary of γ by $\partial\gamma$.

(3.6) DEFINITION. We call a K-non-degenerate family $F(x;t)$ *a strongly K-non-degenerate family* if the corresponding family $\tilde{F}(x;T)$ is K-J_γ-non-degenerate and $F_{\partial\gamma}(x;t)$ is independent of t for any non-compact, non-coordinate face γ of Γ_+.

We denote the function substituting $t_{q,p}$ for t_q of $c_j(t)$ as above by $c_j(T)$.

4. Kuo vector fields and singular Riemannian metrics.

Let $F(x;t)=\sum_{j=0}^k c_j(x;t)x^{i_j}$, $c_j(0;t)\neq 0$ be a real analytic function in a neighbourhood of $\{0\}\times I$, where I is a compact interval in $\boldsymbol{R}$ and $F(0;t)=0$. Namely, $F(x;t)$ is a real analytic family with one parameter.

Assume that $F(x;t)$ is a family with a fixed Newton polygon $\Gamma_+=\Gamma_+(f_t)$.

Let $K\pi: KX\longrightarrow K^n$ be the subblowing-up (resp. the blowing-up) of K^n with center KW where $KW=\{x\in K^n \mid x^{i_0}=x^{i_1}=\cdots=x^{i_k}=0\}$ if $K=\boldsymbol{R}$ (resp. $K=\boldsymbol{C}$).

In this section, we shall prove that there exists an analytic vector field $\tilde{V}$

defined in a neighbourhood of $\boldsymbol{R}\pi^{-1}(0)\times I$ (resp. $(\boldsymbol{C}\pi|\boldsymbol{R})^{-1}(0)\times I$) in $\boldsymbol{R}^n\times \boldsymbol{RP}^k\times \boldsymbol{R}$ if the all $f_t(x)$ are $\boldsymbol{R}$-non-degenerate (resp. $\boldsymbol{C}$-non-degenerate) with the following properties:

($\tilde{V}1$) $\tilde{V}$ is tangent to the analytic manifold $\boldsymbol{R}X_W^*\times \boldsymbol{R}$ and is tangent to the level set of $F\circ(\boldsymbol{R}\pi\times \mathrm{id}_I)$ (resp. $F\circ(\boldsymbol{C}\pi|\boldsymbol{R}\times \mathrm{id}_I)$) at its regular point,

($\tilde{V}2$) the t component of $\tilde{V}$ is $\partial/\partial t$.

Now, let us introduce a singular Riemannian metric on $\boldsymbol{R}^n$. Let $g_p(x)$, $1\leqq p\leqq n$, be analytic functions on $\boldsymbol{R}^n$. Define the inner product by

$$\left\langle g_p\frac{\partial}{\partial x_p},\ g_q\frac{\partial}{\partial x_q}\right\rangle := \delta_{pq}$$

where δ_{pq}, $1\leqq p,\ q\leqq n$, are the Kronecker's symbols. This inner product induces a Riemannian metric on $\boldsymbol{R}^n-\{x\in \boldsymbol{R}^n\,|\,g_1g_2\cdots g_n=0\}$, which we call a *singular Riemannian metric* on $\boldsymbol{R}^n$.

Using this singular metric, we have the following

(4.1) LEMMA.

(4.1.1) $$\mathrm{Grad}_x F=\sum_{p=1}^{n} g_p{}^2\frac{\partial F}{\partial x_p}\frac{\partial}{\partial x_p}.$$

(4.1.2) $$|\mathrm{Grad}_x F|^2=\sum_{p=1}^{n}\left(g_p\frac{\partial F}{\partial x_p}\right)^2.$$

PROOF. Put $\mathrm{Grad}_x F:=\sum_{p=1}^{n} a_p\partial/\partial x_p$. Then, by the definition of $\mathrm{Grad}_x F$, we have:

$$\frac{\partial F}{\partial x_p}=\left\langle \mathrm{Grad}_x F,\ \frac{\partial}{\partial x_p}\right\rangle=\left\langle a_p\frac{\partial}{\partial x_p},\ \frac{\partial}{\partial x_p}\right\rangle=\frac{a_p}{g_p{}^2}.$$

So, $a_p=g_p{}^2\partial F/\partial x_p$. This completes the proof of (4.1.1).

