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Brauer-Thrall type theorem for maximal Cohen-Macaulay modules

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Let k be a perfect valuation field and let R be a local analytic k-algebra, which is always assumed to be Cohen-Macaulay. In the present paper we are interested in the category $C(R)$ of maximal Cohen-Macaulay modules. Hence the objects in $C(R)$ are finitely generated modules M with equality depth$(M)=\dim(R)$. We say that $C(R)$ is of finite representation type provided that there are only finite number of isomorphic classes of indecomposable objects in $C(R)$. Analytic algebras with $C(R)$ of finite representation type are recently studied by various authors and they actually become well-understandable objects in ring theory. In fact, if k is algebraically closed of characteristic 0, then a Gorenstein algebra has $C(R)$ of finite representation type only when it is a simple hypersurface singularity [**10**]. Moreover if R has dimension 2, then the finiteness of representation type of $C(R)$ is equivalent to that R is a quotient singularity. See Artin-Verdier [**1**], Auslander [**4**] and Herzog [**14**]. In the case of dimension 1, such finiteness is characterized by the condition that R dominates a simple plane curve as is shown by Greuel-Knörrer [**13**]. See also Knörrer [**19**] and Kiyek-Steinke [**20**].

In this paper we will give a certain sufficient condition for $C(R)$ to be of finite representation type in the case R has only an isolated singularity. Precisely, if there is an upper bound for multiplicities of indecomposable modules in $C(R)$, then $C(R)$ is of finite representation type. See (1.4). This is, of course, an analogous result to Brauer-Thrall conjecture or Roiter-Auslander theorem for Artin rings. We will also show that the corresponding result of the Auslander-Reiten theory for Artin algebras is valid for the category $C(R)$. See Theorem (1.1). It should be noted that these results will fail unless R is an isolated singularity. (Cf. (1.6).)

Precise statement of our main theorem will be given in Section 1 and the subsequent sections will be devoted to a proof and an application of the theorem.

In Section 2 we will discuss a method which reduces some problems into

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Artinian case. The main tool in this section is the Noether different, by which we can take a good system of parameters making this reduction effective.

In Section 3 we will give a proof of the theorem by using the results obtained in Section 2. Roughly speaking, our method of proof is along the way given by Yamagata [28] for Artinian case.

In Section 4 an application of Theorem (1.1) will be given, where we prove the following: Let R be a 2-dimensional normal Gorenstein local analytic algebra over an algebraically closed field of characteristic 0. Assume that R is not a Klein singularity. Then, for any integer n, there are infinitely many classes of indecomposable reflexive modules of rank n.

In Appendix we summarize some definitions and results from Auslander-Reiten theory for $C(R)$, where we will give a direct and easy proof of Auslander's theorem: $C(R)$ admits Auslander-Reiten sequences if and only if R is an isolated singularity. The author believes that, for commutative algebraists, this appendix would be a good introduction to the Auslander-Reiten theory for the category $C(R)$.

§1. The main theorem.

Let k be a valuation field, i.e. there is a mapping v from k to $\boldsymbol{R}^+$ satisfying $v(0)=0$, $v(ab)=v(a)v(b)$ and $v(a+b)\leqq v(a)+v(b)$ for any $a, b\in k$. The convergent power series ring $k\{x_1, x_2, \cdots, x_n\}$ over k is a k-algebra consisting of power series $\sum a_{i_1i_2\cdots i_n}x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ with the property that there are positive real numbers $r_1, \cdots, r_n$ and M satisfying $v(a_{i_1i_2\cdots i_n})r_1^{i_1}r_2^{i_2}\cdots r_n^{i_n}\leqq M$ for any $i_1, i_2, \cdots, i_n$. Note that if the valuation is trivial, then $k\{x_1, x_2, \cdots, x_n\}$ will be a formal power series ring over k. A local analytic k-algebra is a commutative local ring which is module-finite over a convergent power series ring over k. In the rest of this paper we always assume that k is a perfect field and a local analytic k-algebra R is Cohen-Macaulay with the maximal ideal $\mathfrak{m}$.

For a local analytic k-algebra R we denote the category of maximal Cohen-Macaulay (abbr. MCM) modules over R by $C(R)$ which is a full subcategory of the category of all finitely generated R-modules. Recall that the objects in $C(R)$ are finite R-modules M with equality $\operatorname{depth}(M)=\dim(R)$. Since R is a Hensel ring, a module M in $C(R)$ is indecomposable if and only if $\operatorname{End}_R(M)$ is local. In particular, the category $C(R)$ admits the Krull-Schmidt theorem. $C(R)$ is said to be of finite representation type if there are only a finite number of isomorphic classes of indecomposable objects in $C(R)$. We are mainly concerned with this property of $C(R)$ in this paper.

For a finitely generated R-module M and for a large integer n it is known that the length of $M/\mathfrak{m}^nM$ is a polynomial in n of degree $\dim(M)$ and the coefficient of $n^{\dim(R)}$ is in the form $e(M)/\dim(R)!$ with $e(M)$ an integer. This number $e(M)$ is called the multiplicity of M with respect to the maximal ideal $\mathfrak{m}$. Note that, if R is an integral domain, then one has the equality: $e(M)=e(R)\cdot\mathrm{rank}(M)$. Also note that, if M is an MCM module and if $\mathfrak{X}=\{x_1, x_2, \cdots, x_d\}$ is a system of parameters for R, then it holds that $e(M)\leqq\mathrm{length}(M/\mathfrak{X}M)\leqq n^d e(M)$ where n is the least integer with $\mathfrak{m}^n\subset\mathfrak{X}R$. For more detail see Nagata [22].

Let Γ denote the AR-graph for the category $C(R)$. See Appendix for the definition and the properties of AR-graphs. A subgraph Γ' of Γ is said to be of bounded multiplicity type if there is a bound for multiplicities of indecomposable MCM modules corresponding to vertices in Γ'. Note that, if R is an integral domain, then this is the case only when there is a bound for ranks of such modules. Also note that, for a system of parameters $\mathfrak{X}=\{x_1, x_2, \cdots, x_d\}$, Γ' is of bounded multiplicity type if and only if there is a bound for lengths of $M/\mathfrak{X}M$ for M belonging to Γ'. It is obvious that any finite subgraph of Γ is of bounded multiplicity type.

We are now ready to state our main theorem.

THEOREM (1.1). *Let Γ° be a connected component of Γ. Assume that R has only an isolated singularity and assume that Γ° is of bounded multiplicity type. Then $\Gamma=\Gamma^\circ$ and Γ is a finite graph. In particular, $C(R)$ is of finite representation type.*

REMARK (1.2). Replacing analytic algebras, the category of MCM modules and multiplicities respectively by (noncommutative) Artin algebras, the category of finitely generated modules and lengths, then the corresponding result to (1.1) is known as one of the main results of Auslander-Reiten theory. See Auslander-Reiten [7] or Ringel [24].

As corollaries of this theorem we obtain the following.

COROLLARY (1.3). *Let R be an isolated singularity. If Γ has a finite connected component Γ°, then $\Gamma=\Gamma^\circ$ and hence $C(R)$ is of finite representation type.*

COROLLARY (1.4) (Brauer-Thrall type theorem). *Let R be an isolated singularity as above. If there is a bound for multiplicities of indecomposable* MCM *modules over R, then $C(R)$ is of finite representation type.*

REMARK (1.5). The corresponding result to (1.4) for (noncommutative) Artin algebras is known as Brauer-Thrall conjecture or Roiter-Auslander theorem. Cf. Auslander [2], Ringel [24] and Roiter [25].

Our theorem will not be valid unless R is an isolated singularity. We give an example in the following which makes Corollary (1.4) fail.

