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Expansive homeomorphisms with the pseudo-orbit tracing property on compact surfaces

By Koichi HIRAIDE

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§ 0. Introduction.

It is known that compact surfaces which admit Anosov diffeomorphisms are tori, and that such diffeomorphisms on tori are topologically conjugate to toral automorphisms (see J. Franks [**3**]). The purpose of this paper is to prove in topological setting the following

THEOREM. *Let M^2 be a compact surface and $f: M^2 \to M^2$ be a homeomorphism. If f is expansive and has POTP, then f is topologically conjugate to a hyperbolic toral automorphism.*

REMARK. Let M^2 be as in Theorem. It is known (see T. O'Brien and W. Reddy [**9**]) that if M^2 is orientable and has positive genus, then M^2 admits an expansive homeomorphism. Thus the assumption of POTP in our theorem can not drop.

Let (X, d) be a compact metric space and $f: X \to X$ be a homeomorphism (every homeomorphism means bijective). We say that f is *expansive* if there is $c>0$ such that when $x \neq y$, $d(f^i(x), f^i(y))>c$ for some $i \in \boldsymbol{Z}$ (c is called an *expansive constant* for f). A sequence $\{x_i\}_{i\in Z}$ in X is a *δ-pseudo-orbit* of f if $d(f(x_i), x_{i+1})<\delta$ for all $i \in \boldsymbol{Z}$. A point x in X is said to *ε-trace* $\{x_i\}_{i\in Z}$ if $d(f^i(x), x_i)<\varepsilon$ for all $i \in \boldsymbol{Z}$. We say that f has *POTP* if for every $\varepsilon>0$ there is $\delta>0$ such that every δ-pseudo-orbit of f is ε-traced by some point in X. We remark that expansiveness and POTP are independent of the compatible metrics used, and preserved under topological conjugacy. For materials on topological dynamics on closed manifolds, the reader may refer to A. Morimoto [8].

For $x \in X$, define the *stable* and *unstable sets* $W^s(x)$ and $W^u(x)$ as

$$W^s(x) = \{y \in X:\ d(f^n(x), f^n(y)) \to 0 \quad \text{as } n \to \infty\},$$
$$W^u(x) = \{y \in X:\ d(f^n(x), f^n(y)) \to 0 \quad \text{as } n \to -\infty\}$$

and put

$$\mathcal{F}^\sigma(X, f) = \{W^\sigma(x):\ x \in X\} \quad (\sigma = s, u).$$

Then $\mathcal{F}^\sigma(X, f)$ is a decomposition of X. For $\varepsilon>0$, let $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ be the *local stable* and *unstable sets* defined by

$$W^s_\varepsilon(x) = \{y\in X:\ d(f^n(x), f^n(y))\leqq\varepsilon,\ n\geqq 0\},$$
$$W^u_\varepsilon(x) = \{y\in X:\ d(f^n(x), f^n(y))\leqq\varepsilon,\ n\leqq 0\}.$$

R. Mañé proved in [7] that if $f: X\to X$ is expansive (with expansive constant c), then for every $r>0$ there is $N>0$ such that

$$f^n W^s_c(x)\subset W^s_r(f^n(x)) \quad \text{and} \quad f^{-n}W^u_c(x)\subset W^u_r(f^{-n}(x))$$

for all $n\geqq N$ and all $x\in X$. From this result we can easily see that

$$W^s(x) = \bigcup_{n\geqq 0} f^{-n}W^s_\varepsilon(f^n(x)),\quad W^u(x) = \bigcup_{n\geqq 0} f^n W^u_\varepsilon(f^{-n}(x))$$

for all $x\in X$ and all ε with $0<\varepsilon\leqq c$.

Throughout this paper, the term *surfaces* will be applied only to connected 2-manifolds without boundary. For materials on foliations on surfaces, the reader may refer to G. Reeb [**10**] and G. Hector and U. Hirsch [**5**].

The conclusion of our theorem will be obtained in proving the following two propositions.

PROPOSITION 1. *Uuder the notations and the assumptions in Theorem, the families* $\mathcal{F}^\sigma(M^2, f)$ $(\sigma=s, u)$ *have the following properties;*

(1) $\mathcal{F}^\sigma(M^2, f)$ *is a* C^0 *foliation on* M^2,
(2) *every leaf* $W^\sigma(x)\in\mathcal{F}^\sigma(M^2, f)$ *is homeomorphic to* $\boldsymbol{R}$,
(3) $\mathcal{F}^s(M^2, f)$ *is transverse to* $\mathcal{F}^u(M^2, f)$.

If Proposition 1 holds, then from Poincaré-Kneser's Theorem and Kneser's Theorem together with Proposition 1, we can easily prove the following

COROLLARY. *If* M^2 *is not a 2-torus, then* M^2 *does not admit expansive homeomorphisms with POTP.*

PROPOSITION 2. *Under the notations and the assumptions in Theorem, if* M^2 *is a 2-torus, then* f *is topologically conjugate to a hyperbolic toral automorphism.*

§1. Proof of Proposition 1.

Let (X, d) be a compact metric space and $f: X\to X$ be a homeomorphism.

Hereafter we assume that f is expansive (with expansive constant c) and has POTP. The following results (I), (II) and (III) are proved in [6].

(I) Fix $c>0$ and put $\varepsilon_0=c/4$. Then there is $\delta_0>0$ such that if $d(x, y)<\delta_0$, then $W^s_{\varepsilon_0}(x)\cap W^u_{\varepsilon_0}(y)$ is a single point. We denote the point by $[x, y]$.

(II) Let $\delta_0>0$ be as above and put

$$\Delta(\delta_0)=\{(x, y)\in X\times X:\ d(x, y)<\delta_0\}.$$

Then $[\cdot,\cdot]:\Delta(\delta_0)\to X$ is a continuous map and

$$[x, x]=x,\quad [[x, y], z]=[x, z],\quad [x, [y, z]]=[x, z]$$

whenever the two sides of these relations are defined.

(III) For every δ with $0<\delta<\delta_0/2$, we define

$$W^{\sigma}_{\varepsilon_0,\delta}(x)=\{y\in W^{\sigma}_{\varepsilon_0}(x):\ d(x, y)<\delta\}\quad (\sigma=s, u),$$

$$N_{x,\delta}=[W^{u}_{\varepsilon_0,\delta}(x),\ W^{s}_{\varepsilon_0,\delta}(x)].$$

Then there are $0<\delta_1<\delta_0/2$ and $\rho_0>0$ such that for every $x\in X$

(a) N_{x,δ_1} is open in X and $\mathrm{diam}(N_{x,\delta_1})<\delta_0$,
(b) $[\cdot,\cdot]:W^{u}_{\varepsilon_0,\delta_1}(x)\times W^{s}_{\varepsilon_0,\delta_1}(x)\to N_{x,\delta_1}$ is a homeomorphism,
(c) $N_{x,\delta_1}\supset B_{\rho_0}(x)$ where $B_{\rho_0}(x)$ denotes the closed ball of x with radius ρ_0.