(4.1.2) We have:

$$\begin{aligned}|\mathrm{Grad}_x F|^2&=\left\langle \sum_{p=1}^{n} g_p{}^2\frac{\partial F}{\partial x_p}\frac{\partial}{\partial x_p},\ \sum_{p=1}^{n} g_p{}^2\frac{\partial F}{\partial x_p}\frac{\partial}{\partial x_p}\right\rangle\\&=\sum_{p=1}^{n}\left(g_p\frac{\partial F}{\partial x_p}\right)^2.\end{aligned}$$

Recall so called a Kuo vector field $V(x\,;t)$ ([3]):

$$\begin{aligned}V(x\,;t)&:=\frac{|\mathrm{Grad}_{x,t}F|^2}{|\mathrm{Grad}_x F|^2}\left(\frac{\partial}{\partial t}-\left\langle\frac{\partial}{\partial t},\ \frac{\mathrm{Grad}_{x,t}F}{|\mathrm{Grad}_{x,t}F|}\right\rangle\frac{\mathrm{Grad}_{x,t}F}{|\mathrm{Grad}_{x,t}F|}\right)\\&=\frac{-\partial F/\partial t}{|\mathrm{Grad}_x F|^2}\mathrm{Grad}_x F+\frac{\partial}{\partial t}.\end{aligned}$$

The vector field $V(x;t)$ is tangent to the level set of $F(x;t)$ at its any regular point, by definition. By (4.1), we have the following representation of $V(x;t)$.

(4.2) LEMMA.

$$V(x;t)=-\frac{\partial F}{\partial t}\sum_{p=1}^{n}g_p{}^2\frac{\partial F}{\partial x_p}\frac{\partial}{\partial x_p}\Big/\sum_{p=1}^{n}\Big(g_p\frac{\partial F}{\partial x_p}\Big)^2+\frac{\partial}{\partial t}.$$

Define

$$E_p:=\begin{cases} x_1x_2\cdots x_n/x_p & \text{if } p\notin J,\\ 1 & \text{otherwise.}\end{cases}$$

$$A_p(x,\zeta;t):=\frac{-\Big(\sum_{j=0}^{k}(\partial c_j/\partial t)\zeta_j\Big)E_p{}^2\Big\{\sum_{j=0}^{k}i_{jp}c_j(x;t)\zeta_j+x_p\sum_{j=0}^{k}(\partial c_j/\partial x_p)\zeta_j\Big\}}{\sum_{p=1}^{n}E_p{}^2\Big\{\sum_{j=0}^{k}i_{jp}c_j(x;t)\zeta_j+x_p\sum_{j=0}^{k}(\partial c_j/\partial x_p)\zeta_j\Big\}^2}.$$

(4.3) THEOREM. *Suppose that $F(x;t)$ be a real analytic family of germs of $\boldsymbol{R}$-J-non-degenerate (resp. $\boldsymbol{C}$-J-non-degenerate) functions $f_t(x):=F(x;t)$ with a fixed Newton polygon Γ_+. Then, the vector field defined in $\boldsymbol{R}^n\times U_r\times\boldsymbol{R}=\{(x,\zeta;t)\mid \zeta_r\neq 0\}$*

$$\tilde{V}^r:=\sum_{p=1}^{n}A_p(x,\zeta;t)x_p\frac{\partial}{\partial x_p}+\sum_{j=0}^{k}\sum_{p=1}^{n}A_p(x,\zeta;t)(i_{jp}-i_{rp})\frac{\zeta_j}{\zeta_r}(1-\delta_{jr})\frac{\partial}{\partial(\zeta_j/\zeta_r)}+\frac{\partial}{\partial t}$$

is analytic in a neighbourhood of $\boldsymbol{R}\pi^{-1}(0)\times I$ (resp. $(\boldsymbol{C}\pi\mid\boldsymbol{R})^{-1}(0)\times I$) in $\boldsymbol{R}^n\times\boldsymbol{RP}^k\times\boldsymbol{R}$ and satisfies the conditions ($\tilde{V}$1) and ($\tilde{V}$2).

