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A family of solutions of a nonlinear ordinary differential equation and its application to Painlevé equations (III), (V) and (VI)

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In the neighbourhood of a fixed singular point of the regular type, Painlevé equations (III), (V) and (VI) are transformed into an equation of the form

$$\text{(E)}\quad x(xu')' = F_0(x, e^{-u}, xe^{u})+F_1(x, e^{-u}, xe^{u})(xu')+F_2(x, e^{-u}, xe^{u})(xu')^2 \quad ('=d/dx).$$

Here $F_j(x, \xi, \eta)$'s ($j=0, 1, 2$) are holomorphic and bounded in the polydisk

$$(0.1)\qquad |x| < r_0, \quad |\xi| < r_1, \quad |\eta| < r_1 \qquad (r_0, r_1>0)$$

and satisfy

$$(0.2)\qquad F_j(0, 0, 0) = 0$$

(cf. [6]). In [6] and [7], we constructed a two-parameter family of solutions of equation (E), and obtained families of solutions of Painlevé equations (III), (V) and (VI) expanded into convergent series near the fixed singular point of the regular type. In particular, for the sixth Painlevé equation (VI), our series expansion represents part of solutions studied by R. Garnier [1]. Using Hamiltonian systems (given by K. Okamoto [5]), H. Kimura [3] and K. Takano [9] obtained families of solutions of Painlevé equations (III), (V) and (VI), of which the expressions are different from ours.

In this paper, we show that the series expansion of solutions of equation (E) obtained in [6] and [7] converges in a larger domain, and we give some analytic representations of solutions of Painlevé equations (III), (V) and (VI) near the fixed singular point of the regular type. In some special cases, our representations of solutions of Painlevé equations are valid in the domain where the solutions have infinitely many movable poles and zeroes.

In Section 1, we explain the notation used in this paper. Our main theorems concerning equation (E) are stated in Section 2. In Section 3, preliminary propositions are proved, and in Section 4, using these propositions, we prove the main theorems. We construct a formal series of solutions by iteration, and show the convergence of it using a kind of majorant series. In the final section, we apply the main theorems to Painlevé equations (III), (V) and (VI) and obtain analytic representations of solutions near the fixed singular point.

§ 1. Notation.

Throughout this paper, we use the following notation.

1) $\mathcal{R}_0$ denotes the universal covering of $\boldsymbol{C}-\{0\}$.

2) $\Omega_0=\boldsymbol{C}-(\{\omega\in\boldsymbol{R};\ \omega\leqq 0\}\cup\{\omega\in\boldsymbol{R};\ \omega\geqq 1\})$.

3) Let ε be an arbitrary constant satisfying $0<\varepsilon<1/2$. Then $\Omega(\varepsilon)$ denotes a domain defined by

$$\{\omega\in\boldsymbol{C}\ ;\ \varepsilon<\operatorname{Re}\omega<1-\varepsilon\}\cup\left\{\omega\in\boldsymbol{C};\ |\operatorname{Im}\omega|>\varepsilon\left|\operatorname{Re}\omega-\frac{1}{2}\right|\right\}.$$

Let Ω be an arbitrary domain included in Ω_0, and let r be an arbitrary positive constant.

4) $\Delta(\Omega, r)$ denotes a domain defined by

$$\{(\omega,\kappa,x)\in\Omega\times\boldsymbol{C}\times\mathcal{R}_0\ ;\ |x|<r,\ |e^{-\kappa}x^{\omega}|<r^{1/2},\ |e^{\kappa}x^{1-\omega}|<r^{1/2}\}.$$

5) $\mathfrak{N}$ denotes a set of formal series ϕ expressed as

$$\phi=\sum_{i\geq 1}a_i(\omega)x^i+\sum_{\substack{i\geq 0\\ j\geq 1}}b_{ij}(\omega)x^i(e^{-\kappa}x^{\omega})^j+\sum_{\substack{i\geq 0\\ j\geq 1}}c_{ij}(\omega)x^i(e^{\kappa}x^{1-\omega})^j, \tag{1.1}$$

where $\omega\in\Omega_0$ and $\kappa\in\boldsymbol{C}$ are complex parameters and the coefficients $a_i(\omega)$'s, $b_{ij}(\omega)$'s and $c_{ij}(\omega)$'s belong to $\boldsymbol{C}(\omega)$ (i. e. rational function field in ω).

6) Let ϕ be an element of $\mathfrak{N}$ expressed as (1.1). Then $T:\mathfrak{N}\to\mathfrak{N}$ denotes an operator defined by

$$T[\phi]=\sum_{i\geq 1}\frac{a_i(\omega)}{i}x^i+\sum_{\substack{i\geq 0\\ j\geq 1}}\frac{b_{ij}(\omega)}{i+\omega j}x^i(e^{-\kappa}x^{\omega})^j+\sum_{\substack{i\geq 0\\ j\geq 1}}\frac{c_{ij}(\omega)}{i+(1-\omega)j}x^i(e^{\kappa}x^{1-\omega})^j. \tag{1.2}$$

7) Let ϕ be an element of $\mathfrak{N}$ expressed as (1.1). Then $\|\phi\|$ denotes a function of $(\omega, x)\in\Omega_0\times\boldsymbol{C}$ (which is not necessarily finite valued) defined by

$$\|\phi\|=\sum_{i\geq 1}|a_i(\omega)||x|^i+\sum_{\substack{i\geq 0\\ j\geq 1}}(|b_{ij}(\omega)|+|c_{ij}(\omega)|)|x|^{i+(j/2)}.$$

8) $\mathfrak{N}(\Omega, r)=\{\phi\in\mathfrak{N}\ ;\ \sup\{\|\phi\|\ ;\ |x|<r,\ \omega\in\Omega\}<\infty\}$.

9) $\mathfrak{S}(\Omega, r)$ denotes a set of functions $s(\omega, x, \xi, \eta)$ with the following properties:

a) $s(\omega, x, \xi, \eta)$ is a holomorphic and bounded function of (ω, x, ξ, η) in the domain

$$\{(\omega,x,\xi,\eta)\in\boldsymbol{C}^4\ ;\ \omega\in\Omega,\ |x|<r,\ |\xi|<r^{1/2},\ |\eta|<r^{1/2}\}, \tag{1.3}$$

b) $s(\omega, x, \xi, \eta)$ is represented by a convergent series

(1.4) $$s(\omega, x, \xi, \eta) = \sum_{i\geq 1} A_i(\omega)x^i + \sum_{\substack{i\geq 0\\ j\geq 1}} B_{ij}(\omega)x^i\xi^j + \sum_{\substack{i\geq 0\\ j\geq 1}} C_{ij}(\omega)x^i\eta^j$$

in domain (1.3), where the coefficients $A_i(\omega)$'s, $B_{ij}(\omega)$'s and $C_{ij}(\omega)$'s belong to $\boldsymbol{C}(\omega)$.