EXAMPLE (1.6). Let $R=k\{x, y\}/(x^2)$ and let I_n be an ideal of R generated by $\{x, y^n\}$ for any integer $n>0$ and set $I_0=R$ and $I_\infty=xR$. Then $\{I_n|0\leqq n\leqq\infty\}$ is the complete list of non-isomorphic indecomposable MCM modules over R. In particular $C(R)$ is not of finite representation type. On the other hand, $e(I_n)=2$ for any n.

PROOF. Let $T=k\{y\}$. Then R is a finite T-algebra and any MCM module is free over T. Thus giving an MCM module M over R is equivalent to giving a T-algebra map f_M from R into a matrix algebra over T and this is also equivalent to giving a square matrix A_M with elements in T and with $A_M^2=0$ (by taking $f_M(x)=A_M$). It can be also shown that two MCM modules M and N are R-isomorphic if and only if $A_M=PA_NP^{-1}$ for some invertible matrix with elements in T. Thus the classification of all MCM modules over R is the same as the classification of square-zero matrices over T up to equivalence. Since T is a discrete valuation ring, it is easily seen that any square-zero matrix is equivalent to a matrix of the following form;

$$\begin{pmatrix}
0 & \cdots & 0 & * & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & & \vdots & 0 & * & & & & \vdots \\
\vdots & & \vdots & & & \ddots & & & \vdots \\
\vdots & & \vdots & & & & * & & \vdots \\
\vdots & & \vdots & & & & & * & \vdots \\
\vdots & & \vdots & & & & & 0 & \vdots \\
\vdots & & \vdots & & & & & & \ddots \\
0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & & \vdots & \vdots & & & & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix}.$$

It is therefore easy to see that an MCM module M is indecomposable if and only if A_M is equivalent to one of the following matrices;

$$(0), \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & y^n \\ 0 & 0 \end{pmatrix} \quad \text{for some integer } n>0.$$

These matrices correspond to I_∞, I_0 and I_n respectively.

§2. Noether different and MCM modules.

As in the previous section k always denotes a perfect valuation field and R is a local analytic k-algebra which is Cohen-Macaulay. Taking a system of parameters $\mathcal{X}=\{x_1, x_2, \cdots, x_d\}$ for R, we can form a convergent power series ring $T=k\{x_1, x_2, \cdots, x_d\}$ where d is the dimension of R. Note that R is always module-finite over T and hence it is T-free, for R being Cohen-Macaulay.

Now let R^e be the enveloping algebra $R\otimes_T R$ of R over T and let $\mu: R^e\to R$ be the multiplication mapping. The Noether different $\mathfrak{N}_T^R$ of R over T is defined as follows;

$$\mathfrak{N}_T^R = \mu(\mathrm{Ann}_{R^e}(\mathrm{Ker}(\mu))).$$

If R is reduced (and Cohen-Macaulay) then $\mathfrak{N}_T^R$ coincides with the Dedekind different $\mathfrak{D}_T^R$ which is defined as follows;

$$\mathfrak{D}_T^R = \{f(\mathrm{Sp}) \in R \mid f \in \mathrm{Hom}_R(\mathrm{Hom}_T(R, T), R)\},$$

where Sp denotes the trace map of the total quotient ring $Q(R)$ of R over the quotient field $Q(T)$ of T. This is also equal to the inverse ideal of $\mathcal{C}_T^R = \{x \in Q(R) \mid \mathrm{Sp}(xR) \subset T\}$. For more detail see Scheja-Storch [**27**]. We say that a sytem of parameters $\mathfrak{X}$ is separable if the extension $Q(R)/Q(T)$ is separable. If this is the case, then it is known that $\mathfrak{D}_T^R$ is an ideal of pure height 1 (purity of branch locus), in particular it is non-trivial. Thus we have

(2.1) *If R is a Cohen-Macaulay local analytic k-algebra which is reduced and if $\mathfrak{X} = \{x_1, x_2, \cdots, x_d\}$ is a separable system of parameters for R, then $\mathfrak{N}_T^R$ is an ideal of pure height* 1 *where $T = k\{x_1, x_2, \cdots, x_d\}$.*

We also note here that there always exists a separable system of parameters for a reduced analytic algebra R, for k is perfect. See Scheja-Storch [**26**].

The following is one of the most important property of Noether differents.

(2.2) *Let R be a T-algebra as above and let M be any R^e-module (i.e. R-bimodule with T acting centrally). Then $\mathfrak{N}_T^R$ annihilates the i-th Hochschild cohomology* $\mathrm{H}_T^i(R, M)$ *for $i > 0$.* (For the Hochschild cohomology see Hochschild [**16**] or Pierce [**23**].)

As a special case of this we see the following.

(2.3) *$\mathfrak{N}_T^R$ defines the ramification locus of* Spec(R) *over* Spec(T). *I.e. for a prime ideal P of R, R_P is ramified over $T_{P \cap T}$ if and only if P contains $\mathfrak{N}_T^R$.*

From these observations we have the following

LEMMA (2.4). *Let $\{P_1, P_2, \cdots, P_n\}$ be a set of prime ideals of R with the property that each R_{P_i} $(i = 1, 2, \cdots, n)$ is a regular local ring of same dimension t. Then there is a system of parameters $\{x_1, x_2, \cdots, x_d\}$ for R such that $\mathfrak{N}_{k\{x_1, x_2, \cdots, x_d\}}^R$ is not contained in any P_i $(i = 1, 2, \cdots, n)$.*

PROOF. Let S denote the set $R - \bigcup_i P_i$ which is multiplicatively closed in R. Note that $S^{-1}R$ is a semi-local ring with maximal ideals $P_i(S^{-1}R)$, $1 \leqq i \leqq n$. Since $R_{P_i} = (S^{-1}R)_{P_i(S^{-1}R)}$ is a regular local ring of dimension t, one can choose a set of elements $\{x_1, x_2, \cdots, x_t\}$ in R which forms a regular system of parameters for each R_{P_i}, $1 \leqq i \leqq n$. Thus $(x_1, x_2, \cdots, x_t)R_{P_i} = P_i R_{P_i}$ for any i. Since $R/(x_1, x_2, \cdots, x_t)R$ is a reduced analytic algebra over k and since k is perfect, there is a separable system of parameters $\{x_{t+1}, x_{t+2}, \cdots, x_d\}$ for $R/(x_1, x_2, \cdots, x_t)R$.

Consider the power series ring $T=k\{x_1, x_2, \cdots, x_d\}$. Note that $(x_1, x_2, \cdots, x_t)T = P_i \cap T$ for any i $(1 \leqq i \leqq n)$, which we denote by p. Then it is easy to show that each R_{P_i} $(1 \leqq i \leqq n)$ is unramified over T_p, for R_{P_i}/pR_{P_i} is a separable extension of T_p/pT_p. Thus by (2.3) we conclude that P_i does not contain $\mathfrak{N}_T^R$.

Now we define an ideal of R which seems to be a good invariant of an analytic algebra R.

DEFINITION (2.5). $\mathfrak{N}^R = \sum \mathfrak{N}^R_{k\{x_1, x_2, \cdots, x_d\}}$ where $\{x_1, x_2, \cdots, x_d\}$ ranges through all systems of parameters for R.

As a corollary of Lemma (2.4) we see that $\mathfrak{N}^R$ has the following property.

COROLLARY (2.6). *$\mathfrak{N}^R$ defines the singular locus of* Spec(R), *that is, for a prime ideal P of R, R_P is regular if and only if P does not contain $\mathfrak{N}^R$.*

PROOF. If P does not contain $\mathfrak{N}^R$, then P does not contain $\mathfrak{N}_T^R$ for some $T=k\{x_1, x_2, \cdots, x_d\}$. Thus R_P is unramifie over $T_{P\cap T}$ by (2.3), which implies that R_P is regular. On the other hand, assume that R_P is regular. Then by (2.4) P does not contain $\mathfrak{N}_T^R$ for some $T=k\{x_1, x_2, \cdots, x_d\}$, in particular, it does not contain $\mathfrak{N}^R$.