PROOF. Substituting $x=0$ in the denominator of the coefficient of $\partial/\partial x_p$, we have $\sum_{p\in J}\{\sum_{j=0}^{k}i_{jp}c_j(0;t)\zeta_j\}^2$.

By (3.4), it does not vanish on $\boldsymbol{R}\pi^{-1}(0)\times I$ (resp. $(\boldsymbol{C}\pi\mid\boldsymbol{R})^{-1}(0)\times I$) because the germs $f_t(x)$ are $\boldsymbol{R}$-J-non-degenerate (resp. $\boldsymbol{C}$-J-non-degenerate). So, the vector field $\tilde{V}(x,\zeta;t)$ is analytic in a neighbourhood of $\boldsymbol{R}\pi^{-1}(0)\times I$ (resp. $(\boldsymbol{C}\pi\mid\boldsymbol{R})^{-1}(0)\times I$) in $\boldsymbol{R}^n\times\boldsymbol{RP}^k\times\boldsymbol{R}$, where $\tilde{V}\mid\boldsymbol{R}^n\times U_r\times\boldsymbol{R}:=\tilde{V}^r$.

Now, let (x,ζ) be a point of $\boldsymbol{R}X_S^*$. Then,

$$d(\boldsymbol{R}\pi\times\mathrm{id}_I)(\tilde{V}(x,\zeta;t))=d((\boldsymbol{C}\pi\mid\boldsymbol{R})\times\mathrm{id}_I)(\tilde{V}(x,\zeta;t))$$

$$=\frac{-\Big(\sum_{j=0}^{k}\partial c_j/\partial t\,x^{i_j}\Big)\sum_{p=1}^{n}E_p{}^2\Big\{\sum_{j=0}^{k}i_{jp}c_j(x;t)x^{i_j}+x_p\sum_{j=0}^{k}\partial c_j/\partial x_p\,x^{i_j}\Big\}x_p\partial/\partial x_p}{\sum_{p=1}^{n}E_p{}^2\Big\{\sum_{j=0}^{k}i_{jp}c_j(x;t)x^{i_j}+x_p\sum_{j=0}^{k}\partial c_j/\partial x_p\,x^{i_j}\Big\}^2}+\frac{\partial}{\partial t}$$

$$= \frac{-\partial F/\partial t \sum_{p=1}^{n} E_p{}^2 x_p(\partial F/\partial x_p)x_p\partial/\partial x_p}{\sum_{p=1}^{n} (E_p x_p \partial F/\partial x_p)^2} + \frac{\partial}{\partial t}$$
$$= V(x\,;t)\,.$$

The last equality is proved by the application of (4.2) in which put $g_p := E_p x_p$, $1 \leqq p \leqq n$.

This shows that the vector field $\tilde{V}(x, \zeta\,; t)$ is tangent to the analytic manifold $\boldsymbol{R}X_{\S}^{\sharp}\times\boldsymbol{R}$ and is tangent to the level set of $F\circ(\boldsymbol{R}\pi\times\mathrm{id}_I)$ (resp. $F\circ(\boldsymbol{C}\pi|\boldsymbol{R}\times\mathrm{id}_I)$) at its regular point since the Kuo vector field $V(x\,;t)$ satisfies similar properties. Thus, the vector field $\tilde{V}(x, \zeta\,; t)$ satisfies the condition ($\tilde{V}$1).

The condition ($\tilde{V}$2) is clearly satisfied by the definition of $\tilde{V}(x, \zeta\,; t)$. This completes the proof of (4.3).