§ 2. Main results.

THEOREM 2.1. *Let Ω' be an arbitrary bounded domain of which the closure is included in Ω_0. Then, for a sufficiently small positive constant $r'=r'(\Omega')$, equation* (E) *admits a family of solutions $\{u(\omega, \kappa; x);\ \omega\in\Omega', \kappa\in\boldsymbol{C}\}$ such that*

i) *$u(\omega, \kappa; x)$ is a holomorphic function of (ω, κ, x) in $\Delta(\Omega', r')$,*

ii) *$u(\omega, \kappa; x)$ is expressed as*

(2.1) $$u(\omega, \kappa; x) = -\omega \log x + \kappa + \sigma(\omega, x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega})$$

in $\Delta(\Omega', r')$, where $\sigma(\omega, x, \xi, \eta)$ is some element of $\mathfrak{S}(\Omega', r')$.

THEOREM 2.2. *Assume that $F_1(x, \xi, \eta)=F_2(x, \xi, \eta)\equiv 0$. Let $\varepsilon(<1/2)$ be an arbitrary small positive constant. Then, for a sufficiently small positive constant $\tilde{r}=\tilde{r}(\varepsilon)$, equation* (E) *admits a family of solutions $\{u(\omega, \kappa; x);\ \omega\in\Omega(\varepsilon), \kappa\in\boldsymbol{C}\}$ such that*

i) *$u(\omega, \kappa; x)$ is a holomorphic function of (ω, κ, x) in $\Delta(\Omega(\varepsilon), \tilde{r})$,*

ii) *$u(\omega, \kappa; x)$ is expressed as*

(2.2) $$u(\omega, \kappa; x) = -\omega \log x + \kappa + \tilde{\sigma}(\omega, x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega})$$

in $\Delta(\Omega(\varepsilon), \tilde{r})$, where $\tilde{\sigma}(\omega, x, \xi, \eta)$ is some element of $\mathfrak{S}(\Omega(\varepsilon), \tilde{r})$.

From Theorem 2.1, we immediately obtain

COROLLARY 2.3. *For each $(\omega, \kappa)\in\Omega_0\times\boldsymbol{C}$, equation* (E) *admits a holomorphic solution $u(\omega, \kappa; x)$ expressed as* (2.1) *in the domain defined by*

$$\{x\in\mathfrak{R}_0\ ;\ |x|<r_0',\ |e^{-\kappa}x^{\omega}|<r_0'^{1/2},\ |e^{\kappa}x^{1-\omega}|<r_0'^{1/2}\},$$

where $r_0'=r_0'(\omega)$ is a sufficiently small positive constant.

REMARK 2.1. For every positive constant r'' $(<r')$ (resp. r'' $(<\tilde{r})$), the series $\sigma(\omega, x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega})$ (resp. $\tilde{\sigma}(\omega, x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega})$) in (2.1) (resp. in (2.2)) converges absolutely and uniformly in $\Delta(\Omega', r'')$ (resp. in $\Delta(\Omega(\varepsilon), r'')$) and satisfies

$$\sigma(\omega, x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega}) = O(|x|+|e^{-\kappa}x^{\omega}|+|e^{\kappa}x^{1-\omega}|),$$

$$(\text{resp. } \tilde{\sigma}(\omega, x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega}) = O(|x|+|e^{-\kappa}x^{\omega}|+|e^{\kappa}x^{1-\omega}|))$$

uniformly in the same domain (cf. the definition of $\mathfrak{S}(\Omega, r)$).

§3. Preliminaries.

Throughout this paragraph, Ω denotes a domain included in Ω_0, and r denotes a positive constant.

3.1. First we have

PROPOSITION 3.1. 1) *$\mathfrak{R}$ is a ring.*

2) *An element $\phi\in\mathfrak{R}(\Omega, r)$ represents a holomorphic function of (ω, κ, x) in $\Delta(\Omega, r)$, and satisfies $\|\phi\|=O(|x|^{1/2})$ uniformly for $|x|<r$, $\omega\in\Omega$. Moreover ϕ can be differentiated with respect to (ω, κ, x) term by term in $\Delta(\Omega, r)$.*

3) *For each $\phi\in\mathfrak{R}(\Omega, r)$, there exists an element $s(\omega, x, \xi, \eta)\in\mathfrak{S}(\Omega, r)$ such that $\phi=s(\omega, x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega})$.*

4) *For every positive constant $\tilde{r}$ $(<r)$, each element $s(\omega, x, \xi, \eta)\in\mathfrak{S}(\Omega, r)$ satisfies $s(\omega, x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega})\in\mathfrak{R}(\Omega, \tilde{r})$.*

PROOF. Let χ and ψ be elements of $\mathfrak{R}$. Clearly $\chi+\psi\in\mathfrak{R}$. Note that, for every pair $(I, J)\in(\boldsymbol{N}\cup\{0\})^2-\{(0, 0)\}$, the number of triples $(\alpha, \beta, \gamma)\in(\boldsymbol{N}\cup\{0\})^3$ satisfying $x^{\alpha}(e^{-\kappa}x^{\omega})^{\beta}(e^{\kappa}x^{1-\omega})^{\gamma}=x^I(e^{-\kappa}x^{\omega})^J$ or $=x^I(e^{\kappa}x^{1-\omega})^J$ is finite. This implies that $\chi\psi\in\mathfrak{R}$. Thus assertion 1) is proved. Let ϕ be an element of $\mathfrak{R}(\Omega, r)$ expressed as (1.1). Then, from the definition of $\mathfrak{R}(\Omega, r)$, we obtain estimates $|a_i(\omega)|\leqq Mr^{-i}$ $(i\geqq 1)$, $|b_{ij}(\omega)|\leqq Mr^{-i-(j/2)}$, $|c_{ij}(\omega)|\leqq Mr^{-i-(j/2)}$ $(i\geqq 0, j\geqq 1)$ for $\omega\in\Omega$, where $M=\sup\{\|\phi\|; |x|<r, \omega\in\Omega\}$. Hence, for every positive constant $\tilde{r}<r$, the series $\phi\in\mathfrak{R}(\Omega, r)$ converges absolutely and uniformly in $\Delta(\Omega, \tilde{r})$, and satisfies $\|\phi\|=O(|x|^{1/2})$ uniformly for $|x|<r$, $\omega\in\Omega$. Using Weierstrass's theorem, we can easily verify assertion 2). From assertion 2) and the definition of $\mathfrak{R}(\Omega, r)$, assertion 3) follows immediately. Let $s(\omega, x, \xi, \eta)$ be an element of $\mathfrak{S}(\Omega, r)$ represented by (1.4). By the boundedness of s in domain (1.3), estimates $|A_i(\omega)|\leqq Lr^{-i}$ $(i\geqq 1)$, $|B_{ij}(\omega)|\leqq Lr^{-i-(j/2)}$, $|C_{ij}(\omega)|\leqq Lr^{-i-(j/2)}$ $(i\geqq 0, j\geqq 1)$ are valid uniformly for $\omega\in\Omega$, where L is some positive constant independent of i and j. From this fact, assertion 4) follows.