REMARK (2.7). If $R=T[x]/(f(x))$, then it is known that $\mathfrak{N}_T^R$ is an ideal generated by the derivative $f'(x)$. Thus in the case R is a hypersurface $k\{x_1, x_2, \cdots, x_{d+1}\}/(f)$, $\mathfrak{N}^R$ is generated by derivatives $\partial f/\partial x_i$, $1 \leqq i \leqq d+1$. By this fact we are able to see the assumption that k is a perfect field is indispensable in Corollary (2.6). For example, if there is an element a in k which is not in k^p where p is the characteristic of k, then consider a k-algebra $R= k\{x, y\}/(x^p+ay^p)$. In this example, R is an integral domain of dimension 1, hence an isolated singularity, though $\mathfrak{N}^R=0$.

COROLLARY (2.8). *If R has only an isolated singularity, then one can choose a system of parameters $\mathfrak{X}=\{x_1, x_2, \cdots, x_d\}$ for R which satisfies the condition;*

(2.8.*) *for any i $(1 \leqq i \leqq d)$ there is a regular subring T_i of R on which R is finite and x_i belongs to the Noether different $\mathfrak{N}^R_{T_i}$.*

PROOF. By the induction on j $(1 \leqq j \leqq d)$, one can choose a part of system of parameters $\{x_1, x_2, \cdots, x_j\}$ for R, such that x_i belongs to $\mathfrak{N}^R_{T_i}$ for some T_i $(1 \leqq i \leqq j)$. This is obvious if $j=1$ since $\mathfrak{N}_T^R$ is an ideal of pure height 1 for some regular subring T of R. For $2 \leqq j < d$, assume that $\{x_1, x_2, \cdots, x_{j-1}\}$ are already chosen. Then by (2.4) there is some T_j with the property that $\mathfrak{N}^R_{T_j}$ is not contained in any minimal prime ideals of $(x_1, x_2, \cdots, x_{j-1})R$. Thus there is an elelment x_j in $\mathfrak{N}^R_{T_j}$, such that, $\{x_1, x_2, \cdots, x_j\}$ form a subsystem of parameters for R.

Such a system of parameters satisfying the condition (2.8.$*$) will play a central role in the rest of this paper. The following proposition will be a key for using these parameters.

PROPOSITION (2.9). *Let $\mathcal{X}=\{x_1, x_2, \cdots, x_d\}$ be a system of parameters for R satisfying the condition* (2.8.$*$) *and let M, N be* MCM *modules over R. We denote by $\mathcal{X}^{(n)}$ the ideal of R generated by $\{x_1^n, x_2^n, \cdots, x_d^n\}$. Then for any R-homomorphism φ from $M/\mathcal{X}^{(2)}M$ to $N/\mathcal{X}^{(2)}N$, there exists an R-homomorphism ϕ from M to N such that $\varphi\otimes R/\mathcal{X}R=\phi\otimes R/\mathcal{X}R$.*

PROOF. By the induction on i $(0\leqq i\leqq d)$ we prove the following

(2.9.i) there is an R-homomorphism φ_i from $M/(x_1^2, \cdots, x_i^2)M$ to $N/(x_1^2, \cdots, x_i^2)N$ such that $\varphi_i\otimes R/\mathcal{X}R=\varphi\otimes R/\mathcal{X}R$.

$\varphi_d=\varphi$ is given and there is nothing to prove for $i=d$. Assume that φ_{i+1} is already constructed $(0\leqq i\leqq d-1)$. It is enough to show the existence of φ_i from $M/(x_1^2, \cdots, x_i^2)M$ to $N/(x_1^2, \cdots, x_i^2)N$ satisfying $\varphi_i\otimes R/(x_1^2, \cdots, x_i^2, x_{i+1})R=\varphi_{i+1}\otimes R/(x_1^2, \cdots, x_i^2, x_{i+1})$. For the simplicity we denote an ideal generated by $\{x_1^2, \cdots, x_i^2\}$ (respectively $\{x_1^2, \cdots, x_i^2, x_{i+1}\}$) by $\mathcal{Y}_i$ (respectively $\mathcal{Z}_i$). Since N is an MCM module over R, we have the commutative diagram with exact rows;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N/\mathcal{Y}_iN & \xrightarrow{x_{i+1}^2} & N/\mathcal{Y}_iN & \longrightarrow & N/\mathcal{Y}_{i+1}N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle x_{i+1}} & & \| & & \downarrow & & \\
0 & \longrightarrow & N/\mathcal{Y}_iN & \xrightarrow{x_{i+1}} & N/\mathcal{Y}_iN & \longrightarrow & N/\mathcal{Z}_iN & \longrightarrow & 0.
\end{array}$$

Applying the functor $\mathrm{Hom}_{T_{i+1}}(M,\ \ )$ to this diagram where T_{i+1} being as in (2.8.$*$), we obtain the following diagram;

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Hom}_{T_{i+1}}(M, N/\mathcal{Y}_iN) & \to & \mathrm{Hom}_{T_{i+1}}(M, N/\mathcal{Y}_iN) & \to & \mathrm{Hom}_{T_{i+1}}(M, N/\mathcal{Y}_{i+1}N) & \to & 0 \\
 & & \downarrow{\scriptstyle x_{i+1}} & & \| & & \downarrow & & \\
0 & \to & \mathrm{Hom}_{T_{i+1}}(M, N/\mathcal{Y}_iN) & \to & \mathrm{Hom}_{T_{i+1}}(M, N/\mathcal{Y}_iN) & \to & \mathrm{Hom}_{T_{i+1}}(M, N/\mathcal{Z}_iN) & \to & 0
\end{array}$$

where the rows are exact, since M is a free T_{i+1}-module. Note that these rows are also exact sequences of R-bimodules. (The left (resp. right) action of R on $\mathrm{Hom}(M, N')$ is given as the one induced from the action on N' (resp. M).) Noting that $\mathrm{H}^0_{T_{i+1}}(R, \mathrm{Hom}_{T_{i+1}}(M, N'))=\mathrm{Hom}_R(M, N')$ for any R-modules M and N', we now get the commutative diagram with exact rows by taking Hochschild cohomology functor;

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Hom}_R(M, N/\mathcal{Y}_iN) & \to & \mathrm{Hom}_R(M, N/\mathcal{Y}_iN) & \to & \mathrm{Hom}_R(M, N/\mathcal{Y}_{i+1}N) & \to & \mathrm{H}^1_{T_{i+1}}(R, \mathrm{Hom}_{T_{i+1}}(M, N/\mathcal{Y}_iN)) \\
 & & \downarrow & & \| & & \downarrow & & \downarrow{\scriptstyle x_{i+1}} \\
0 & \to & \mathrm{Hom}_R(M, N/\mathcal{Y}_iN) & \to & \mathrm{Hom}_R(M, N/\mathcal{Y}_iN) & \to & \mathrm{Hom}_R(M, N/\mathcal{Z}_iN) & \to & \mathrm{H}^1_{T_{i+1}}(R, \mathrm{Hom}_{T_{i+1}}(M, N/\mathcal{Y}_iN)).
\end{array}$$

By (2.2) and by our choice of x_{i+1} and T_{i+1} in (2.8.*) we show that x_{i+1} on the right vertical arrow induces the trivial map. Therefore an easy diagram chasing shows that for any φ_i in $\mathrm{Hom}_R(M, N/\mathcal{Y}_{i+1}N)$ there is φ_{i+1} in $\mathrm{Hom}_R(M, N/\mathcal{Y}_iN)$ such that $\varphi_i \otimes R/\mathcal{Z}_iR = \varphi_{i+1}\otimes R/\mathcal{Z}_iR$. This completes the proof of the proposition.