5. Proof of main Theorems.

PROOF OF (1.2). The proof of Theorem (1.2) in the case of $\boldsymbol{C}$-non-degeneracy is completely parallel to the proof of Theorem (1.2) in the case of $\boldsymbol{R}$-non-degeneracy, using the blowing-up $\boldsymbol{C}\pi|\boldsymbol{R}\colon \boldsymbol{C}X|\boldsymbol{R} \longrightarrow \boldsymbol{R}^n$ of $\boldsymbol{R}^n$ with center $\boldsymbol{R}W$ instead of the subblowing-up $\boldsymbol{R}\pi\colon \boldsymbol{R}X \longrightarrow \boldsymbol{R}^n$ of $\boldsymbol{R}^n$ with center $\boldsymbol{R}W$.

Now, let us prove Theorem (1.2) in the case of $\boldsymbol{R}$-non-degeneracy.

Let $F(x\,;t)\colon \boldsymbol{R}^n\times I \longrightarrow \boldsymbol{R}$ be a real analytic family of germs of $\boldsymbol{R}$-non-degenerate real analytic functions with a fixed Newton polygon $\Gamma_+ = \Gamma_+(f_t)$ where $I = \times_{q=1}^{m}[a_q, b_q]$ be a compact cube in $\boldsymbol{R}^m$.

Let $\tilde{V}_q$ be an analytic vector field in a neighbourhood of $\boldsymbol{R}\pi^{-1}(0)\times I$ in $\boldsymbol{R}^n\times \boldsymbol{RP}^k\times\boldsymbol{R}^m$ with the properties:

($\tilde{V}_q$1) $\tilde{V}_q$ is tangent to the analytic manifold $\boldsymbol{R}X_{W}^{\sharp}\times\boldsymbol{R}$ and is tangent to the level set of $F\circ(\boldsymbol{R}\pi\times\mathrm{id}_I)$ at its regular point,

($\tilde{V}_q$2) the t component of $\tilde{V}_q$ is $\partial/\partial t_q$ for $1\leqq q\leqq m$.

By (4.3) in which put $J := \{1, 2, \cdots, n\}$, $\partial/\partial t := \partial/\partial t_q$, the existence of $\tilde{V}_q$ is guaranteed. Let $\phi_q(t_q\,; x, \zeta, c)$ be the trajectory of $\tilde{V}_q$ with $\phi_q(0\,; x, \zeta, c) = (x, \zeta, c)$. Then,

$$\tilde{H}(x, \zeta, t) := \phi_m(t_m - a_m\,; \phi_{m-1}(\cdots\,; \phi_1(t_1 - a_1\,; x, \zeta, a)\cdots))$$

is an analytic isomorphism between two neighbourhoods of $\boldsymbol{R}\pi^{-1}(0)\times I$ in $\boldsymbol{R}^n\times \boldsymbol{RP}^k\times\boldsymbol{R}$.

Since the vector field $\tilde{V}_q$ has the properties ($\tilde{V}_q$1) and ($\tilde{V}_q$2), the restriction of $\tilde{H}(x, \zeta\,; t)$ to $\boldsymbol{R}X\times\boldsymbol{R}$ induces an analytic isomorphism between two neighbourhoods of $\boldsymbol{R}\pi^{-1}(0)\times I$ in $\boldsymbol{R}X\times\boldsymbol{R}$ and the function $F\circ(\boldsymbol{R}\pi\times\mathrm{id}_I)\circ\tilde{H}(x, \zeta\,; t)$ is independent of t. This completes the proof of Theorem (1.2) in the case of

$\boldsymbol{R}$-non-degeneracy.

PROOF OF (1.3). It is sufficient to prove Theorem (1.3) in the case of $\boldsymbol{R}$-non-degeneracy by the same reason in the proof of (1.2).

In this proof, we make use of some results in Section 2, and so we assume that the subblowing-up of $\boldsymbol{R}^n$ with center $\boldsymbol{R}W$ is the $\boldsymbol{R}X_l$ defined by the monomials x^{i_j}, $0 \leqq j \leqq l$ satisfying the conditions (2.4.1) and (2.4.2).