PROPOSITION 3.2. *Let ϕ and ψ be elements of $\mathfrak{R}(\Omega, r)$.*

0) *If $\|\phi\|\equiv 0$ for $|x|<r$, $\omega\in\Omega$, then $\phi=0$.*

1) *If $c(\omega)\in\boldsymbol{C}(\omega)$, then $\|c(\omega)\phi\|=|c(\omega)|\,\|\phi\|$ for $|x|<r$, $\omega\in\Omega$.*

2) *$\|\phi+\psi\|\leqq\|\phi\|+\|\psi\|$ for $|x|<r$, $\omega\in\Omega$.*

3) *$\|\phi\psi\|\leqq\|\phi\|\|\psi\|$ for $|x|<r$, $\omega\in\Omega$.*

PROOF. Assertions 0), 1) and 2) are clear. Assertion 3) follows immediately from the facts that, for $\phi\in\mathfrak{R}(\Omega, r)$ expressed as (1.1),

$$\|\phi\| = \sum_{i\geqq 1}\|a_i(\omega)x^i\|+\sum_{\substack{i\geqq 0\\ j\geqq 1}}(\|b_{ij}(\omega)x^i(e^{-\kappa}x^{\omega})^j\|+\|c_{ij}(\omega)x^i(e^{\kappa}x^{1-\omega})^j\|),$$

and that, for every quadruplet $(i, j, k, l)\in(\boldsymbol{N}\cup\{0\})^4$,

$$\|x^i(e^{-\kappa}x^{\omega})^j\cdot x^k(e^{\kappa}x^{1-\omega})^l\|=\|x^i(e^{-\kappa}x^{\omega})^j\|\|x^k(e^{\kappa}x^{1-\omega})^l\|.$$

Let $f(y_1, \cdots, y_n)$ be a power series

$$f(y_1, \cdots, y_n)=\sum_{p,\cdots,q\geqq 0}c_{p\cdots q}y_1^p\cdots y_n^q$$

$(c_{p\cdots q}\in\boldsymbol{C})$ which converges for $|y_\nu|<R_\nu$ $(1\leqq\nu\leqq n)$ and satisfies $c_{0\cdots 0}=0$. Then we have

PROPOSITION 3.3. 1) *If* $\phi_\nu\in\mathfrak{R}$ $(1\leqq\nu\leqq n)$, *then* $f(\phi_1, \cdots, \phi_n)\in\mathfrak{R}$.

2) *Assume that* ϕ_ν*'s* $(\in\mathfrak{R}(\Omega, r))$ $(1\leqq\nu\leqq n)$ *satisfy* $\|\phi_\nu\|<R_\nu$ *for* $|x|<r$, $\omega\in\Omega$. *Then*

$$\|f(\phi_1, \cdots, \phi_n)\|\leqq|f|(\|\phi_1\|, \cdots, \|\phi_n\|)$$

for $|x|<r$, $\omega\in\Omega$, *where*

$$|f|(y_1, \cdots, y_n)=\sum_{p,\cdots,q\geqq 0}|c_{p\cdots q}|y_1^p\cdots y_n^q.$$

PROOF. Note that, for every n-tuple $(p, \cdots, q)\in(\boldsymbol{N}\cup\{0\})^n-\{(0, \cdots, 0)\}$, $c_{p\cdots q}\phi_1^p\cdots\phi_n^q\in\mathfrak{R}$ and that the lowest term of the series $\|c_{p\cdots q}\phi_1^p\cdots\phi_n^q\|$ is of order $O(|x|^{(p+\cdots+q)/2})$ (as $|x|\to 0$). Hence, for every pair $(I, J)\in(\boldsymbol{N}\cup\{0\})^2-\{(0, 0)\}$, the number of n-tuples $(p, \cdots, q)$ such that $c_{p\cdots q}\phi_1^p\cdots\phi_n^q$ contains the term $x^I(e^{-\kappa}x^{\omega})^J$ or the term $x^I(e^{\kappa}x^{1-\omega})^J$ is finite. From this fact and the condition $c_{0\cdots 0}=0$, assertion 1) follows immediately. Next assume that $\|\phi_\nu\|<R_\nu$ for $|x|<r$, $\omega\in\Omega$. Then, using assertion 1) and Proposition 3.2, we have

$$\begin{aligned}\|f(\phi_1, \cdots, \phi_n)\|&\leqq\sum_{p,\cdots,q\geqq 0}|c_{p\cdots q}|\|\phi_1\|^p\cdots\|\phi_n\|^q\\&=|f|(\|\phi_1\|, \cdots, \|\phi_n\|).\end{aligned}$$

3.2. We consider an operator $T:\mathfrak{R}\to\mathfrak{R}$ defined by (1.2). Clearly T is a $\boldsymbol{C}(\omega)$-linear operator. By Proposition 3.1, 2), we have

PROPOSITION 3.4. *If* $T[\phi]\in\mathfrak{R}(\Omega, r)$, *then* $(d/dx)T[\phi]=x^{-1}\phi$ *in* $\Delta(\Omega, r)$.

