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L^p-boundedness of pseudo-differential operators satisfying Besov estimates I

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0. Introduction.

There have already been many results on sufficient conditions for the $L^2(\boldsymbol{R}^n)$-boundedness of pseudo-differential operators; for example, Hörmander [6], Calderón-Vaillancourt [2], Cordes [5], Childs [3], Kato [7], Coifman-Meyer [4]. In many papers, it is the main problem to what degree we can relax regularity conditions for symbols. In this direction, the most fundamental result is the following theorem, due to Calderón-Vaillancourt [2].

THEOREM A. *Let $\sigma(x, \xi)$ be a function which satisfies $\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi) \in L^\infty(\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n)$ for $\alpha, \beta \in \{0, 1, 2, 3\}^n$. Then the pseudo-differential operator*

$$\sigma(X, D)f(x) = (2\pi)^{-n} \int_{\boldsymbol{R}^n} e^{ix\cdot\xi} \sigma(x, \xi) \hat{f}(\xi) d\xi, \quad f \in \mathcal{S}(\boldsymbol{R}^n)$$

is $L^2(\boldsymbol{R}^n)$-bounded, that is, it can be extended to a bounded operator on $L^2(\boldsymbol{R}^n)$.

Here we have used the usual notation of multi-indices, the Schwartz space $\mathcal{S}=\mathcal{S}(\boldsymbol{R}^n)$ of rapidly decreasing smooth functions, and the Fourier transformation $\hat{f}$ of f.

Furthermore, various sufficient conditions for the $L^2(\boldsymbol{R}^n)$-boundedness have been obtained. We list some of them in the following:

THEOREM B. *Let $\sigma(x, \xi)$ be a function which satisfies one of the following five conditions:*

i) (Cordes [5; Theorem B_1']) $\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi) \in L^\infty(\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n)$ *(and is continuous) for $|\alpha|, |\beta| \leqq [n/2]+1$.*

ii) (Cordes [5; Theorem D]) *There exist real numbers $\lambda, \lambda' > n/2$ such that $\langle D_x \rangle^\lambda \langle D_\xi \rangle^{\lambda'} \sigma(x, \xi) \in L^\infty(\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n)$.*

iii) (Childs [3]) *There exist a real number $\lambda > 1/2$ and a constant C such that $\|\Delta_x^\alpha(h) \Delta_\xi^\beta(h') \sigma(x, \xi)\|_{L^\infty(\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n)} \leqq C \prod_{i,j=1}^n |h_i|^{\alpha_i \lambda} |h_j'|^{\beta_j \lambda}$ holds for all $\alpha, \beta \in \{0, 1\}^n$ and all $h=(h_1, \cdots, h_n)$, $h'=(h_1', \cdots, h_n') \in \boldsymbol{R}^n$.*

iv) (Coifman-Meyer [**4**; Théorème 2, Corollaire 3]) $\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi) \in L^\infty(\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n)$ *(and is continuous) for* $|\alpha| \leqq [n/2]+1$, $\beta \in \{0, 1, 2\}^n$.

v) (Coifman-Meyer [**4**; Théorème 4]) *There exists a real number* $2 \leqq q < \infty$ *such that* $\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi) \in L^q(\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n)$ *for* $|\alpha| \leqq [n(1/2-1/q)]+1$, $|\beta| \leqq 2n$.

Then the operator $\sigma(X, D)$ *is* $L^2(\boldsymbol{R}^n)$*-bounded.*

Here we have used the following notations.

$[s]$ is the integral part of a real number s.

$\langle D_x \rangle^\lambda = \mathcal{F}_y^{-1}(1+|y|^2)^{\lambda/2}\mathcal{F}_x$, where $\mathcal{F}$ (resp. $\mathcal{F}^{-1}$) is the Fourier (resp. inverse Fourier) transformation with respect to the suffixed variable. $\langle D_\xi \rangle^{\lambda'}$ is defined in the same way.

$\Delta_x^\alpha(h) = \Delta_{x_1}^{\alpha_1}(h_1) \cdots \Delta_{x_n}^{\alpha_n}(h_n)$; $x = (x_1, \cdots, x_n) \in \boldsymbol{R}^n$, where $\Delta_{x_i}^{\alpha_i}(h_i)$ $(i=1, \cdots, n)$ denotes the α_i-th order $(\alpha_i = 0, 1)$ difference operator with respect to the variable x_i, that is, $\Delta_{x_i}^0(h_i)\sigma(x, \xi) = \sigma(x, \xi)$ and $\Delta_{x_i}^1(h_i)\sigma(x, \xi) = \sigma(x+h_i e_i, \xi) - \sigma(x, \xi)$. ($e_i$ denotes the element in $\boldsymbol{R}^n$ with all its entries equal to 0 except the i-th, which is equal to 1.) $\Delta_\xi^\beta(h')$ is defined in the same way.

REMARK. 1) Theorem A is contained in Theorem B with condition iii).

2) We need not necessarily assume the continuity of derivatives of symbols in conditions i) and iv).

The purpose of the present paper is to unify all these results in a *single* form in terms of Besov spaces, and to give a *sharper* condition for L^2-boundedness. Roughly speaking, condition i) in Theorem B implies that the symbols are of class $C^{[n/2]+1}$ with respect to x and ξ separately. ($C^{[n/2]+1}$ is the class of all the functions that are ($[n/2]+1$)-times bounded-continuously differentiable.) We remark that the order "$[n/2]+1$" is critical; in fact, there exists a counterexample in the space $C^{[n/2]}$; see Coifman-Meyer [**4**; p. 12].

On the other hand, it is known that the proper inclusions

$$C^{[n/2]+1} \subsetneqq B_{\infty,\infty}^{(n/2)+\varepsilon} \subsetneqq B_{\infty,1}^{n/2} \subsetneqq C^{[n/2]}$$

hold for sufficiently small $\varepsilon > 0$. Here $B_{p,q}^\lambda$ denotes the Besov space, and is an extension of Lipschitz spaces; see, for example, Bergh-Löfström [**1**], Peetre [**12**] or Triebel [**14**]. Recently, Miyachi [**10**] has shown that the symbols in $B_{\infty,\infty}^{(n/2)+\varepsilon}$, $\varepsilon > 0$, (with respect to x and ξ separately) also generate $L^2(\boldsymbol{R}^n)$-bounded pseudo-differential operators, and this result is equivalent to Theorem B with condition ii) ([**10**; Theorem B]). We shall show that it can be extended to the space $B_{\infty,1}^{n/2}$ (Theorem 2.1.2; $q=\infty$). In other words, Theorem B remains valid if we replace the "order" $[n/2]+1$ or $(n/2)+\varepsilon$ in condition i) or ii) by $n/2$ in the sense of Besov spaces.

Furthermore, the same conclusions are true for conditions iii) and iv) in Theorem B, if we appropriately decompose variables $(x, \xi)\in \boldsymbol{R}_x^n\times\boldsymbol{R}_\xi^n$ into $(x, \xi)=(x_1, \cdots, x_N, \xi_1, \cdots, \xi_{N'})\in\boldsymbol{R}_{x_1}^{n_1}\times\cdots\times\boldsymbol{R}_{x_N}^{n_N}\times\boldsymbol{R}_{\xi_1}^{n'_1}\times\cdots\times\boldsymbol{R}_{\xi_{N'}}^{n'_{N'}}$ $(n=n_1+\cdots+n_N=n'_1+\cdots+n'_{N'})$ and consider Besov spaces with respect to each decomposed variable separately. For example, condition iii) of Theorem B is in the case $N=N'=n$, and this condition says that a symbol $\sigma(x, \xi)$ belongs to the space $B_{\infty,\infty}^{(1/2)+\varepsilon}$, $\varepsilon>0$, with respect to each variable separately. (See Remark 2.1.2.) We shall show that the order $(1/2)+\varepsilon$ can be replaced by $1/2$.

By this method mentioned above, we can unify conditions i) through iv) in Theorem B in a single form. In order to state our results precisely, we shall introduce Besov spaces on the product space $\boldsymbol{R}_{x_1}^{n_1}\times\cdots\times\boldsymbol{R}_{x_N}^{n_N}\times\boldsymbol{R}_{\xi_1}^{n'_1}\times\cdots\times\boldsymbol{R}_{\xi_{N'}}^{n'_{N'}}$. They are extensions of Lipschitz spaces on the product space $\boldsymbol{R}_x^n\times\boldsymbol{R}_\xi^n$, which were introduced for the first time by Miyachi [**10**; Section 2]. In the framework of Lipschitz spaces, we can treat only bounded symbols. But in the framework of our Besov spaces, we can treat symbols belonging to the spaces L^q $(q\neq\infty)$. Particularly, our theory allows us to treat condition v) of Theorem B in our theory.

The main result is the following:

MAIN THEOREM (Theorem 2.1.2). *Let* $2\leqq q\leqq\infty$. *Then there exists a constant* C *such that the estimate*

$$\|\sigma(X, D)f\|_{L^2(R^n)}\leqq C\|\sigma(x, \xi)\|_{B_{q,1}^{(1/2-1/q)(\boldsymbol{n},\boldsymbol{n}')}}\|f\|_{L^2(R^n)}$$

holds for all σ *in* $B_{q,1}^{(1/2-1/q)(\boldsymbol{n},\boldsymbol{n}')}$ *and all* f *in* $\mathcal{S}(\boldsymbol{R}^n)$.