As a direct consequence of Proposition (2.9) we obtain the following

PROPOSITION (2.10). *Let $\mathcal{X}=\{x_1, x_2, \cdots, x_d\}$ be a system of parameters for R which satisfies the condition* (2.8.*) *and let M be an* MCM *module over R. Then M is an indecomposable R-module if and only if $M/\mathcal{X}^{(2)}M$ is indecomposable.*

PROOF. If M is decomposable, then it is obviously true that $M/\mathcal{X}^{(2)}M$ is also decomposable. Conversely assume that M is indecomposable. Take an idempotent e in $\mathrm{End}_R(M/\mathcal{X}^{(2)}M)$. We want to prove that either $e=1$ or 0. There is a commutative diagram of natural ring homomorphisms;

$$\begin{tikzcd} \mathrm{End}_R(M) \arrow[r, "\otimes R/\mathcal{X}^{(2)}R"] \arrow[dr, "\alpha=\otimes R/\mathcal{X}R"'] & \mathrm{End}_R(M/\mathcal{X}^{(2)}M) \arrow[d, "\beta=\otimes R/\mathcal{X}R"] \\ & \mathrm{End}_R(M/\mathcal{X}M) \end{tikzcd} \quad .$$

Now we denote by A the image of α which is also local, for it is a homomorphic image of the local algebra $\mathrm{End}(M)$. It then follows from Proposition (2.9) that $\beta(e)$ belongs to A. Since $e^2=e$, $\beta(e)$ is also an idempotent of A, hence either $\beta(e)=1$ or 0. If $\beta(e)=0$, then $e(M/\mathcal{X}^{(2)}M)\subset\mathcal{X}(M/\mathcal{X}^{(2)}M)$ hence $e=0$, for $e^2=e$. If $\beta(e)=1$, then $\beta(1-e)=0$ hence by the above $e=1$.

We also obtain from (2.9) the following

PROPOSITION (2.11). *Let $\mathcal{X}=\{x_1, x_2, \cdots, x_d\}$ be as in* (2.10) *and let $s: 0\to N \xrightarrow{q} E \xrightarrow{p} M \to 0$ be an exact sequence in the category $C(R)$. Denote by $\tilde{s}$ the sequence obtained from s by tensoring $R/\mathcal{X}^{(2)}R$:*

$$\tilde{s}: 0 \longrightarrow N/\mathcal{X}^{(2)}N \xrightarrow{\tilde{q}} E/\mathcal{X}^{(2)}E \xrightarrow{\tilde{p}} M/\mathcal{X}^{(2)}M \longrightarrow 0.$$

(*Note that $\tilde{s}$ is also exact, for $\mathcal{X}$ is a regular sequence on M.*) *If $\tilde{s}$ is split, then so is s.*

PROOF. Assume $\tilde{s}$ is split, that is, there is f in $\mathrm{Hom}_R(M/\mathcal{X}^{(2)}M, E/\mathcal{X}^{(2)}E)$ such that $\tilde{p}\cdot f$ is the identity on $M/\mathcal{X}^{(2)}M$. Proposition (2.9) shows that there is g in $\mathrm{Hom}_R(M, E)$ such that $g\otimes R/\mathcal{X}R = f\otimes R/\mathcal{X}R$. Thus $(p\cdot g)\otimes R/\mathcal{X}R = (p\otimes R/\mathcal{X}R)\cdot(g\otimes R/\mathcal{X}R)$ is the identity mapping on $M/\mathcal{X}M$. In particular we see by Nakayama lemma that $p\cdot g$ is an epimorphism, then it must be an automorphism on M. This shows s being a split sequence.

The next proposition will be useful later.

PROPOSITION (2.12). *Let $\mathfrak{X}=\{x_1, x_2, \cdots, x_d\}$ be as in* (2.9) *and let M, N be indecomposable* MCM *modules over R. If $M/\mathfrak{X}^{(2)}M$ is isomorphic to $N/\mathfrak{X}^{(2)}N$, then M is isomorphic to N.*

PROOF. Let $\tilde{f}$ be an isomorphism from $M/\mathfrak{X}^{(2)}M$ onto $N/\mathfrak{X}^{(2)}N$. Then by (2.9) we have a homomorphism f from M to N such that $f\otimes R/\mathfrak{X}R=\tilde{f}\otimes R/\mathfrak{X}R$. In particular f is epimorphic by Nakayama lemma. Thus we obtain an exact sequence; $0\to\mathrm{Ker}(f)\to M\xrightarrow{f}N\to 0$, where it is easily seen that $\mathrm{Ker}(f)$ is also an MCM module. Tensoring $R/\mathfrak{X}R$ with this sequence we have an exact sequence;

$$0\longrightarrow\mathrm{Ker}(f)\otimes R/\mathfrak{X}R\longrightarrow M\otimes R/\mathfrak{X}R\xrightarrow{f\otimes R/\mathfrak{X}R}N\otimes R/\mathfrak{X}R\longrightarrow 0.$$

Since $f\otimes R/\mathfrak{X}R$ is an isomorphism, we see that $\mathrm{Ker}(f)\otimes R/\mathfrak{X}R=0$, hence $\mathrm{Ker}(f)=0$ again by Nakayama lemma. Thus f gives an isomorphism between M and N.

REMARK (2.13). The above proof also shows the following: If f is an R-homomorphism between MCM modules M and N, and if $f\otimes R/\mathfrak{X}^{(2)}R$ gives an isomorphism, then f is also an isomorphism.

§3. Proof of the main theorem.

In this section we prove our main theorem (1.1). For this purpose we need some lemmas which modify the ones in [2] or [28]. We begin with the following

LEMMA (3.1) (Harada-Sai lemma for MCM modules). *Let M_i, $0\leqq i\leqq 2^n$, be indecomposable* MCM *modules over R and let $\mathfrak{X}=\{x_1, x_2, \cdots, x_d\}$ be a system of parameters which satisfies the condition* (2.8.*). *Let $f_i: M_{i-1}\to M_i$, $1\leqq i\leqq 2^n$, be non-isomorphic homomorphisms. Assume that $\mathrm{length}(M_i/\mathfrak{X}^{(2)}M_i)\leqq n$ for $0\leqq i\leqq 2^n$. Then we have*

$$(f_{2^n}\cdots f_2\cdot f_1)\otimes R/\mathfrak{X}^{(2)}R=0.$$

PROOF. We denote $M/\mathfrak{X}^{(2)}M$, $f\otimes R/\mathfrak{X}^{(2)}R$ respectively by $\tilde{M}$, $\tilde{f}$. Then by (2.10) and (2.13) $\tilde{R}$-modules $\tilde{M}_i$ and $\tilde{R}$-homomorphisms $\tilde{f}_i$ satisfy the following conditions;

(a) $\tilde{M}_i$, $0\leqq i\leqq 2^n$, are indecomposable,

(b) $\mathrm{length}(\tilde{M}_i)\leqq n$ for all i, and

(c) $\tilde{f}_i$, $1\leqq i\leqq 2^n$, are all non-isomorphic.

Then Harada-Sai lemma ([18; Lemma 12] or [24]) shows that the composition $\tilde{f}_{2^n}\cdots\tilde{f}_2\cdot\tilde{f}_1$ is trivial, which proves the lemma.

In the case that we are given a sequence of irreducible maps;

$$(*)\quad M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \cdots \xrightarrow{f_n} M_n$$

with all M_i indecomposable MCM modules, then we will call this a chain of irreducible morphisms of length n. (For the defintion of irreducible maps, see (A.15).) A chain $(*)$ of irreducible morphisms is said to be non-trivial with respect to a system of parameters $\mathfrak{X}=\{x_1, x_2, \cdots, x_d\}$ provided that $(f_n \cdot f_{n-1} \cdots f_1) \otimes R/\mathfrak{X}R$ is a nontrivial homomorphism. The following is the corresponding result to [**24**; 2.1. lemma].