Furthermore we have

PROPOSITION 3.5. *Let* ε $(<1/2)$ *be an arbitrary small positive constant. Then every element* $\phi\in\mathfrak{R}(\Omega(\varepsilon), r)$ *satisfies*

$$\|T[\phi]\|\leqq 2\varepsilon^{-1}\int_0^{|x|}t^{-1}\|\phi\|(t)dt$$

for $|x|<r$, $\omega\in\Omega(\varepsilon)$, *where* $\|\phi\|(t)$ *denotes a series obtained from* $\|\phi\|$ *by replacing* $|x|$ *with* t.

In the proof of this proposition, we use the following lemma which will be proved afterward.

LEMMA 3.6. *We have*

$$\lambda(i, j)=\frac{i+(j/2)}{|i+\omega j|}\leqq 2\varepsilon^{-1}, \quad \mu(i, j)=\frac{i+(j/2)}{|i+(1-\omega)j|}\leqq 2\varepsilon^{-1}$$

for every $(i, j)\in(\boldsymbol{N}\cup\{0\})\times\boldsymbol{N}$ *and for every* $\omega\in\Omega(\varepsilon)$.

PROOF OF PROPOSITION 3.5. By Lemma 3.6, we have

$$\frac{|x|^{i+(j/2)}}{|i+\omega j|}\leqq 2\varepsilon^{-1}\int_0^{|x|}t^{i+(j/2)-1}dt \qquad (i\geqq 0,\ j\geqq 1),$$

$$\frac{|x|^{i+(j/2)}}{|i+(1-\omega)j|}\leqq 2\varepsilon^{-1}\int_0^{|x|}t^{i+(j/2)-1}dt \qquad (i\geqq 0,\ j\geqq 1)$$

for $|x|<r$, $\omega\in\Omega(\varepsilon)$. Assume that $\phi\in\mathfrak{R}$ is expressed as (1.1). Then, from these inequalities, we derive

$$\begin{aligned}\|T[\phi]\| &\leqq 2\varepsilon^{-1}\int_0^{|x|}t^{-1}\Big(\sum_{i\geqq 1}|a_i(\omega)|t^i+\sum_{\substack{i\geqq 0\\ j\geqq 1}}(|b_{ij}(\omega)|+|c_{ij}(\omega)|)t^{i+(j/2)}\Big)dt\\ &=2\varepsilon^{-1}\int_0^{|x|}t^{-1}\|\phi\|(t)dt\end{aligned}$$

for $|x|<r$, $\omega\in\Omega(\varepsilon)$. This completes the proof.

PROOF OF LEMMA 3.6. Assume that $\omega\in\Omega(\varepsilon)$. If $\varepsilon<\operatorname{Re}\omega<1-\varepsilon$, then, for $(i, j)\in(\boldsymbol{N}\cup\{0\})\times\boldsymbol{N}$, we have

$$\lambda(i, j),\ \mu(i, j)\leqq\frac{i+(j/2)}{i+\varepsilon j}=\frac{(i/j)+(1/2)}{(i/j)+\varepsilon}\leqq(2\varepsilon)^{-1}.$$

Next consider the case where ω satisfies $|\operatorname{Im}\omega|>\varepsilon|\operatorname{Re}\omega-(1/2)|$. Note that, for every $a\in\boldsymbol{R}-\{0\}$,

$$a\Big(\omega-\frac{1}{2}\Big)\in\Omega_1(\varepsilon)=\{\tau;\ |\operatorname{Im}\tau|>\varepsilon|\operatorname{Re}\tau|\}.$$

Using this fact, we have

$$\lambda(i, j)=\left|1+\frac{\omega-(1/2)}{(i/j)+(1/2)}\right|^{-1}\leqq(\operatorname{dist}(\{-1\}, \Omega_1(\varepsilon)))^{-1}<2\varepsilon^{-1},$$

$$\mu(i, j)=\left|1+\frac{(1/2)-\omega}{(i/j)+(1/2)}\right|^{-1}\leqq(\operatorname{dist}(\{-1\}, \Omega_1(\varepsilon)))^{-1}<2\varepsilon^{-1},$$

for $(i, j)\in(\boldsymbol{N}\cup\{0\})\times\boldsymbol{N}$. Thus we obtain the lemma.

§ 4. Proof of main results.

We may assume that the constants r_0 and r_1 in (0.1) satisfy

$$\rho_0 = \log(r_1 r_0^{-1/2}) > 0. \tag{4.1}$$

4.1. Using (0.2), by a simple computation, we have

PROPOSITION 4.1. *For* $(\omega, \kappa, x) \in \Delta(\Omega_0, r_0)$, *by the transformation* $u = -\omega \log x + \kappa + v$, *equation* (E) *is changed into an equation of the form*

$$x(xv')' = H(x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega}) + K(x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega}, v, xv') \tag{E$'$}$$

with the following properties:

1) $H(x, \xi, \eta)$ *is holomorphic for* $|x|<r_0$, $|\xi|<r_0^{1/2}$, $|\eta|<r_0^{1/2}$, *and* $K(x, \xi, \eta, v, w)$ *is holomorphic for* $|x|<r_0$, $|\xi|<r_0^{1/2}$, $|\eta|<r_0^{1/2}$, $|v|<\rho_0$, $|w|<\rho_0$,

2) $H(0, 0, 0)=0$, $K(x, \xi, \eta, 0, 0)\equiv 0$ *and* $K(0, 0, 0, v, w)\equiv 0$.

REMARK 4.1. H and K are written as

$$H(x, \xi, \eta) = G_0(x, \xi, \eta),$$

$$K(x, \xi, \eta, v, w) = G_0(x, \xi e^{-v}, \eta e^{v}) - G_0(x, \xi, \eta) + G_1(x, \xi e^{-v}, \eta e^{v})w + G_2(x, \xi e^{-v}, \eta e^{v})w^2,$$

where $G_0 = F_0 - \omega F_1 + \omega^2 F_2$, $G_1 = F_1 - 2\omega F_2$, $G_2 = F_2$.