Here $B_{q,1}^{(1/2-1/q)(\boldsymbol{n},\boldsymbol{n}')}$ is one of the Besov spaces on product spaces which are introduced in Section 1; see Definition 1.1.3. Any condition in Theorem B implies that $\sigma\in B_{q,1}^{(1/2-1/q)(\boldsymbol{n},\boldsymbol{n}')}$ for some q, $\boldsymbol{n}$, $\boldsymbol{n}'$, but conversely, for any q, $\boldsymbol{n}$, $\boldsymbol{n}'$, the condition $\sigma\in B_{q,1}^{(1/2-1/q)(\boldsymbol{n},\boldsymbol{n}')}$ does not necessarily imply some condition in Theorem B; hence this theorem is a proper extension of Theorem B (Remark 2.1.2).

Recently, Muramatu [**11**] has also discussed the L^2-boundedness of pseudo-differential operators whose symbols belong to the Besov spaces on the product space $\boldsymbol{R}_x^n\times\boldsymbol{R}_\xi^n$, and obtained, in a different way from ours, some results similar to our main theorem; see Muramatu [**11**; Theorem 3.1]. His results contain Theorem B with conditions i), ii), and v) while they do not necessarily contain Theorem B with conditions iii) and iv).

Our Besov spaces are extensions of the symbol class $S_{0,0}^0$ in the sense of Kumano-go [8] (Remark 2.1.3). On the other hand, Muramatu [**11**] also discusses a Besov space version of the symbol class $S_{\rho,\delta}^m$ $(0\leqq\delta\leqq\rho\leqq1)$, and proves the L^2-boundedness of pseudo-differential operators whose symbols belong to this class; see [**11**; Theorem 4.5, 4.6].

There are two methods of treating Besov spaces; the method of "Fourier transformation" and the method of "regularization of distribution". Our treatment follows the first one, while Muramatu [**11**] follows the second.

The contents of the subsequent sections are the following. In Section 1, we introduce the weighted Besov spaces on product spaces. These are an extension of the ordinary Besov spaces. In Section 2, we show some boundedness theorems (including Main Theorem) in terms of these Besov spaces. In the present paper, we mainly discuss the $L^2(\boldsymbol{R}^n)$-boundedness, but the general $L^p(\boldsymbol{R}^n)$-boundedness ($p\neq 2$) will be discussed in the forthcoming paper (Sugimoto [**13**]).

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NOTATIONS. We shall explain the notations used in this paper.

$\boldsymbol{R}^n$ denotes the Euclidean space of dimension n, and $\boldsymbol{N}$ denotes the set of all natural numbers.

Multi-index notation follows Kumano-go [8; p. 6]. For $\boldsymbol{\lambda}=(\lambda_1, \cdots, \lambda_N)$ and $\boldsymbol{\sigma}=(\sigma_1, \cdots, \sigma_N)\in\boldsymbol{R}^N$, $\boldsymbol{\lambda}\geqq\boldsymbol{\sigma}$ means $\lambda_r\geqq\sigma_r$ $(r=1, \cdots, N)$.

$\mathcal{S}=\mathcal{S}(\boldsymbol{R}^n)$ is the Schwartz space of rapidly decreasing smooth functions, and $\mathcal{S}'=\mathcal{S}'(\boldsymbol{R}^n)$ is the dual space of $\mathcal{S}=\mathcal{S}(\boldsymbol{R}^n)$.

Let $f(x)$ be a function in $\mathcal{S}(\boldsymbol{R}^n_x)$, and $\sigma(x, \xi)$ in $\mathcal{S}(\boldsymbol{R}^n_x\times\boldsymbol{R}^n_\xi)$. Then the Fourier transformations of f and σ are defined respectively by the following formulae:

$$\hat{f}(y)=\mathcal{F}f(y)=\mathcal{F}_xf(y)=\int_{\boldsymbol{R}^n}e^{-ix\cdot y}f(x)dx,$$

$$\hat{f}^*(y)=\mathcal{F}^{-1}f(y)=\mathcal{F}_x^{-1}f(y)=(2\pi)^{-n}\int_{\boldsymbol{R}^n}e^{ix\cdot y}f(x)dx,$$

$$\mathcal{F}\sigma(y, \eta)=\mathcal{F}_{x,\xi}\sigma(y, \eta)=\iint_{\boldsymbol{R}^n\times\boldsymbol{R}^n}e^{-i(x\cdot y+\xi\cdot\eta)}\sigma(x, \xi)dxd\xi,$$

$$\mathcal{F}^{-1}\sigma(x, \xi)=\mathcal{F}_{y,\eta}^{-1}\sigma(x, \xi)=(2\pi)^{-2n}\iint_{\boldsymbol{R}^n\times\boldsymbol{R}^n}e^{i(x\cdot y+\xi\cdot\eta)}\sigma(y, \eta)dyd\eta.$$

They can be extended to the dual space $\mathcal{S}'$ as usual.

If $\sigma(x, \xi)$ is a function on $\boldsymbol{R}^n_x\times\boldsymbol{R}^n_\xi$, then the pseudo-differential operator $\sigma(X, D)$ can be defined by

$$\sigma(X, D)f(x)=\int e^{ix\cdot\xi}\sigma(x, \xi)\hat{f}(\xi)đ\xi, \quad f\in\mathcal{S}(\boldsymbol{R}^n)$$

$(đ\xi=(2\pi)^{-n}d\xi)$ if the integral exists.

Throughout the paper the letter "C" denotes a constant which may be different in each occasion.

1. Weighted Besov spaces on product spaces.

In this section, we shall introduce the weighted Besov spaces on product spaces. Although these spaces are a slight extension of the ordinary Besov spaces discussed in Peetre [**12**], Bergh-Löfström [**1**] or Triebel [**14**], these generalized spaces play an important role in Section 2. We shall extend Triebel's discussion ([**14**]) of the ordinary Besov spaces.

1.1. Definitions. Following Triebel [**14**], we shall introduce some classes of partition of unity.

DEFINITION 1.1.1. Let $\mathcal{A}(\boldsymbol{R}^n)$ be the collection of all the systems $\Theta=\{\Theta_j(y)\}_{j=0}^{\infty}\subset\mathcal{S}(\boldsymbol{R}^n)$ that satisfy the following conditions:

i) $\operatorname{supp}\Theta_0\subset\{y\,;\,|y|\leqq 2\}$, $\operatorname{supp}\Theta_j\subset\{y\,;\,2^{j-1}\leqq|y|\leqq 2^{j+1}\}$ if $j=1, 2, \cdots$.

ii) $\sum_{j=0}^{\infty}\Theta_j(y)=1$.

iii) For every multi-index α, there exists a positive number C_α such that the estimate $2^{j|\alpha|}|\partial^\alpha\Theta_j(y)|\leqq C_\alpha$ holds for all $j=0, 1, 2, \cdots$ and all $y\in\boldsymbol{R}^n$.

Next, we shall introduce the polynomially weighted L^p-spaces.

DEFINITION 1.1.2. Let $0<p\leqq\infty$, $n=n_1+\cdots+n_N$; $n_r\in\boldsymbol{N}$ $(r=1, \cdots, N)$, and let $\boldsymbol{n}=(n_1, \cdots, n_N)$, $\boldsymbol{\rho}=(\rho_1, \cdots, \rho_N)\in\boldsymbol{R}^N$, $x=(x_1, \cdots, x_N)$; $x_r\in\boldsymbol{R}^{n_r}$ $(r=1, \cdots, N)$. Then we set

$$\omega_{\boldsymbol{\rho}}^{\boldsymbol{n}}(x)=\langle x_1\rangle^{\rho_1}\cdots\langle x_N\rangle^{\rho_N}, \quad \|f\|_{L_{\boldsymbol{\rho}}^p(\boldsymbol{R}_{\boldsymbol{n}})}=\|\omega_{\boldsymbol{\rho}}^{\boldsymbol{n}}\cdot f\|_{L^p(\boldsymbol{R}^n)},$$
$$L_{\boldsymbol{\rho}}^p(\boldsymbol{R}_{\boldsymbol{n}})=\{f\in\mathcal{S}'\,;\,\|f\|_{L_{\boldsymbol{\rho}}^p(\boldsymbol{R}_{\boldsymbol{n}})}<+\infty\}.$$

Here $\|\cdot\|_{L^p(\boldsymbol{R}^n)}$ means usual L^p-(quasi-)norm on $\boldsymbol{R}^n$ with respect to Lebesgue measure, and $\langle\cdot\rangle=(1+|\cdot|^2)^{1/2}$.

Now, we can define the weighted Besov spaces on product spaces following Triebel [**14**].