Let ε_0 and δ_1 be as above. We denote by D^{σ}_x the connected component of x in $W^{\sigma}_{\varepsilon_0,\delta_1}(x)$ for $\sigma=s, u$, and put

$$N_x=[D^u_x,\ D^s_x].$$

PROPOSITION 1.1 (Local product structure). *If (X, d) is a compact connected locally connected metric space and $f:X\to X$ is expansive and has POTP, then for every $x\in X$*

(a) *N_x is connected and open in X and* $\mathrm{diam}(N_x)<\delta_0$,
(b) *$[\cdot,\cdot]:D^u_x\times D^s_x\to N_x$ is a homeomorphism,*
(c) *there is $0<\rho<\varepsilon_0$ such that $N_x\supset B_\rho(x)$ for all $x\in X$,*
(d) *$D^{\sigma}_x\supsetneqq\{x\}$ for $\sigma=s, u$.*

PROOF. Let $\delta_1>0$ be as above. Obviously $D^u_x\times D^s_x$ is the connected component of (x, x) in $W^u_{\varepsilon_0,\delta_1}(x)\times W^s_{\varepsilon_0,\delta_1}(x)$. By (III)(b) we have that N_x is the connected component of x in N_{x,δ_1}. Since X is locally connected and N_{x,δ_1} is open in X ((III)(a)), N_x is open in X and therefore (a) holds. (b) is obtained by (III)(b). Let $\rho_0>0$ be as in (III). Then there is a finite open cover $\{U_i:1\leqq i\leqq k\}$ of X such that $\mathrm{diam}(U_i)<\rho_0$ $(1\leqq i\leqq k)$ and each U_i is connected. Let ρ $(<\varepsilon_0)$ be a Lebesgue number of $\{U_i:1\leqq i\leqq k\}$. Then for $x\in X$ there is U_i such that $B_\rho(x)\subset U_i$. Since $U_i\subset B_{\rho_0}(x)$, we have $U_i\subset N_{x,\delta_1}$ (by (III)(c)) and so $U_i\subset N_x$ since U_i is connected. Therefore $B_\rho(x)\subset N_x$.

It remains to prove (d). Let $X^s=\{x\in X: D^s_x=\{x\}\}$ and take $x\in X^s$. Since $N_x=[D^u_x, D^s_x]$, we have $N_x=D^u_x\subset W^u_{\varepsilon_0}(x)$ and so $N_x\subset W^u_{2\varepsilon_0}(y)$ for $y\in N_x$. Since f is expansive and $\varepsilon_0=c/4$, we have

$$y \in D_y^s \cap N_x \subset W_{\varepsilon_0}^s(y) \cap W_{2\varepsilon_0}^u(y) = \{y\}$$

and so $D_y^s=\{y\}$. This implies that $y \in X^s$ for $y \in N_x$, i.e., $N_x \subset X^s$. Therefore X^s is open by (a). (c) ensures that X^s is closed. If $X^s \neq \emptyset$, then $X^s=X$ since X is connected. In this case we can choose a finite set $\{x_i: 1 \leqq i \leqq l\}$ in X such that

$$X=\bigcup_{i=1}^{l} N_{x_i}=\bigcup_{i=1}^{l} D_{x_i}^u .$$

Since $D_{x_i}^u \subset W_{\varepsilon_0}^u(x_i)$, using the result of Mañé stated in Introduction we can find $N>0$ such that $f^{-N}(X) \subsetneqq X$. But this is impossible since f is surjective. Therefore $X^s=\emptyset$. In the same way, we have that $D_x^u \supsetneqq \{x\}$ for all $x \in X$.

Let X and Y be non-trivial topological spaces and $X \times Y$ be the product space with product topology.

LEMMA 1.2. *Under the above notations, if $X \times Y$ is a connected 2-manifold, then X and Y are 1-manifolds.*

PROOF. Since $X \times Y$ is a manifold, X and Y are Hausdorff spaces and satisfy the second axiom of countability. To obtain the conclusion, it is enough to prove that X and Y are locally homeomorphic to either the line or the half line. We shall show this for X. Then the conclusion for Y is obtained in the same way.

For simplicity we denote by I, I^0 and I^1 the intervals $[0,1]$, $(0,1)$ and $[0,1)$ respectively. If $x \neq y$ $(x, y \in X)$, then there is a continuous map $\alpha: I \to X$ corresponding to a path of x to y since $X \times Y$ is connected. It is clear that $\alpha(I)$ is a Peano space. Hence $\alpha(I)$ is arcwise connected ([4]). So we can find an arc of x to y in $\alpha(I)$. Since x and y are arbitrary in X, X must be arcwise connected.

Let A_X (resp. A_Y) be defined as the set of all injective continuous maps from I to X (resp. Y). For $\alpha \in A_X$, $\alpha: I^0 \to \alpha(I^0)$ is a homeomorphism since α is injective. We shall show that $\alpha(I^0)$ is open in X. To do this, take and fix $\beta \in A_Y$. If (U, φ) denotes a chart of $X \times Y$, then the map $\varphi \circ (\alpha \times \beta)$

$$I^0 \times I^0 \cap (\alpha \times \beta)^{-1}(U) \xrightarrow{\alpha \times \beta} U \xrightarrow{\varphi} \boldsymbol{R}^2$$

is injective and continuous. By Brouwer's Theorem the image of $\varphi \circ (\alpha \times \beta)$ is open in $\boldsymbol{R}^2$. This shows that $(\alpha \times \beta)(I^0 \times I^0)$ is open in $X \times Y$ since (U, φ) is arbitrary, and so $\alpha(I^0)$ is open in X.

Put $V_X=\bigcup_{\alpha \in A_X} \alpha(I^0)$. If $V_X=X$ then we have that X is a 1-manifold without boundary. For the case when $V_X \neq X$, we prove that X is homeomorphic to I or I^1.

Take and fix $x \in X \setminus V_X$ and define

$$A'_X = \{\alpha \in A_X : \alpha(0) = x\}.$$

Then we claim that there is the implication between $\alpha_1(I)$ and $\alpha_2(I)$ for $\alpha_1, \alpha_2 \in A'_X$. Indeed, assume that $\alpha_1(I) \not\subset \alpha_2(I)$. Then we have that $\alpha_2(I) \subset \alpha_1(I)$. For, let $J = \alpha_1(I) \setminus \alpha_2(I)$. Since $\alpha_1(0) = \alpha_2(0) = x$, $J \subsetneqq \alpha_1(I)$ and so $\alpha_1^{-1}(J) \subsetneqq I$. Since $J \neq \emptyset$, there is a boundary point b of $\alpha_1^{-1}(J)$. Since J is open in $\alpha_1(I)$, $\alpha_1^{-1}(J)$ is open in I. Hence $b \notin \alpha_1^{-1}(J)$ and so $\alpha_1(b) \notin J$. This implies that $\alpha_1(b) \in \alpha_2(I)$. But $\alpha_2(I^0)$ is open in X. Since b is a boundary point of $\alpha_1^{-1}(J)$, we have that $\alpha_1(b) = \alpha_2(0)$ or $\alpha_1(b) = \alpha_2(1)$. If $\alpha_1(b) = \alpha_2(0)$, then we must have $b = 0$ (since $\alpha_1, \alpha_2 \in A'_X$). In this case there is $c \in (0, 1]$ such that $\alpha_1([0, c]) \cap \alpha_2([0, 1]) = \{x\}$. This shows that there is in X an arc of $\alpha_1(c)$ to $\alpha_2(1)$ through the point x. By the definition we have $x \in V_X$, thus contradicting $x \notin V_X$. Therefore $\alpha_1(b) = \alpha_2(1)$. Since b is arbitrary in the boundary set of $\alpha_1^{-1}(J)$, the boundary set of $\alpha_1^{-1}(J)$ is a single point. Therefore $\alpha_1^{-1}(J)$ is an open interval in I and $\alpha_1^{-1}(J) \ni 1$, from which we see that $\alpha_1^{-1}(J) = (b, 1]$ and hence $\alpha_1([0, b]) \subset \alpha_2([0, 1])$. Since $\alpha_1(0) = \alpha_2(0)$ and $\alpha_1(b) = \alpha_2(1)$, obviously $\alpha_1([0, b]) = \alpha_2([0, 1])$ and therefore $\alpha_2(I) \subset \alpha_1(I)$. We proved that $\{\alpha(I) : \alpha \in A'_X\}$ is a totally ordered set under inclusion.