We have

$$K(x, \xi, \eta, v_2, w_2) - K(x, \xi, \eta, v_1, w_1)$$
$$= (v_2 - v_1)K_1(x, \xi, \eta, v_1, v_2, w_1, w_2) + (w_2 - w_1)K_2(x, \xi, \eta, v_1, v_2, w_1, w_2)$$

for $(x, \xi, \eta, v_1, v_2, w_1, w_2)$ satisfying

$$|x| < r_0, \quad |\xi| < r_0^{1/2}, \quad |\eta| < r_0^{1/2}, \quad |v_i| < \rho_0, \quad |w_i| < \rho_0 \quad (i=1, 2). \tag{4.2}$$

Here $K_i(x, \xi, \eta, v_1, v_2, w_1, w_2)$'s $(i=1, 2)$ are holomorphic in domain (4.2), and satisfy $K_i(0, 0, 0, v_1, v_2, w_1, w_2)\equiv 0$ $(i=1, 2)$. Note that K and K_i's are polynomials in ω of degree 2. Under the assumption that Ω'' is bounded, applying Propositions 3.2 and 3.3 to each coefficient of ω^i $(i=0, 1, 2)$, we have

PROPOSITION 4.2. *Let* Ω'' $(\subset \Omega_0)$ *be a bounded domain and* r *be an arbitrary positive constant satisfying* $r<r_0/2$. *If* $\phi_i, \psi_i \in \mathfrak{R}(\Omega'', r)$ $(i=1, 2)$ *satisfy* $\|\phi_i\|<\rho_0/2$, $\|\psi_i\|<\rho_0/2$ *for* $|x|<r$, $\omega \in \Omega''$, *then*

$$\|K(x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega}, \phi_2, \psi_2) - K(x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega}, \phi_1, \psi_1)\|$$
$$\leqq c|x|^{1/2}(\|\phi_2 - \phi_1\| + \|\psi_2 - \psi_1\|)$$

uniformly for $|x|<r$, $\omega\in\Omega''$, *where* $c=c(\Omega'')$ (>0) *is some constant independent of* r $(<r_0/2)$.

Since H is a polynomial in ω of degree 2, we have

PROPOSITION 4.3. *Under the same assumptions as above, we have*

$$\|H(x,\ e^{-\kappa}x^{\omega},\ e^{\kappa}x^{1-\omega})\| \leqq c_0|x|^{1/2}$$

uniformly for $|x|<r$, $\omega\in\Omega''$, *where* $c_0=c_0(\Omega'')$ (>0) *is some constant independent of* r $(<r_0/2)$.

REMARK 4.2. In case $F_1=F_2\equiv 0$, the functions H and K_i's $(i=1, 2)$ are independent of ω (cf. Remark 4.1), and, in Propositions 4.2 and 4.3, the assumption that Ω'' is bounded is unnecessary. Therefore, in such a case, a bounded domain Ω'' $(\subset\Omega_0)$ in these propositions can be replaced by an arbitrary domain Ω $(\subset\Omega_0)$, and the constants c and c_0 can be taken independently of Ω.

4.2. Consider equation (E′) (cf. Proposition 4.1) and the corresponding formal integral equations

$$xv' = T[H(x,\ e^{-\kappa}x^{\omega},\ e^{\kappa}x^{1-\omega})+K(x,\ e^{-\kappa}x^{\omega},\ e^{\kappa}x^{1-\omega},\ v,\ xv')]\,,$$
$$v = T[xv']\,.$$

By Propositions 4.1 and 3.3, 1), we can define sequences $\{v_n(x)\in\mathfrak{R};\ n\geqq 0\}$, $\{w_n(x)\in\mathfrak{R};\ n\geqq 0\}$ recursively by

$$v_0(x) = w_0(x) = 0\,,$$

$$(4.3)\quad \begin{cases} w_n(x) = T[H(x,\ e^{-\kappa}x^{\omega},\ e^{\kappa}x^{1-\omega})]+T[K(x,\ e^{-\kappa}x^{\omega},\ e^{\kappa}x^{1-\omega},\ v_{n-1}(x),\ w_{n-1}(x))], \\ v_n(x) = T[w_n(x)] \qquad (n\geqq 1)\,. \end{cases}$$

Furthermore we put, for $n\geqq 1$,

$$(4.4)\qquad V_n(x) = v_n(x)-v_{n-1}(x)\,,\quad W_n(x) = w_n(x)-w_{n-1}(x)\,.$$

Let Ω' be an arbitrary bounded domain of which the closure is included in Ω_0. Clearly there exists a positive constant $\varepsilon'=\varepsilon'(\Omega')$ $(<1/2)$ such that $\Omega'\subset\Omega(\varepsilon')$. Then estimates of these sequences are given by

PROPOSITION 4.4. *There exists a positive constant* $r'=r'(\Omega')$ *such that*

$$(4.5,\ n)\qquad \|v_n(x)\| < \rho_0/3,\quad \|w_n(x)\| < \rho_0/3 \qquad (n\geqq 0)\,,$$

$$(4.6,\ n)\qquad \|V_n(x)\| \leqq C_n|x|^{n/2},\quad \|W_n(x)\| \leqq C_n|x|^{n/2} \qquad (n\geqq 1)$$

$$(C_n=\tilde{c}_0(2\tilde{c})^{-1}(32\tilde{c}\varepsilon'^{-2})^n(n\,!)^{-1})\,,$$

$$(4.7) \qquad \sum_{n\geq 1} C_n |x|^{n/2} < \rho_0/4\,,$$

for $|x|\leqq r'$, $\omega\in\Omega'$. *Here* $\tilde{c}=c(\Omega')$ *and* $\tilde{c}_0=c_0(\Omega')$ *are some positive constants* (*cf. Propositions* 4.2 *and* 4.3).

We shall prove Proposition 4.4 afterward. Using these estimates, we prove Theorem 2.1.

PROOF OF THEOREM 2.1. By (4.6, n), in the expressions of $V_n(x)$, $W_n(x)$ ($\in\mathfrak{R}(\Omega', r')$), the coefficients of x^i's, $x^i(e^{-\kappa}x^{\omega})^j$'s and $x^i(e^{\kappa}x^{1-\omega})^j$'s vanish for i satisfying $0<i<n/2$, and for (i, j) satisfying $0<i+(j/2)<n/2$. Therefore, if we consider formal series $v(x)=\sum_{n\geq 1}V_n(x)$, $w(x)=\sum_{n\geq 1}W_n(x)$, then $v(x)\in\mathfrak{R}$, $w(x)\in\mathfrak{R}$. Furthermore, by (4.6, n) and (4.7),

$$(4.8) \qquad \|v(x)\| < \rho_0/3, \quad \|w(x)\| < \rho_0/3\,,$$

$$(4.9) \qquad \|v(x)-v_N(x)\| \leqq M_1|x|^{N/2}, \quad \|w(x)-w_N(x)\| \leqq M_1|x|^{N/2}$$