Since $F(x\,;t)$ is an $\boldsymbol{R}$-non-degenerate real analytic family, the hypothesis of Theorem (1.2) is satisfied. So, there exists an analytic isomorphism $\tilde{H}(x, \zeta\,; t)$ proving (1.2). For the proof of (1.3), it is sufficient to show that the analytic vector field $\tilde{V}_q(x, \zeta\,; t)$ in the proof of (1.2) is tangent to $\boldsymbol{R}\pi^{-1}(x^0) \times \boldsymbol{R}$ for any $x^0 \in \boldsymbol{R}W$ whose inverse image of $\boldsymbol{R}\pi$ is not a one point set. In fact, if the all vector fields $\tilde{V}_q$, $1 \leqq q \leqq m$, have these properties, then the analytic isomorphism $\tilde{H}$ induces the homeomorphism H between two neighbourhoods of $\{0\} \times I$ in $\boldsymbol{R}^n \times \boldsymbol{R}^m$ proving (1.3).

(5.1) LEMMA. *The coefficient of $\partial/\partial x_p$ in $\tilde{V}_q(0, \zeta\,; t)$ vanishes for any p, q.*

PROOF. This is clear by the definition of $\tilde{V}_q$.

(5.2) LEMMA. *If $\#\boldsymbol{R}\pi_l^{-1}(x^0)>1$ and $x^0 \neq 0$, then $\tilde{V}_q(x^0, \zeta^0\,; t)=\partial/\partial t_q$, $1 \leqq q \leqq m$, for $(x^0, \zeta^0) \in \boldsymbol{R}X_l$.*

PROOF. By (2.11.1), the point (x^0, ζ^0) is supported by a non-compact, non-coordinate face $\gamma=\gamma(x^0, \zeta^0)$ of the Newton polygon Γ_+.

Then,

$$\sum_{j=0}^{k} \frac{\partial c_j}{\partial t_q}(x^0\,; t)\zeta_j^0 = \sum_{i_j \in \gamma} \frac{\partial c_j}{\partial t_q}(x^0\,; t)\zeta_j^0$$

by (2.5). Note that

$$F_\gamma(x^0\,; t) = \sum_{i_j \in \gamma} c_j(x^0\,; t)(x^0)^{i_j}.$$

Because we may assume that the exponent of any monomial of $c_j(x\,; t)x^{i_j}$, $i_j \in \gamma$, lies on the face γ after changing (if necessary) the formulation $F(x\,; t)= \sum_{j=0}^{k} c_j(x\,; t)x^{i_j}$ without change of $\boldsymbol{R}\pi$ up to isomorphism. $F_\gamma(x\,; t)$ is independent of t, the hypothesis of (1.3).

Hence,

$$\sum_{i_j \in \gamma} \frac{\partial c_j}{\partial t_q}(x^0\,; t)\zeta_j^0 = 0\,.$$

Since $\sum_{j=0}^{k} \partial c_j/\partial t_q(x\,; t)\zeta_j$ is a factor of the coefficient of $\partial/\partial x_p$ and $\partial/\partial(\zeta_j/\zeta_r)$ (see the definition of the vector field $\tilde{V}_q$ in (4.3)), the coefficient of $\partial/\partial x_p$ and $\partial/\partial(\zeta_j/\zeta_r)$ in the vector field $\tilde{V}_q(x^0, \zeta^0\,; t)$ is zero. So,

$$\tilde{V}_q(x^0, \zeta^0; t) = \partial/\partial t_q.$$

So, the vector field $\tilde{V}_q(x^0, \zeta^0; t)$ is equal to $\partial/\partial t_q$. This completes the proof of (5.2).

By (5.1) and (5.2), the vector field $\tilde{V}_q(x^0, \zeta^0; t)$ is tangent to $\boldsymbol{R}\pi_l^{-1}(x^0)\times\boldsymbol{R}$.