On the other hand, since X is arcwise connected, we have

$$X = \bigcup_{\alpha \in A'_X} \alpha(I).$$

These two facts show that X is homeomorphic to either I or I^1. The proof is completed.

Assume that $f : M^2 \to M^2$ is expansive and has POTP.

LEMMA 1.3. *Under the above assumptions, for every* $x \in M^2$, D_x^σ *is homeomorphic to* $\boldsymbol{R}$ $(\sigma = s, u)$.

PROOF. Let ρ be as in Proposition 1.1 (c). First we prove that for every $x \in M^2$

$$W_\rho^\sigma(x) \subset D_x^\sigma \quad (\sigma = s, u). \tag{1.1}$$

We give the proof for $\sigma = s$ and then the case when $\sigma = u$ is easily obtained in the quite same way. Since $W_\rho^s(x) \subset B_\rho(x)$ and $B_\rho(x) \subset N_x$ by Proposition 1.1 (c), we have that $y \in N_x$ when $y \in W_\rho^s(x)$. Since $N_x = [D_x^u, D_x^s]$ by the definition, clearly $y = [u, v]$ for some $u \in D_x^u$ and $v \in D_x^s$. Hence $[x, y] = [x, [u, v]] = [x, v]$ by (II). Since $\rho < \varepsilon_0$ (see Proposition 1.1), we have $W_\rho^s(x) \subset W_{\varepsilon_0}^s(x)$ and hence $y \in W_{\varepsilon_0}^s(x)$. By the definition of $[\cdot, \cdot]$, $y = [x, y] \in W_{\varepsilon_0}^s(x) \cap W_{\varepsilon_0}^u(y)$. Since $v \in D_x^s \subset W_{\varepsilon_0}^s(x)$, we have also $[x, v] = v$. From the above calculation we have

$y=v\in D_x^s$.

Combining Proposition 1.1 (a), (b) and Lemma 1.2, we see that each D_x^s is a connected 1-manifold without boundary. Hence D_x^s is homeomorphic to $\boldsymbol{R}$ or S^1.

To obtain the conclusion of the lemma, assume that D_x^s is homeomorphic to S^1. Since $D_x^s\subset W_{\varepsilon_0}^s(x)$, by using Mañé's result we have that $f^N(D_x^s)\subset W_\rho^s(f^N(x))$ for some $N>0$. Hence $f^N(D_x^s)\subset D_{f^N(x)}^s$ by (1.1). Since every 1-manifold has no circles as proper subsets, we must have $f^N(D_x^s)=D_{f^N(x)}^s$ and so $D_{f^N(x)}^s \subset W_\rho^s(f^N(x))\subset B_\rho(f^N(x))$. Since ρ is chosen small enough, we can choose $\rho>0$ such that $B_\rho(f^N(x))$ is homeomorphic to the unit disk. For simplicity we identify $B_\rho(f^N(x))$ with the unit disk, and define

$$F\colon D_{f^N(x)}^s\times[0, 1] \longrightarrow D_{f^N(x)}^s$$

by putting $F(z, t)=[f^N(x), tz]$. Then F is a homotopy between a constant map and the identity map. Hence $D_{f^N(x)}^s$ is contractible. But this is a contradiction since $D_{f^N(x)}^s=f^N(D_x^s)$ is homeomorphic to S^1. Therefore each D_x^s is homeomorphic to $\boldsymbol{R}$.

PROOF OF PROPOSITION 1. We first show that each $W^\sigma(x)$ is connected for $\sigma=s, u$. Using Mañé's result, we have

$$(1.2)\qquad W^s(x)=\bigcup_{n\geqq 0} f^{-n}W_\rho^s(f^n(x)),\quad W^u(x)=\bigcup_{n\geqq 0} f^n W_\rho^u(f^{-n}(x))$$

and $N>0$ such that

$$(1.3)\qquad f^N(D_x^s)\subset W_\rho^s(f^N(x)),\quad f^{-N}(D_x^u)\subset W_\rho^u(f^{-N}(x)).$$

Put $g=f^N$ and define

$$s_n(x)=g^{-n}(D_{g^n(x)}^s)\quad\text{and}\quad u_n(x)=g^n(D_{g^{-n}(x)}^u)\quad (n\geqq 0).$$

By (1.1) and (1.3) we have

$$s_n(x)\subset g^{-n-1}W_\rho^s(g^{n+1}(x))\subset s_{n+1}(x),$$
$$u_n(x)\subset g^{n+1}W_\rho^u(g^{-n-1}(x))\subset u_{n+1}(x).$$

So $W^\sigma(x)=\bigcup_{n\geqq 0}\sigma_n(x)$ ($\sigma=s, u$) by (1.2). Using Lemma 1.3, we have that each $\sigma_n(x)$ is homeomorphic to $\boldsymbol{R}$ for $\sigma=s, u$. These facts ensure that for every $x\in M^2$ there is an injective continuous map $j_x^\sigma\colon \boldsymbol{R}\to M^2$ ($\sigma=s, u$) such that $j_x^\sigma(\boldsymbol{R})=W^\sigma(x)$ and for every $n\geqq 0$, $j_x^\sigma\colon(-n-1, n+1)\to\sigma_n(x)$ is a homeomorphism. Therefore each $W^\sigma(x)$ is connected.

For $\sigma=s, u$, denote by h_x^σ a homeomorphism of $\boldsymbol{R}$ onto D_x^σ and define a continuous map $\varphi_x\colon \boldsymbol{R}^2\to N_x$ by

$$\varphi_x=[\cdot, \cdot]\circ(h_x^u\times h_x^s).$$

We see by Proposition 1.1 (b) that φ_x is a homeomorphism. By (II), φ_x^{-1} is of the form

$$\varphi_x^{-1}(w)=((h_x^u)^{-1}\times(h_x^s)^{-1})([w, x], [x, w]). \tag{1.4}$$

Remark that $x\in N_x$ and by Proposition 1.1 (a) each N_x is open in M^2. Hence we see that $\{(N_x, \varphi_x^{-1})\}_{x\in M^2}$ is an atlas of M^2.

We first claim that $N_x=\bigcup_{z\in D_x^u} D_{x,z}^s$ where $D_{x,z}^s=N_x\cap W_{\varepsilon_0}^s(z)$ for $z\in D_x^u$. Indeed, for $w\in N_x$, we have $w=[z, v]$ for some $z\in D_x^u$ and $v\in D_x^s$. Since $w=[z, v]\in W_{\varepsilon_0}^s(z)\cap W_{\varepsilon_0}^u(v)$, we have $w\in N_x\cap W_{\varepsilon_0}^s(z)=D_{x,z}^s$ and so $N_x\subset\bigcup_{z\in D_x^u} D_{x,z}^s$.

Put $D_{x,z}^u=N_x\cap W_{\varepsilon_0}^u(z)$ for $z\in D_x^s$. Then we have $N_x=\bigcup_{z\in D_x^s} D_{x,z}^u$ as above.