($N\geqq 1$) uniformly for $|x|<r'$, $\omega\in\Omega'$, where M_1 is some positive constant independent of N. By (4.8), (4.9) and Propositions 3.3, 4.1, 4.2 and 3.5 (with $\varepsilon=\varepsilon'(\Omega')$), we have

$$\|T[K(x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega}, v(x), w(x))]$$
$$-T[K(x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega}, v_{N-1}(x), w_{N-1}(x))]\| \leqq M_2|x|^{N/2}\,,$$

and hence

$$\|w_N(x)-T[H(x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega})]$$
$$-T[K(x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega}, v(x), w(x))]\| \leqq M_2|x|^{N/2}$$

uniformly for $|x|<r'$, $\omega\in\Omega'$, where M_2 is some positive constant independent of N. Combining this estimate with (4.9), and using (4.8) and Proposition 3.2, 0), we have

$$w(x) = T[H(\cdots)+K(\cdots, v(x), w(x))] \in \mathfrak{R}(\Omega', r')\,.$$

Similarly

$$v(x) = T[w(x)] \in \mathfrak{R}(\Omega', r')\,.$$

By Proposition 3.4, the function $v(x)$ satisfies

$$xv'(x) = w(x)\,,$$
$$x(xv'(x))' = H(\cdots)+K(\cdots, v(x), xv'(x))$$

($'=d/dx$) in $\Delta(\Omega', r')$, namely $v(x)$ is a solution of equation (E$'$). By Proposition 3.1, 3), there exists an element $\sigma(\omega, x, \xi, \eta)\in\mathfrak{S}(\Omega', r')$ such that $v(x)=\sigma(\omega, x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega})$. Thus we obtain a solution $u(\omega, \kappa; x)=-\omega\log x+\kappa+\sigma(\omega, x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega})$ of equation (E) with properties i) and ii) in Theorem 2.1.

4.3. It remains to prove Proposition 4.4.

PROOF OF PROPOSITION 4.4. Take a constant $r'=r'(\Omega')$ $(<r_0/2)$ so small that

$$\sum_{n\geq 1} C_n r'^{n/2} = \tilde{c}_0(2\tilde{c})^{-1}(\exp(32c\varepsilon'^{-2}r'^{1/2})-1) < \rho_0/4 .$$

Then (4.7) is valid for $|x|<r'$. We verify $(4.5, n)$, $(4.6, n)$ by induction on n. Using Propositions 4.3 and 3.5 (with $\varepsilon=\varepsilon'(\Omega')$), we have, for $|x|\leqq r'<r_0/2$, $\omega\in\Omega'$,

$$\|W_1(x)\| = \|w_1(x)\| = \|T[H(x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega})]\| \leqq 4\tilde{c}_0\varepsilon'^{-1}|x|^{1/2},$$

so that $w_1(x)=T[H(x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega})]\in\mathfrak{R}(\Omega', r')$. Using Proposition 3.5 again, we have

$$\|V_1(x)\| = \|v_1(x)\| = \|T[w_1(x)]\| \leqq 16\tilde{c}_0\varepsilon'^{-2}|x|^{1/2},$$

for $|x|\leqq r'$, $\omega\in\Omega'$. This implies that (4.5, 1) and (4.6, 1) are true (note that $4\varepsilon'^{-1}>1$). Suppose that $(4.5, n)$ and $(4.6, n)$ are true for $n\leqq N-1$. Using Propositions 3.5 and 4.2, we derive from (4.3) that $w_N(x)$, $v_N(x)\in\mathfrak{R}(\Omega', r')$, and that

$$\|W_N(x)\| \leqq 2\varepsilon'^{-1}\int_0^{|x|}\tilde{c}t^{-1/2}(\|V_{N-1}(x)\|(t)+\|W_{N-1}(x)\|(t))dt$$

$$\leqq 8\tilde{c}\varepsilon'^{-1}N^{-1}C_{N-1}|x|^{N/2}\leqq C_N|x|^{N/2},$$

$$\|V_N(x)\| \leqq 2\varepsilon'^{-1}\int_0^{|x|}t^{-1}\|W_N(x)\|(t)dt$$

$$\leqq 32\tilde{c}\varepsilon'^{-2}N^{-2}C_{N-1}|x|^{N/2}\leqq C_N|x|^{N/2}$$

for $|x|\leqq r'$, $\omega\in\Omega'$. This implies that $(4.5, N)$ and $(4.6, N)$ are true. Thus we have proved that $(4.5, n)$ and $(4.6, n)$ are true for $n\geqq 1$.

4.4. Let ε $(<1/2)$ be an arbitrary positive constant. In case $F_1=F_2\equiv 0$, the constants c and c_0 (in Propositions 4.2 and 4.3) can be taken independently of $\Omega(\varepsilon)$ (cf. Remark 4.2). Therefore, using Proposition 3.5, we obtain inequalities $(4.5, n)$, $(4.6, n)$ and (4.7) which are valid uniformly for $|x|\leqq\tilde{r}$, $\omega\in\Omega(\varepsilon)$, where $\tilde{r}=\tilde{r}(\varepsilon)$ is a sufficiently small positive constant. Using these inequalities, we can prove Theorem 2.2 in a similar way.

§ 5. Application to Painlevé equations (III), (V) and (VI).

5.1. Consider Painlevé equations

$$\text{(III)} \qquad y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{1}{x}(\alpha y^2+\beta)+\gamma y^3+\frac{\delta}{y},$$

(V) $$y''=\left(\frac{1}{2y}+\frac{1}{y-1}\right)y'^2-\frac{y'}{x}+\frac{(y-1)^2}{x^2}\left(\alpha y+\frac{\beta}{y}\right)+\frac{\gamma y}{x}+\frac{\delta y(y+1)}{y-1},$$

(VI) $$y''=\frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)y'^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)y'$$
$$+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha+\frac{\beta x}{y^2}+\frac{\gamma(x-1)}{(y-1)^2}+\frac{\delta x(x-1)}{(y-x)^2}\right).$$

These equations have a fixed singular point at $x=0$. By a straightforward computation, we have

PROPOSITION 5.1. *By $y=x^{-1}e^{-u}$, $x^2=t$, equation* (III) *is transformed into*

(5.1) $$t(tu')'=-\frac{1}{4}(\alpha e^{-u}+\beta te^{u}+\gamma e^{-2u}+\delta t^2e^{2u})\qquad \left('=\frac{d}{dt}\right).$$