Therefore, the real analytic family $F(x; t)$ admits an $\boldsymbol{R}\pi_l$-MAT along I and so admits an $\boldsymbol{R}\pi$-MAT along I because the subblowing-up $\boldsymbol{R}\pi_l$ of $\boldsymbol{R}^n$ is analytically isomorphic to the subblowing-up $\boldsymbol{R}\pi$ of $\boldsymbol{R}^n$. This completes the proof of (1.3).

PROOF OF (1.4). It is sufficient to prove (1.4) in the case of $\boldsymbol{R}$-non-degeneracy by the same reason in the proof of (1.2).

Since $F(x; t)$ is a subfamily of the corresponding family $\tilde{F}(x; T)$, it is sufficient to prove that $\tilde{F}(x; T)$ admits an $\boldsymbol{R}\pi$-MAT along $\tilde{I}$.

Let $J_{q,r}$ be the subset of $\{1, 2, \cdots, n\}$ defined as follows:

$$J_{q,r} := \begin{cases} J_{\gamma_r} & \text{if } \gamma_r \text{ is a non-compact, non-coordinate face,} \\ \{1, 2, \cdots, n\} & \text{otherwise.} \end{cases}$$

Let $\tilde{V}_{q,r}(x, \zeta; T)$ be an analytic vector field defined by (4.3) in which put $J := J_{q,r}$, $\partial/\partial t := \partial/\partial t_{q,r}$ and $F(x; t) := \tilde{F}(x; T)$.

The residual part of the proof is completely parallel to the proof of (1.3) except making use of the following Lemma (5.3) instead of (5.2).

(5.3) LEMMA. *If* $\#\boldsymbol{R}\pi_l^{-1}(x^0)>1$ *and* $x^0\neq 0$, *then the coefficient of* $\partial/\partial x_p$ *in* $\tilde{V}_{q,r}(x^0, \zeta^0; T)$ *vanishes for any* p.

PROOF. By (2.11.1), the point (x^0, ζ^0) is supported by a non-compact, non-coordinate face γ of the Newton polygon Γ_+.

At first, suppose $\gamma_r = \gamma$. Then, $x_p^0 = 0$ for any $p \in J_{\gamma_r}$ by (2.8). So the coefficient of $\partial/\partial x_p$ in the vector field $\tilde{V}_{q,r}(x^0, \zeta^0; T)$ vanishes if $p \in J_{\gamma_r}$.

If $p \notin J_{\gamma_r}$, then the function $E_p x_p = x_1 x_2 \cdots x_n$ is a factor of the coefficient of $\partial/\partial x_p$ in the vector field $\tilde{V}_{q,r}(x, \zeta; T)$ and so the coefficient of $\partial/\partial x_p$ in $\tilde{V}_{q,r}(x^0, \zeta^0; T)$ vanishes because $\#\boldsymbol{R}\pi^{-1}(x^0)>1$ and $x^0 \in \boldsymbol{R}S$.

Nextly, suppose $\gamma_r \neq \gamma$. Then,

$$\sum_{j=0}^{k} \frac{\partial c_j}{\partial t_{q,r}}(x^0; T)\zeta_j^0 = \sum_{i_j\in\gamma} \frac{\partial c_j}{\partial t_{q,r}}(x^0; T)\zeta_j^0$$

by (2.5).

Here, if $i_j \in \mathrm{Int}(\gamma)$, then $c_j(x; T)$ is independent of $t_{q,r}$ for $\gamma_r \neq \gamma$. We may assume that the exponent of each monomial of $c_j(x; t)$, $i_j \in \partial\gamma$, lies on $\partial\gamma$ after changing (if necessary) the formulation $F(x; t) = \sum_{j=0}^{k} c_j(x; t)x^{i_j}$ without change $\boldsymbol{R}\pi$ up to isomorphism. So, if $i_j \in \partial\gamma$, then $c_j(x; T)$ is independent of $t_{q,r}$ because

$F(x;t)$ is strongly $\boldsymbol{R}$-non-degenerate, the hypothesis of (1.4).