We next claim that $[D_{x,z}^s, x]=\{z\}$ and $[x, D_{x,z}^s]=D_x^s$. Indeed, since we have

$$\begin{aligned}[D_{x,z}^s, x] &= \bigcup_{w\in D_{x,z}^s} W_{\varepsilon_0}^s(w)\cap W_{\varepsilon_0}^u(x) && \text{(by the definition of } [\cdot,\cdot])\\ &\subset W_{2\varepsilon_0}^s(z)\cap W_{\varepsilon_0}^u(x) && (\text{since } W_{\varepsilon_0}^s(x)\subset W_{2\varepsilon_0}^s(z))\\ &= \{z\} && (\text{since } 2\varepsilon_0<c),\end{aligned}$$

the first assertion holds. If $w\in D_{x,z}^s$, then $[x, w]\in N_x$ and so $[x, w]=[u, v]$ for some $u\in D_x^u$ and $v\in D_x^s$. (II) ensures that

$$[x, w]=[x, [x, w]]=[x, [u, v]]=[x, v]=v,$$

so that $[x, w]\in D_x^s$. Consequently $[x, D_{x,z}^s]\subset D_x^s$. If $w\in D_x^s$, then $[z, w]\in N_x$ and $[z, w]\in W_{\varepsilon_0}^s(z)$. Hence $[z, w]\in D_{x,z}^s$. Since $[x, [z, w]]=[x, w]=w$, we have $[x, D_{x,z}^s]\supset D_x^s$. Therefore the second assertion holds.

Combining the above claims and (1.4), we have

$$\begin{aligned}\varphi_x^{-1}(D_{x,z}^s) &= ((h_x^u)^{-1}\times(h_x^s)^{-1})([D_{x,z}^s, x], [x, D_{x,z}^s])\\ &= ((h_x^u)^{-1}\times(h_x^s)^{-1})(\{z\}\times D_x^s)\\ &= \{(h_x^u)^{-1}(z)\}\times\boldsymbol{R}.\end{aligned} \tag{1.5}$$

This implies that φ_x^{-1} sends $D_{x,z}^s$ onto a vertical line $\{a\}\times\boldsymbol{R}$ for some $a\in\boldsymbol{R}$. We can easily show by the above way that φ_x^{-1} sends $D_{x,z}^u$ onto a horizontal line $\boldsymbol{R}\times\{b\}$ for some $b\in\boldsymbol{R}$.

From now on, we prove that $\mathcal{F}^\sigma(M^2, f)$ is a C^0 foliation on M^2 for $\sigma=s, u$. Remark that $\{(N_y, \varphi_y^{-1})\}_{y\in M^2}$ is an atlas of M^2 and $\varphi_y^{-1}(D_{y,z}^s)=\{a\}\times\boldsymbol{R}$ for some $a\in\boldsymbol{R}$ (and also $\varphi_y^{-1}(D_{y,z}^u)=\boldsymbol{R}\times\{b\}$ for some $b\in\boldsymbol{R}$). To obtain that $\mathcal{F}^\sigma(M^2, f)$ is a C^0 foliation on M^2, it is enough to show that each connected component of $W^\sigma(x)\cap N_y$ is of the form $D_{y,z}^\sigma$.

Let $w\in W^s(x)\cap N_y$. Since $N_y=\bigcup_{z\in D_y^u} D_{y,z}^s$, we have $w\in D_{y,z}^s$ for some $z\in D_y^u$. Since $D_{y,z}^s\subset W_{\varepsilon_0}^s(z)$, there is $n(z)>0$ such that $g^{n(z)}(D_{y,z}^s)\subset W_{\rho/2}^s(g^{n(z)}(w))$. On the other hand, since $w\in W^s(x)$, by the definition of $W^s(x)$ we have

$g^{n(z)}(w)\in W^s_{\rho/2}(g^{n(z)}(x))$ for some $n(z)$. Therefore

$$g^{n(z)}(D^s_{y,z})\subset W^s_\rho(g^{n(z)}(x))\subset D^s_{g^{n(z)}(x)}$$

and so $D^s_{y,z}\subset s_{n(z)}(x)$. Since $s_{n(z)}(x)\subset W^s(x)$, we have $w\in D^s_{y,z}\subset W^s(x)$. We showed that there are the index set Λ and a subset $\{z_\lambda:\lambda\in\Lambda\}$ of D^u_y such that

$$W^s(x)\cap N_y=\bigcup_{\lambda\in\Lambda}D^s_{y,z_\lambda}\quad\text{(disjoint union).}$$

We claim that Λ is at most countable. Indeed, since $D^s_{y,z_\lambda}\subset s_{n(z_\lambda)}(x)$ and both D^s_{y,z_λ} and $s_{n(z_\lambda)}(x)$ are homeomorphic to $\boldsymbol{R}$, D^s_{y,z_λ} is open in $s_{n(z_\lambda)}(x)$. Since $j^s_x:(-n(z_\lambda)-1, n(z_\lambda)+1)\to s_{n(z_\lambda)}(x)$ is a homeomorphism, $(j^s_x)^{-1}(D^s_{y,z_\lambda})$ is open in $\boldsymbol{R}$. Since $\{(j^s_x)^{-1}(D^s_{y,z_\lambda}):\lambda\in\Lambda\}$ is mutually disjoint, we have that Λ is at most countable.

Therefore each D^s_{y,z_λ} is a connected component of $W^s(x)\cap N_y$, from which we see that $\mathcal{F}^s(M^2, f)$ is a C^0 foliation on M^2. It follows from the quite same technique that $\mathcal{F}^u(M^2, f)$ is a C^0 foliation on M^2.

Obviously the topology of $W^\sigma(x)$ induced by j^σ_x coincides with the leaf topology. Therefore each $W^\sigma(x)\in\mathcal{F}^\sigma(M^2, f)$ is homeomorphic to $\boldsymbol{R}$. As saw above φ_x^{-1} sends $D^s_{x,z}$ onto a vertical line $\{a\}\times\boldsymbol{R}$ and $D^u_{x,z}$ onto a horizontal line $\boldsymbol{R}\times\{b\}$. Therefore $\mathcal{F}^s(M^2, f)$ is transverse to $\mathcal{F}^u(M^2, f)$. The proof is completed.

§2. Proof of Proposition 2.

Before starting the proof we shall investigate some properties of C^0 foliations on $\boldsymbol{R}^2$.

LEMMA 2.1 (G. Hector and U. Hirsch [5]). *If $\mathcal{F}$ is a C^0 foliation on $\boldsymbol{R}^2$ then the following hold;*

(a) *every leaf of $\mathcal{F}$ is homeomorphic to $\boldsymbol{R}$,*
(b) *$\mathcal{F}$ is orientable,*
(c) *if $\mathcal{F}^\phi$ is a C^0 foliation on $\boldsymbol{R}^2$ transverse to $\mathcal{F}$ then for every $L\in\mathcal{F}$, L intersects all leaves of $\mathcal{F}^\phi$ in at most one point,*
(d) *if L is a leaf of $\mathcal{F}$ then $\boldsymbol{R}^2\setminus L$ is decomposed into two connected components.*

Hereafter we denote by $P:\boldsymbol{R}^2\to\boldsymbol{T}^2$ $(=\boldsymbol{R}^2/\boldsymbol{Z}^2)$ the natural projection. The following plays important roles in the proof of Proposition 2.