By $y=\mathrm{th}^2(u/2)$, equation (V) *is transformed into*

(5.2) $$x(xu')'=\left(\alpha\,\mathrm{th}\frac{u}{2}+\beta\,\mathrm{th}^{-3}\frac{u}{2}\right)\mathrm{ch}^{-2}\frac{u}{2}+\frac{\gamma}{2}x\,\mathrm{sh}u-\frac{\delta}{4}x^2\mathrm{sh}2u\,.$$

By $y=\mathrm{ch}^{-2}(u/2)$, equation (VI) *is transformed into*

(5.3) $$x(xu')'=-\frac{xC(u)S(u)}{2(1-xC(u)^2)}(xu')^2-\left(\frac{x}{x-1}+\frac{xC(u)^2}{1-xC(u)^2}\right)(xu')$$
$$+\frac{(1-xC(u)^2)S(u)C(u)}{(x-1)^2}\left(\frac{\alpha}{C(u)^4}+\beta x+\frac{\gamma(x-1)}{S(u)^4}+\frac{\delta x(x-1)}{(1-xC(u)^2)^2}\right).$$

Here $C(u)=\mathrm{ch}(u/2)=(e^{u/2}+e^{-u/2})/2$, $S(u)=\mathrm{sh}(u/2)=(e^{u/2}-e^{-u/2})/2$, $\mathrm{th}(u/2)=(e^{u/2}-e^{-u/2})/(e^{u/2}+e^{-u/2})$.

REMARK 5.1. For equation (V) the transformation $y=\mathrm{th}^2(u/2)$ (or $y=\mathrm{th}^2u$) is used in [4] and [8].

Equations (5.1), (5.2) and (5.3) are written in the form (E). For example, equation (5.3) corresponds to equation (E) with F_j's given by

$$F_0(x,\xi,\eta)=\frac{(4-2x-x\xi-\eta)}{(x-1)^2}\left(\frac{\alpha\xi(1-\xi)}{(1+\xi)^3}+\frac{\beta(\eta-x\xi)}{16}\right.$$
$$\left.+\frac{\gamma(x-1)\xi(1+\xi)}{(1-\xi)^3}+\frac{\delta(x-1)(\eta-x\xi)}{(4-2x-x\xi-\eta)^2}\right),$$

$$F_1(x,\xi,\eta)=-\left(\frac{x}{x-1}+\frac{2x+x\xi+\eta}{4-2x-x\xi-\eta}\right),$$

$$F_2(x,\xi,\eta)=\frac{x\xi-\eta}{2(4-2x-x\xi-\eta)}\,.$$

$F_j(x,\xi,\eta)$'s are holomorphic for $|x|<1$, $|\xi|<1$, $|\eta|<1$, and satisfy $F_j(0,0,0)=0$.

Moreover F_j's in equations corresponding to (5.1) and (5.2) satisfy $F_1(x, \xi, \eta) = F_2(x, \xi, \eta) \equiv 0$.

Using Propositions 3.1 and 3.3, we easily have

PROPOSITION 5.2. *Let $g(z)$ be a function which satisfies $g(0)=0$ and is holomorphic for $|z|<R$, and let $s(\omega, x, \xi, \eta)$ be an element of $\mathfrak{S}(\Omega, r)$, where R and r are positive constants and Ω is a domain included in Ω_0. Then, for a sufficiently small positive constant r' $(<r)$, $g(s(\omega, x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega}))$ is a holomorphic function of (ω, κ, x) in $\Delta(\Omega, r')$ and is represented by*

$$g(s(\omega, x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega})) = s_0(\omega, x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega}) \tag{5.4}$$

in $\Delta(\Omega, r')$, where $s_0(\omega, x, \xi, \eta)$ is some element of $\mathfrak{S}(\Omega, r')$.

Applying Theorem 2.1 to equation (5.3), we obtain a solution $y_{\mathrm{VI}}(\omega, \kappa; x) = \mathrm{ch}^{-2}(u(\omega, \kappa; x)/2)$. Note that $\mathrm{ch}^{-2}(u/2) = 4e^{-u}(1+e^{-u})^{-2}$ and that $\exp(-u(\omega, \kappa; x)) = e^{-\kappa}x^{\omega}\exp(-\sigma(\omega, x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega}))$. Using Proposition 5.2, we have

THEOREM 5.3. *Let Ω' be an arbitrary bounded domain of which the closure is included in Ω_0. Then, for a sufficiently small positive constant $r_6 = r_6(\Omega')$, equation* (VI) *admits a family of solutions $\{y_{\mathrm{VI}}(\omega, \kappa; x); \omega \in \Omega', \kappa \in \boldsymbol{C}\}$ such that*

i) *$y_{\mathrm{VI}}(\omega, \kappa; x)$ is a holomorphic function of (ω, κ, x) in $\Delta(\Omega', r_6)$,*

ii) *$y_{\mathrm{VI}}(\omega, \kappa; x)$ is expressed as*

$$y_{\mathrm{VI}}(\omega, \kappa; x) = 4e^{-\kappa}x^{\omega}(1+s_6(\omega, x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega}))$$

in $\Delta(\Omega', r_6)$, where $s_6(\omega, x, \xi, \eta)$ is some element of $\mathfrak{S}(\Omega', r_6)$.

Applying Theorem 2.2 to equations (5.1) and (5.2) and using Proposition 5.2, we have

THEOREM 5.4. *Let ε $(<1/2)$ be an arbitrary small positive constant. Then, for sufficiently small positive constants $r_5 = r_5(\varepsilon)$ and $r_3 = r_3(\varepsilon)$, equations* (V) *and* (III) *admit families of solutions $\{y_{\mathrm{V}}(\omega, \kappa; x); \omega \in \Omega(\varepsilon), \kappa \in \boldsymbol{C}\}$ and $\{y_{\mathrm{III}}(\omega, \kappa; x); \omega \in \Omega(\varepsilon), \kappa \in \boldsymbol{C}\}$ respectively such that*

i) *$y_{\mathrm{V}}(\omega, \kappa; x)$ is a holomorphic function of (ω, κ, x) in $\Delta(\Omega(\varepsilon), r_5)$ and $y_{\mathrm{III}}(\omega, \kappa; x)$ is a holomorphic function of (ω, κ, x) in*

$$\Delta'(\Omega(\varepsilon), r_3) = \{(\omega, \kappa, x); (\omega, \kappa, x^2) \in \Delta(\Omega(\varepsilon), r_3)\},$$

ii) *$y_{\mathrm{V}}(\omega, \kappa; x)$ and $y_{\mathrm{III}}(\omega, \kappa; x)$ are expressed as*

$$y_{\mathrm{V}}(\omega, \kappa; x) = 1 - 4e^{-\kappa}x^{\omega}(1+s_5(\omega, x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega})),$$

$$y_{\mathrm{III}}(\omega, \kappa; x) = e^{-\kappa}x^{2\omega-1}(1+s_3(\omega, x^2, e^{-\kappa}x^{2\omega}, e^{\kappa}x^{2(1-\omega)}))$$

in $\Delta(\Omega(\varepsilon), r_5)$ and in $\Delta'(\Omega(\varepsilon), r_3)$ respectively, where $s_j(\omega, x, \xi, \eta)$'s $(j=5, 3)$ are *some elements of* $\mathfrak{S}(\Omega(\varepsilon), r_j)$.