Hence,

$$\sum_{i_j\in r}\frac{\partial c_j}{\partial t_{q,r}}(x^0;T)\zeta_j^0=0.$$

Since $\sum_{j=0}^k \partial c_j/\partial t_{q,r}(x;T)\zeta_j$ is a factor of the coefficient of $\partial/\partial x_p$ (see the definition of the vector field $\tilde{V}_{q,r}$ in (4.3)), the coefficient of $\partial/\partial x_p$ in the vector field $\tilde{V}_{q,r}(x^0,\zeta^0;T)$ is zero.

This completes the proof of (5.3).

This completes the proof of (1.4).

Note that the corresponding family $\tilde{F}(x;T)$ admits an $\boldsymbol{R}\pi$-MAT along $\tilde{I}$ by the proof of (1.4).

6. Corollaries.

(6.1) COROLLARY. *Let $F(x;t)$ be an $\boldsymbol{R}$ (resp. $\boldsymbol{C}$)-non-degenerate real analytic family. Assume that*

$$\boldsymbol{R}W=\{x\in\boldsymbol{R}^n\,|\,x^{i_0}=x^{i_1}=\cdots=x^{i_k}=0\}=\{0\}.$$

Then the family $F(x;t)$ admits an $\boldsymbol{R}\pi$-MAT (resp. $\boldsymbol{C}\pi|\boldsymbol{R}$-MAT) along I.

PROOF. (6.1) follows the proof of (1.2). In fact, the additional hypothesis of (1.3) compared with the hypothesis of (1.2) is needed to prove (5.2). But we are not in need of (5.2) to prove (6.1) by the assumption $\boldsymbol{R}W=\{0\}$ of (6.1).

(6.2) COROLLARY. *Assume that a real analytic function $f(x)$ is $\boldsymbol{R}$-non-degenerate. Then, the $\boldsymbol{R}\pi$-MAT type (and so the local topological type) of $f(x)$ is determined by $f_{\partial\Gamma_+(f)}(x)$.*

PROOF. Let $g(x)$ be a real analytic function at the origin such that $\Gamma_+(g)=\Gamma_+(f)$ and $g_{\partial\Gamma_+}(x)=f_{\partial\Gamma_+}(x)$. Then the analytic family $F(x;t):=(1-t)f(x)+tg(x)$ satisfies the hypothesis of (1.3). So, the family $F(x;t)$ admits an $\boldsymbol{R}\pi$-MAT along the interval $[0,1]$ and $F(x;0)=f(x)$, $F(x;1)=g(x)$. This completes the proof.

7. Examples.

(7.1) EXAMPLE. Let

$$F(x;t):=x_1^4+(1+tx_3^k)x_1^2x_2^3+x_2^8+x_2^6x_3^2,\qquad k\geqq 1.$$

As we studied in (3.3), $F(x;t)$ is a family of $\boldsymbol{R}$-{1, 2}-non-degenerate real analytic functions. Let I be a parameter space, an any compact interval in $\boldsymbol{R}$. Let $\boldsymbol{R}\pi: \boldsymbol{R}X \longrightarrow \boldsymbol{R}^3$ be the subblowing-up of $\boldsymbol{R}^3$ with center

$$\boldsymbol{R}W=\{x\in\boldsymbol{R}^3 \mid x_1^4=x_1^2x_2^3=x_2^8=x_2^6x_3^2=0\}.$$

(7.1.1) *The family $F(x;t)$ admits an $\boldsymbol{R}\pi$-MAT along I.*

PROOF. For any non-compact, non-coordinate face γ_4, γ_{14} or γ_{24}, the transversal direction is {1, 2}. The corresponding family $\tilde{F}(x;T)$ is equal to $F(x;t)$. The family $F(x;t)$ (and so $\tilde{F}(x;T)$) is $\boldsymbol{R}$-J_γ-non-degenerate. And

$$F_{\partial\gamma}(x;t)=x_1^4+x_1^2x_2^3+x_2^6x_3^2$$

is independent of t. Namely, $F(x;t)$ is strongly $\boldsymbol{R}$-non-degenerate.