LEMMA (3.2). *Let R be an isolated singularity and let $\mathfrak{X}=\{x_1, x_2, \cdots, x_d\}$ be a system of parameters for R with the condition* (2.8.$*$). *Let M, N be indecomposable* MCM *modules over R. Assume that there is a morphism φ from M into N satisfying $\varphi \otimes R/\mathfrak{X}^{(2)}R \neq 0$, and also assume that there exists no chain of irreducible morphisms from M to N of* length$<n$ *which is nontrivial with respect to $\mathfrak{X}^{(2)}$. Then*

(a) *there exist a chain of irreducible morphisms;*

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \cdots \longrightarrow M_{n-1} \xrightarrow{f_n} M_n$$

and a morphism $g: M_n \to N$ with $(g \cdot f_n \cdot f_{n-1} \cdots f_1) \otimes R/\mathfrak{X}^{(2)}R \neq 0$

and

(b) *there exist a chain of irreducible morphisms;*

$$N_n \xrightarrow{g_n} N_{n-1} \longrightarrow \cdots \longrightarrow N_1 \xrightarrow{g_1} N_0 = N$$

and a morphism $f: M \to N_n$ with $(g_1 \cdot g_2 \cdots g_n \cdot f) \otimes R/\mathfrak{X}^{(2)}R \neq 0$.

PROOF. We only prove (b), for (a) would be obtained by the dual argument of (b). The proof proceeds by induction on n. For $n=0$, there is nothing to prove. Assume $n>0$. Then by the induction hypothesis we have irreducible morphisms $g_i: N_i \to N_{i-1}$, $1 \leqq i \leqq n-1$, with $N_0=N$ and a morphism $f: M \to N_{n-1}$ such that $(g_1 \cdot g_2 \cdots g_{n-1} \cdot f) \otimes R/\mathfrak{X}^{(2)}R \neq 0$. Our assumption implies that f can never be an isomorphism. We consider two cases: First let N_{n-1} be isomorphic to R. In this case by (A.19) we have an MCM module L and a morphism h from L to N_{n-1} such that f can be factored through h;

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "h'"'] & & N_{n-1}=R \\ & L \arrow[ur, "h"'] & \end{tikzcd} .$$

We decompose L into a direct sum of indecomposable MCM modules L_i, and also decompose h into a direct sum of h_i, and h' into a sum of h'_i. Note that h_i are irreducible and $g=\sum h_i\cdot h'_i$. Now, since $(g_1\cdot g_2\cdots g_{n-1}\cdot f)\otimes R/\mathfrak{X}^{(2)}R\neq 0$, it follows that $(g_1\cdot g_2\cdots g_{n-1}\cdot h_i\cdot h'_i)\otimes R/\mathfrak{X}^{(2)}R\neq 0$ for some i. Letting $N_n=L_i$ and $g_n=h_i$, the lemma follows from this in this case.

In the second case assume that N_{n-1} is not isomorphic to R. Thus there exists an AR-sequence; $0\to L'\to L\xrightarrow{h}N_{n-1}\to 0$. (See Proposition (A.10).) Let $L=\sum L_i$ with L_i indecomposable, and let $h=(h_i)$. Again h_i are irreducible. By the property of AR-sequences one can lift f to L, thus f will be factored again in the form $f=\sum h_i\cdot h'_i$. In the same way as above, one obtains $(g_1\cdot g_2\cdots g_{n-1}\cdot h_i\cdot h'_i)\otimes R/\mathfrak{X}^{(2)}R\neq 0$ for some i, and the proof is completed.

(3.3) Now we proceed to the proof of Theorem (1.1). Let R be an isolated singularity as in the theorem and let Γ° be a connected component of the AR-graph Γ. Assume that all indecomposable MCM modules in Γ° are of multiplicity$\leqq a$. Let $\mathfrak{X}=\{x_1, x_2, \cdots, x_d\}$ be a system of parameters for R which satisfies the condition (2.8.$*$). Note that for a module in Γ° we have $\mathrm{length}(M/\mathfrak{X}^{(2)}M)\leqq m$ where $m=a\cdot b^d$ with b being the least integer satisfying $\mathfrak{m}^b\subset\mathfrak{X}^{(2)}R$.

Let M, N be two indecomposable MCM modules with the property that there is a morphism f from M to N such that $f\otimes R/\mathfrak{X}^{(2)}R\neq 0$. Assume that M belongs to Γ°. First of all we want to prove that there is a chain of irreducible morphisms from M to N of length$<2^m$ which is nontrivial with respect to $\mathfrak{X}^{(2)}$, thus N is also in Γ°. For, otherwise, by (3.2) there is a chain of irreducible morphisms; $M=M_0\xrightarrow{f_1}M_1\to\cdots\to M_{n-1}\xrightarrow{f_n}M_n$ and a morphism $g: M_n\to N$ with $(g\cdot f_n\cdots f_2\cdot f_1)\otimes R/\mathfrak{X}^{(2)}R\neq 0$, where $n=2^m$. Here we note that M_i are all in Γ°, for M_i being connected with M in Γ. In particular, we have $\mathrm{length}(M/\mathfrak{X}^{(2)}M)\leqq m$ by the assumption. Then Lemma (3.1) shows that $(f_n\cdot f_{n-1}\cdots f_1)\otimes R/\mathfrak{X}^{(2)}R=0$, which is a contradiction. Summarizing the above, we have obtained the following

(3.3.1) *Let M and N be indecomposable* MCM *modules with the property that there is a homomorphism f from M to N satisfying $f\otimes R/\mathfrak{X}^{(2)}R\neq 0$. If M belongs to Γ°, then there is a chain of irreducible morphisms from M to N of length$<$ $n(=2^m)$. In particular N also belongs to Γ°.*

The dual argument shows the dual statement of the above.

(3.3.2) *Let M and N be as in* (3.3.1). *If N belongs to Γ°, then there is a chain of irreducible morphisms from M to N of length$<n$. In particular M also belongs to Γ°.*

Now let M be any indecomposable MCM module over R. Then there is a map $f: R \to M$ with $f \otimes R/\mathfrak{X}^{(2)}R \neq 0$. (It is enough to take an element x in M which is not in $\mathfrak{X}^{(2)}M$ and to define $f(r)=r \cdot x$.) Taking M from the vertices in Γ°, one can show by (3.3.2) that R belongs to Γ°. It then follows from (3.3.1) that any M from Γ belongs to Γ°. Thus we proved that $\Gamma=\Gamma^\circ$. And at the same time we showed that any vertex in Γ is connected with R by a directed path of length at most n. On the other hand we know from (A.18) that the graph Γ is locally finite. Hence Γ must be a finite graph and the proof is finished.

§ 4. An application.

In this section we assume that R is a normal Gorenstein local analytic domain of dimension 2 over an algebraically closed field k of characteristic 0. Notice that objects in $C(R)$ are reflexive modules over R in this case. It is known by Auslander [4] that $C(R)$ is of finite representation type if and only if R is one of the Klein singularities. Thus if R is not a Klein singularity, then R admits infinitely many indecomposable MCM modules, more strongly by Corollary (1.4) there are no bounds for multiplicities (or ranks) of indecomposable MCM modules.

As an application of Theorem (1.1) we can prove a stronger result than this one.