DEFINITION 1.1.3. Let $0<p, q\leqq\infty$, $n=n_1+\cdots+n_N=\tilde{n}_1+\cdots+\tilde{n}_{\tilde{N}}$; $n_r, \tilde{n}_s\in\boldsymbol{N}$ $(r=1, \cdots, N;\ s=1, \cdots, \tilde{N})$, and let $\boldsymbol{n}=(n_1, \cdots, n_N)$, $\tilde{\boldsymbol{n}}=(\tilde{n}_1, \cdots, \tilde{n}_{\tilde{N}})$, $\boldsymbol{\lambda}=(\lambda_1, \cdots, \lambda_N)\in\boldsymbol{R}^N$, $\boldsymbol{\rho}=(\rho_1, \cdots, \rho_{\tilde{N}})\in\boldsymbol{R}^{\tilde{N}}$. Then we set

$$B_{p,q,\boldsymbol{\rho}}^{\boldsymbol{\lambda}}(\boldsymbol{R}_{\tilde{\boldsymbol{n}}}^{\boldsymbol{n}})=\{f\in\mathcal{S}'(\boldsymbol{R}^n)\,;\,\|f\|_{B_{p,q,\boldsymbol{\rho}}^{\boldsymbol{\lambda}}(\boldsymbol{R}_{\tilde{\boldsymbol{n}}}^{\boldsymbol{n}})}<+\infty\},$$

where

$$\|f\|_{B^{\boldsymbol{\lambda}}_{p,q,\boldsymbol{\rho}}(\boldsymbol{R}^{\boldsymbol{n}}_{\tilde{\boldsymbol{n}}})} = \left\| \|2^{\boldsymbol{j}\cdot\boldsymbol{\lambda}} \mathcal{F}^{-1} \Phi^{\boldsymbol{n}}_{\boldsymbol{j}} \mathcal{F} f\|_{L^p_{\boldsymbol{\rho}}(\boldsymbol{R}_{\tilde{\boldsymbol{n}}})} \right\|_{l^q}$$

$$= \left\{ \sum_{\boldsymbol{j}\geq 0} \left(\int_{R^n} |\omega^{\tilde{\boldsymbol{n}}}_{\boldsymbol{\rho}}(x) \cdot 2^{\boldsymbol{j}\cdot\boldsymbol{\lambda}} \mathcal{F}^{-1} \Phi^{\boldsymbol{n}}_{\boldsymbol{j}} \mathcal{F} f(x)|^p dx \right)^{q/p} \right\}^{1/q}$$

(with a slight modification in the case of $p=\infty$ and/or $q=\infty$). Here $\boldsymbol{j}=(j_1, \cdots, j_N)$ (non-negative integer vector) and $\Phi^{\boldsymbol{n}}_{\boldsymbol{j}}(y)=\Phi^{\boldsymbol{n}}_{\boldsymbol{j}}(y_1, \cdots, y_N)=\Theta_{j_1}(y_1) \cdots \Theta_{j_N}(y_N)$; $\{\Theta_{j_r}(y_r)\}_{j_r=0}^{\infty} \in \mathcal{A}(\boldsymbol{R}^{n_r})$ $(r=1, \cdots, N)$. In the case $\boldsymbol{\rho}=\boldsymbol{0}$, we use the abbreviation $B^{\boldsymbol{\lambda}}_{p,q}(\boldsymbol{R}^{\boldsymbol{n}})$.

It is convenient to use the following notations.

DEFINITION 1.1.4. Let n, $\boldsymbol{n}$, $\tilde{\boldsymbol{n}}$, $\boldsymbol{\lambda}$, and $\boldsymbol{\rho}$ be the same as in Definition 1.1.3. Then we set for $f \in \mathcal{S}'(\boldsymbol{R}^n)$

$$I^{\boldsymbol{n}}_{\boldsymbol{\lambda}} f = \mathcal{F}^{-1} \omega^{\boldsymbol{n}}_{\boldsymbol{\lambda}} \mathcal{F} f \quad \text{(Bessel transformation)},$$

$$J^{\tilde{\boldsymbol{n}}}_{\boldsymbol{\rho}} f = \omega^{\tilde{\boldsymbol{n}}}_{\boldsymbol{\rho}} \cdot f.$$

DEFINITION 1.1.5. Let n, $\boldsymbol{n}$, $\tilde{\boldsymbol{n}}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ be the same as in Definition 1.1.3, and let $1 \leq p \leq \infty$.

i) (Bessel potential space) We set

$$H^{\boldsymbol{\lambda}}_{p,\boldsymbol{\rho}}(\boldsymbol{R}^{\boldsymbol{n}}_{\tilde{\boldsymbol{n}}}) = \{f \in \mathcal{S}'(\boldsymbol{R}^n);\ \|f\|_{H^{\boldsymbol{\lambda}}_{p,\boldsymbol{\rho}}(\boldsymbol{R}^{\boldsymbol{n}}_{\tilde{\boldsymbol{n}}})} < +\infty\},$$

where

$$\|f\|_{H^{\boldsymbol{\lambda}}_{p,\boldsymbol{\rho}}(\boldsymbol{R}^{\boldsymbol{n}}_{\tilde{\boldsymbol{n}}})} = \|I^{\boldsymbol{n}}_{\boldsymbol{\lambda}} f\|_{L^p_{\boldsymbol{\rho}}(\boldsymbol{R}_{\tilde{\boldsymbol{n}}})}.$$

ii) Let $\boldsymbol{\lambda}$ be a non-negative integer vector. Then we set

$$C^{\boldsymbol{\lambda}}_{\boldsymbol{\rho}}(\boldsymbol{R}^{\boldsymbol{n}}_{\tilde{\boldsymbol{n}}}) = \{f \in C(\boldsymbol{R}^n);\ \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_N}_{x_N} f \in C(\boldsymbol{R}^n) \ \text{ for } |\alpha_r| \leq \lambda_r \ (r=1, \cdots, N) \ \text{ and } \|f\|_{C^{\boldsymbol{\lambda}}_{\boldsymbol{\rho}}(\boldsymbol{R}^{\boldsymbol{n}}_{\tilde{\boldsymbol{n}}})} < +\infty\},$$

where $C(\boldsymbol{R}^n)$ is the set of all continuous functions on $\boldsymbol{R}^n$ and

$$\|f\|_{C^{\boldsymbol{\lambda}}_{\boldsymbol{\rho}}(\boldsymbol{R}^{\boldsymbol{n}}_{\tilde{\boldsymbol{n}}})} = \sum_{\substack{|\alpha_r| \leq \lambda_r \\ (r=1,\cdots,N)}} \|\partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_N}_{x_N} f\|_{L^\infty_{\boldsymbol{\rho}}(\boldsymbol{R}_{\tilde{\boldsymbol{n}}})}$$

(classical derivative) $(x_r \in \boldsymbol{R}^{n_r};\ r=1, \cdots, N)$.

iii) (Sobolev space) Let $\boldsymbol{\lambda}$ be a non-negative integer vector. Then we set

$$W^{\boldsymbol{\lambda}}_{p,\boldsymbol{\rho}}(\boldsymbol{R}^{\boldsymbol{n}}_{\tilde{\boldsymbol{n}}}) = \{f \in \mathcal{S}'(\boldsymbol{R}^n);\ \|f\|_{W^{\boldsymbol{\lambda}}_{p,\boldsymbol{\rho}}(\boldsymbol{R}^{\boldsymbol{n}}_{\tilde{\boldsymbol{n}}})} < +\infty\},$$

where

$$\|f\|_{W^{\boldsymbol{\lambda}}_{p,\boldsymbol{\rho}}(\boldsymbol{R}^{\boldsymbol{n}}_{\tilde{\boldsymbol{n}}})} = \sum_{\substack{|\alpha_r| \leq \lambda_r \\ (r=1,\cdots,N)}} \|\partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_N}_{x_N} f\|_{L^p_{\boldsymbol{\rho}}(\boldsymbol{R}_{\tilde{\boldsymbol{n}}})}$$

(distributional derivative) $(x_r \in \boldsymbol{R}^{n_r};\ r=1, \cdots, N)$.

In the case $\boldsymbol{\rho}=\mathbf{0}$, we use the abbreviation $H_p^\lambda(\boldsymbol{R}^n)$, $C^\lambda(\boldsymbol{R}^n)$ and $W_p^\lambda(\boldsymbol{R}^n)$.

$H_{p,\boldsymbol{\rho}}^\lambda(\boldsymbol{R}_{\tilde{n}}^n)$ (resp. $C_{\boldsymbol{\rho}}^\lambda(\boldsymbol{R}_{\tilde{n}}^n)$, $W_{p,\boldsymbol{\rho}}^\lambda(\boldsymbol{R}_{\tilde{n}}^n)$) is a normed space with norm (quasi-norm if $0<p\leqq 1$) $\|\cdot\|_{H_{p,\boldsymbol{\rho}}^\lambda(\boldsymbol{R}_{\tilde{n}}^n)}$ (resp. $\|\cdot\|_{C_{\boldsymbol{\rho}}^\lambda(\boldsymbol{R}_{\tilde{n}}^n)}$, $\|\cdot\|_{W_{p,\boldsymbol{\rho}}^\lambda(\boldsymbol{R}_{\tilde{n}}^n)}$)

In the rest of this section, we shall abbreviate the above notations by omitting $\boldsymbol{n}$, $\tilde{\boldsymbol{n}}$, $\boldsymbol{R}^n$, $\boldsymbol{R}_{\tilde{n}}$ and $\boldsymbol{R}_{\tilde{n}}^n$. For example, we shall abbreviate $\omega_{\boldsymbol{\rho}}^n$ and $B_{p,q,\boldsymbol{\rho}}^\lambda(\boldsymbol{R}_{\tilde{n}}^n)$ to $\omega_{\boldsymbol{\rho}}$ and $B_{p,q,\boldsymbol{\rho}}^\lambda$ respectively.

REMARK 1.1.1. If $N=\tilde{N}=1$ and $\boldsymbol{\rho}=\mathbf{0}$, $B_{p,q}^\lambda$ in Definition 1.1.3 is the ordinary one; see [**1**], [**12**] and [**14**]. Miyachi [**10**] defines and uses some spaces which correspond to $B_{\infty,\infty,(\rho,\rho')}^{(\lambda,\lambda')}(\boldsymbol{R}_{(n,n)}^{(n,n)})$ in our notation.

REMARK 1.1.2. Triebel [**14**] and Löfström [**9**] also discuss the weighted Besov spaces (not on product spaces). For example, Löfström [**9**] discusses more general weight functions which include our polynomially weight functions.