REMARK 5.2. The domains of convergence in these theorems are larger than those in [6; Theorem 3.1] and [7]. In fact, if $|e^{-\kappa}|$ is sufficiently small, then equation (VI) admits a solution

$$y_{\mathrm{VI}}(\sqrt{-1}, \kappa; x) = 4e^{-\kappa}(\cos(\log x)+\sqrt{-1}\sin(\log x))(1+O(|x|+|e^{-\kappa}|))$$

as $x\to+0$ along $\boldsymbol{R}^+=\{x\in\boldsymbol{R};\ x>0\}$.

5.2. Under the condition $\alpha=\gamma=0$, by the transformation $u/2=w+(1/2)\log x$, equation (5.3) is changed into an equation of the form (E) (with the unknown variable w) in which

$$F_2(x, \xi, \eta) = \frac{\xi^2-\eta^2}{4-(\xi+\eta)^2},$$

$$F_1(x, \xi, \eta) = -\Big(\frac{x}{x-1}+\frac{(\xi+\eta)^2}{4-(\xi+\eta)^2}\Big)+\frac{1}{2}F_2(x, \xi, \eta),$$

$$F_0(x, \xi, \eta) = \frac{(\eta^2-\xi^2)}{32(x-1)^2}\Big(\beta(4-(\xi+\eta)^2)+\frac{16\delta(x-1)}{4-(\xi+\eta)^2}\Big)+\frac{1}{2}F_1(x, \xi, \eta).$$

Under the condition $\alpha=\beta=0$, by the same transformation, equation (5.2) is changed into an equation of the form (E) in which

$$F_0(x, \xi, \eta) = \frac{(\eta^2-\xi^2)}{16}(2\gamma-\delta(\eta^2+\xi^2)),$$

$$F_1(x, \xi, \eta) = F_2(x, \xi, \eta) \equiv 0.$$

Applying Theorems 2.1 and 2.2 to these equations, we have

THEOREM 5.5. *Assume that* $\alpha=\gamma=0$ *in* (VI). *Let* Ω' *be an arbitrary bounded domain of which the closure is included in* Ω_0. *Then, for a sufficiently small positive constant* $r_6'=r_6'(\Omega')$, *equation* (VI) *admits a family of solutions* $\{Y_{\mathrm{VI}}(\omega, \kappa; x);\ \omega\in\Omega', \kappa\in\boldsymbol{C}\}$ *such that*

i) $Y_{\mathrm{VI}}(\omega, \kappa; x)$ *is a holomorphic function of* (ω, κ, x) *in* $\Delta(\Omega', r_6')$,

ii) $Y_{\mathrm{VI}}(\omega, \kappa; x)$ *is expressed as*

$$Y_{\mathrm{VI}}(\omega, \kappa; x) = \mathrm{ch}^{-2}\Big(\Big(\frac{1}{2}-\omega\Big)\log x+\kappa+\sigma_6(\omega, x, e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega})\Big)$$

in $\Delta(\Omega', r_6')$, *where* $\sigma_6(\omega, x, \xi, \eta)$ *is some element of* $\mathfrak{S}(\Omega', r_6')$.

THEOREM 5.6. *Assume that* $\alpha=\beta=0$ *in* (V). *Let* ε $(<1/2)$ *be an arbitrary positive constant. Then, for a sufficiently small positive constant* $r_5'=r_5'(\varepsilon)$, *equation* (V) *admits a family of solutions* $\{Y_{\mathrm{V}}(\omega, \kappa; x);\ \omega\in\Omega(\varepsilon), \kappa\in\boldsymbol{C}\}$ *such that*

i) $Y_{\mathrm{V}}(\omega, \kappa; x)$ *is a holomorphic function of* (ω, κ, x) *in* $\Delta(\Omega(\varepsilon), r_5')$,

ii) $Y_{\mathrm{V}}(\omega, \kappa; x)$ *is expressed as*

$$Y_{\mathrm{V}}(\omega, \kappa; x) = \mathrm{th}^2\Big(\Big(\frac{1}{2}-\omega\Big)\log x+\kappa+\sigma_5(\omega, x; e^{-\kappa}x^{\omega}, e^{\kappa}x^{1-\omega})\Big)$$

in $\Delta(\Omega(\varepsilon), r_5')$, *where* $\sigma_5(\omega, x, \xi, \eta)$ *is some element of* $\mathfrak{S}(\Omega(\varepsilon), r_5')$.

If $\mathrm{Im}\,\omega\neq 0$, the expressions given above are valid in the domains in which the solutions have infinitely many movable poles and zeroes. In fact, if we put $\omega=(1/2)-\lambda\sqrt{-1}$, $\kappa=\mu\sqrt{-1}$ ($\lambda\in\boldsymbol{R}-\{0\}$, $\mu\in\boldsymbol{R}$), then

$$Y_{\mathrm{VI}}(\omega, \kappa; x) = \cos^{-2}(\lambda\log x+\mu+O(x^{1/2}))\,, \tag{5.5}$$

$$Y_{\mathrm{V}}(\omega, \kappa; x) = -\tan^2(\lambda\log x+\mu+O(x^{1/2})) \tag{5.6}$$

as $x\to+0$ through the sector $|\arg x|<\varepsilon$, where ε is an arbitrary small positive constant. Y_{VI} has infinitely many poles and Y_{V} has infinitely many poles and zeroes in this sector. These expressions cannot be obtained from Theorems 5.3 and 5.4. For equation (V) with $\alpha=\beta=\gamma=0$, $\delta=-2$, asymptotic representation (5.6) is obtained by R. Garnier [**2**; 23].

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