PROPOSITION (4.1). *Let R be as above. Assume that R is none of the Klein singularities. Then, for any positive integer n, there exist infinitely many classes of indecomposable* MCM *modules of rank n.*

PROOF. Let Γ be the AR-graph of $C(R)$. Since R is not a Klein singularity, note that the divisor class group of R has infinitely many elements. Equivalently there are infinitely many isomorphism classes of MCM modules of rank 1 (= divisorial ideals). Now let $\mathfrak{a}$ be an MCM module of rank 1 and let $\Gamma_\mathfrak{a}$ be the connected component of Γ containing $\mathfrak{a}$. Since R is a Gorenstein ring of dimension 2, (A.14) shows that the AR-translation τ is given by $\tau(M)=M$. (See also [4; Corollary (6.2)].) Then by the similar argument as in [9] we see that the rank, $\mathrm{rk}(M)$ for each vertex M of Γ, gives an additive function on Γ, that is, for each M in Γ the equality $2 \cdot \mathrm{rk}(M)=\sum \mathrm{rk}(N)$ holds, where N runs through vertices in Γ which are incident to M. Then the underlying undirected graph $|\Gamma_\mathfrak{a}|$ of $\Gamma_\mathfrak{a}$ should look like one of the graphs in the list [17; p. 282]. Notice that, in our case, the rank function gives an unbounded function on $\Gamma_\mathfrak{a}$. For, otherwise, $\Gamma=\Gamma_\mathfrak{a}$ and Γ is finite by Theorem (1.1) and consequently R would be a Klein singularity by [4; (4.9)]. It hence follows from [17; Section 2, Theorem] that $|\Gamma_\mathfrak{a}|$ is of the type A_∞ and that an additive function on $\Gamma_\mathfrak{a}$ is essentially unique which looks like

$$\overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{2}{\circ}\!\!-\!\!\!-\!\!\overset{3}{\circ}\!\!-\!\!\!-\!\!\overset{4}{\circ}\!\!-\!\!\!-\!\!\overset{5}{\circ}\!\!-\!\!\!-\cdots.$$

Note that the leftmost vertex (which has the least rank) should be the class of $\mathfrak{a}$. And it is easy to prove that $\Gamma_{\mathfrak{a}}\cap\Gamma_{\mathfrak{b}}=\emptyset$ whenever $\mathfrak{b}$ is a divisorial ideal and is not isomorphic to $\mathfrak{a}$. For any n, this shows the existence of indecomposable MCM modules of rank n which, of course, correspond to the vertices with valuation n in $\Gamma_{\mathfrak{a}}$ with $\mathfrak{a}$ a divisorial ideal.

Other examples and applications of our theorem will be discussed in our forthcoming paper [**29**].

Appendix. Auslander-Reiten theory for MCM modules.

In this section we summarize some definitions and facts from the Auslander-Reiten theory (for short AR-theory) for the category of MCM modules. Cf. Auslander [**3**], [**5**], Auslander-Reiten [**7**], Pierce [**23**] and Yamagata [**28**]. For the rest of this paper R is always a Cohen-Macaulay Hensel local ring with the maximal ideal $\mathfrak{m}$ and $C(R)$ is the category of MCM modules over R. Note that $M\in C(R)$ is indecomposable if and only if $\mathrm{End}_R(M)$ is local.

DEFINITION (A.1). For an indecomposable module M in $C(R)$, $S(M)$ is the set of non-split exact sequences $s: 0\to N_s\to E_s\to M\to 0$ in $C(R)$ in which N_s is also indecomposable. Hence $s\in S(M)$ is a nontrivial element of $\mathrm{Ext}^1_R(M, N_s)$. For two elements s and t in $S(M)$ we denote $s\leqq t$ provided that there is an R-homomorphism f from N_t to N_s such that $\mathrm{Ext}^1(M, f)(t)=s$.

LEMMA (A.2). *Let M be an indecomposable* MCM *module over R and let s and t be elements in $S(M)$. If $s\leqq t$ and $t\leqq s$, then there is an isomorphism f from N_s onto N_t such that* $\mathrm{Ext}^1(M, f)(s)=t$.

To prove this, it is obviously enough to verify the following

LEMMA (A.3). *Let s be in $S(M)$ and h in* $\mathrm{End}_R(N_s)$. *If* $\mathrm{Ext}^1(M, h)(s)=s$, *then h is an automorphism on N_s.*

PROOF. Suppose that h is not an automorphism. Then h is in the radical of the local ring $\mathrm{End}_R(N_s)$, hence some power of h is contained in $\mathfrak{m}\mathrm{End}_R(N_s)$. We may assume that h is in $\mathfrak{m}\mathrm{End}_R(N_s)$. Particularly for any integer n we may write $h^n=\sum a_i^{(n)}g_i^{(n)}$ $(a_i^{(n)}\in\mathfrak{m}^n,\ g_i^{(n)}\in\mathrm{End}_R(N_s))$. Thus $s=\mathrm{Ext}^1(M, h^n)(s)=\sum a_i^{(n)}\mathrm{Ext}^1(M, g_i^{(n)})(s)$ and this belongs to $\mathfrak{m}^n\mathrm{Ext}^1(M, N_s)$. This holds for any integer n, hence we see that $s=0$ which is a contradiction.

LEMMA (A.4). *For any two elements s and t in $S(M)$ there exists an element u in $S(M)$ such that $u\leqq s$ and $u\leqq t$.*

PROOF. Let s (resp. t) be the sequence; $0 \to N_s \to E_s \xrightarrow{p_s} M \to 0$ (resp. $0 \to N_t \to E_t \xrightarrow{p_t} M \to 0$) and let E denote the direct sum of E_s and E_t and $p=(p_s, p_t)$. Then the sequence; $0 \to L \to E \xrightarrow{p} M \to 0$ is also non-split, where L is the kernel of p. Decompose L into a sum of indecomposable modules L_i and consider the exact sequences; $u_i: 0 \to L_i \to E/\sum_{j \neq i} L_j \to M \to 0$. Then one of the u_i is non-split and hence is in $S(M)$. It is trivial that this u_i is less than s and t.

COROLLARY (A.5). *If s is minimal in $S(M)$, then it is minimum in $S(M)$.*

DEFINITION (A.6). For an indecomposable MCM module M, the minimum (or minimal) element in $S(M)$ (if it exists) is called the AR-sequence ending in M. If there exists an AR-sequence; $0 \to N \to E \to M \to 0$ ending in M, then N is uniquely determined by M. This N is denoted by $\tau(M)$ and τ is called the AR-translation.

AR-sequences have the following property.

LEMMA (A.7). *For an exact sequence $s: 0 \to N_s \to E_s \xrightarrow{p_s} M \to 0$ in $S(M)$, the following conditions are equivalent.*

(a) *s is the AR-sequence ending in M.*

(b) *For any homomorphism $p_1: M_1 \to M$ in $C(R)$ which is not a split epimorphism, there exists a homomorphism $f: M_1 \to E_s$ such that $p_s \cdot f = p_1$.*

PROOF. (b) implies (a). Let $t: 0 \to N_t \to E_t \xrightarrow{p_t} M \to 0$ be an element in $S(M)$ such that $t \leqq s$. We want to show that $s \leqq t$. Since p_t is not a split epimorphism, (b) shows that there is a morphism f from E_t to E_s such that $p_s \cdot f = p_t$. If g denotes the restriction of f on N_t, then it is easy to see that $\mathrm{Ext}^1(M, g)(t) = s$, hence $s \leqq t$.

(a) implies (b). Let $p_1: M_1 \to M$ be as in (b). Consider the exact sequence; $u: 0 \to Q \to E_s \oplus M_1 \xrightarrow{p} M \to 0$, where $p=(p_s, p_1)$ and $Q = \mathrm{Ker}(p)$. Note that u is not split and that there is a morphism h from N_s to Q such that $\mathrm{Ext}^1(M, h)(s) = u$. (In fact h is the restriction of the natural injection of E_s into $E_s \oplus M_1$.) Decompose Q into indecomposable modules Q_i. Then $\mathrm{Ext}^1(M, Q) = \sum \mathrm{Ext}^1(M, Q_i)$ and $u = (u_i)$ along this decomposition. Since u is nontrivial in $\mathrm{Ext}^1(M, Q)$, one of the u_i is a non-split sequence. Denote this sequence by t. Then t belongs to $S(M)$ and it satisfies the inequality $t \leqq s$. Thus we know that $s \leqq t$ for s is minimum. This shows the existence of a homomorphism g from Q to N_s such that $\mathrm{Ext}^1(M, g)(u) = s$. Thus there is a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & Q \arrow[r] \arrow[d, "g"] & E_s \oplus M_1 \arrow[r, "p"] \arrow[d, "f"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N_s \arrow[r] & E_s \arrow[r, "p_s"] & M \arrow[r] & 0, \end{tikzcd}$$

where $f=(f_s, f_1)$. This f_1 obviously lifts the map p_1.