REMARK 1.1.3. In the next section, we only use the case $\boldsymbol{\rho}=\mathbf{0}$. But the case $\boldsymbol{\rho}\neq\mathbf{0}$ plays an important role in Sugimoto [**13**].

1.2. Fundamental inequalities. We shall show several fundamental inequalities which are used to construct the theory of Besov spaces systematically. Throughout this subsection, we always decompose variables in $\boldsymbol{R}^n$ in such a way that

$$x=(x_1,\cdots,x_N),\quad x_r\in\boldsymbol{R}^{n_r}\ (r=1,\cdots,N),$$
$$=(x_1,\cdots,x_{\tilde{N}}),\quad x_s\in\boldsymbol{R}^{\tilde{n}_s}\ (s=1,\cdots,\tilde{N}).$$

The next lemma states the most important property of our weight functions.

LEMMA 1.2.1. *There exists a constant C such that the inequality*

$$\omega_{\boldsymbol{\rho}}(x+u)\leqq C\omega_{\boldsymbol{\rho}}(x)\omega_{\boldsymbol{\rho}^*}(u) \tag{1}$$

holds for all $x, u\in\boldsymbol{R}^n$. *Here* $\boldsymbol{\rho}^*=(|\rho_1|,\cdots,|\rho_{\tilde{N}}|)$ *for* $\boldsymbol{\rho}=(\rho_1,\cdots,\rho_{\tilde{N}})$.

PROOF. This lemma can be proved immediately from the facts $\langle x_r+u_r\rangle\leqq 2\langle x_r\rangle\langle u_r\rangle$, $\langle x_r+u_r\rangle^{-1}\leqq 2\langle x_r\rangle^{-1}\langle u_r\rangle$, $(r=1,\cdots,\tilde{N})$.

With this lemma, we can reform the fundamental inequalities which are used in Triebel [**14**].

LEMMA 1.2.2 (cf. [**14**; p. 18 (5)], [**12**; p. 54 (12), (13)]). *Let* α_r *be a multi-index of dimension* n_r $(r=1,\cdots,N)$. *Then it holds that*

$$\|f\|_{L_{\boldsymbol{\rho}}^q}\leqq C(b_1^{n_1}\cdots b_N^{n_N})^{(1/p-1/q)}\|f\|_{L_{\boldsymbol{\rho}}^p}\qquad \textit{if } 0<p\leqq q\leqq\infty$$

(*Nikol'skij inequality*) *and*

$$\|\partial_{x_1}^{\alpha_1}\cdots\partial_{x_N}^{\alpha_N}f\|_{L^p_{\boldsymbol\rho}}\leqq Cb_1^{|\alpha_1|}\cdots b_N^{|\alpha_N|}\|f\|_{L^p_{\boldsymbol\rho}}\qquad if\ 1\leqq p\leqq\infty$$

(Bernstein inequality) for all $b_1,\cdots,b_N\geqq 1$ *and all* $f\in L^p_{\boldsymbol\rho}$ *such that* $\operatorname{supp}\hat f\subset\{y;|y_1|\leqq b_1,\cdots,|y_N|\leqq b_N\}$. *Here* C *is a constant independent of* $b_1,\cdots,b_N$ *and* f.

LEMMA 1.2.3 (Fourier multiplier theorem; cf. [**14**; p. 28 (13)]). *Let* $0<p\leqq\infty$, *and let* $s_{p,\boldsymbol\rho}=n(1/\min(p,1)-1/2)+|\boldsymbol\rho|$ $(|\boldsymbol\rho|=|\rho_1|+\cdots+|\rho_{\bar N}|)$. *If* $s>s_{p,\boldsymbol\rho}$, *then there exists a constant* C *such that the estimate*

$$\|\mathcal{F}^{-1}M\mathcal{F}f\|_{L^p_{\boldsymbol\rho}}\leqq C\|M(b_1y_1,\cdots,b_Ny_N)\|_{H_2^s(\boldsymbol R^n)}\|f\|_{L^p_{\boldsymbol\rho}}$$

holds for all $b_1,\cdots,b_N\geqq 1$, *all* $M\in H_2^s(\boldsymbol R^n)$ *(ordinary Sobolev space), and all* $f\in L^p_{\boldsymbol\rho}$ *such that* $\operatorname{supp}\hat f\subset\{y;|y_1|\leqq b_1,\cdots,|y_N|\leqq b_N\}$.

The proof of these lemmas is essentially the same as that in Triebel [**14**].

PROOF OF LEMMA 1.2.2. We shall prove the lemma only for $f\in\mathcal{S}$. By taking limit, we have the general case; see [**14**; Theorem 1.4.1]. Moreover, it suffices to show the inequalities

(2) $$\|\omega_{\boldsymbol\rho,\boldsymbol b}\cdot f\|_{L^q}\leqq C\|\omega_{\boldsymbol\rho,\boldsymbol b}\cdot f\|_{L^p}\quad(0<p\leqq q\leqq\infty)$$

(3) $$\|\omega_{\boldsymbol\rho,\boldsymbol b}\cdot\partial_{x_1}^{\alpha_1}\cdots\partial_{x_N}^{\alpha_N}f\|_{L^p}\leqq C\|\omega_{\boldsymbol\rho,\boldsymbol b}\cdot f\|_{L^p}\quad(1\leqq p\leqq\infty)$$

for all $\boldsymbol b=(b_1,\cdots,b_N)$ such that $b_1,\cdots,b_N\geqq 1$ and all $f\in\mathcal{S}$ such that $\operatorname{supp}\hat f\subset\{y;|y_1|\leqq 1,\cdots,|y_N|\leqq 1\}$. Here $\omega_{\boldsymbol\rho,\boldsymbol b}(x)=\omega_{\boldsymbol\rho}(b_1^{-1}x_1,\cdots,b_N^{-1}x_N)$.

We can write $f=\mathcal{F}^{-1}\phi\mathcal{F}f=\hat\phi^{*}*f$ for some $\phi\in\mathcal{S}(\boldsymbol R^n)$ such that $\phi=1$ on $\{y;|y_1|\leqq 1,\cdots,|y_N|\leqq 1\}$, hence $\partial_{x_1}^{\alpha_1}\cdots\partial_{x_N}^{\alpha_N}f=(\partial_{x_1}^{\alpha_1}\cdots\partial_{x_N}^{\alpha_N}\hat\phi^{*})*f$. Then we have by Lemma 1.2.1

(4) $$|(\omega_{\boldsymbol\rho,\boldsymbol b}\cdot\partial_{x_1}^{\alpha_1}\cdots\partial_{x_N}^{\alpha_N}f)(x)|\leqq C\int|(\omega_{\boldsymbol\rho^*}\cdot\partial_{x_1}^{\alpha_1}\cdots\partial_{x_N}^{\alpha_N}\hat\phi^{*})(x-u)|\cdot|(\omega_{\boldsymbol\rho,\boldsymbol b}\cdot f)(u)|du,$$

where we use the fact that $\omega_{\boldsymbol\rho^*,\boldsymbol b}(x-u)\leqq\omega_{\boldsymbol\rho^*}(x-u)$ holds for all $b_1,\cdots,b_N\geqq 1$. (3) is an easy consequence of (4). (2) is also obtained from (4) by the same argument as in [**14**; Proposition 1.3.2].

PROOF OF LEMMA 1.2.3. It suffices to show the inequality

(5) $$\|\omega_{\boldsymbol\rho,\boldsymbol b}\cdot\mathcal{F}^{-1}M\mathcal{F}f\|_{L^p}\leqq C\|M\|_{H_2^s(\boldsymbol R^n)}\|\omega_{\boldsymbol\rho,\boldsymbol b}\cdot f\|_{L^p}$$

for all $f\in L^p_{\boldsymbol\rho}$ such that $\operatorname{supp}\hat f\subset\{y;|y_1|\leqq 1,\cdots,|y_N|\leqq 1\}$ and all $\boldsymbol b=(b_1,\cdots,b_N)$ such that $b_1,\cdots,b_N\geqq 1$. Here $\omega_{\boldsymbol\rho,\boldsymbol b}(x)=\omega_{\boldsymbol\rho}(b_1^{-1}x_1,\cdots,b_N^{-1}x_N)$. Moreover (5) is reduced to the following two inequalities:

(6) $$\|\omega_{\boldsymbol\rho,\boldsymbol b}\cdot\mathcal{F}^{-1}M\mathcal{F}f\|_{L^p}\leqq C\|\omega_{\boldsymbol\rho^*}\cdot\mathcal{F}^{-1}M\|_{L^{\bar p}}\|\omega_{\boldsymbol\rho,\boldsymbol b}\cdot f\|_{L^p}$$

(7) $\|\omega_{\rho*}\cdot \mathcal{F}^{-1}M\|_{L^{\tilde{p}}} \leqq C\|M\|_{H_2^s(R^n)}$

for all $b_1, \cdots, b_N \geqq 1$, all $f\in L_\rho^p$ such that $\mathrm{supp}\hat{f}\subset\{y; |y_1|\leqq 1, \cdots, |y_N|\leqq 1\}$, and all $M\in H_2^s(R^n)$ such that $\mathrm{supp}\, M\subset\{y; |y_1|\leqq 2, \cdots, |y_N|\leqq 2\}$. Here $\tilde{p}=\min(1, p)$ and ρ^* is the same as in Lemma 1.2.1.

(6) and (7) are easy consequences of Lemmas 1.2.1 and 1.2.2 by the same argument as in the proof of [**14**; Proposition 1.5.1, Theorem 1.5.2]. We shall omit the details.