LEMMA 2.2. *Let $\mathcal{F}$ and $\mathcal{F}^\phi$ be transverse C^0 foliations on $\boldsymbol{T}^2$. Denote by $\bar{\mathcal{F}}$ and $\bar{\mathcal{F}}^\phi$ the lifts of $\mathcal{F}$ and $\mathcal{F}^\phi$ by P respectively. Assume that every leaf which is in $\mathcal{F}$ and $\mathcal{F}^\phi$ is homeomorphic to $\boldsymbol{R}$. Then every leaf of $\bar{\mathcal{F}}$ intersects all leaves of $\bar{\mathcal{F}}^\phi$ in exactly one point.*

PROOF. We orient $\mathcal{F}$ and $\mathcal{F}^{\phi}$ according to Lemma 2.1 (b) and denote by $s(x)$ (resp. $u(x)$) the leaf of $\mathcal{F}^{\phi}$ (resp. $\mathcal{F}$) through $x\in \boldsymbol{R}^2$. Let $y_1, y_2\in\sigma(x)$ for $\sigma=s, u$. We write $y_1\leqq_\sigma y_2$ if either $y_1=y_2$ or the arc in $\sigma(x)$ from y_1 to y_2 has the same orientation as that of $\sigma(x)$. If $y_1\leqq_\sigma y_2$ and $y_1\neq y_2$, we write $y_1<_\sigma y_2$. For the case when $y_1<_\sigma y_2$, the subsets of $\sigma(x)$ are defined as follows.

$$[y_1, y_2]_\sigma=\{y\in\sigma(x): y_1\leqq_\sigma y\leqq_\sigma y_2\},$$

$$[y_1, y_2)_\sigma=\{y\in\sigma(x): y_1\leqq_\sigma y<_\sigma y_2\},$$

$$[y_1, +\infty)_\sigma=\{y\in\sigma(x): y_1\leqq_\sigma y\}.$$

Such subsets are called here intervals under the relation $\leqq_\sigma$.

We define

$$\varDelta=\{(x, y)\in\boldsymbol{R}^2\times\boldsymbol{R}^2: s(x)\cap u(y)\neq\emptyset\}.$$

By Lemma 2.1 (c) we have that $s(x)\cap u(y)$ is a single point for $(x, y)\in\varDelta$, and denote the point by $\alpha(x, y)$. Since $\mathcal{F}^{\phi}$ is transverse to $\mathcal{F}$, we have that $\alpha:\varDelta\to\boldsymbol{R}^2$ is a continuous map. The following is obtained easily; Let $x, y\in\boldsymbol{R}^2$. If I is an interval of $s(x)$ such that $\{y\}\times I\subset\varDelta$, then we have

(1) $\alpha(y,\cdot):I\to s(y)$ is injective and preserves the relation $\leqq_s$,
(2) $\alpha(\{y\}\times I)$ is an interval of $s(y)$.

If I is an interval of $u(x)$ and satisfies $I\times\{y\}\subset\varDelta$, then we have

(3) $\alpha(\cdot, y):I\to u(y)$ is injective and preserves the relation $\leqq_u$,
(4) $\alpha(I\times\{y\})$ is an interval of $u(y)$.

If $\varDelta=\boldsymbol{R}^2\times\boldsymbol{R}^2$ holds, then the conclusion of the lemma is obtained. Thus we prove that $\varDelta=\boldsymbol{R}^2\times\boldsymbol{R}^2$. To do this, put

$$X_s=\{x\in\boldsymbol{R}^2: s\cap u(x)\neq\emptyset\}\quad(s\in\mathcal{F}^{\phi}).$$

If we establish that $X_s=\boldsymbol{R}^2$ for all $s\in\mathcal{F}^{\phi}$, then we have $\varDelta=\boldsymbol{R}^2\times\boldsymbol{R}^2$. Therefore we assume $X_s\subsetneqq\boldsymbol{R}^2$ and prepare some claims to derive a contradiction.

CLAIM I. *Let $I_x=X_s\cap s(x)$ for $x\in X_s$. Then I_x is an open interval of $s(x)$.*

PROOF. By transversality condition we have that

$$X_s \text{ is open in } \boldsymbol{R}^2. \tag{2.1}$$

Hence I_x is open in $s(x)$. If $b\in s(x)$ is a boundary point of I_x, then $b\notin I_x$ and so $b\notin X_s$. This implies that $s\cap u(b)=\emptyset$. We remark that $\boldsymbol{R}^2\setminus u(b)$ is decomposed into two connected components U_1 and U_2 by Lemma 2.1 (d). Since $s\cap u(b)=\emptyset$, we can assume that $s\subset U_1$. Then $I_x\subset U_1$ since $s\cap u(y)\neq\emptyset$ if $y\in I_x\subset X_s$. Since $I_x\subset s(x)=s(b)$, we have $I_x\subset U_1\cap s(b)$ where $U_1\cap s(b)$ is a connected component of $s(x)\setminus\{b\}$ $(=s(b)\setminus\{b\})$. Since b is arbitrary, I_x is an interval of $s(x)$.

Claim I implies that I_x is of the form

$$I_x=(a(x),\ b(x))_s.$$

Since $X_s\subsetneqq \boldsymbol{R}^2$ by the assumption, there is a boundary point b of X_s. But $b\notin X_s$ by (2.1). By transversality condition we have $(x_0,\ b)\in\Delta$ when $x_0\in X_s$ is near b. By the definition of α, $\alpha(x_0,\ b)\notin X_s$ since $b\notin X_s$. And we have $I_{x_0}\subsetneqq s(x_0)$. Hence we can assume that $b(x_0)\in s(x_0)$.

Now put $\{x_2\}=s\cap u(x_0)$. Then $I_{x_2}=s$ since $s(x_2)=s$ and so $x_0\neq x_2$. Hence we have either $x_0<_u x_2$ or $x_2<_u x_0$. We deal with the case $x_0<_u x_2$ since the case $x_2<_u x_0$ is done in the same way. Since $u(x_0)\subset X_s$, if

$$J=\{x\in u(x_0):\ b(x)=+\infty\},$$

then $x_0\notin J$ and $x_2\in J$. We see by transversality condition that $u(x_0)\setminus J$ is open in $u(x_0)$, and so J is closed in $u(x_0)$. Hence there is $x_1\in(x_0,\ x_2]_u\cap J$ such that $b(x)\in s(x)$ for every $x\in[x_0,\ x_1)_u$.

CLAIM II. $[x_0,\ x_1]_u\times[x_0,\ b(x_0)]_s\setminus(x_1,\ b(x_0))\subset\Delta$.

PROOF. By the definition of Δ we have that $\{x_0\}\times[x_0,\ b(x_0)]_s\subset\Delta$ and $[x_0,\ x_1]_u\times\{x_0\}\subset\Delta$. First we check that $(x_0,\ x_1)_u\times(x_0,\ b(x_0))_s\subset\Delta$. Let $x\in(x_0,\ x_1)_u$ and $y\in(x_0,\ b(x_0))_s$. We remark that $\boldsymbol{R}^2\setminus s(x)$ is decomposed into two connected components by Lemma 2.1 (d). Then one of the components contains $s(x_0)$ and the other one contains $s(x_2)=s$ since $x_0<_u x<_u x_2$. Since $y\in I_{x_0}\subset X_s$, we have $u(y)\cap s\neq\emptyset$ and since $y\in s(x_0)$, $s(x)\cap u(y)\neq\emptyset$. Hence $(x,\ y)\in\Delta$. The inclusion $\{x_1\}\times(x_0,\ b(x_0))_s\subset\Delta$ is obtained in the same way.