Let K be the canonical module of R. We denote the functor $\mathrm{Hom}_R(\ ,K)$ from $C(R)$ to itself by $(\ )'$. Note that this functor gives the duality on $C(R)$, that is, $M''=M$ for any M in $C(R)$. For the detail of the canonical modules the reader will refer to Herzog-Kunz [**15**]. Dualizing everything in the above from (A.1) to (A.7) we may have the definitions and lemmas in the dual state. For instance, an exact sequence $s:0\to N\to E\to M\to 0$ is the AR-sequence starting from N if and only if its dual $s':0\to M'\to E'\to N'\to 0$ is the AR-sequnce ending in N'.

Now the following is almost trivial.

LEMMA (A.8). *Let $s:0\to N\to E\to M\to 0$ be an exact sequence in $C(R)$ with M and N being indecomposable. Then s is the* AR-*sequence ending in M if and only if s is the* AR-*sequence starting from N.*

DEFINITION (A.9). We say that $C(R)$ admits AR-sequences if, for any indecomposable MCM module M over R which is not isomorphic to R, there exists the AR-sequence ending in M. Remark that this is equivalent to saying that, for any indecomposable MCM module N over R which is not isomorphic to the canonical module K, there exists the AR-sequence starting from N.

The following is due to Auslander [**5**], and we will give an outline of a new proof for it.

PROPOSITION (A.10). *For a Cohen-Macaulay Hensel local ring, the following conditions are equivalent.*

(a) *$C(R)$ admits* AR-*sequences.*

(b) *R has only an isolated singularity, that is, for any non-maximal prime ideal P of R, R_P is a regular local ring.*

Before giving a proof of this proposition, we remark the following fact, whose proof is almost trivial and is left to the reader.

LEMMA (A.11). *The following are equivalent.*

(1) *R has only an isolated singularity.*

(2) $\mathrm{Ext}^1(M, N)$ *is of finite length for any M and N in $C(R)$.*

Now we can prove one implication in Proposition (A.10).

(a) implies (b). Supposed that R is not an isolated singularity. Then by Lemma (A.11) there would be a non-maximal prime ideal P and M, N in $C(R)$ such that $\mathrm{Ext}^1(M, N)_P\neq 0$. Taking indecomposable direct summands of M and N if necessary, we may assume that both M and N are indecomposable. It can be seen that there is an element s in $\mathrm{Ext}^1(M, N)$ and r in $\mathfrak{m}$ but not in P

such that $r^n s \neq 0$ for any integer n. Note that s belongs to $S(M)$. By the assumption there exists the AR-sequence t ending in M, and hence $t \leqq r^n s$ in $S(M)$. Thus there is a homomorphism $f^{(n)}$ from N to N_t with $\mathrm{Ext}^1(M, f^{(n)})(r^n s) = t$ for any integer n. This implies that t is in $r^n \mathrm{Ext}^1(M, N_t)$ for any n which finally shows that $t=0$. This clearly contradicts t being non-split.

Before proceeding to the proof of the implication from (b) to (a), it will be necessary to introduce some notion.

DEFINITION (A.12). (i) Let M be an element of $C(R)$. Assume that M has the following presentation by free modules; $F_1 \xrightarrow{f} F_0 \to M \to 0$ which is minimal. We define $\mathrm{tr}(M) = \mathrm{Coker}(f^*)$ where $*$ denotes the functor $\mathrm{Hom}_R(\ , R)$. And tr is called the transpose.

(ii) Let M be a finitely generated R-module and let $0 \to N \to F \to M \to 0$ be an exact sequence of R-modules with F free. Decompose N into a direct sum $L \oplus G$ where G is free and L has no free direct summand. We define L is the first syzygy module $\mathrm{Syz}_1 M$ of M. Note that $\mathrm{Syz}_1 M$ is uniquely determined by M. And inductively we define $\mathrm{Syz}_n M = \mathrm{Syz}_{n-1}(\mathrm{Syz}_1 M)$ and call it the n-th syzygy module of M.

(iii) Let M and N be in $C(R)$. We define $\underline{\mathrm{Hom}}_R(M, N) = \mathrm{Hom}_R(M, N)/P(M, N)$ where $P(M, N)$ is the set of homomorphisms from M to N which factor through free modules. We also denote $\mathrm{End}_R(M)/P(M, M)$ by $\underline{\mathrm{End}}_R(M)$. Note that $\underline{\mathrm{End}}_R(M)$ is a local ring whenever M is indecomposable. It is known by [**7**; 2.2] and is easy to prove that $\underline{\mathrm{End}}_R(M)$ is isomorphic to $\mathrm{Tor}_1^R(\mathrm{tr}(M), M)$ as an $\underline{\mathrm{End}}_R(M)$-bimodule.

To complete the proof of Proposition (A.10) we need a lemma.

LEMMA (A.13). *Let R be an isolated singularity of dimension d with the canonical module K and let M be an* MCM *module over R. If we denote the ring $\underline{\mathrm{End}}_R(M)$ by A, then there is an isomorphism of A-bimodules*; $\mathrm{Ext}_R^d(A, K) \cong \mathrm{Ext}_R^1(M, (\mathrm{Syz}_d \mathrm{tr}(M))')$.

PROOF. We have the following two spectral sequences converging to the same module;

$$'\mathrm{E}_2^{pq} = \mathrm{Ext}_R^p(\mathrm{tr}(M), \mathrm{Ext}_R^q(M, K)) \Rightarrow H_n$$
$$''\mathrm{E}_2^{pq} = \mathrm{Ext}_R^p(\mathrm{Tor}_q^R(\mathrm{tr}(M), M), K) \Rightarrow H_n .$$

Since we have that $\mathrm{Ext}_R^q(M, K)=0$ for $q>0$ by the local duality, we see that $'\mathrm{E}_2^{pq}=0$ for $q>0$ and it follows from this that $H_n = \mathrm{Ext}_R^n(\mathrm{tr}(M), M')$. On the other hand we know that $\mathrm{Tor}_q^R(\mathrm{tr}(M), M)$ has a finite length, since M_P is free over R_P for any non-maximal prime ideal P, for R being an isolated singularity. It therefore turns out that $''\mathrm{E}_2^{pq}=0$ for $q>0$ and $p \neq d$, hence we have that

$H_{d+1}=\mathrm{Ext}_R^d(\mathrm{Tor}_1^R(\mathrm{tr}(M), M), K)$ and this is isomorphic to $\mathrm{Ext}_R^d(A, K)$. (See (A.12) (iii).) By the above we showed that $\mathrm{Ext}^d(A, K)\cong\mathrm{Ext}^{d+1}(\mathrm{tr}(M), M')$. On the other hand, one sees inductively that $\mathrm{Ext}^{d+1}(\mathrm{tr}(M), M')\cong\mathrm{Ext}^d(\mathrm{Syz}_1\mathrm{tr}(M), M')$ $\cong\mathrm{Ext}^{d-1}(\mathrm{Syz}_2\mathrm{tr}(M), M')\cong\cdots\cong\mathrm{Ext}^1(\mathrm{Syz}_d\mathrm{tr}(M), M')$ and this last one is isomorphic to $\mathrm{Ext}^1(M, (\mathrm{Syz}_d\mathrm{tr}(M))')$ by the duality. This completes the proof of the lemma.