1.3. Basic properties. We shall show some basic properties of the weighted Besov spaces on product spaces. Their proof is carried out essentially in the same way as in Triebel [14] and Peetre [12] with the aid of the preceding lemmas. We shall leave most of them to the reader as exercises.

THEOREM 1.3.1 (cf. [**14**; Theorem 2.3.3]). *Let* $0<p, q\leqq\infty$, $\boldsymbol{\lambda}\in \boldsymbol{R}^N$ *and* $\boldsymbol{\rho}\in \boldsymbol{R}^{\tilde{N}}$. *Then it holds that*

i) $B_{p,q,\rho}^{\lambda}$ *is a quasi-Banach space.* (*Banach space if* $1\leqq p, q\leqq\infty$).
ii) $B_{p,q,\rho}^{\lambda}$ *does not depend on the choice of* $\{\Phi_j\}$ *in Definition* 1.1.3.
iii) $\mathcal{S}\subset B_{p,q,\rho}^{\lambda}\subset\mathcal{S}'$ (*continuous embedding*).
iv) *In particular,* $\mathcal{S}$ *is dense in* $B_{p,q,\rho}^{\lambda}$ *if* $0<p, q<\infty$.

THEOREM 1.3.2 (cf. [**14**; Proposition 2.3.2]). *Let* $0<p\leqq\infty$, $\boldsymbol{\lambda}, \boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1\in \boldsymbol{R}^N$ *and* $\boldsymbol{\rho}, \boldsymbol{\rho}_0, \boldsymbol{\rho}_1\in \boldsymbol{R}^{\tilde{N}}$. *Then we have the following continuous inclusions:*

i) $B_{p,q_0,\rho_0}^{\lambda}\subset B_{p,q_1,\rho_1}^{\lambda}$ *if* $0<q_0\leqq q_1\leqq\infty$ *and* $\boldsymbol{\rho}_1\leqq\boldsymbol{\rho}_0$.
ii) $B_{p,q_0,\rho}^{\lambda_0}\subset B_{p,q_1,\rho}^{\lambda_1}$ *if* $0<q_0, q_1\leqq\infty$ *and* $\boldsymbol{\lambda}_1<\boldsymbol{\lambda}_0$.

THEOREM 1.3.3 (cf. [**12**; Chapter 5, Theorem 6]). *Let* $1\leqq p_0, p_1, q_0, q_1\leqq\infty$ (*excluding the cases* $p_0=p_1=\infty$ *and* $q_0=q_1=\infty$), $\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1\in \boldsymbol{R}^N$, $\boldsymbol{\rho}_0, \boldsymbol{\rho}_1\in \boldsymbol{R}^{\tilde{N}}$ *and* $0<\theta<1$. *Then it holds that*

$$[B_{p_0,q_0,\rho_0}^{\lambda_0}, B_{p_1,q_1,\rho_1}^{\lambda_1}]_\theta = B_{p,q,\rho}^{\lambda} \quad (\textit{complex interpolation}).$$

Here $1/p=(1-\theta)/p_0+\theta/p_1$, $1/q=(1-\theta)/q_0+\theta/q_1$, $\boldsymbol{\lambda}=(1-\theta)\boldsymbol{\lambda}_0+\theta\boldsymbol{\lambda}_1$, *and* $\boldsymbol{\rho}=(1-\theta)\boldsymbol{\rho}_0+\theta\boldsymbol{\rho}_1$.

REMARK 1.3.1. Löfström [**9**] proves Theorem 1.3.3 in the case $N=1$, but deals with more general weight functions satisfying the inequality (1) in Lemma 1.2.1; see Löfström [**9**; Theorem 4].

THEOREM 1.3.4 (lifting property; cf. [**14**; Theorem 2.3.8]). *Let* $0<p, q\leqq\infty$, $\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1\in \boldsymbol{R}^N$ *and* $\rho\in \boldsymbol{R}^{\tilde{N}}$. *Then the mapping*

$$I_{\lambda_1}: B_{p,q,\rho}^{\lambda_0}\longrightarrow B_{p,q,\rho}^{\lambda_0-\lambda_1}$$

is isomorphism.

THEOREM 1.3.5 (cf. [**14**; Section 2.5.6, 2.5.7]). *Let* $0<p\leqq\infty$, $\boldsymbol{\lambda}\in\boldsymbol{R}^N$ *and* $\boldsymbol{\rho}\in\boldsymbol{R}^{\tilde{N}}$. *Then we have the following continuous embeddings:*

i) $B^{\boldsymbol{\lambda}}_{\infty,1,\boldsymbol{\rho}}\subset C^{\boldsymbol{\lambda}}_{\boldsymbol{\rho}}\subset B^{\boldsymbol{\lambda}}_{\infty,\infty,\boldsymbol{\rho}}$ *if* $\boldsymbol{\lambda}$ *is a non-negative integer vector.*

ii) $B^{\boldsymbol{\lambda}}_{p,1,\boldsymbol{\rho}}\subset H^{\boldsymbol{\lambda}}_{p,\boldsymbol{\rho}}\subset B^{\boldsymbol{\lambda}}_{p,\infty,\boldsymbol{\rho}}$ *if* $1\leqq p\leqq\infty$.

iii) $H^{\boldsymbol{\lambda}}_p=W^{\boldsymbol{\lambda}}_p$ *if* $\boldsymbol{\lambda}$ *is a non-negative integer vector and* $1<p<\infty$.

THEOREM 1.3.6 (pointwise multiplier theorem; cf. [**14**; p. 143 (24)]). *Let* $1\leqq p\leqq\infty$, $0<q\leqq\infty$, $\boldsymbol{0}<\boldsymbol{\lambda}\in\boldsymbol{R}^N$ *and* $\boldsymbol{\rho}_0$, $\boldsymbol{\rho}_1\in\boldsymbol{R}^{\tilde{N}}$. *Then there exists a constant C such that the estimate*

$$\|f\cdot g\|_{B^{\boldsymbol{\lambda}}_{p,q,\boldsymbol{\rho}_0+\boldsymbol{\rho}_1}}\leqq C\|f\|_{B^{\boldsymbol{\lambda}}_{p,q,\boldsymbol{\rho}_0}}\|g\|_{B^{\boldsymbol{\lambda}}_{\infty,q,\boldsymbol{\rho}_1}}$$

holds for all f in $B^{\boldsymbol{\lambda}}_{p,q,\boldsymbol{\rho}_0}$ *and all g in* $B^{\boldsymbol{\lambda}}_{\infty,q,\boldsymbol{\rho}_1}$.

REMARK 1.3.2. The condition $\boldsymbol{0}<\boldsymbol{\lambda}$ yields $f\in L^p_{\boldsymbol{\rho}_0}$ and $g\in L^\infty_{\boldsymbol{\rho}_1}$ by virtue of Theorems 1.3.2 and 1.3.5, hence pointwise multiplication $f\cdot g$ makes sense.

THEOREM 1.3.7. *Let* $1\leqq p\leqq\infty$, $0<q\leqq\infty$, $\boldsymbol{0}<\boldsymbol{\lambda}\in\boldsymbol{R}^N$ *and* $\boldsymbol{\rho}_0$, $\boldsymbol{\rho}_1\in\boldsymbol{R}^{\tilde{N}}$. *Then the mapping*

$$J_{\boldsymbol{\rho}_1}:B^{\boldsymbol{\lambda}}_{p,q,\boldsymbol{\rho}_0}\longrightarrow B^{\boldsymbol{\lambda}}_{p,q,\boldsymbol{\rho}_0-\boldsymbol{\rho}_1}$$

is isomorphism.

REMARK 1.3.3. Theorem 1.3.7 is an easy consequence of Theorems 1.3.2, 1.3.5 and 1.3.6.

THEOREM 1.3.8 (cf. [**14**; Theorem 2.3.9]). *Let* $1\leqq p_0$, $p_1\leqq\infty$, $0<q_0$, $q_1\leqq\infty$, $\boldsymbol{0}<\boldsymbol{\lambda}_0$, $\boldsymbol{\lambda}_1\in\boldsymbol{R}^N$ *and* $\boldsymbol{\rho}\in\boldsymbol{R}^{\tilde{N}}$. *Then* $B^{\boldsymbol{\lambda}_0}_{p_0,q_0,\boldsymbol{\rho}}=B^{\boldsymbol{\lambda}_1}_{p_1,q_1,\boldsymbol{\rho}}$ *if and only if* $p_0=p_1$, $q_0=q_1$, *and* $\boldsymbol{\lambda}_0=\boldsymbol{\lambda}_1$.

REMARK 1.3.4. Triebel [14] essentially proved this theorem in the case $\boldsymbol{\rho}=\boldsymbol{0}$. Theorem 1.3.7 yields the general case $\boldsymbol{\rho}\neq\boldsymbol{0}$.

We shall define the operator $\delta_s:\boldsymbol{R}^N\to\boldsymbol{R}^{N-1}$ $(s=1,\cdots,N-1)$ by $\delta_s(\boldsymbol{\lambda})=(\lambda_1,\cdots,\lambda_{s-1},\lambda_s+\lambda_{s+1},\lambda_{s+2},\cdots,\lambda_N)$, where $\boldsymbol{\lambda}=(\lambda_1,\cdots,\lambda_N)$. Then we have the following typical property of Besov spaces on product spaces.