It remains to check that $[x_0,\ x_1)_u\times\{b(x_0)\}\subset\Delta$. If $x\in[x_0,\ x_1)_u$ then $b(x)\in s(x)$. By transversality condition $b(x')\in u(b(x))$ when $x'\in[x_0,\ x_1)_u$ is near enough to x in $u(x_0)$. This shows that $b(x)\in u(b(x_0))$ for all $x\in[x_0,\ x_1)_u$.

If $x\in[x_0,\ x_1)_u$, then $\alpha(x,\ x_0)=x$ and $\alpha(x,\ b(x_0))=b(x)$ (since $b(x)\in u(b(x_0))$ as saw above). By Claim II we have

$$\alpha(\{x\}\times[x_0,\ b(x_0)]_s)=[x,\ b(x)]_s\quad (x\in[x_0,\ x_1)_u). \tag{2.2}$$

If $y\in[x_0,\ b(x_0))_s$, obviously $\alpha(x_0,\ y)=y$. By Claim II we have

$$\alpha([x_0,\ x_1]_u\times\{y\})=[y,\ c(y)]_u\quad (y\in[x_0,\ b(x_0))_s) \tag{2.3}$$

where $c(y)=\alpha(x_1,\ y)$. Since $\alpha(x_1,\ x_0)=x_1$, by Claim II there is $d\in s(x_1)$ or $d=+\infty$ such that

$$\alpha(\{x_1\}\times[x_0,\ b(x_0))_s)=[x_1,\ d)_s. \tag{2.4}$$

CLAIM III. *d is plus infinite* ($d=+\infty$).

PROOF. If $d\in s(x_1)$, then $[x_1, d]_s\subset X_s$ since $x_1\in J$. Hence X_s is a neighborhood of $[x_1, d]_s$ in $\boldsymbol{R}^2$ by (2.1). If $x\in[x_0, x_1)_u$ is near enough to x_1 in $u(x_0)$, then we have $[x, b(x)]_s\subset X_s$ by (2.2), (2.4) and transversality condition. This contradicts $b(x)\notin X_s$.

Let G denote the group of all covering transformations for $P:\boldsymbol{R}^2\to\boldsymbol{T}^2$. It is clear that $\beta(\bar{\mathcal{F}})=\bar{\mathcal{F}}$ and $\beta(\bar{\mathcal{F}}^\phi)=\bar{\mathcal{F}}^\phi$ for $\beta\in G$.

CLAIM IV. *If $\beta\in G$ is not the identity, then $\beta(\sigma(x))\cap\sigma(x)=\emptyset$ for $\sigma=s, u$.*

PROOF. If $\beta(s(x))\cap s(x)\neq\emptyset$, then $\beta(s(x))=s(x)$. In this case β is a covering transformation for $P: s(x)\to P(s(x))\in\mathcal{F}$. By the assumption of Lemma 2.2 $P(s(x))$ is homeomorphic to $\boldsymbol{R}$, from which we see that β is the identity. We obtain the conclusion in the same way for $\sigma=u$.

Let G^0 denote the set of $\beta\in G$ which preserves the orientations of $\bar{\mathcal{F}}$ and $\bar{\mathcal{F}}^\phi$. It follows then that G^0 is a subgroup of G with finite index. Hence we have the orbit space $\boldsymbol{R}^2/G^0$ which is a finite cover of $\boldsymbol{T}^2$ ($=\boldsymbol{R}^2/G$). We denote by $\mathcal{F}^0$ (resp. $\mathcal{F}^{0\phi}$) the foliation on $\boldsymbol{R}^2/G^0$ induced by lifting $\mathcal{F}$ (resp. $\mathcal{F}^\phi$). By the assumption of Lemma 2.2 we have that all leaves of $\mathcal{F}^0$ and $\mathcal{F}^{0\phi}$ are homeomorphic to $\boldsymbol{R}$. Let $Q:\boldsymbol{R}^2\to\boldsymbol{R}^2/G^0$ denote the natural projection. Obviously the lift of $\mathcal{F}^0$ (resp. $\mathcal{F}^{0\phi}$) by Q coincides with $\bar{\mathcal{F}}$ (resp. $\bar{\mathcal{F}}^\phi$).

Now we give the proof of Lemma 2.2 under the above preparations. Since $\boldsymbol{R}^2/G^0$ is compact and every leaf of $\mathcal{F}^{0\phi}$ are homeomorphic to $\boldsymbol{R}$, by recurrent property of leaf we can choose $\{n_i\}_{i=0}^\infty\subset[x_1, +\infty)_s$ and a compact interval I of $u(n_0)$ such that $n_i\to+\infty$ ($i\to\infty$) and $Q(n_i)\in Q(I)$ for all i. Combining (2.4) and Claim III, there is $\{m_i\}_{i=0}^\infty\subset[x_0, b(x_0))_s$ such that $\alpha(x_1, m_i)=n_i$ for all i. Since $\alpha(x_1, m_i)=c(m_i)$, by (2.3) we have

$$\alpha([x_0, x_1]_u\times\{m_i\})=[m_i, n_i]_u. \tag{2.5}$$

From the facts that $n_i\to+\infty$ ($i\to\infty$) and $\{m_i\}_{i=0}^\infty$ is bounded in $\boldsymbol{R}^2$, it follows that

$$\operatorname{diam}([m_i, n_i]_u)\longrightarrow\infty \quad \text{as } i\to\infty \tag{2.6}$$

where the diameter is taken under Euclidean metric. Hence we can assume that $x_0=m_0$, $x_1=n_0$ and $I=[x_0, r]_u$ for some $r\in u(x_0)$. Since $Q(n_i)\in Q(I)=Q([x_0, r]_u)$, there is $\{\beta_i\}_{i=0}^\infty\subset G^0$ such that $\beta_i(n_i)\in[x_0, r]_u$ for all i. Then $\beta_i([m_i, n_i]_u)\subset u(x_0)$ since $\beta_i(\bar{\mathcal{F}})=\bar{\mathcal{F}}$. Since β_i preserves the orientation of $\bar{\mathcal{F}}$, we have

$$\beta_i([m_i, n_i]_u)=[\beta_i(m_i), \beta_i(n_i)]_u. \tag{2.7}$$

Since β_i is isometry under Euclidean metric, by (2.6) we have

$$\operatorname{diam}(\beta_i([m_i, n_i]_u)) \longrightarrow \infty \quad \text{as } i \to \infty .$$

Since $\beta_i(n_i) \in [x_0, r]_u$, this fact ensures the existence of i_0 such that

$$x_0 \in \beta_{i_0}([m_{i_0}, n_{i_0}]_u) . \tag{2.8}$$

Since i_0 is taken sufficiently large, we see that β_{i_0} is not the identity.