Thus, the family $F(x;t)$ satisfies the hypothesis of (1.4) and admits an $\boldsymbol{R}\pi$-MAT along I.

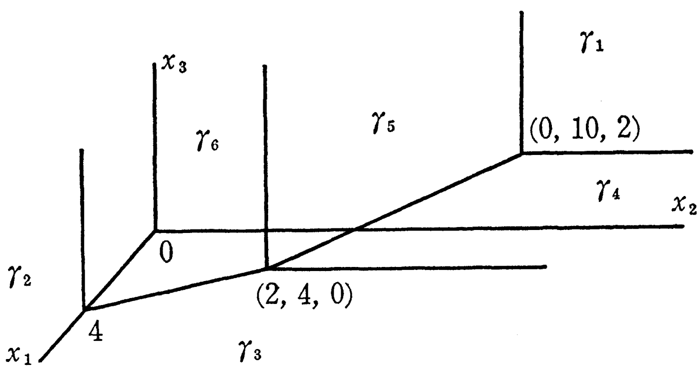
(7.2) EXAMPLE. Let

$$F(x;t):=x_1^4+x_1^2x_2^4+x_2^{10}x_3^2+tx_1x_2^7x_3^2+tx_1x_2^8x_3+tx_1^3x_2x_3.$$

(7.2.1) *The family $F(x;t)$ is strongly $\boldsymbol{R}$-non-degenerate.*

PROOF. The figure of the Newton polygon Γ_+ of $F(x;t)$ is denoted as follows:

(7.2.2) Figure.



The corresponding family $\tilde{F}(x;T)$ to $F(x;t)$ is defined as follows:

$$\tilde{F}(x;T)=x_1^4+x_1^2x_2^4+x_2^{10}x_3^2+t_1x_1x_2^7x_3^2+t_2x_1x_2^8x_3+t_3x_1^3x_2x_3.$$

Note that $\tilde{F}_\gamma=F_\gamma$ for any compact face γ. Calculating the polynomials $x_p\partial F_\gamma/\partial x_p$ for the all compact faces γ of Γ_+, we have the following table (7.2.3).

(7.2.3) Table.

compact face γ	$x_1\dfrac{\partial F_\gamma}{\partial x_1}$	$x_2\dfrac{\partial F_\gamma}{\partial x_2}$	$x_3\dfrac{\partial F_\gamma}{\partial x_3}$
γ_{45}	$2x_1^2x_2^4$	$4x_1^2x_2^4+10x_2^{10}x_3^2$	$2x_2^{10}x_3^2$
γ_{36}	$4x_1^4+2x_1^2x_2^4$	$4x_1^2x_2^4$	0
γ_{145}	0	$10x_2^{10}x_3^2$	$2x_2^{10}x_3^2$
γ_{345}	$2x_1^2x_2^4$	$4x_1^2x_2^4$	0
γ_{236}	$4x_1^4$	0	0

This calculation implies that the family $\tilde{F}(x;T)$ is $\boldsymbol{R}$-$\{1, 2\}$-non-degenerate and $\boldsymbol{R}$-$\{1, 3\}$-non-degenerate. For any non-compact, non-coordinate face $\gamma=\gamma_4$, γ_5, γ_6, γ_{14}, γ_{15}, γ_{26}, γ_{34}, γ_{56}, the transversal direction J_γ is $\{1, 2\}$ or $\{1, 3\}$ and it is clear that $F_{\partial\gamma}$ is independent of t. Thus, the family $F(x;t)$ is strongly $\boldsymbol{R}$-non-degenerate and so admits an $\boldsymbol{R}\pi$-MAT along any compact interval in $\boldsymbol{R}$ by (1.4).

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