THEOREM 1.3.9. *Let* $1\leqq p\leqq\infty$, $0<q\leqq\infty$, $\boldsymbol{\lambda}=(\lambda_1,\cdots,\lambda_N)\in\boldsymbol{R}^N$, λ_s, $\lambda_{s+1}>0$, *and* $\boldsymbol{\rho}\in\boldsymbol{R}^{\tilde{N}}$, *and let* $\boldsymbol{n}$ *and* $\tilde{\boldsymbol{n}}$ *be the same as in Definition* 1.1.3. *Then we have the following proper inclusion:*

$$B^{\delta_s(\boldsymbol{\lambda})}_{p,q,\boldsymbol{\rho}}(\boldsymbol{R}^{\delta_s(\boldsymbol{n})}_{\tilde{\boldsymbol{n}}})\subsetneqq B^{\boldsymbol{\lambda}}_{p,q,\boldsymbol{\rho}}(\boldsymbol{R}^{\boldsymbol{n}}_{\tilde{\boldsymbol{n}}}).$$

PROOF OF THEOREM 1.3.9. For the sake of simplicity, we shall assume $s=1$. For a function $\Phi^{\boldsymbol{n}}_j(y)=\Phi^{\boldsymbol{n}}_j(y_1,\cdots,y_N)=\Theta_{j_1}(y_1)\cdots\Theta_{j_N}(y_N)$ (see Definition 1.1.3), we shall define $\{\Theta'_j(y_1,y_2)\}_{j=0}^\infty$ and $\{\Theta'_{j_r}(y_r)\}_{j_r=0}^\infty$ $(r=3,\cdots,N)$ by $\Theta'_0=\Theta_0+\Theta_1+\Theta_2$, $\Theta'_j=\Theta_{j-1}+\Theta_j+\Theta_{j+1}+\Theta_{j+2}$ $(j\in\boldsymbol{N})$. Then we have

$$\Theta_{j_1}(y_1)\cdot\Theta_{j_2}(y_2)=\Theta'_{\max\{j_1,j_2\}}(y_1, y_2)\cdot\Theta_{j_1}(y_1)\cdot\Theta_{j_2}(y_2),$$
$$\Theta_{j_r}(y_r)=\Theta'_{j_r}(y_r)\cdot\Theta_{j_r}(y_r) \qquad (r=3, \cdots, N).$$

Hence, Fourier multiplier Theorem (Lemma 1.2.3) gives

$$\|\mathcal{F}^{-1}\Phi^n_j(y)\mathcal{F}f\|_{L^p_\rho}\leqq CA_{\max\{j_1,j_2\},j_3,\cdots,j_N}, \tag{1}$$

where $A_{j,j_3,\cdots,j_N}=\|\mathcal{F}^{-1}\Theta'_j(y_1, y_2)\cdot\Theta_{j_3}(y_3)\cdots\Theta_{j_N}(y_N)\mathcal{F}f\|_{L^p_\rho}$. On the other hand, we can easily see

$$\begin{aligned}&\{\textstyle\sum_{j_1,j_2,\cdots,j_N}(2^{j_1\lambda_1+j_2\lambda_2+\cdots+j_N\lambda_N}A_{\max\{j_1,j_2\},j_3,\cdots,j_N})^q\}^{1/q}\\&\leqq C\,\{\textstyle\sum_{j,j_3,\cdots,j_N}(2^{j(\lambda_1+\lambda_2)+j_3\lambda_3+\cdots+j_N\lambda_N}A_{j,j_3,\cdots,j_N})^q\}^{1/q}\end{aligned} \tag{2}$$

(with a slight modification in the case of $q=\infty$). From the inequalities (1) and (2), we have the desired inclusion. By virtue of Theorem 1.3.8, we can easily show the existence of a function which belongs to $B^{\lambda}_{p,q,\rho}$ while not to $B^{\delta_1(\lambda)}_{p,q,\rho}$. (Consider a function of the type $f(x_1)g(x_2)h(x_3, \cdots, x_N)$.)

From these theorems, we can easily obtain the following corollaries.

COROLLARY 1.3.1. *Let* $1<p<\infty$, $0<q\leqq\infty$, $\boldsymbol{\lambda}$, $\boldsymbol{\sigma}\in\boldsymbol{R}^N$, $\boldsymbol{\rho}\in\boldsymbol{R}^{\tilde{N}}$ *and let* $\mathbf{0}<\boldsymbol{\lambda}<\boldsymbol{\sigma}$. *Then we have the following proper inclusions:*

i) $C^{\boldsymbol{\sigma}}_{\boldsymbol{\rho}}\subsetneqq B^{\boldsymbol{\lambda}}_{\infty,q,\boldsymbol{\rho}}$ *if* $\boldsymbol{\sigma}\in\boldsymbol{N}^N$.

ii) $W^{\boldsymbol{\sigma}}_p\subsetneqq B^{\boldsymbol{\lambda}}_{p,q}$ *if* $\boldsymbol{\sigma}\in\boldsymbol{N}^N$.

iii) $H^{\boldsymbol{\sigma}}_{\infty,\boldsymbol{\rho}}\subsetneqq B^{\boldsymbol{\lambda}}_{\infty,q,\boldsymbol{\rho}}$.

COROLLARY 1.3.2. *Let* $1\leqq p\leqq\infty$, $0<q\leqq\infty$, $\boldsymbol{\lambda}\in\boldsymbol{R}^N$, $\boldsymbol{\sigma}\in\boldsymbol{N}^N$, $\boldsymbol{\rho}_0$, $\boldsymbol{\rho}_1\in\boldsymbol{R}^{\tilde{N}}$ *and let* $\mathbf{0}<\boldsymbol{\lambda}<\boldsymbol{\sigma}$. *Then it holds that*

$$\|\phi\cdot f\|_{B^{\boldsymbol{\lambda}}_{p,q,\boldsymbol{\rho}_0+\boldsymbol{\rho}_1}}\leqq C\|\phi\|_{C^{\boldsymbol{\sigma}}_{\boldsymbol{\rho}_0}}\|f\|_{B^{\boldsymbol{\lambda}}_{p,q,\boldsymbol{\rho}_1}}.$$

Here C *is a constant independent of* f *and* ϕ.

2. L^p-estimates for pseudo-differential operators.

In this section, we shall show some L^p-estimates for pseudo-differential operators with symbols in the Besov spaces which were introduced in the preceding section.

2.1. Main results. In order to state the main results, we use "unweighted" Besov spaces on product spaces of dimension "$2n$". Throughout this section, we take $n=n_1+\cdots+n_N=n'_1+\cdots+n'_{N'}$ $(n_r, n'_s\in\boldsymbol{N};\ r=1, \cdots, N,\ s=1, \cdots, N')$ and use the following notations:

Let $0<p,\ q\leqq\infty$, $\boldsymbol{\lambda}=(\lambda_1,\cdots,\lambda_N)\in\boldsymbol{R}^N$, $\boldsymbol{\lambda}'=(\lambda'_1,\cdots,\lambda'_{N'})\in\boldsymbol{R}^{N'}$, $\boldsymbol{n}=(n_1,\cdots,n_N)$ and $\boldsymbol{n}'=(n'_1,\cdots,n'_{N'})$. Then we set

$$B_{p,q}^{(\boldsymbol{\lambda},\boldsymbol{\lambda}')}=B_{p,q}^{(\boldsymbol{\lambda},\boldsymbol{\lambda}')}(\boldsymbol{R}^{(\boldsymbol{n},\boldsymbol{n}')})=\{\sigma\in\mathcal{S}'(\boldsymbol{R}^{2n});\ \|\sigma\|_{B_{p,q}^{(\boldsymbol{\lambda},\boldsymbol{\lambda}')}}<+\infty\},$$

where

$$\|\sigma\|_{B_{p,q}^{(\boldsymbol{\lambda},\boldsymbol{\lambda}')}}=\left\|\|2^{\boldsymbol{j}\cdot\boldsymbol{\lambda}+\boldsymbol{k}\cdot\boldsymbol{\lambda}'}\mathcal{F}^{-1}\Phi_{\boldsymbol{j},\boldsymbol{k}}\mathcal{F}\sigma\|_{L^p(\boldsymbol{R}^{2n})}\right\|_{l^q}$$
$$=\left\{\sum_{\boldsymbol{j},\boldsymbol{k}\geq 0}\left(\iint|2^{\boldsymbol{j}\cdot\boldsymbol{\lambda}+\boldsymbol{k}\cdot\boldsymbol{\lambda}'}\mathcal{F}^{-1}\Phi_{\boldsymbol{j},\boldsymbol{k}}\mathcal{F}\sigma(x,\xi)|^p dxd\xi\right)^{q/p}\right\}^{1/q}$$

(with a slight modification in the case of $p=\infty$ and/or $q=\infty$). Here $\boldsymbol{j}=(j_1,\cdots,j_N)$, $\boldsymbol{k}=(k_1,\cdots,k_{N'})$ (non-negative integer vectors) and $\Phi_{\boldsymbol{j},\boldsymbol{k}}(y,\eta)=\Phi_{\boldsymbol{j}}(y)\Phi'_{\boldsymbol{k}}(\eta)=\Phi^{\boldsymbol{n}}_{\boldsymbol{j}}(y)\Phi^{\boldsymbol{n}'}_{\boldsymbol{k}}(\eta)$; see Definition 1.1.3.

Furthermore, we always decompose variables y, η in such a way that $y=(y_1,\cdots,y_N)$, $\eta=(\eta_1,\cdots,\eta_{N'})$; $y_r\in\boldsymbol{R}^{n_r}$, $\eta_s\in\boldsymbol{R}^{n'_s}$ $(r=1,\cdots,N;\ s=1,\cdots,N')$.

Now, we shall give the main results of the present paper.