Remark that $\beta_{i_0}(n_{i_0}) \in u(x_0) = u(x_1)$. If $x_1 \leqq_u \beta_{i_0}(n_{i_0})$, then $[x_0, x_1]_u \subset \beta_{i_0}([m_{i_0}, n_{i_0}]_u)$ by (2.7) and (2.8). In this case, we have

$$[x_0, x_1]_u \subset \beta_{i_0} \circ \alpha([x_0, x_1]_u \times \{m_{i_0}\})$$

by (2.5) and hence

$$\beta_{i_0} \circ \alpha(\cdot, m_{i_0}) : [x_0, x_1]_u \longrightarrow u(x_0)$$

has a fixed point p. This implies that $\beta_{i_0}(s(p)) = s(p)$ which contradicts Claim IV (since β_{i_0} is not the identity). Therefore $\beta_{i_0}(n_{i_0}) <_u x_1$ and so by (2.7) and (2.8)

$$\beta_{i_0}(n_{i_0}) \in [x_0, x_1)_u . \tag{2.9}$$

Since $\beta_{i_0}^{-1}(x_0) \in [m_{i_0}, n_{i_0}]_u$ by (2.8), we see by (2.5) that there is $x \in [x_0, x_1]_u$ such that $\beta_{i_0}^{-1}(x_0) = \alpha(x, m_{i_0})$. Using (2.2), we have

$$\beta_{i_0}^{-1}(x_0) \in [x, b(x)]_s . \tag{2.10}$$

Since β_{i_0} preserves the orientation of $\mathcal{F}^{\phi}$, we have $x_0 \leqq_s \beta_{i_0}(b(x))$. Remark that $\beta_{i_0}^{-1}(b(x_0)) \in s(x)$. If $b(x) <_s \beta_{i_0}^{-1}(b(x_0))$, then $\beta_{i_0}(b(x)) \in [x_0, b(x_0)]_s$. From (2.2), (2.3) and (2.9) we have

$$[\beta_{i_0}(n_{i_0}), b(\beta_{i_0}(n_{i_0}))]_s \cap [\beta_{i_0}(b(x)), c(\beta_{i_0}(b(x)))]_u \neq \emptyset$$

and hence

$$[n_{i_0}, +\infty)_s \cap u(b(x)) \neq \emptyset$$

which contradicts $b(x) \notin X_s$ since $[n_{i_0}, +\infty)_s \subset s$. Therefore $\beta_{i_0}^{-1}(b(x_0)) \leqq_s b(x)$, and so $\beta_{i_0}^{-1}([x_0, b(x_0)]_s) \subset [x, b(x)]_s$ by (2.10). Hence we have

$$[x_0, b(x_0)]_s \subset \beta_{i_0} \circ \alpha(\{x\} \times [x_0, b(x_0)]_s)$$

by (2.2) and so

$$\beta_{i_0} \circ \alpha(x, \cdot) : [x_0, b(x_0)]_s \longrightarrow s(x_0)$$

has a fixed point q. This implies that $\beta_{i_0}(u(q)) = u(q)$, which contradicts Claim IV (since β_{i_0} is not the identity). We arrived at a contradiction.

Hereafter $f : \boldsymbol{T}^2 \to \boldsymbol{T}^2$ is a homeomorphism with expansiveness and POTP. Define $\mathcal{F}^s(\boldsymbol{T}^2, f)$ and $\mathcal{F}^u(\boldsymbol{T}^2, f)$ as in Introduction. We see that $\mathcal{F}^s(\boldsymbol{T}^2, f)$ and

$\mathcal{F}^u(\boldsymbol{T}^2, f)$ satisfy all the assertions of Proposition 1. As before, let $P:\boldsymbol{R}^2\to\boldsymbol{T}^2$ denote the natural projection and $\bar{\mathcal{F}}^\sigma(\boldsymbol{T}^2, f)$ $(\sigma=s, u)$ denote the lift of $\mathcal{F}^\sigma(\boldsymbol{T}^2, f)$ by P. If $\overline{W}^\sigma(x)$ is the leaf of $\bar{\mathcal{F}}^\sigma(\boldsymbol{T}^2, f)$ through $x\in\boldsymbol{R}^2$, then $\overline{W}^s(x)\cap\overline{W}^u(y)$ is a single point by Lemma 2.2 and so we denote it by $\alpha(x, y)$. Then $\alpha:\boldsymbol{R}^2\times\boldsymbol{R}^2\to\boldsymbol{R}^2$ is a continuous map since $\mathcal{F}^s(\boldsymbol{T}^2, f)$ is transverse to $\mathcal{F}^u(\boldsymbol{T}^2, f)$. Obviously

$$\alpha_0=\alpha\Big|_{\overline{W}^u(0)\times\overline{W}^s(0)}:\ \overline{W}^u(0)\times\overline{W}^s(0)\longrightarrow\boldsymbol{R}^2$$

is bijective.

Let $A:\boldsymbol{T}^2\to\boldsymbol{T}^2$ be the group automorphism such that

$$f_*=A_*:\ H_1(\boldsymbol{T}^2;\boldsymbol{R})\longrightarrow H_1(\boldsymbol{T}^2;\boldsymbol{R}).$$

Obviously A is homotopic to f (since $\boldsymbol{T}^2$ is $K(\boldsymbol{Z}^2, 1)$).

LEMMA 2.3. *A is hyperbolic.*

PROOF. Since f has POTP, the set of periodic points, $\mathrm{Per}(f)$, is non-empty (see [**1**]). We may assume that the identity is a periodic point of f. Then there is the lift $\bar{g}:\boldsymbol{R}^2\to\boldsymbol{R}^2$ of f^n such that $\bar{g}(0)=0$. Since $f(\mathcal{F}^\sigma(\boldsymbol{T}^2, f))=\mathcal{F}^\sigma(\boldsymbol{T}^2, f)$ for $\sigma=s, u$, we have $\bar{g}(\bar{\mathcal{F}}^\sigma(\boldsymbol{T}^2, f))=\bar{\mathcal{F}}^\sigma(\boldsymbol{T}^2, f)$. Let α_0 be as above. Then $\bar{g}\circ\alpha_0(x, y)=\alpha_0(\bar{g}(x), \bar{g}(y))$ and hence $\mathrm{Per}(\bar{g})=\{0\}$. Denote by $\bar{B}:\boldsymbol{R}^2\to\boldsymbol{R}^2$ the linear map which covers A^n. Obviously $\bar{B}|_{Z^2}=\bar{g}|_{Z^2}$ since A^n is homotopic to f^n, and hence $\mathrm{Per}(\bar{B})=\{0\}$. This shows that no eigenvalues of $\bar{B}$ are roots of 1. Since $\bar{B}$ is an integer matrix with determinant 1, it is easily checked that $\bar{B}$ is hyperbolic, and therefore so is A.

LEMMA 2.4. *$f:\boldsymbol{T}^2\to\boldsymbol{T}^2$ has fixed points.*

PROOF. Denote by λ_1 and λ_2 the eigenvalues of the linear map $A_*:H_1(\boldsymbol{T}^2;\boldsymbol{R})\to H_1(\boldsymbol{T}^2;\boldsymbol{R})$. Since $f_*=A_*$, we have that the Lefschetz number of f is

$$L(f)=(1-\lambda_1)(1-\lambda_2).$$

Since A is hyperbolic by Lemma 2.3, it follows that $L(f)\neq0$, and therefore the conclusion is obtained by the fixed point theorem.

PROOF OF PROPOSITION 2. Let $x_0\in\boldsymbol{T}^2$ be a fixed point of f (see Lemma 2.4). Combining Lemma 2.3 and Proposition 2.1 of [**3**], we see that there is a continuous map $h:\boldsymbol{T}^2\to\boldsymbol{T}^2$ such that h is homotopic to the identity id, $h(x_0)=P(0)$ and $A\circ h=h\circ f$. Therefore it is enough to prove that $h:\boldsymbol{T}^2\to\boldsymbol{T}^2$ is bijective. To do this, we take the lifts by P, $\bar{f}$, $\bar{A}$ and $\bar{h}$, such that $\bar{A}\circ\bar{h}=\bar{h}\circ\bar{f}$ holds.