Now we will be back to the proof of Proposition (A.10). Let R be an isolated singularity and let M be an indecomposable MCM module over R which is non-free. We want to show that there is the AR-sequence ending in M. Let A be the local ring $\underline{\mathrm{End}}_R(M)$ and let J be the Jacobson radical of A. Note that A is an Artin local ring. It is easily seen by the duality that the injective hull $\mathrm{E}_A(A/J)$ of A/J as a left A-module is isomorphic to $\mathrm{Ext}_R^d(A, K)$. It then follows from Lemma (A.13) that there is an isomorphism of left A-modules; $\mathrm{E}_A(A/J)\cong\mathrm{Ext}_R^1(M, (\mathrm{Syz}_d\mathrm{tr}(M))')$. Take an element s in $\mathrm{Ext}^1(M, (\mathrm{Syz}_d\mathrm{tr}(M))')$ which corresponds by this isomorphism to a generator of the socle of $\mathrm{E}_A(A/J)$. If we know that $(\mathrm{Syz}_d\mathrm{tr}(M))'$ is indecomposable, then the same argument in [**7**; Proof of Proposition (4.1)], together with the isomorphism given in Lemma (A.13), shows that the exact sequence s is the AR-sequence starting from $(\mathrm{Syz}_d\mathrm{tr}(M))'$.

Now it remains to prove that $(\mathrm{Syz}_d\mathrm{tr}(M))'$ is indecomposable. It suffices to prove that $\mathrm{Syz}_d\mathrm{tr}(M)$ is indecomposable. We assume that $d\geqq 2$. (In the case $d\leqq 1$ it is rather easy to prove this and it is left to the reader.) In this case R is a normal domain and any MCM modules are reflexive. It is also known by Auslander-Bridger ([**6**] or [**12**]) that for any finitely generated module N it is an MCM module if and only if $\mathrm{Ext}_R^i(\mathrm{tr}(N), R)=0$ for $1\leqq i\leqq d$. We first claim that (*) $\mathrm{Ext}^i((\mathrm{Syz}_d\mathrm{tr}(M))^*, R)=0$ for $1\leqq i\leqq d-2$. In fact this follows from the fact that $\mathrm{Syz}_d\mathrm{tr}(M)$ is the d-th syzygy module of $\mathrm{tr}(M)$ and that $\mathrm{Ext}^i(\mathrm{tr}(M), R)=0$ for $1\leqq i\leqq d$. Now supposed that $\mathrm{Syz}_d\mathrm{tr}(M)$ is decomposed into $X_1\oplus X_2$. If the projective dimension $\mathrm{pd}(X_i^*)$ is less than $d-1$, then $\mathrm{Ext}^i(X_i^*, R)\neq 0$ for some $1\leqq i\leqq d-2$ and this leads the contradiction to (*). Thus both X_i^* have projective dimension $\geqq d-1$. In particular $\mathrm{Syz}_j(X_i^*)$, $1\leqq j\leqq d-1$, are non-trivial. Since $\mathrm{Ext}^i(\mathrm{tr}(M), R)=0$, $1\leqq i\leqq d$, it follows that $(\mathrm{tr}(M))^*$ is the d-th syzygy module of $(\mathrm{Syz}_d\mathrm{tr}(M))^*$ and thus by the exact sequence; $0\to(\mathrm{tr}(M))^*\to F_1\to F_0\to M\to 0$ we have that M is the $(d-2)$-th syzygy module of $(\mathrm{Syz}_d\mathrm{tr}(M))^*$. Consequently one obtains that $M\cong\mathrm{Syz}_{d-2}((\mathrm{Syz}_d\mathrm{tr}(M))^*)\cong\mathrm{Syz}_{d-2}(X_1^*)\oplus\mathrm{Syz}_{d-2}(X_2^*)$ which contradicts the indecomposability of M. This contradiction shows that $\mathrm{Syz}_d\mathrm{tr}(M)$ is indecomposable, and the proof is completed.

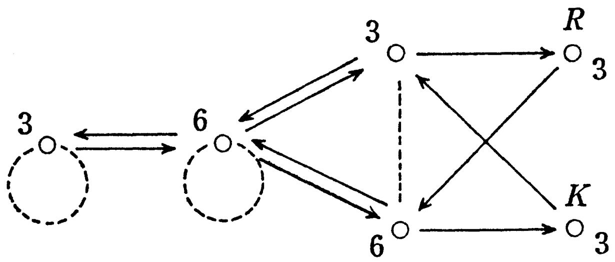
In this proof we have proved the following

REMARK (A.14). If R is an isolated singularity, then $C(R)$ admits AR-sequences and the AR-translation τ is given by $\tau(M)=(\mathrm{Syz}_d\mathrm{tr}(M))'$ for any indecomposable MCM module M.

DEFINITION (A.15). (a) Let M and N be indecomposable MCM modules over a Cohen-Macaulay Hensel local ring. A homomorphism f from M to N is said to be irreducible if the following conditions hold: (1) f is not an isomorphism, and (2) given any factorization $f=g\cdot h$ in the category $C(R)$, g is a split epimorphism or h is a split monomorphism.

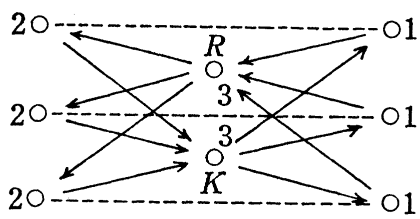
(b) The AR-graph of $C(R)$ (or simply the AR-graph of R) is a directed graph which has as vertices the isomorphic classes of indecomposable MCM modules over R, and there is an arrow from the isomorphic class of M to that of N provided there is an irreducible morphism from M to N.

EXAMPLE (A.16). (a) If R is a reduced local ring of dimension 1 given by $k\{x, y, z\}/(x^3-yz, y^2-xz, z^2-x^2y)$, then the AR-graph of R is given as follows;



where each integer indicates the multiplicities of modules corresponding to each vertex and the dotted lines show the AR-translation.

(b) If $R=k\{x, y, z\}/(xy, yz, zx)$, then the AR-graph is given by



(c) Other examples of AR-graphs of analytic algebras can be found in Auslander-Reiten [**9**], Dieterich-Wiedemann [**11**], Knörrer [**19**] and Yoshino-Kawamoto [**29**].

The following is proved in [8; 2.4].

LEMMA (A.17). *Let* $0\to N\xrightarrow{q}E\xrightarrow{p}M\to 0$ *be the* AR-*sequence ending in* M *and let* f *be any irreducible morphism from* L *to* M *(resp.* N*) where all modules are in* $C(R)$*. Then there is a split monomorphism (resp. a split epimorphism)* g *from* L *(resp.* E*) into* E *(resp.* L*) such that* $f=p\cdot g$ *(resp.* $f=g\cdot q$*). Thus if there is an* AR-*sequence ending in* M *(resp. starting from* N*), then only a finite number* *of arrows in the* AR-*graph ending in the vertex M (resp. starting from the vertex N) can occur.*

Combining this with the previous proposition we have

PROPOSITION (A.18). *Let R be an isolated singularity and let Γ be the* AR-*graph of R. Then Γ is a locally finite graph, that is, each vertex may be incident to only a finite number of other vertices.*

PROOF. If M is an indecomposable MCM module which is not isomorphic to R (resp. K), then arrows ending in M (resp. starting from N) are finite by Proposition (A.10) and Lemma (A.17). The problem is to show the finiteness of arrows ending in R and arrows starting from K. By the duality it is sufficient to show the finiteness of arrows ending in R. However this is a direct consequence of the following fact proved by Auslander [**4**]:

FACT (A.19). *Let M be any finitely generated module (not necessary* MCM). *Then there is a homomorphism f from an* MCM *module L to M such that any homomorphism from any* MCM *module to M is factored through f.*

In fact by this fact one sees that there is an MCM module L and a homomorphism f from L to $\mathfrak{m}$ factoring any homomorphism from any MCM module to $\mathfrak{m}$. Decomposing $L=\sum L_i$ with L_i indecomposable, then this means that these L_i are just the set of indecomposable modules from which there are irreducible morphisms to R. Hence they are finite.

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