THEOREM 2.1.1. *Let $1\leqq p\leqq 2$. Then there exists a constant C such that the estimate*

$$\|\sigma(X,D)f\|_{L^p(\boldsymbol{R}^n)}\leqq C\|\sigma(x,\xi)\|_{B^0_{p,1}}\|f\|_{L^p(\boldsymbol{R}^n)}$$

holds for all $\sigma\in B^0_{p,1}$ and all f in $\mathcal{S}(\boldsymbol{R}^n)$.

THEOREM 2.1.2. *Let $2\leqq q\leqq\infty$. Then there exists a constant C such that the estimate*

$$\|\sigma(X,D)f\|_{L^2(\boldsymbol{R}^n)}\leqq C\|\sigma(x,\xi)\|_{B_{q,1}^{(1/2-1/q)(\boldsymbol{n},\boldsymbol{n}')}}\|f\|_{L^2(\boldsymbol{R}^n)}$$

holds for all σ in $B_{q,1}^{(1/2-1/q)(\boldsymbol{n},\boldsymbol{n}')}$ and all f in $\mathcal{S}(\boldsymbol{R}^n)$.

REMARK 2.1.1. In Theorem 2.1.2 with $2<q\leqq\infty$, we can have the sharpest estimates if we take $N=N'=n$, and the other case is a corollary of this special case by virtue of Theorem 1.3.9.

REMARK 2.1.2. Theorem 2.1.2 is an extension of Theorem B in Section 0. The conditions i) and ii) (resp. iii), iv), v)) of Theorem B are in the case $q=\infty$, $N=N'=1$ (resp. $q=\infty$, $N=N'=n$; $q=\infty$, $N=1$, $N'=n$; $2\leqq q<\infty$, $N=N'=1$). In fact, we can easily see that Theorem B with conditions i), ii), iv), and v) is contained in Theorem 2.1.2, by virtue of Corollary 1.3.1. On the other hand, condition iii) of Theorem B says that $\sigma(x,\xi)\in B_{\infty,\infty}^{((1/2)+\varepsilon,\cdots,(1/2)+\varepsilon)}$, $\varepsilon>0$, by virtue of the ordinary argument; see, for example, Triebel [14; Section 2]. Hence Theorem B with condition iii) is contained in Theorem 2.1.2 by virtue of Theorem 1.3.2.

REMARK 2.1.3. The symbol class $S^0_{0,0}$, in the sense of Kumano-go [8], is contained in $B^{(n/2,n'/2)}_{\infty,1}$ by Corollary 1.3.1. Hence Theorem 2.1.2 with $q=\infty$ states that the symbols belonging to the class $S^0_{0,0}$ generate $L^2(\boldsymbol{R}^n)$-bounded pseudo-differential operators.

REMARK 2.1.4. Muramatu [**11**] also proves Theorem 2.1.2 in the case of $N=N'=1$ in a way different from ours.

REMARK 2.1.5. The sharpness of the order $(1/2-1/q)\boldsymbol{n}$ in Theorem 2.1.2 was essentially discussed in Coifman-Meyer [**4**; p. 24]. The sharpness of the order $\boldsymbol{n}/2$ and $\boldsymbol{n}'/2$ in Theorem 2.1.2 with $q=\infty$ was also discussed in Miyachi [**10**; Section 5]. But the present author does not know whether the suborder "1" can be relaxed or not.

2.2. Lemmas. In order to prove the main results, we shall show some lemmas.

LEMMA 2.2.1. *There exists a pair of functions $\phi, \chi\in\mathcal{S}(\boldsymbol{R}^n)$ which satisfies the following two conditions:*

i) $\int\phi\chi(\xi)d\xi=1.$

ii) $\mathrm{supp}\,\phi\subset\{\xi\,;\,|\xi|<1\}$, $\mathrm{supp}\,\hat{\chi}\subset\{\eta\,;\,|\eta|<1\}$.

PROOF. Let χ be a function in $\mathcal{S}(\boldsymbol{R}^n)$ which satisfies $\mathrm{supp}\,\hat{\chi}\subset\{\eta\,;\,|\eta|<1\}$ and $\chi(0)=(2\pi)^{-n}\int\hat{\chi}(\eta)d\eta=1$. Then we can take such a function $\phi_0\in\mathcal{S}(\boldsymbol{R}^n)$ as satisfies $\mathrm{supp}\,\phi_0\subset\{\xi\,;\,|\xi|<1\}$ and $\int\phi_0\chi(\xi)d\xi=C\neq0$. Set $\phi=\phi_0/C$ and this proves the lemma.

In the following two lemmas $|\Omega|$ denotes the Lebesgue measure of a subset $\Omega\subset\boldsymbol{R}^n$.

LEMMA 2.2.2. *Let $g_\tau(x)=g(x,\tau)$ be as follows:*

i) $g(x,\tau)\in L^2(\boldsymbol{R}^n_x\times\boldsymbol{R}^n_\tau)$.

ii) $\sup_x\|g(x,\tau)\|_{L^1(\boldsymbol{R}^n_\tau)}<+\infty$.

iii) $\mathrm{supp}\,\mathcal{F}_x g_\tau(y)\subset\Omega$, *where Ω is a compact subset of $\boldsymbol{R}^n$ independent of τ.*

If $h(x)=\int e^{ix\cdot\tau}g(x,\tau)d\tau$, then it holds that

$$\|h\|_{L^2(\boldsymbol{R}^n)}\leqq C|\Omega|^{1/2}\|g(x,\tau)\|_{L^2(\boldsymbol{R}^n_x\times\boldsymbol{R}^n_\tau)}.$$

Here C is a constant independent of g and Ω.

PROOF. By the Fourier transformation, we formally have

$$\hat{h}(y)=\int \mathcal{F}_x g_\tau(y-\tau)d\tau=\int(\mathcal{F}_x g_\tau\cdot\chi_\Omega)(y-\tau)d\tau, \tag{1}$$

where χ_Ω denotes the characteristic function of Ω. Then, we have by Schwarz's inequality,

$$|\hat{h}(y)|^2 \leqq C|\Omega|\int|\mathcal{F}_x g_\tau(y-\tau)|^2 d\tau. \tag{2}$$

Integrating both sides of (2) with respect to y, we have

$$\begin{aligned}\|\hat{h}(y)\|_{L^2(R^n)}{}^2 &\leqq C|\Omega|\iint|\mathcal{F}_x g_\tau(y-\tau)|^2 dy d\tau \\ &= C|\Omega|\int\|\mathcal{F}_x g_\tau\|_{L^2(R_y^n)}{}^2 d\tau.\end{aligned} \tag{3}$$

By Plancherel's theorem, (3) implies the desired inequality. Formula (1) can be easily justified with the aid of ii), and we shall omit the details.

LEMMA 2.2.3. *Let $2\leqq p\leqq\infty$ and let $\sigma_x(\xi)=\sigma(x,\xi)$ be as follows:*

i) $\sigma_x(\xi)\in L^1(\boldsymbol{R}_\xi^n)$ *and* $\in L^2(\boldsymbol{R}_\xi^n)$ *for all* $x\in\boldsymbol{R}^n$.

ii) $\operatorname{supp}\mathcal{F}_\xi\sigma_x(\eta)\subset\Omega$, *where Ω is a compact subset of $\boldsymbol{R}^n$ independent of x.*

Then it holds that

$$\|\sigma(X,D)f\|_{L^p(R^n)}\leqq C|\Omega|^{1/2}\sup_x\|\sigma(x,\xi)\|_{L^2(R_\xi^n)}\|f\|_{L^p(R^n)}$$

for $f\in\mathcal{S}(\boldsymbol{R}^n)$. Here C is a constant independent of f, σ and Ω.

PROOF. Set $K(x,\eta)=\mathcal{F}_\xi\sigma_x(\eta)$. Then we have

$$\begin{aligned}\sigma(X,D)f(x)&=(2\pi)^{-n}\int K(x,\eta-x)f(\eta)d\eta \\ &=(2\pi)^{-n}\int K(x,\eta-x)\chi_\Omega(\eta-x)f(\eta)d\eta.\end{aligned}$$

Here χ_Ω denotes the characteristic function of Ω. Then by Schwarz's inequality and Plancherel's theorem, we have

$$|\sigma(X,D)f(x)|\leqq C\|\sigma(x,\xi)\|_{L^2(R_\xi^n)}\cdot(|\chi_{-\Omega}|^2*|f|^2(x))^{1/2}. \tag{4}$$

By Young's inequality, (4) implies

$$\begin{aligned}\|\sigma(X,D)f\|_{L^p(R^n)}&\leqq C\sup_x\|\sigma(x,\xi)\|_{L^2(R_\xi^n)}\cdot\||\chi_{-\Omega}|^2*|f|^2\|_{L^{p/2}(R^n)}{}^{1/2} \\ &\leqq C|\Omega|^{1/2}\sup_x\|\sigma(x,\xi)\|_{L^2(R_\xi^n)}\|f\|_{L^p(R^n)},\end{aligned}$$

which is the desired inequality.