Let $\mathcal{F}_A^s$ and $\mathcal{F}_A^u$ denote the stable and unstable foliations of A and $\bar{\mathcal{F}}_A^\sigma$ $(\sigma=s, u)$ denote the lifts of $\mathcal{F}_A^\sigma$ by P. Then we remark that $\bar{\mathcal{F}}_A^\sigma$ is the family of translations of the eigen space since A is hyperbolic. Since $A\circ h=h\circ f$, it is clear that $h(\mathfrak{F}^{\sigma}(\boldsymbol{T}^2, f))=\mathfrak{F}_A^{\sigma}$, and hence $\bar{h}(\bar{\mathfrak{F}}^{\sigma}(\boldsymbol{T}^2, f))=\bar{\mathfrak{F}}_A^{\sigma}$. If $L^{\sigma}(x)$ denotes the leaf of $\bar{\mathfrak{F}}_A^{\sigma}$ through $x\in\boldsymbol{R}^2$, then we have

$$\bar{h}(\overline{W}^{\sigma}(x)) = L^{\sigma}(\bar{h}(x)) \quad (\sigma=s, u). \tag{2.11}$$

Since h is homotopic to id, there is $K>0$ such that for every $x\in\boldsymbol{R}^2$

$$\|\bar{h}(x)-x\| < K \tag{2.12}$$

where $\|\cdot\|$ denotes Euclidean norm. For $\varepsilon>0$, let $N_{\varepsilon}(L^{\sigma}(x))$ be the ε-neighborhood of $L^{\sigma}(x)$ in $\boldsymbol{R}^2$. By (2.11) and (2.12) we have

$$\overline{W}^{\sigma}(x) \subset N_{2K}(L^{\sigma}(x)) \quad (\sigma=s, u). \tag{2.13}$$

We define

$$I_{x,y}^{\sigma} = \bar{h}^{-1}(x)\cap\overline{W}^{\sigma}(y) \quad (y\in\bar{h}^{-1}(x)).$$

It is checked that $\alpha(I_{x,y}^{u}\times I_{x,y}^{s})=\bar{h}^{-1}(x)$ where α is as above. By (2.11) we have that for $u, v\in\bar{h}^{-1}(x)$

$$\begin{aligned}\{\bar{h}\circ\alpha(u, v)\} &= \bar{h}(\overline{W}^{s}(u)\cap\overline{W}^{u}(v)) \\ &\subset \bar{L}^{s}(\bar{h}(u))\cap\bar{L}^{u}(\bar{h}(v)) \\ &= \{x\},\end{aligned}$$

from which $\alpha(u, v)\in\bar{h}^{-1}(x)$, and so $\alpha(I_{x,y}^{u}\times I_{x,y}^{s})=\bar{h}^{-1}(x)$.

Since $\overline{W}^{\sigma}(x)$ is homeomorphic to $\boldsymbol{R}$, we can take the minimal connected subset $\bar{I}_{x,y}^{\sigma}$ of $\overline{W}^{\sigma}(y)$ such that $I_{x,y}^{\sigma}\subset\bar{I}_{x,y}^{\sigma}$. Since $\operatorname{diam}(\bar{h}^{-1}(x))<2K$ by (2.12), we have

$$\bar{I}_{x,y}^{s} \subset \alpha(y, B_{2k}(y)) \quad\text{and}\quad \bar{I}_{x,y}^{u} \subset \alpha(B_{2K}(y), y) \tag{2.14}$$

where $B_{2K}(y)=\{z\in\boldsymbol{R}^2 : \|y-z\|\leqq 2K\}$. It is clear that there is $N>0$ such that if $\|y-z\|\leqq 6K$ then

$$L^{s}(y)\cap L^{u}(z) \subset B_{N}(y)\cap B_{N}(z). \tag{2.15}$$

Hence we have by (2.13) that

$$\begin{aligned}\alpha(y, B_{2K}(y)) &= \bigcup_{z\in B_{2K}(y)} \overline{W}^{s}(y)\cap\overline{W}^{u}(z) \\ &\subset \bigcup_{z\in B_{2K}(y)} N_{2K}(L^{s}(y))\cap N_{2K}(L^{u}(z)) \quad \text{(by (2.13))} \\ &\subset B_{N+4K}(y) \quad \text{(by (2.15))}\end{aligned}$$

and also $\alpha(B_{2K}(y), y)\subset B_{N+4K}(y)$. Therefore $\bar{I}_{x,y}^{\sigma}\subset B_{N+4K}(y)$ by (2.14).

On the other hand, we have $\bar{f}\circ\bar{h}^{-1}(x)=\bar{h}^{-1}\circ\bar{A}(x)$ since $\bar{A}\circ\bar{h}=\bar{h}\circ\bar{f}$, and hence

$$\bar{f}^n(\bar{I}^{\sigma}_{x,y}) = \bar{I}^{\sigma}_{\bar{A}^n(x),\bar{f}^n(y)} \quad (\sigma = s,\ u).$$

By using the above facts we show that each $\bar{I}^{\sigma}_{x,y}$ is a single point. If this is checked for $\sigma=u$, then the case $\sigma=s$ is done in the same way. So we assume that $\bar{I}^u_{x,y}$ is not a single point. Then $P(\bar{I}^u_{x,y})$ is an interval of $W^u(P(y))$. Hence for $k>0$ there are $\varepsilon>0$ and $y_i \in P(\bar{I}^u_{x,y})$ $(i=1, 2, \cdots, k)$ such that $P(\bar{I}^u_{x,y}) \supset \bigcup_{i=1}^k W^u_{\varepsilon}(y_i)$ (disjoint union). Mañé's result ensures the existence of $n \geqq 0$ such that $f^n \circ P(\bar{I}^u_{x,y}) \supset \bigcup_{i=1}^k W^u_{\varepsilon_0}(z_i)$ (disjoint union) where $z_i = f^n(y_i)$. Putting $\bar{z}_i = P|_{\bar{W}^u(\bar{f}^n(y))}{}^{-1}(z_i)$, we have that $\bar{z}_i \in \bar{I}_{\bar{A}^n(x),\bar{f}^n(y)}$ for all i. Denote by $\bar{N}_{\bar{z}_i}$ the lift of N_{z_i} by P such that $\bar{z}_i \in \bar{N}_{\bar{z}_i}$. Then we have $\bar{N}_{\bar{z}_i} \cap \bar{N}_{\bar{z}_j} = \emptyset$ $(i \neq j)$. Hence $B_{\rho}(\bar{z}_i) \cap B_{\rho}(\bar{z}_j) = \emptyset$ $(i \neq j)$ by Proposition 1.5 (c). Since $\bar{I}^u_{\bar{A}^n(x),\bar{f}^n(y)} \subset B_{N+4K}(\bar{f}^n(y))$, we have $B_{\rho}(\bar{z}_i) \subset B_{N+4K+\rho}(\bar{f}^n(y))$ for all i. This is impossible since k is arbitrary. Therefore each $\bar{I}^u_{x,y}$ is a single point.

Since $\bar{h}^{-1}(x) \subset \alpha(\bar{I}^u_{x,y} \times \bar{I}^s_{x,y})$, $\bar{h}^{-1}(x)$ is a single point and hence $\bar{h}$ is injective. Since $\bar{h}$ is surjective by (2.12), we have that $\bar{h}$ is bijective and therefore h is bijective.

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Koichi HIRAIDE
Kagoshima National College of Technology
Shinko 1460-1, Hayato-cho, Aira-gun
Kagoshima 899-51
Japan