2.3. Proofs of the main results. Using the preceding lemmas, we can easily prove the main results.

(PROOF OF THEOREM 2.1.1)

Step 1 (The case $\sigma\in\mathcal{S}(\boldsymbol{R}^{2n})$). We can decompose σ in such a way that

$$\sigma(x,\xi)=\sum_{j,k}\sigma_{j,k}(x,\xi)\quad(\sigma_{j,k}=\mathcal{F}^{-1}\Phi_{j,k}\mathcal{F}\sigma)\quad\text{in }\mathcal{S}(\boldsymbol{R}^{2n}),$$

and have

(1) $$\sigma(X,D)f(x)=\sum_{j,k}\sigma_{j,k}(X,D)f(x)\quad\text{for } f\in\mathcal{S}(\boldsymbol{R}^n).$$

Setting $K_{j,k}(x,\eta)=\mathcal{F}_\xi[\sigma_{j,k}(x,\xi)](\eta)$, we can write

(2) $$\sigma_{j,k}(X,D)f(x)=(2\pi)^{-n}\int K_{j,k}(x,\eta-x)f(\eta)d\eta.$$

By Hölder's inequality and the Hausdorff-Young inequality, (2) implies

(3) $$\begin{aligned}|\sigma_{j,k}(X,D)f(x)|&\leqq C\|K_{j,k}(x,\eta)\|_{L^q(\boldsymbol{R}^n_\eta)}\|f\|_{L^p(\boldsymbol{R}^n)}\\&\leqq C\|\sigma_{j,k}(x,\xi)\|_{L^p(\boldsymbol{R}^n_\xi)}\|f\|_{L^p(\boldsymbol{R}^n)}.\\&\qquad(1=1/p+1/q;\ 1\leqq p\leqq 2)\end{aligned}$$

Then we have

(4) $$\|\sigma_{j,k}(X,D)f\|_{L^p(\boldsymbol{R}^n)}\leqq C\|\sigma_{j,k}\|_{L^p(\boldsymbol{R}^n_x\times\boldsymbol{R}^n_\xi)}\|f\|_{L^p(\boldsymbol{R}^n)}.$$

Combining (1) and (4), we have the desired result.

Step 2 (The general case). Let $\sigma(x,\xi)$ be in $B^0_{p,1}$. Then there exists a sequence $\{\sigma_\nu\}_{\nu=1}^\infty\subset\mathcal{S}(\boldsymbol{R}^{2n})$ such that $\sigma_\nu\to\sigma$ in $B^0_{p,1}$ $(\nu\to\infty)$ (Theorem 1.3.1, iv)). Notice that

$$\begin{aligned}&\|(\sigma(X,D)-\sigma_\nu(X,D))f\|_{L^p(\boldsymbol{R}^n)}\\&\leqq C\Big\|\|(\sigma(x,\xi)-\sigma_\nu(x,\xi))\hat{f}(\xi)\|_{L^1(\boldsymbol{R}^n_\xi)}\Big\|_{L^p(\boldsymbol{R}^n_x)}\\&\leqq C\|\sigma-\sigma_\nu\|_{L^p(\boldsymbol{R}^n_x\times\boldsymbol{R}^n_\xi)}\|\hat{f}\|_{L^q(\boldsymbol{R}^n)}\qquad(1=1/p+1/q)\\&\leqq C\|\sigma-\sigma_\nu\|_{B^0_{p,1}}\|\hat{f}\|_{L^q(\boldsymbol{R}^n)}\qquad(\text{Theorem 1.3.5})\\&\to 0\quad(\nu\to\infty).\end{aligned}$$

The general case of the theorem is obtained from this and Step 1.

(PROOF OF THEOREM 2.1.2)

Step 0. Theorem 2.1.2 with $q=2$ is Theorem 2.1.1 with $p=2$. By virtue of the interpolation theorem (Theorem 1.3.3), it suffices to show Theorem 2.1.2 with $q=\infty$.

Step 1. Let σ be in $B^{(n/2,n'/2)}_{\infty,1}$ and f in $\mathcal{S}$. From Lemma 2.2.1, we have $\int \phi\chi(\xi-\tau)d\tau=1$ for all ξ, and this implies

$$h(x) = \sigma(X, D)f(x) = \int d\tau \int e^{ix\cdot\xi}\sigma(x, \xi)\phi\chi(\xi-\tau)\hat{f}(\xi)d\xi = \int e^{ix\cdot\tau}g(x, \tau)d\tau \qquad (\xi\to\xi+\tau), \tag{5}$$

where

$$g(x, \tau) = \sigma_\tau(X, D)\chi(D)f_\tau(x), \tag{6}$$

$$\sigma_\tau(x, \xi) = \sigma(x, \xi+\tau) \in B^{(n/2,n'/2)}_{\infty,1}, \tag{7}$$

$$\hat{f}_\tau(\xi) = \phi(\xi)\hat{f}(\xi+\tau). \tag{8}$$

We can decompose σ_τ in such a way that

$$\sigma_\tau(x, \xi) = \sum_j \sigma_{\tau,j}(x, \xi) = \sum_{j,k} \sigma_{\tau,j,k}(x, \xi) \qquad \text{(uniform convergence)}, \tag{9}$$

where

$$\begin{aligned} \sigma_{\tau,j}(x, \xi) &= \mathcal{F}_y^{-1}\Phi_j\mathcal{F}_x\sigma_\tau(x, \xi) \\ \sigma_{\tau,j,k}(x, \xi) &= \mathcal{F}_\eta^{-1}\Phi'_k\mathcal{F}_\xi\sigma_{\tau,j}(x, \xi) \\ &= \mathfrak{F}^{-1}\Phi_{j,k}\mathfrak{F}\sigma_\tau(x, \xi) \\ &= \mathfrak{F}^{-1}\Phi_{j,k}\mathfrak{F}\sigma(x, \xi+\tau). \end{aligned} \tag{10}$$

We define g_j and $g_{j,k}$ in the same way as in (6) with σ_τ replaced by $\sigma_{\tau,j}$ and $\sigma_{\tau,j,k}$ respectively. We define h_j and $h_{j,k}$ in the same way as in (5) with g replaced by g_j and $g_{j,k}$ respectively. Then we have

$$h(x) = \sum_j h_j(x), \quad g_j(x, \tau) = \sum_k g_{j,k}(x, \tau). \tag{11}$$

Step 2. By Lemma 2.2.2, we have

$$\|h_j\|_{L^2(R^n)} \leq C2^{j\cdot n/2}\|g_j(x, \tau)\|_{L^2(R^n_x\times R^n_\tau)}. \tag{12}$$

In fact, noticing

$$\mathcal{F}_x[e^{ix\cdot\xi}\sigma_{\tau,j}(x, \xi)\chi(\xi)\hat{f}_\tau(\xi)](y) = \Phi_j(y-\xi)\mathcal{F}_x\sigma_\tau(y-\xi, \xi)\chi(\xi)\phi(\xi)\hat{f}(\xi+\tau) \qquad \text{(by (8) and (10))},$$

we have

$$\operatorname*{supp}_{y} \mathcal{F}_x[g_j(x, \tau)](y) \subset \{y ; y-\xi \in \operatorname{supp} \Phi_j \quad \text{for some } \xi \in \operatorname{supp} \phi\}$$
$$\subset \{y ; |y_1| \leqq 2^{j_1+1}+1, \cdots, |y_N| \leqq 2^{j_N+1}+1\}$$

by Definitions 1.1.1, 1.1.3, and Lemma 2.2.1. (12) is easily obtained from this.

Step 3. By Lemma 2.2.3, we have

$$\text{(13)} \qquad \|g_{j,k}(x, \tau)\|_{L^2(R^n_x)} \leqq C\, 2^{k\cdot n'/2} \sup_x \|\sigma_{\tau,j,k}(x, \xi)\chi(\xi)\|_{L^2(R^n_\xi)} \|f_\tau\|_{L^2(R^n)}$$
$$\leqq C \left\{\sup_{\tau\in R^n} \|2^{k\cdot n'/2}\sigma_{\tau,j,k}\|_{L^\infty(R^n_x\times R^n_\xi)}\right\} \cdot \|f_\tau\|_{L^2(R^n)}.$$

In fact, it holds that

$$\operatorname*{supp}_{\eta} \mathcal{F}_\xi[\sigma_{\tau,j,k}(x, \xi)\chi(\xi)](\eta) \subset \operatorname*{supp}_{\eta} \mathcal{F}_\xi[\sigma_{\tau,j,k}(x, \xi)](\eta) + \operatorname{supp}\hat{\chi}$$
$$\subset \operatorname{supp}\Phi'_k + \operatorname{supp}\hat{\chi} \qquad \text{(by (10))}$$
$$\subset \{\eta ; |\eta_1| \leqq 2^{k_1+1}+1, \cdots, |\eta_{N'}| \leqq 2^{k_{N'}+1}+1\},$$

which implies (13). (The second inequality of (13) is trivial.) Squaring and integrating both sides of (13) with respect to τ, we easily have

$$\text{(14)} \qquad \|g_{j,k}(x, \tau)\|_{L^2(R^n_x\times R^n_\tau)} \leqq C \sup_{\tau\in R^n} \|2^{k\cdot n'/2}\sigma_{\tau,j,k}\|_{L^\infty(R^n_x\times R^n_\xi)} \|f\|_{L^2(R^n)}$$

by (8) and Plancherel's theorem.

Step 4. Combining (11), (12), and (14), we have

$$\|h\|_{L^2(R^n)} \leqq C \sum_{j,k} \sup_{\tau\in R^n} \|2^{j\cdot n/2+k\cdot n'/2}\sigma_{\tau,j,k}\|_{L^\infty(R^n_x\times R^n_\xi)} \|f\|_{L^2(R^n)}$$
$$= C \sum_{j,k} \|2^{j\cdot n/2+k\cdot n'/2}\mathcal{F}^{-1}\Phi_{j,k}\mathcal{F}\sigma\|_{L^\infty(R^n_x\times R^n_\xi)} \|f\|_{L^2(R^n)}$$
$$\text{(by (10))}.$$

This implies the desired result.

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